A geometric derivation of the theoretical limit $T_2 \le 2T_1$.

As magnetization decays, we know that $M_z = 1 - \exp(-t/T_1)$ and $M_\rho = \exp(-t/T_2)$ and we require that $M_z^2 + M_\rho^2 \le 1$ during the decay, which says the M vector cannot go outside the unit sphere. So define f(t):

$$f(t) = M_z^2 + M_\rho^2 - 1 = [\,1 - \exp(-t/T_1)]^{\,2} + [\exp(-t/T_2)]^2 - 1$$

$$= \exp(-2t/T_1) + \exp(-2t/T_2) - 2\exp(-t/T_1).$$

We know that $f(0) = f(\infty) = 0$ and we are wondering if f(t) ever goes positive in between, meaning M goes outside the unit sphere (puncturing the surface). Change to a different variable:

$$x = \exp(-t/T_1) \quad \Rightarrow \quad f(t) = (x^2 - 2x) + x^{2a} \equiv F(x) \text{ where } a = T_1/T_2$$

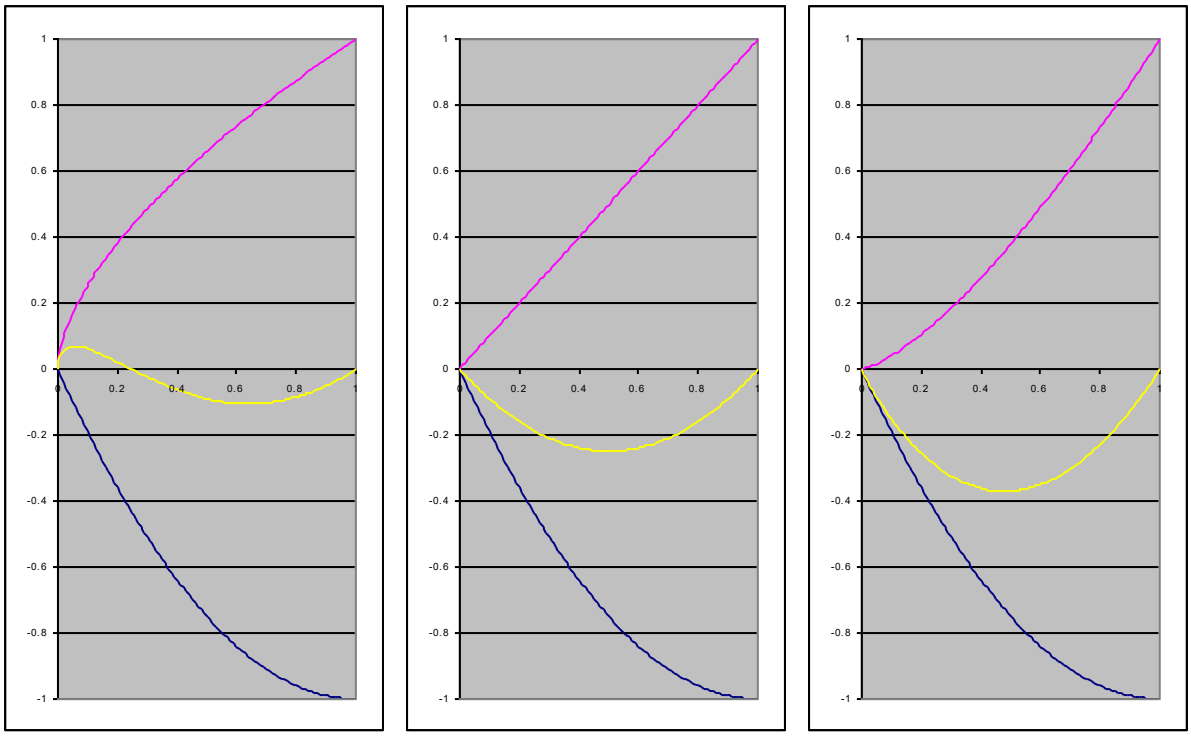
This is the sum of a negative parabola and a positive power curve in the range [0,1]. This curve goes positive near the x=0 end if the slope is positive there:

$$F'(x) = 2x - 2 + 2ax^{2a-1} \qquad \Rightarrow \qquad F'(0) \approx -2 + 2ax^{2a-1}$$

If $a \ge 1/2$, $\lim_{x\to 0} x^{2a-1} = 0$ so slope is -2. If $a < 1/2$, $\lim_{x\to 0} x^{2a-1} = +\infty$. so slope is positive. The conclusion is that we must have $a \ge 1/2$, otherwise F(x) = f(t) goes positive and our magnetization vector punctures the unit sphere. But $a \ge 1/2$ means $(T_1/T_2) \ge 1/2$ or $T_2 \le 2T_1$ which is the "theoretical limit". This of course is within the context of our simple model of exponential relaxation decay.

Here are some plots from Excel where the negative parabola is blue, the positive power curve purple, and the sum curve F(x) is yellow. The a=.3 case shows unit sphere puncturing at the left end of the interval, which is x near 0 or t = large, meaning the end of the relaxation period. The unit sphere is closing down, and the magnetization vector pushes through it near the end of its spiral run.

a = .3 a = .5 a = .7


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1