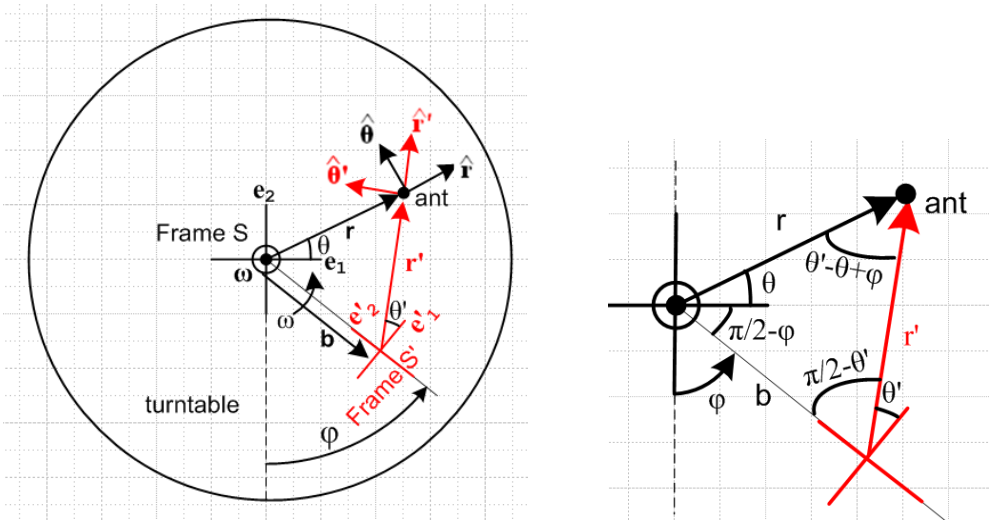
This is installed, do not edit here

Kinematics common to all Ant Problems

Consider a turntable occupied by an ant as shown in this drawing. Here Frame S is a fixed frame with origin at the turntable center, while Frame S' (glued to the turntable surface) is a rotating frame.



(15.1)

In Frame S, the vector **r** has coordinates (r,θ) in standard polar coordinates.
In Frame S', the vector **r'** has coordinates (r',θ') in standard polar coordinates.

When φ = 0, red Frame S' lies directly *under* black Frame S and the axes line up. For any angle φ one has **b** = -b **e'**$_2$. Since the rotation axis goes through the origin of Frame S, the turntable problems fall into Special Case #1 of Section 4.4. Basis vectors **e**$_3$ = **e'**$_3$ (not labeled) point to the viewer as does the **ω** vector for ω>0.

The relation between the three angles θ, θ' and φ is complicated and can be indirectly obtained by writing the laws of sines and cosines for the triangle shown on the right above. The left and bottom internal triangle angles are obvious. The top one is then

$$\pi - (\theta + \pi/2 - \varphi) - (\pi/2 - \theta') = \theta'-\theta+\varphi \qquad (15.2)$$

None of this geometric detail will be needed below (except in a Reader Exercise).

In the first two Problems considered below, an ant executes some crawling motion on the turntable as described by certain **r'**, **v'**, and **a'** in Frame S'. Our task in each problem is to use our Section 12 summary results to compute **r**, **v**, and **a** as seen in Frame S and to plot some trajectories **r**(t).

In the third problem, the ant becomes a flying ant doing a straight-line fly-by at constant velocity in Frame S just over the turntable surface, which fly-by is described by some **r**, **v**, and **a**. This is an example of the Inverse Problem discussed in Section 13 and our goal here is to compute **r'**, **v'**, and **a'** in Frame S' using the equations provided in Section 13.3.

In each frame we define Cartesian and cylindrical coordinates and unit vectors as follows:

Frame S	$r_i = x,y,z$	basis vectors	$\mathbf{e}_i = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$	
	$\xi_i = r,\theta,z$	basis vectors	$\hat{\boldsymbol{e}}_i = \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$.	
Frame S'	$r'_i = x',y',z'$	basis vectors	$\mathbf{e}'_i = \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$	
	$\xi'_i = r',\theta',z'$	basis vectors	$\hat{\boldsymbol{e}}'_i = \hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\mathbf{z}}'$	(15.3)

We are thus providing a specific example of (14.1) concerning general curvilinear coordinates in two frames of reference.

What do we know about all the basis vectors?

In order to illustrate some of the work of Section 1.1, we provide the reader with a complete set of socket wrenches even though only a few of these tools will actually be used below. All of the following relations can be obtained by inspection from the above figure:

Within Frame S we have

$\hat{\boldsymbol{e}}_i = R_z(\theta)\mathbf{e}_i$. for example $\hat{\mathbf{r}} = R_z(\theta)\,\hat{\mathbf{x}}$ (a)

A corresponding equation applies in Frame S' ,

$\hat{\boldsymbol{e}}'_i = R_z(\theta')\mathbf{e}'_i$. for example $\hat{\mathbf{r}}' = R_z(\theta')\,\hat{\mathbf{x}}'$ (b)

The relation between the Frame S and Frame S' Cartesian unit vectors is

$\mathbf{e}'_i = R_z(\varphi)\mathbf{e}_i$. for example $\hat{\mathbf{x}}' = R_z(\varphi)\,\hat{\mathbf{x}}$ (c)

The relation between the Frame S and Frame S' Cylindrical unit vectors is

$\hat{\boldsymbol{e}}'_i = R_z(\varphi)\hat{\boldsymbol{e}}_i$. for example $\hat{\mathbf{r}}' = R_z(\varphi)\,\hat{\mathbf{r}}$ (d)

Relations (d) and (a) can be combined to get

$\hat{\boldsymbol{e}}'_i = R_z(\varphi+\theta)\mathbf{e}_i$. for example $\hat{\mathbf{r}}' = R_z(\varphi+\theta)\,\hat{\mathbf{x}}$ (e) (15.4)

Writing the basis vector relations in matrix notation

Recall now that it was shown in (1.1.1) and (1.1.2) that (we use dummy names $\mathbf{a}_n$ and $\mathbf{a}'_n$) ,

$$\mathbf{a}_n = R\,\mathbf{a}'_n \quad \Leftrightarrow \quad \mathbf{a}_n = \Sigma_m (R^{-1})_{nm}\,\mathbf{a}'_m \tag{15.5}$$

On the left, we rotate vector $\mathbf{a'_n}$ to get vector $\mathbf{a_n}$.
On the right, we express $\mathbf{a_n}$ as a linear combination of the vectors $\mathbf{a'_n}$.
Remember that the subscripts on the **a** and **a'** are labels, not components!

Suppose we take the k^{th} component of the equation on the right of (15.5),

$$[\mathbf{a_n}]_k = \Sigma_m (R^{-1})_{nm} [\mathbf{a'_m}]_k \quad . \tag{15.6a}$$

One can write this as

$$A_{nk} = \Sigma_m (R^{-1})_{nm} \; A'_{mk} \qquad \text{where} \quad A_{nk} = [\mathbf{a_n}]_k \;\text{ and }\; A'_{nk} = [\mathbf{a'_n}]_k \; . \tag{15.6b}$$

For a matrix A_{nk} one knows that n is the row index and k is the column index. Therefore, saying $A_{nk} = [\mathbf{a_n}]_k$ is the same as saying that the vector $\mathbf{a_n}$ is the n^{th} row of matrix A. Thus we can write the above matrix equation in this manner

$$\begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \mathbf{a_3} \end{pmatrix} = R^{-1} \begin{pmatrix} \mathbf{a'_1} \\ \mathbf{a'_2} \\ \mathbf{a'_3} \end{pmatrix} \qquad \Leftrightarrow \qquad \mathbf{a_n} = R\, \mathbf{a'_n} \quad n = 1,2,3 \quad . \tag{15.7}$$

All our rotations of interest in (15.4) are z-rotations which from (E.1) have the form

$$R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.1}_z$$

Example 1: Apply (15.7) to (15.4e) which says $\hat{\boldsymbol{e}}'_i = R_z(\varphi+\theta)\mathbf{e_i}$:

$$\begin{pmatrix} \hat{\boldsymbol{e}}'_1 \\ \hat{\boldsymbol{e}}'_2 \\ \hat{\boldsymbol{e}}'_3 \end{pmatrix} = R_z(-\theta' - \varphi) \begin{pmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix} = \begin{pmatrix} \cos(\theta'+\varphi) & \sin(\theta'+\varphi) & 0 \\ -\sin(\theta'+\varphi) & \cos(\theta'+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{\mathbf{r}}' \\ \hat{\boldsymbol{\theta}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = \begin{pmatrix} \cos(\theta'+\varphi) & \sin(\theta'+\varphi) & 0 \\ -\sin(\theta'+\varphi) & \cos(\theta'+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \quad .$$

Writing out the linear combinations, one gets

$$\begin{aligned} \hat{\mathbf{r}}' &= \cos(\theta'+\varphi)\, \hat{\mathbf{x}} + \sin(\theta'+\varphi)\, \hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}}' &= -\sin(\theta'+\varphi)\, \hat{\mathbf{x}} + \cos(\theta'+\varphi)\, \hat{\mathbf{y}} \\ \hat{\mathbf{z}}' &= \hat{\mathbf{z}} \quad . \end{aligned} \tag{15.8}$$

Example 2: Apply (15.8) to (15.4c) which says $\mathbf{e}'_i = R_z(\varphi)\mathbf{e_i}$:

$$\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \\ \mathbf{e'}_3 \end{pmatrix} = R_z(-\varphi)\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

Writing out the linear combinations, one gets

$$\begin{aligned} \hat{\mathbf{x}}' &= \cos\varphi\, \hat{\mathbf{x}} + \sin\varphi\, \hat{\mathbf{y}} \\ \hat{\mathbf{y}}' &= -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \qquad // \ \mathbf{e'}_2 = -\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2 \\ \hat{\mathbf{z}}' &= \hat{\mathbf{z}} \ . \end{aligned} \tag{15.9}$$

Relation between Frame S and Frame S'

Assume at time t = 0 we have $\varphi = \varphi_0$ in Fig (15.1).

If the rotation follows some angular velocity profile $\omega = \omega(t)$, and since $\omega = d\varphi/dt$, we have

$$d\varphi/dt = \omega(t) \qquad \Rightarrow \qquad \varphi(t) = \varphi_0 + \int_0^t \omega(\tau)d\tau \ . \tag{15.10}$$

For simplicity, we shall assume constant ω in which case we have

$$\varphi(t) = \varphi_0 + \omega t \ . \tag{15.11}$$

Motion of vector **b**

From the picture and from (15.9) we have,

$$\mathbf{b}(t) = -b\, \mathbf{e'}_2 = -b\,[-\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}}] = b\sin\varphi\, \hat{\mathbf{x}} - b\cos\varphi\, \hat{\mathbf{y}} \ . \tag{15.12}$$