This was installed on 3.1.17 without archiving the old.

Within Frame S we have (each of these equations has the (1.1.30) template form $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$) :

$\hat{\boldsymbol{e}}_i = R_z(\theta)\mathbf{e}_i$. for example $\hat{\mathbf{r}} = R_z(\theta)\,\hat{\mathbf{x}}$ (a)

A corresponding equation applies in Frame S' ,

$\hat{\boldsymbol{e}}'_i = R_z(\theta')\mathbf{e}'_i$. for example $\hat{\mathbf{r}}' = R_z(\theta')\,\hat{\mathbf{x}}'$ (b)

The relation between the Frame S and Frame S' Cartesian unit vectors is

$\mathbf{e}'_i = R_z(\varphi)\mathbf{e}_i$. for example $\hat{\mathbf{x}}' = R_z(\varphi)\,\hat{\mathbf{x}}$ (c)

The relation between the Frame S and Frame S' cylindrical unit vectors is

$\hat{\boldsymbol{e}}'_i = R_z(\varphi)\hat{\boldsymbol{e}}_i$. for example $\hat{\mathbf{r}}' = R_z(\varphi)\,\hat{\mathbf{r}}$ (d)

Relations (d) and (a) can be combined to get

$\hat{\boldsymbol{e}}'_i = R_z(\varphi+\theta)\mathbf{e}_i$. for example $\hat{\mathbf{r}}' = R_z(\varphi+\theta)\,\hat{\mathbf{x}}$ (e) (15.4)

Writing the basis vector relations in matrix notation

Recall the Basis Theorem of (1.1.30) : (we use dummy basis vector names $\mathbf{a}_n$ and $\mathbf{a}'_n$) ,

$\mathbf{a}'_n = R^{-1}\,\mathbf{a}_n \quad \Leftrightarrow \quad \mathbf{a}_n = \Sigma_m (R^{-1})_{nm}\,\mathbf{a}'_m$ (15.5)

On the left, we rotate vector $\mathbf{a}_n$ by R^{-1} to get vector $\mathbf{a}'_n$.
On the right, we express $\mathbf{a}_n$ as a linear combination of the basis vectors $\mathbf{a}'_n$.
Remember that the subscripts on the **a** and **a'** are labels, not components!

Suppose we take the kth component of the equation on the right of (15.5),

$[\mathbf{a}_n]_k = \Sigma_m (R^{-1})_{nm}\,[\mathbf{a}'_m]_k$. (15.6a)

One can write this as

$A_{nk} = \Sigma_m (R^{-1})_{nm}\; A'_{mk}$ where $A_{nk} = [\mathbf{a}_n]_k$ and $A'_{nk} = [\mathbf{a}'_n]_k$. (15.6b)

For a matrix A_{nk} one knows that n is the row index and k is the column index. Therefore, saying $A_{nk} = [\mathbf{a}_n]_k$ is the same as saying that the vector $\mathbf{a}_n$ is the nth row of matrix A. Thus we can write (15.6a) in this manner,

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = R^{-1} \begin{pmatrix} \mathbf{a'}_1 \\ \mathbf{a'}_2 \\ \mathbf{a'}_3 \end{pmatrix} \qquad \Leftrightarrow \qquad \mathbf{a}_n = R^{-1}\,\mathbf{a'}_n \quad n = 1,2,3 \ . \tag{15.7a}$$

Inverting,

$$\begin{pmatrix} \mathbf{a'}_1 \\ \mathbf{a'}_2 \\ \mathbf{a'}_3 \end{pmatrix} = R \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \ . \tag{15.7b}$$

We have thus provided an interpretation for the "alternative notation" shown in (1.1.32) : the three vectors in a column can be regards as rows of a matrix.

All our rotations of interest in (15.4) are z-rotations which, from (A.1), have the form

$$\mathrm{R}_{\mathbf{z}}(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.1}_{\mathbf{z}}$$

Example 1: Apply (15.7b) to (15.4e) which says $\hat{\boldsymbol{e}}'_{\mathbf{i}} = \mathrm{R}_{\mathbf{z}}(\varphi+\theta)\mathbf{e}_{\mathbf{i}} = R^{-1}\mathbf{e}_{\mathbf{i}}$ so $R = \mathrm{R}_{\mathbf{z}}(-\theta-\varphi)$:

$$\begin{pmatrix} \hat{\boldsymbol{e}}'_1 \\ \hat{\boldsymbol{e}}'_2 \\ \hat{\boldsymbol{e}}'_3 \end{pmatrix} = \mathrm{R}_{\mathbf{z}}(-\theta-\varphi) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta+\varphi) & \sin(\theta+\varphi) & 0 \\ -\sin(\theta+\varphi) & \cos(\theta+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad \text{// see (A.1)}$$

or

$$\begin{pmatrix} \hat{\mathbf{r}}' \\ \hat{\boldsymbol{\theta}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\varphi) & \sin(\theta+\varphi) & 0 \\ -\sin(\theta+\varphi) & \cos(\theta+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \ .$$

Writing out the linear combinations, one gets

$$\begin{aligned} \hat{\mathbf{r}}' &= \cos(\theta+\varphi)\,\hat{\mathbf{x}} + \sin(\theta+\varphi)\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}}' &= -\sin(\theta+\varphi)\,\hat{\mathbf{x}} + \cos(\theta+\varphi)\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}}' &= \hat{\mathbf{z}} \ . \end{aligned} \tag{15.8}$$

Example 2: Apply (15.7b) to (15.4c) which says $\mathbf{e}'_{\mathbf{i}} = \mathrm{R}_{\mathbf{z}}(\varphi)\mathbf{e}_{\mathbf{i}} = R^{-1}\mathbf{e}_{\mathbf{i}}$ so $R = \mathrm{R}_{\mathbf{z}}(-\varphi)$:

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = \mathrm{R}_{\mathbf{z}}(-\varphi) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \ .$$

Writing out the linear combinations, one gets

$$\begin{aligned}
\hat{\mathbf{x}}' &= \cos\varphi\, \hat{\mathbf{x}} + \sin\varphi\, \hat{\mathbf{y}} \\
\hat{\mathbf{y}}' &= -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \qquad // \ \mathbf{e}'_2 = -\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2 \\
\hat{\mathbf{z}}' &= \hat{\mathbf{z}} \ .
\end{aligned} \tag{15.9}$$

We could reduce our 3x3 matrix work to 2x2 for the turntable examples, but other problems require the full 3x3 notation so we maintain it throughout.