Limiting calculation. Consider

$$r_{cog} = \sqrt{GM_E M/F}$$

$$= \frac{1}{[\ (\mu_1/r'_1{}^2)^2 + (\mu_2/r'_2{}^2)^2 + 2\,(\mu_1/r'_1{}^3)\,(\mu_2/r'_2{}^3)(b^2 + b(r_1-r_2)\cos\theta - r_1 r_2)\ ]^{1/4}} \qquad \text{(D.2.21)}$$

$$r'_1{}^2 = r_1{}^2 + b^2 + 2r_1 b\cos\theta$$
$$r'_2{}^2 = r_2{}^2 + b^2 - 2r_2 b\cos\theta \qquad \textit{(D.2.4)}$$

This is in notation where r_1 and r_2 are the *short* ones.

Suppose the two masses are equal AND things are aligned with m_1 on the top. Then we know $r_1 = r_2$ and we know $r'_1 = b + r_1$ and $r'_2 = b - r_1$. Then $\mu_1 = \mu_2 = 1/2$ and the above becomes

$$r_{cog} = \frac{1}{[\ \frac{1}{4}\frac{1}{r'_1{}^4} + \frac{1}{4}\frac{1}{r'_2{}^4} + \frac{2}{4}\frac{1}{r'_1{}^3 r'_2{}^3}(b^2 - r_1 r_2)\ ]^{1/4}}$$

$$= \frac{r'_1 r'_2}{[\ \frac{1}{4}r'_2{}^4 + \frac{1}{4}r'_1{}^4 + \frac{2}{4}\,r'_1 r'_2(b^2 - r_1{}^2)\ ]^{1/4}}$$

$$= \frac{r'_1 r'_2}{[\ \frac{1}{4}r'_2{}^4 + \frac{1}{4}r'_1{}^4 + \frac{2}{4}\,r'_1 r'_2 r'_1 r'_2\ ]^{1/4}}$$

$$= \frac{r'_1 r'_2}{[\ \frac{1}{4}r'_2{}^4 + \frac{1}{4}r'_1{}^4 + \frac{2}{4}\,r'_1{}^2 r'_2{}^2\ ]^{1/4}}$$

$$= (1/\sqrt{2})\ \frac{r'_1 r'_2}{[\ r'_2{}^4 + r'_1{}^4 + 2\,r'_1{}^2 r'_2{}^2\ ]^{1/4}}$$

$$= (1/\sqrt{2})\ \frac{r'_1 r'_2}{[\ (r'_1{}^2 + r'_2{}^2)^2\ ]^{1/4}}$$

$$= (1/\sqrt{2})\ \frac{r'_1 r'_2}{[\ (r'_1{}^2 + r'_2{}^2)\ ]^{1/2}}$$

Now

$$r'_1{}^2 + r'_2{}^2 = (b+r_1)^2 + (b-r_1)^2 = 2b^2 + 2r_1{}^2$$

$$r'_1 r'_2 = (b+r_1)\,(b-r_1) = b^2 - r_1{}^2$$

so then

$$r_{cog} = (1/\sqrt{2}) \frac{b^2-r_1^2}{[\ (2b^2+2r_1^2)\ ]^{1/2}} = (1/2) \frac{b^2-r_1^2}{[\ (b^2+r_1^2)\ ]^{1/2}}$$

$$= (1/2) \frac{b^2(1-(r_1/b)^2)}{[\ b(1+(r_1/b)^2)\ ]^{1/2}}$$

$$= (1/2)\, b \frac{(1-(r_1/b)^2)}{[\ (1+(r_1/b)^2)\ ]^{1/2}}$$

Agrees with (D.4.5).

********** this from the main text tether section. Added eq in App D.

Digresson on Center of Gravity

The phrase "center of gravity" is often used as a synomym for "center of mass" which complicates searching for information about the former concept. Consider a system of masses m_i each of which experiences some force $\mathbf{F}_i$. We define $\mathbf{F}_T = \Sigma_i \mathbf{F}_i$, $\mathbf{N}_T^{(R)} = \Sigma_i\, (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i$ and $\mathbf{N}_T^{(0)} = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i$, the latter being total system torques with respect to point **R** and point **0**. The **center of gravity** is a point **R** such that the total torque on a system measured *with respect to* point **R** vanishes, so $\mathbf{N}_T^{(R)} = 0$. This is exactly the same as saying $\mathbf{N}_T^{(0)} = \mathbf{R} \times \mathbf{F}_T$. Dotting with $\mathbf{F}_T$ shows that one must have $\mathbf{N}_T^{(0)} \bullet \mathbf{F}_T = 0$ as a *condition* for **R** to even exist! This notion of "center of gravity" really has nothing to do with gravity, but can be applied to a situation involving gravity.

For a weird-shaped asteroid near the Earth one has $\mathbf{N}_T^{(0)} = 0$ (origin at Earth center) since $\mathbf{r}_i \times \mathbf{F}_i = 0$ for each m_i of the asteroid. Thus the condition is met and **R** does exist. Here is a method for computing **R**, a variation of Symon p 258. One first computes $\mathbf{F}_T = \Sigma_i \mathbf{F}_i$ (summed over all points in the asteroid) and one then knows the direction of $\mathbf{F}_T$. One draws a line along $\mathbf{F}_T$ and then translates it parallel to $\mathbf{F}_T$ so the line contains the center of the Earth. The center of gravity lies on that translated line a distance R from the center of the Earth such that $GM_E M_{asteroid}/R^2 = F_T$. One then has R and $\hat{\mathbf{R}}$ so **R** is determined. This **R** solves $\mathbf{N}_T^{(0)} = \mathbf{R} \times \mathbf{F}_T$ because the left side is 0 and **R** is parallel to $\mathbf{F}_T$. The solution **R** is not unique, and in this example **R** = 0 also works, but is not very interesting.

For such an asteroid, **R** always lies closer to the Earth than the center-of-mass point, and that is the case for our tethered satellite shown in Fig 8.7. Unlike the center of mass, the center of gravity changes position within the asteroid as its orientation changes.

Reader Exercise: Show that if the masses are radially aligned as in Fig 8.7, one has

$$R_{cms} = \frac{r_1 m_1 + r_2 m_2}{m_1 + m_2} \qquad \text{and} \qquad R_{cog} = \sqrt{\frac{m_1+m_2}{(m_1/r_1^2)+(m_2/r_2^2)}}\ . \qquad \text{(none)}$$

Show that $R_{cog} < R_{cms}$ (this takes a bit of work). These results are not needed below.