Optics Notes from Carlson, Jackson, Goodman and Portis PhL 11.28.02

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10. Goodman Chapter 4: the Fresnel and Fraunhofer Approximations. He makes the connection between Fraunhofer and 2D Fourier Transform, writes the more general Fresnel formula, does some basic Fraun examples, then a detailed close-field Fresnel example. Fresnel Number and Talbot Images.

11. Waves Book: Comments. Babinet and diffraction from opaque non-metal objects.

12. Carlson Chapter 2. Similar to Goodman, Fresnel and Fraunhofer stuff, beam arrays.

13. Portis Chapter 14 on Scattering Applications. Thomson, Rayleigh = dielectric dipole, etc.

14. Cornu Spiral. Way to see graphically integrals of $F(t) = C(t) + i\, S(t)$.

15. Goodman Chapter 5: Wave Optics. Long set of notes. You follow the quadratic phases of wavefronts through a lens system and see what happens. Focal plane has Fourier Transform of image, no matter where you put your image transparency. Instant computing!

1. Maxwell's Equations, wave equation, plane waves, dispersion relation, index, velocity ψ

If you start with Maxwell's equations with the four fields and if you assume that μ and ε don't vary in space (are uniform), you can reduce the field count to two, such as E and H. If you further assume no charges or currents in some region, you get the four homogeneous field equations. Next, if you assume the usual exp(iωt) time dependence, you get the time-independent set of equations where time derivatives are replaced by -iω By applying the usual vector identities, you can then show that all field components must satisfy the **Helmholtz wave equation** in which k^2 is equal to ω^2 μ ε. ["vector wave equation"]

As a separate thread, you can just think about a field component of the form A exp (i[kz-ωt]) where A is a constant amplitude. This form describes a "**plane wave**", that is, a wave with a completely flat wave front in the z plane, which wave moves to the right (+z) at a certain **phase velocity**. In time dt it must move dz to keep a constant phase, and you get **v = dz/dt = ω/k.** Symbol ω is the radians per second in time, and k is the radians per meter in space (the **wave number**).

Now it happens that a field component of this plane wave form solves the wave equation provided k and ω are related as noted above, so the wave equation imposes a **dispersion relation** on ω and k, relating them to each other. In fact what is really happening is that the wave equation is imposing a phase velocity on the plane wave which is v = ω/k = 1/sqrt(με). For free space, v = c, so Helmholtz tells you that there are plane wave solutions but they must move at the speed of light. For glass or the like, this speed is modified by the index n, so you have v = c/n.

Are there other interesting solutions to the wave equation besides plane waves? We know in spherical coordinates we can have spherical plane waves, for example, exp(i[kr-ωt]). In this case, the wave fronts are curved (spherical surfaces), yet all rays still travel in a straight line. This last fact is I think a characteristic of having constant μ and ε. How do I prove this? I do not have a proof at this point!

About exp(ikr)/r . Note that exp(ikr)/r is really the time-independent Green's function Jackson p541, a solution to the *scalar* wave equation driven by a delta function source at the origin. But **E** and **B** are *vector* quantities. If you make a point dipole radiating source, you will get **E** and **B** having this exp(ikr)/r as a *factor*, but there will be other factors as well. Here in fact is the B field from a point dipole source **p**,

$$\mathbf{A(r)} = -ik\frac{e^{ikr}}{r}\mathbf{p} \qquad \mathbf{B} = k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{p} \qquad |\mathbf{B}| = k^2 p\frac{e^{ikr}}{r}\sin\theta$$

and this satisfies Maxwell's equations as written only for large r due to an approximation in doing the curl to get B from A (a dropped term we could have shown above) . If you restrict interest to rays near θ = 90 degrees, you might approximate as $\mathbf{B} = k^2\frac{e^{ikr}}{r}\hat{\mathbf{y}}$ if $\mathbf{p} = p\,\hat{\mathbf{x}}$. This might be useful to represent a point source off to the left, say, in a diffraction problem. The sinθ variation is going to be very slow compared to any interference/diffraction phase variation, so sinθ can in this sense be neglected in the general Fraunhofer forward direction regime. The smaller r becomes, the more this solution for B violates Maxwell's equations. It is only meant for distant r use.

2. The Eikonal Equation and the Ray Equation: mirage example

Now, we have to make several assumptions to get to the eikonal approximation. The first one is that the quantities μ and ε vary only slowly relative to wavelength. This vague assumption then allows you to

ignore certain spatial derivatives of μ and ε, and, in so ignoring, you still get the wave equation. The idea is that with slightly varying μ or ε, you know the wave equation is not exact, but you accept it as an approximate equation describing the field components. The second assumption is that, in a amplitude/phase representation of the field component, the amplitude varies slowly in space again relative to wavelength, while the phase is still allowed to have a fast variation. This phase is called **"the eikonal"**, usually labeled S. With this second assumption, you drop various terms from the amplitude/phase wave equations, and you end up with "**the eikonal equation**" which simply says this: $|\nabla S| = n$. [Detail: most sources write the phase factor as $\exp(ik_0 S)$ where $k_0 = \omega/c$ = the wave number in free space. This causes S to have the dimensions of distance, so that ∇S is then dimensionless, so that $|\nabla S| = n$ makes sense.] What this equation really says is that any solution of the wave equation with our listed approximations behaves *locally* as a plane wave traveling in the direction ∇S with speed c/n. In other words, locally we have $S = (\mathbf{k}/k_0) \bullet r$, so $\nabla S = (\mathbf{k}/k_0)$. We have to integrate out the local solution to see what happens globally.

So, at this point, you assume that your solution for phase S is described by a wavefront of some smooth but arbitrary shape that moves along, perhaps changing shape and/or orientation as it moves. If you trace a point on the wavefront in time, the path it leaves is called a **"ray"**. If the wavefront is in fact changing shape or orientation as it moves, the "ray" will likely NOT be going in a straight line. If you draw a differential picture of this situation, where ds describes distance along a short piece of "ray", you can use the eikonal equation just mentioned to derive what is called **"the ray equation"**. In this equation, dr/ds is really a unit vector u along the ray, as shown in the equation to the right below, which of course is just a restatement of the eikonal equation quoted above.

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \nabla n \quad \text{(ray eqn)} \qquad\qquad \nabla S = n\vec{u} = n\frac{d\vec{r}}{ds} \quad \text{(eikonal eqn)}$$

The vector r is "the ray" and you think of it as a function of position, r = r(x,y,z). This equation describes the way in which the ray r changes direction in a local sense (ie, it is a differential equation). If you integrate this equation, you can solve for r(x,y,z) which is "the ray" in a non-local sense. Of course you have to assume some initial direction for your ray. One of my PDF documents uses this equation to analyze how rays bend due to a **heated road surface**. In this example, air temperature drops going up, causing density to increase going up, causing the index to slightly increase as you go up. You end up with rays having a quadratic shape y = f(z) and they tend to bend up into the region of larger index n. Just looking at the above equation, in this example the right side has an up-directed gradient, so crudely the unit vector is going to get rotated up as you move along in s ~ z.

Now, if n is absolutely constant, the RHS of the ray equation is 0, and we conclude that as we move in some direction s ~ z along a ray, the ray has zero curvature. This means that the ray has to go in a straight line.

So the above ray equation is useful in tracing light or radio rays through situations where you have a very slowly varying index, such as the example sited above. I am sure this same idea is used to trace non-EM rays in many physical situations which are also described by a wave equation. Another EM example is that **radio waves** on line of site sometimes travel 500 miles instead of the usual 75 because they are **bent** back down by the reduction in index as you go up in the atmosphere.

Now, when we talk about conventional "ray optics" with test benches and lenses and so on, we don't really have any region where n varies slowly. Index n is a constant in all regions, and at the interface between lens and air, we cannot use the eikonal idea because n varies violently there. Instead, we have to

use Snell's law. So all we really get from the "ray equation" for conventional optics is that "rays travel in straight lines" away from the lens interfaces. [Well, we could use the eikonal equation to get the **saltus condition** at the surface which yields Snell's Law, as I think Jones showed.]

The **word eikonal** comes from eikon or icon, meaning image. In the ray approximation, the pieces of a wavefront maintain their coherent relative position more or less as the wavefront moves, and it you think of the wavefront as carrying an image in the amplitude function, then we are describing how this image propagates in space. I guess this is the origin of the name. This word eikonal appears in no dictionaries I know of, including the OED.

3. Diffraction from Jackson: the Theory (topics from Chapter 6 and Chapter 9)

By "diffraction" at this point I mean the following: something is "happening" in some small region to the left, and it radiates and causes a field pattern somewhere to the right. Perhaps a better title for this section would be "radiation from some localized area".

The usual way this is approached is as follows (see figure page 282 of green Jackson). You assume one of the **two geometries shown in Figure 9.5**. The left geometry is useful for flat sources such as apertures on open surface S1. The right geometry is useful for 3D sources surrounded by closed surface S1. In either case, your "receiving" point is somewhere in region II to the "right" of the sources. Also in both cases, contributions to integrals we are soon to discuss vanish on the infinite hemisphere (left), or sphere (right).

Green's Function (time-dependent). Let's start on page 183 of Jackson. He writes the time-dependent wave equation with a source f, defines the Green's function by replacing f with a delta function, transforms to k/ω space and easily finds the solution Green's function there, which has the conventional pole form of a k-space propagator (6.59). He then transforms back into spacetime to get the **spacetime Green's function (6.64).** Notice there is a 1/R factor, but no exp(ikR) factor here. The general solution for field component ψ is then given in (6.65) where the sources f appear inside the integral with the Green's function. Integrating over time gives (6.66) which sure looks simple.

Poisson solution. Jackson then notes at the start of section 6.7 that (6.66) is really the "particular integral" solution and that in general you have to add to this a solution of the no-sources wave equation if you are going to satisfy some boundary conditions. Remember now that Green's Theorem relates a volume integral of something to a closed surface integral around that volume of a related something, see (6.67) for a very general statement. He applies this to our case and ends up with messy equation (6.73) where D and F are the boundary conditions at t=0, and where he has placed the "observation point" at the origin, hence ψ(0,t) on the left side. He is just trying to show that in theory, you can compute your field component by doing a messy integral, and all you need to know is your boundary condition field values. (This was done in the electrostatics section quite a bit.) The general idea of Green is that you can find the value of a field at some point in a volume if you know the field along a bounding surface -- this is related to analyticity I think. Note that you can use this equation with or without sources f being present. Jackson calls this **(6.73) the Poisson Solution**, and we are then referred to other books for more details.

Time-dependent Kirchhoff. Next, Jackson changes the scenario a bit. Suppose there are no sources, and suppose the initial value of our field component (and derivative) are zero, perhaps at some long time in the past. Then we get the much simpler equation (6.74) which relates a field component to a surface integral of it and G. But we know G, so we can jam it in and do the time integral, with the **result**

(6.76). This is called the Kirchhoff Integral Representation, an exact result I think. This is not a "solution" of anything, it is just a way to replace our differential wave equation with an integral equation. The unknown quantity appears inside the integral. If you could somehow know the solution at all surface points on S, then you would also know via this equation the solution everywhere inside.

Time-independent Kirchhoff. We now suddenly **jump to page 281 in Jackson Chapter 9** where (9.62) restates this Kirchhoff integral form. Notice again that the 1/R is present, but not the exp(ikR). If you assume the usual expo time dependence exp(-iωt)and add back the time dt' integration with the δ(t' - [t - R/c]) to express the retarded time business, then you find exp(-iωt') factoring out from the integrand, and the time integral with the delta function then gives you exp(-iωt)* exp(iωR/c). Recall from the wave equation that ω/c = k, and boom, there is your famous exp(ikR) factor, so you now have factor exp(ikR)/R in the **time-independent** form of the **Kirchhoff-Huygens integral representation, and this is stated as (9.63).** By the usual arguments, we throw out contributions from the infinite spherical regions, and we end up with **(9.65)**, which is the same as (9.63) apart from a sign change due to changing the definition of the normal as now pointing "out" of the surface S1 into our "enclosed volume".

So after all this work, we have arrived at (9.65) which is the time-independent Kirchhoff integral formula which again tells you the field component anywhere if you know it everywhere on local surface S1.

Vector version of Kirchhoff Integral. [See expanded note on this subject in a separate document] In the next section 9.6 Jackson rewrites the scalar Kirchhoff equation as (9.67), where he puts back in the symbol G to represent the (time-independent) Green's function written out in (9.66). He has done nothing really. He then rewrites this for all 6 field components in vector form, such as (9.68). Now E is an integral of E, and B an integral of B. Again, nothing has really been done. He then messes around with vector identities and shows how certain integral terms vanish, uses Maxell's equations, and ends up finally with the two equations on top of page 285. Now E is an integral of E and B, and similarly for B. The result looks messy, but the main point is that he has eliminated any gradients of fields on the right. So he regards these two equations as **"the vector versions" of Kirchhoff.** In (9.77) he makes the further assumption that everywhere on the integration surface the E and B fields have their usual free-space directional relationship. Now the integral is only over the local surface S1.

Apertures in flat plates. For the special case of apertures in a flat conducting surface, Jackson is able to simplify the vector Kirchhoff equation to **the form (9.82)** where now E at any point to the right can be computed as an integral of just the tangential E field integrated over the apertures. [I see how you would apply this to an aperture in a conducting metal plate, which plate cannot support a tangential E field except in the aperture. If you are talking light and a hole in a piece of paper, I guess you can say that the tangential E field just to the right of the sheet of opaque paper is 0, and the same result applies.]

Babinet's Principle (section 9.7). Jackson first proves this only within the "**Kirchhoff approximation**". This "approximation" refers not to the Kirchhoff integral form which is exact, but rather to the assumption you make regarding the field in an aperture. You assume (and this is of course not exact) that the field in an aperture is exactly the same as it would be if the metal plate were not present! You imagine some illumination from the left here. Now, with this assumption, you can use 9.65 to prove Babinet as follows. Think of surface S as being the left surface S1 in Figure 9.5, page 282. Write this as S = Sa + Sb, where Sa and Sb are the complementary surfaces as shown page 288. With no metal plates present at all, surface S = S1 still exists of course (nothing is there, however), and you say that ψ is given by integration of the field over S. Now if Sa is present, you say that ψa is the integral over Sb (the aperture of situation Sa). And similarly for ψb. If you add the two equations you get ψa + ψb = ψ !! The only reason this works is that you are saying that the field ψ inside the integral is the SAME in all three

situations, which is to say, the field in an aperture is unaffected by the presence of the metal plate which makes the aperture, which is the Kirchhoff Approximation.

How might we apply this? A circular disk makes ψb. On the right you see your original radiation whatever it is along with a diffraction pattern. A circular hole makes just the diffraction pattern with the opposite sign on the field. Add them up and the diffraction patterns (fields, not intensities!) cancel and you just get the incident pattern. So the point is that the "interesting part" of the diffraction *field* is the SAME for both disk and circular hole, and this is true regardless of the details of the incident wave, but so far we have only proved this in the context of the K approximation. [Ah, but phase interference can make "difference" not what you might expect it to be! For example, $|\ 1 - \text{Airy}\ |^2$ differs a lot from $|\text{Airy}|^2$]

Jackson goes on to **prove Babinet without making the K approximation**, and in a full vector sense. The full statement is given on page 291. As you switch from Sa to Sb, you also have to "rotate" the incoming fields to get the "complimentary" situation. In the case of a plane wave incident, you maintain the result that the intensity (!) of the transmitted diffracted patterns is the same for both Sa and Sb. As an application, Jackson notes this: a thin slot aperture in a waveguide with H along the slot gives the same intensity pattern as a thin wire with E along the wire.

Vector Potential. Since this is so closely related to the above discussion, I throw in a few notes on it here. Back on page 179, Jackson restates Maxwell's equations in terms of A and Φ. In (6.29) and (6.31) he shows how you recover the physical fields from these potentials. [Together these of course form the vector photon field A^μ , but we are doing classical stuff here so we forget that fact.] Jackson shows that you can always shift the potentials by a gauge transformation without affecting the E and B fields, so you are free to choose a gauge in which $\partial_\mu A^\mu = 0$, which is the **Lorentz gauge**. In this case (only), you find that all four components of A^μ satisfy the same old wave equation we have been discussing above (!), but with sources on the right hand side, see 6.38. [Jackson notes that within the Lorentz Gauge you are still free to do further gauge transformations by a function which satisfies the wave equation, and you stay in the Lorentz gauge by so doing.]

Another gauge is the **Coulomb gauge**, aka the transverse gauge where **div(A)=0.** In this gauge, you get Poisson's electrostatic equation for Φ 6.44 instead of a wave equation, and we know how to integrate Poisson from earlier work to get the very intuitive 6.45. The only problem with this thing is the instantaneous nature implied by the equation which Jackson should be saying something about, but he remains silent. In this same gauge you find that A still satisfies the wave equation driven now only by the transverse current.

Now finally, we can jump ahead to page 269 where Jackson starts talking about "radiation", and right off the bat we see in equation 9.3 the solution for the vector field A as an integral over the current density J. In this integral, we see the same exp(ikR)/R propagator factor that we saw in all the Kirchhoff stuff above, simply because A satisfies the wave equation as we just showed (we assume the Lorentz gauge). So make some kind of antenna, write down the current J, and you can compute everything. The scalar component Φ is zero since we assume our radiating system has no bare charge density ρ.

On page 269 Jackson then describes three "zones": near, induction, and radiation. In all three cases, you must have your distance r >> d of the radiator. This distinction between the three zones depends on the relation between r and λ. [Also, he always assumes d << λ!]

In the **near zone**, r << λ and then you can ignore the phasor part of 98.3 and you get a Coulomb like result. In 9.6 Jackson writes the multipole solution in this case, which arises from a spherical harmonic expansion of the Coulomb 1/R.

In the **radiation zone**, r >> λ , you can pull out the 1/R in 9.3 and approximate R in the phase by R = r - n•x' and this gives you the general form 9.8 which shows a simple spherical wave going out. If λ >> d

as well, you can expand this approximated expo as in 9.9, and this is what gives you the traditional far-zone multipoles like E1, M1 and E2, and so on, which are further discussed in this chapter.

In the **induction zone** we have r ~ λ so we have a harder problem. Here, Jackson rolls out the full-bore multipole expansion. This thing is based on an expansion of exp(ikR)/R in terms of sphericals and various kinds of Bessel functions as treated on page 541 of the multipole fields chapter, and the result is quoted as 9.11. He then says how this can be further expanded, but I don't care right now.

Now finally we are ready to consider Jackson's circular aperture discussion section 9.8. The **Fraunhofer zone** corresponds to the radiation zone defined above. If you have r ~ d with r >> λ, you are in the **Fresnel zone**, a zone not really considered above.

4. Jackson circular aperture.

(in the Kirchhoff approximation, so a >> λ, meaning large ka, so we have a "large" aperture; also we are in the radiation far-field limit) To make things messy, Jackson of course has the incident plane wave come in at some angle α off normal. He assumes the incoming beam has its E field in the plane of incidence (the other choice would have been an E field tangential to the surface, but he does not do this case). In a confusing to me way, he uses ε1, ε2 and ε3 instead of x,y,z unit vectors. He first does the "vector" Kirchhoff case (of course with the Kirchhoff approx's) which is now 9.95 in the far zone. He jams in the field and does the integration over the disk and gets the result 9.102 where "screw" is some complicated combination of the angle α with the usual spherical harmonic angles of the observation point. From this he computes the angular power distribution (in all directions). Integrating this over half a sphere gives the transmission T, and he comments a lot on this. He then starts over and redoes the entire computation using the "scalar" Kirchhoff form 9.95 where he assumes that ψ is |E| (have to assume something here!), and the result is 9.111 with power given by 9.112. The results are NOT the same in terms of angles, as you might expect. However, if α = 0 where polarization is intuitively less important, vector and scalar give the same power result which is 9.113. This also happens to be the result I am really interested in. J1(x) is a sin(x) like function. In this calculation, we used the two different versions of the Kirchhoff integral (vector and scalar), but in both cases we used **the "Kirchhoff approximation"** which says that the field in the aperture is just that of the incident wave. We trust this approximation only in the case that a >> λ (ka >>1), so we don't really trust the computed results for ka ~ 1 nor for ka <<1 (but it still might be semi reasonable). In the case of a blood cell of diameter 8 μ and λ = 0.5 μ, ka = 2πa/λ = 48, so Jackson's result should be correct. In this case, even for small θ the J1 argument will be large, so we will pick up oscillations from it.

What does Jackson's result look like? I did it exactly in Excel which does know about Bessel functions, and that showed me the way to understand the result. For large ka, the J1(x) function oscillates with first peak at 0.6, and later peaks smaller, sort of a modified sin(x) variation, so the numerator never gets large. For small angles that make the denominator small compared to 1 (the central peak), J1(x) ~ x/2, so we get the amplitude factor peaking at 0.5. Then when ka sin(θ) ~ π (say), the amplitude is down to 1/π and power is down to about 1/10, so that angle is roughly the size of the pattern. sin(θ) = π/ka = λ/2a = λ/D. In our case D = 8 μ and λ = .5 μ, so sin(θ) = 1/16, θ = 4 degrees. I have confirmed this with Excel. So as you make the disk larger, the diffraction ring gets smaller. Note that 4 degrees also appears on my Dr. Call page. For all values of D in the range 2 to 100μ, the height of the first off-center maximum is very small compared to the central peak (looking at intensity), and further out ones are not even visible on my Excel scale. Remember that this solution only applies in the far-field limit.

5. **Jackson on small apertures**.

This is the subject of page 297. Obviously in this case the Kirchhoff approximation does not apply at all (d is << λ here). Jackson quotes work of Bethe in 1942 that showed an easy way to solve the problem in this case. You poke a small hole in a metal plate, and you realize that the fields near the hole are going to be "near fields" which you know are electrostatic like at least for the E field. You can then solve the electrostatic problem to find the E field that would exist in the hole, in a quasi-static sense, as suggested by the picture on page 298. In particular, there are tangential electric fields in the hole area that are not just the "constant" you assume in the Kirchhoff approximation. In fact, 9.114 gives the exact tangential E field (note that it is a function of radius ρ within the hole, and also depends on E0 and B0 to the left of the hole, which you could assign to be plane wave values for the simple case). Now you can stick this Et expression into vector Kirchhoff 9.82 and get the diffraction pattern. Jackson does not however quote the result, it is left as a chapter-end problem.

6. Jackson on scattering by conducting sphere

On page 299 Jackson does this for a large sphere relative to λ, then on page 569 he does the general case and also extracts the small-sphere limit.

For the first small λ solution, his method is truly fascinating. You imagine the sphere almost optically as having an illuminated half and a shadowed half. There is, however, a hazy area between these two halves. In each of these two areas, he makes assumptions about what the fields must be doing on the conducting surface. He is going to use the "vector Kirchhoff approximation" formula 9.77, but only in the radiation zone limit. Even in this limit, of course, he has to come up with fields on the surface so he can do the Kirchhoff surface integration. This formula 9.77 is general and applies to diffraction sources of any 3D shape bounded by a surface S1 as shown in the right side of figure 9.5. We have already done an example of a flat diffractor as in the left side of figure 9.5. The field assumptions at the surface are shown top of page 300. In the shadow region, you have to assume that the scattered field is canceling the incoming field, which is why you have "shadow". In the illuminated region, you make assumptions appropriate for a mirror surface. We now have estimated values for the E and B fields at all points on the sphere. Jamming this stuff into 9.77 in the radiation zone limit, we end up with two integrals, the shadow one and the illuminated one. Jackson then comments on the nature of these two contributions. He is able to identify the shadow region integral as the cause of the diffraction pattern (the transmitted wave), and the illuminated integral as the reflected wave. He puts the boundary between the regions right at π/2 in polar angle, and does the shadow integral and ends up with 9.124 for the diffraction pattern. For small θ, the result is basically identical to the result for the flat disk! The illumination integral gives a result that is basically isotropic scattering and corresponds to the optical reflection in all directions from such a sphere, at least in the reflection half sphere. So the result is then summarized in 9.133, and he gets the famous result that the total cross section is twice the area of a disk the diameter of the sphere. Half comes from the reflection, and half from the diffraction.

Now on to the full solution on page 569. Here we have the full bore multipole expansion for the scattered fields in terms of the "vector spherical harmonics" **X**lm which you get by applying the vector angular momentum operator $\mathbf{L} = \mathbf{r} \times \nabla$ (see p 542) to the normal Ylm functions. Also appearing here are the Hankel functions you get in doing the multipole expansion, and then the "coefficients" contain your

solution to the problem. The boundary conditions on the sphere are pretty simple 16.142, we don't have any shadow side now or anything like that. We grind through and the answer is given by 16.151 in terms of the coefficients called δ which are now all known (albeit messy). In the limit of a small sphere relative to λ, the angular behavior is 16.160, plotted on page 573. This is not the optical limit I am interested in, it is a small sphere with long λ and you find mostly backscatter only, with a tiny weak forward peak. The cross section shows the **ω^4 behavior known as Rayleigh's law which arises from any dipole scatterer.** An example would be scattering by air molecules at sunset. The higher ω scatter more, causing what is left to be red. And of course the sky is blue because you are looking from the side and seeing all scatter.

7. Carlson's book from SLC library, Chapter 1.

Introduction to Applied Optics for Engineers, F. Paul Carlson, 1977 (Academic Press) 621.36 C 284 in

At first I liked this book better than Goodman due to its notation, but then Goodman won me over with his more powerful descriptions, better pictures, deeper probing. Also, Goodman's book is newer, being 1996. As a result, I ended up copying the first few chapters of Goodman, but none of Carlson. I did verify that the key results are the same in both books, such as the Fresnel and Fraunhofer formulas, etc. I did copy Carlson's contents however.

This is a real optics book, by the way, not a general E&M text like Jackson. We start with Maxwell's equations and get the wave equation for isotropic media. Plane waves and "eta" is used as index n. He goes through the **eikonal** business with S scaled as exp(iS) and gets the eiconal formula (as he spells it) which is 1-25. This is all described in detail above, so enough.

Right now, I think of **Fermat's Principle** as an oddity. Using the eiconal thing, you first show that if you integrate phase along some (possibly curved) ray, the total phase you get is the same as just integrating the index n along this path (apart from constant factors). Fermat's Principle is a statement about the path that light follows. The original statement was that the path from A to B of a ray is whatever path minimizes the integral of the index along that path, which is the same as saying that you have minimized the time it takes a wavefront to move from A to B. The correct statement is that the correct ray is the one that makes the integral stationary, allowing mins, maxes, and saddles. Web site uses this in trivial way to prove Snell's law of refraction and reflection. I suppose the mirage thing could be worked in this manner as well.

On page 9 Carlson starts dealing with the "scattering" problem in general terms, stuff on the left makes a field at the right. He quickly gets to the Kirchhoff equation with a full surface integral and with the usual two terms: (G grad(u) - u grad(G))• n . Carlson then tries to find a solution with G=0 on the surface, so only the u grad(G) term survives (he has picked Dirichlet boundary conditions). He then goes on to write G(r) in the usual form exp(ikr)/r. In order to force G=0, he puts an "image source" on the left of the aperture plane and will take the limit later to the plane, an interesting trick. The next task is to compute grad(G) which really means dG/dz at z=0, z being the diffraction central axis as usual. This derivative is roughly (z'/R) * exp(ikR) * (jkR - 1). The first factor is the famous cosine factor that you see in Jackson as n • R. The bracket is also familiar from Jackson, the two "terms". Here is Carlson's result: (called the Sommerfeld-Rayleigh formula)

$$u(r') = 1/(2\pi) * \int dxdy\, u(r)\, (z'/R)\, (1/R - ik)\, \exp(ikR)/R \,] \qquad R = | r - r' |$$

This agrees with Jackson's scalar Kirchhoff 9.65 with the Jackson's first term set to 0. In Jackson, this would mean that in the aperture, the scalar field does not have a normal gradient component. In Carlson, we killed this first term by forcing G = 0. In any event, the above equation will be Carlson's main tool. There are two further approximations he will often make: (1) ignore the 1/R term in (1/R - ik) as long as $R >> \lambda$. (2) ignore the z'/R cosine factor if you are only interested in small angles. [all quite clear]

Now we can comment finally on **Huygen's Principle** (1690). The principle was (is) that you can treat something like a slot as if it were filled with a continuum of spherical radiators. In the above integral, if we ignore the 1/R term and if we ignore that cosine factor, we get exactly what Huygen's said: we are adding up spherical radiators over the surface, carefully tracking the phases with exp(ikR). We might even get the cosine too if we think about the incident wave as being polarized across the slot. In our plane wave incident cases, we would treat u(r) inside the integral as a constant across the slot, we might extract the 1/R factor as just 1/r, and then we are doing nothing more than adding up the exp(ikR) contributions. The point is that we have a formal mathematical derivation of Huygen's principle that he made with no math at all around 1680. Newton had just started inventing calculus around this time. Around 1816 (a full century + later) Fresnel wrote Huygen's principle as the above integration. In 1823 Fraunhofer did his theory of diffraction. In 1835 Airy did the circular aperture. In 1865 we got Maxwell's equations. In 1887 I think Kirchhoff did a rigorous derivation of the above integral that Fresnel postulated, and so we now call it the Fresnel-Kirchhoff integral. Green was somewhere early 1880's. This was the Big Time in optics and much of physics.

8. Simple Huygen's Examples.

You have very very small slot, small compared to λ. The result is a hemi-spherical and uniform diffraction pattern that is fairly isotropic. There are no interference effects at all. At least this is what Huygen's and Kirchhoff above would say. You should get that cosine factor too. However, we know that Kirchhoff does not apply in this regime, so this result may not be rigorously correct. Jackson showed that we need to use the near-field method to get the fields in such a small aperture, then use them with the integral to get the right result. [Another idea to keep in mind is the photon uncertainty principle. With a super narrow slot, you try to localize in space, so the result is that momentum is uncertain sideways after the slot.]

Next, take a wide slot. We add up the spherical radiators pairwise, and deduce a full minimum when $d/2 * \sin(\theta) = \lambda/2, 3\lambda/2$, etc. Also maxima in between. My Waves Berkeley book adds up a discrete number N of such radiators in a slot and gets a sinc-like result, then takes a limit and gets a real sinc function. I know sinc is the Fourier transform of a box amplitude (a slot), but I have not been able to show how this arises from the Huygen's integral. For example, the Fourier integral goes to infinity, but my slot integration goes -a/2 to a/2. Perhaps some clever change of variables is needed to show this. I hope I will run into the answer to this somewhere along the way here. [The answer is given below!]

9. Goodman Chapter 3: Foundations of Scalar Diffraction Theory

First, doing roughly what Jackson does, Goodman arrives at what he calls the "integral theorem of Helmholtz and **Kirchhoff**". This is the thing that has the both terms and requires both the field and its

gradient to be defined on the surface. In the usual "K approx" we say the field and derivative are zero away from the apertures, and are the same as the initial field in the apertures.

The K theory has two problems: (1) the K approx is of course just an approx; (2) it has a theoretical problem that you are required to specify a field and its normal derivative as being zero on the "metal plate". But we know that this rigorously implies the field is 0 everywhere, which is then nonsense, so the theory is somehow "theoretically inconsistent". However, calculations come out very accurately.

Sommerfeld must have come after Kirchhoff and he (with Rayleigh perhaps) is credited with solving this inconsistency by doing the little image limit derivation, as reported in Carlson's book. In fact Carlson's is really the "first" Sommerfeld solution, where we make G=0 on the plane. If you *add* the two Green's terms instead of subtract them, you get the "second" Sommerfeld where the gradient of G vanishes instead on the plane. These two solutions each knock out one of the "terms" in the Kirchhoff formula that Jackson gives.

Goodman is now so good as to **compare the two "theories"** in his Section 3.6: the Kirchhoff vs the Sommerfelds. The conclusion is that the Kirchhoff is the average of the two Sommerfeld solutions! The three solutions differ only in the cosine factor business, which is called the **obliquity factor**. Goodman shows these three angular formulas first in the case that the source on the left is a point source, and second if the source is instead a plane wave. In the plane wave case we get $\cos(\theta)$ for Sommerfeld I, 1 for Sommerfeld II, and half the sum of these for Kirchhoff. The three theories are the same for small angles (very distant screen), but definitely differ as you get close to the aperture. Goodman notes that the Kirchhoff retains the advantage of applying to non-planar sources, whereas the Sommerfeld's are only for planes, due to the image business used in the derivation.

Goodman on page 52 gives a nice "**physical interpretation**" of the Sommerfeld I formula. He even comments on the $1/j\lambda$ factor out front, which implies an extra 90 degree phase shift you might not guess from the Huygen principle, and also a λ dependence on the amplitude out front. The explanation of this υ dependence is just that the imagined Huygens sources radiate by an amount related to time derivative of the field, so that accounts for the $j\lambda$. Goodman admits that there is no physical explanation for the cosine factor, and reminds us that this factor is not even the same in the various theories. [In my later notes, I see how the radiation from a flat metal object approach gives this single ik due to the B = curl A op.]

10. Goodman Chapter 4: the Fresnel and Fraunhoffer Approximations

We now skip ahead in Goodman to page 66 where he discusses **"the Fresnel approximation"**. This is just a simple first-term power series expansion of the R in the expo factor of the Sommerfeld formula, done in 2D of course. The approx is therefore really $z >\sim$ all transverse dimensions (on source and screen). The trick here is to extend those integration limits to infinity and put the aperture boundary into the internal field function which he calls U. [I dimly remember this idea from my diagonalization with dg in group theory!] By messing around, Goodman is able to show that the resulting field due to Fresnel diffraction is ALMOST the double Fourier Transform of the field under the integral, something I have been looking for! The problem is that the thing being transformed also includes a phase factor that depends on the integration coordinates. BUT, if we further assume that $z >> D*D/\lambda$, then this phase factor becomes 1, and we get the result I want (where D here is the largest radius of the aperture). This is equation 4.17, I should maybe copy this page. Don't miss the main point here: Fraunhofer Limit is the *same* as doing a double-Fourier transform, apart from some external phase factors.

Now if I put a box function in for both x and y, we get a product of sinc(ax) * sinc(by) as the diffraction pattern of a rectangular aperture a x b. Now, although we are assuming z is "large" in our

Fresnel approximation, a study of where the integration actually gets contributions shows that the result is accurate much closer to the aperture than you would think, so in general one thinks of **Fresnel as a "near field" approximation**, but of course not "too" near.

Next, on page 73 we get "**the Fraunhofer approximation**". This is exactly what I just discussed above. Now you take the Fresnel formula and assume also that $z >> D*D/\lambda$, then you get the direct Fourier Transform result, apart from a phase. For human size apertures like 1 cm with light, you get 1 cm * 1 cm / 0.5 μ = 200 m, which is pretty darn far away (although result is good closer anyway). But for my eye problem with D = 4 μ and λ = .5 μ, we get z = 32 μ, so we are easily in the Fraunhofer limit ! So finally I have an explanation of my Fourier transform mystery!

On page 75 Goodman follows my lead, boom we have the sinc sinc solution for a box, and we get a nice plot of what this looks like on a screen!

Next, he does the **circular aperture** in Fraunhofer limit. Earlier in the book Goodman shows how you can deal with a double x-y Fourier Transform if the aperture function is radially symmetric. The result is that the transform is also radially symmetric, and is given by doing a simple 1D Fourier-Bessel Transform of the aperture function, equation is 2-32. When you jam a simple constant over a circle into this transform, you have a simple integral of J0 which ends up being J1, as shown in 2-35. This is the sort of radial version of the sinc function, and it is what Jackson got in his analysis. My old Schaum book does not give Fourier-Bessel Transforms. Goodman refers to the Bessel sinc thing as the "**jinc**" or "**besinc**" function, I am happy to see it has a name! The argument of J1(x)/x is x = ka r/z = ka sin(θ), just as it appears in Jackson page 296. Jackson gets that extra Kirchhoff obliquity factor, whereas in our case here we just get cos(θ) = 1, so we have no factor in the Sommerfeld form. Of course Jackson's is the same in this limit. The circular diffraction pattern of this "jinc" function is called the **Airy** pattern, he first derived it in 1835 (as noted above!). The pattern is shown on page 79, this is what I made Excel do. He reminds us that the zeros and maxima are not evenly spaced as they are in the regular sinc function!

He goes on to treat two more interesting situations. The first is a sine *amplitude* **grating**, which makes side peaks as you might expect, whose positions depend on λ. The second is a sine *phase* grating, which makes a spiky result. The nice thing is that this kind of grating absorbs no power. The point of these gratings of course is to separate light into its color components!

Now as a great tour de force, Goodman reconsiders the **square aperture** not in the easy Fraunhofer limit, but in the Fresnel case, which allows you to see the "near zone" as well as the far zone. Here we don't have the simple Fourier transform, so we have to do more brute force work. The integral is written, and then we change variables in a fancy way and end up with a very complex result for the intensity pattern shown in 4-47, where we are using the C and S standard "Fresnel" integrals. These are integrals of $\cos(x^2)\ dx$ and $\sin(x^2)\ dx$ with 0 lower endpoint and arbitrary upper endpoint. The arguments of these Fresnel integrals involve the scaled coordinates X and Y, along with Fresnel Numbers. Recall the Fraunhofer condition above, which I sated as $z >> D*D/\lambda$. The **Fresnel number** is $N_F = w*w/(z*\lambda)$ and is a measure of how far you are into the Fraunhofer limit. Small N_F means large z and yes Fraunhofer like. Page 86 then shows great graphical solutions for one dimension of the square aperture as a function of N_F. The nice thing is that the pattern changes in a gradual, if messy, fashion from the Fraunhofer sinc function shown at the top at .01 (which makes a pattern much wider than the aperture), to the geometric shadow limit shown at the bottom with 10 (which is only as wide as the aperture). Goodman notes that he is omitting the graphical **Cornu spiral** that usually appears in this discussion, I will find it elsewhere.

Goodman's last gasp in this Chapter 4 is to treat the sine amplitude grating in the Fresnel case and he shows that there are an infinite series of alternating exact images of the grating and inverted images that appear on the right, known as **Talbot images**. That is, as you move your screen away, you will see these images come into focus at the right spacings.

This was an excellent chapter.

11. Waves Book: Comments

This book reminds us that there really is a physical explanation for all this stuff. When you have a hole in a metal plate or in a cardboard opaque sheet, in both cases there is a large shadow area to the right. In both cases, the cause of this shadow is that the sheet adds an equal and opposite field to the incident field, it moves the electrons in exactly the right way to make this exactly happen. This is analogous to the way charges move on the surface of a sheet of metal to make the E field be zero inside everywhere. Now, when you remove a "plug" to make a hole, the diffraction pattern you see on the right is really being made by the rest of the metal plate adding its field to the incident plane wave. You can think of the hole as making the pattern if you want. If you have just the plug, then IT makes the field, but the result is the same field for the interesting part. The only difference between a metal plug and an opaque plug is on the left side, not on the right side! For metal, you get a reflected field to the left, whereas the **opaque** plug gets hot and makes no reflection. In his conducting sphere example, Jackson computed this reflected field from the "illuminated" side. For the metal disk, the reflected field going off to the left would surely have the same "beam" shape as the shadow field you see on the right. No doubt the energy subtracted on the right because of the diffraction shadow matches the energy reflected to the left. I would guess that the diffraction pattern on the right from an **opaque sphere** would be exactly the same as for the conducting sphere, and we have seen how at small angles this is the same as for the disk. So I think I now know how to deal with spherical 3D opaque floaters!

12. Carlson Chapter 2.

In Chapter 1 he came up the Sommerfeld integral, and here his first act is to do that approximation for R and the first term there yields the Fresnel approximation 2-5, just as done in Goodman above. He notes that you almost have a Fourier Transform except for that phase thing. Then he notes that if you make that phase 1, you do get the Fourier, and then you are in the Fraunhofer limit. So nothing new. He then does the square aperture in Fraunhofer and gets the sinc sinc answer, and has some non-exciting plots of intensity. The circular disk Airy case comes next, but he just does it with no mention of the formal Bessel-Fourier transform.

His next example is two rectangular apertures, and here we do learn something new. The result is a product of the two sinc functions that you would get from one aperture, then the whole result has another cosine factor called the "array factor" that of course arises from the spacing between the two apertures. This cosine factor tends to make the central diffraction hump narrower in the array direction than it would be with just one aperture. This suggests the idea that you might take a whole **array of apertures** to form a narrow beam. If these are antennas, you get the idea of repeating elements **to make a narrow beam**.

Next example is a Fraunhofer rectangular aperture but you insert some kind of linear phase shift medium in there so you get a linear phase shift going across either way -- he calls this a **"wedge"** so I guess you could make it with a wedge shaped piece of glass in the aperture. The result is that this shifts the location of the central hump, as you would expect from Huygens.

Next example is a full Fresnel calculation this time with a quadratic phase shift wedge, perhaps you do this with a parabolic piece of glass (a "thin lens"). Here we get our first hint of lens optics. The Fresnel result is very complex, unless you happen to choose z' = the lens focal plane. In this case, the integral

simplifies and you just get the Fourier Transform of the source image function! This suggests to me that if you have a real image at a focal plane of a lens, the "image" appearing down the center of the thin lens will be the Fourier transform image of that image! (We shall see). He goes on to do the general solution just to show off the Fresnel integral functions as Goodman did. He does draw one more lens conclusion which I did not follow.

13. Portis Chapter 14: Scattering Applications

This is the second last chapter in Portis's book, he is finally getting around to diffraction. He starts with the usual idea of an incident E-field inducing a dipole into a *free* electron, and that electron moves and makes a scattered field, and this is **Thomson scattering**. The result is that a free electron acts as a small disk with radius very small, the classical electron radius. The cross section has no dependence on ω. In the particle approach, this is photon-electron scattering, called **Compton scattering**, and I recall once doing this complete calculation I think in Bjorken & Drell, getting the Klein-Nishina formula.

On the web http://farside.ph.utexas.edu/~rfitzp/teaching/jk1/lectures/node86.html, someone repeats this Thomson calculation, then goes on to do the computation again but with a **bound** electron with a restoring force. The result here (which I think is in my waves book) is the same, but now there is an ω dependent factor which includes $\omega 0$ and γ, the resonant frequency and damping factor. Scattering is most near resonance. If you are well below resonance $\omega << \omega 0$ (long wavelength), then you get the ω^4 factor which and this is **Rayleigh (blue-sky) scattering**. I guess the *electronic* "resonance" levels of air atoms are in the UV, so visible light is then long wavelength. For clouds, drops are big and you are then in the other limit where there is no ω dependence, hence **clouds are white**. Portis does some Rayleigh on page 561. [Scattering from small dipole objects like N2 molecules gives the k^2 factor, hence Rayleigh. Scattering from large dielectric spheres gives no k factor when ka >>1, hence white clouds. Droplets in clouds are 10-15 μ]

Jackson got this Rayleigh ω^4 result in his treatment of scattering from a conducting sphere. He claims this arises wherever a "dipole" is involved. I think a better answer is when a dipole is involved with a restoring force, which implies a + and a - next to each other, as when you polarize an atom or a sphere.

Portis's next adventure is "scattering by a dielectric sphere". We don't have any diffraction integral formulas here at all. Instead, the sphere has some polarization P proportional to E, and we have a formula for B field of a dipole which leads to a simple integral over the volume of the sphere. When this is done in the Fraunhofer limit, we get a sinc-like result on page 543. Whatever it is, it is the Fourier transform in 3D of the unit sphere.

In similar fashion, he next treats "scattering by a dielectric disc". Instead of using Kirchhoff or Sommerfeld stuff, Portis just assumes the disk has a uniform induced P polarization, and reapplies his standard formula for B field from a dipole. In this way, he reproduces the Airy result involving J1. In a way, this is a little closer perhaps to my floater situation.

In his next section he makes the **dieletric disk** so thick it is **opaque**. In this case, he gets the same angular factor, only the overall intensity is affected. He then goes to the **mirror** case by assuming things about the media properties, and this just adjusts the scaling of the result again, and now he refers to the result as the Fresnel-Kirchhoff diffraction formula! It is true that the K theory is for conducting screens.

Well, OK, enough for Portis. I don't think I am a big fan of this book, but I am not sure why. Yes, I never used it. It does touch on many "practical" applications of E&M, that is true. I was hoping to see the Cornu friends, but I did not!

14. Cornu Spiral.

(Also known as Euler's Spiral). Define the complex Fresnel function as F(t) = C(t) + i S(t), although I just think of this as a single complex integral from 0 to t of exp($i\pi/2 * t^2$) dt. The spiral is simply a plot in the complex plane of the value of the integral F(t). Parameter t moves along the curve. At t=0 of course the integral is 0. As you move to + infinite, you spiral around and eventually reach the point 1/2, 1/2 in the upper right, the upper "eye" of the spiral, and go the other way to reach the other eye. Now, when you take the Kirchhoff or Sommerfeld formula and expand the phase factor R in the usual power series, you get the Fresnel formula, and this gives you exactly this type of quadratic phase factor. That is to say, you get exp(i K η^2) = cos(K η^2) + isin(K η^2) that you need to integrate from 0 to some finite t, see page 300 of A&S. If you have a constant source, such as a disk or half plane or whatever, you will always get the results in terms of Fresnel integrals with strange endpoints of the F(t) function, pretty simple. You can write such an integral between two endpoints as F(t1) - F(t2). So if you find the parameter locations t1 and t2 on the Cornu curve, the line segment joining them is this difference, and that was the graphical idea. As you move the transverse observation point X, you typically move both endpoints as in Goodman. As these points spiral around the eyes, the difference segment sort of oscillates in length, and that is precisely the intensity varying! Just a way to visualize what is happening, perhaps also to put some bounds on things.

15. Goodman Chapter 5: Wave Optics

This is all new to me! You have geometric optics based on rays, which does not account for diffraction effects, and you have wave optics which does include diffraction! He opens by describing the effect of a thin lens of arbitrary shape on a wavefront, 5.2, very clear. The phase at each point is adjusted by the lens. He now abuts two spherical lens surfaces with a central slab, and computes the thickness function. He then limits interest to paraxial rays (near center), and can then power expand square roots, and we end up with parabolic surfaces so the function is now as in 5-8, all fine. Notice that R2 is negative by convention. The lens has to be "thin" so there is no transverse displacement of the beam in going through the lens, this is a major assumption. If we identify the focal length f in the usual way and drop the constant overall phase, our phase shift function is the simply exp[- i (k/2f) ($x^2 + y^2$)]. { This is a very important and basic result, remember it well! }

We then get to a main point: such a lens maps a flat wavefront into a spherical wavefront, at least for the paraxial rays. I knew this, just nice to see it clearly stated. It is noted that even if surfaces are really spherical, lens thickness causes aberration from this spherical fact, and you can tune the lens surface shape to optimize this conversion of flat to spherical waves (correcting a lens).

Case (a): put a transparency just to the left of your lens, and compute the field at a plane through the focal point on the right. First, map the transparency through the lens by adding the lens phase factor. Then put this result into the usual Sommerfeld formula in the Fresnel approximation. If you set z=f, that nasty extra phase inside the integral gets cancelled out, and the result is this: the field on the focal plane is precisely the 2D Fourier transform of the source transparency image (apart from a phase that does not matter if you only care about intensity, see 5-15). This is an amazing fact to me! What you see on the focal plane is the Fraunhofer diffraction pattern from the transparency source, but you don't have to go "far away" to get it!

{ Digression back to Goodman page 55 on the "angular spectrum". }

I got stopped cold on Case 2 because I did not earlier read this section, so we handle it now! I will try to rephrase this somewhat obscure section in my own words. Stare at a 2D Fourier expansion of some U(x,y) into its Fourier frequency components which he calls A(fx,fy). You can think of this A as the amplitude of a Fourier plane wave that is travelling in a *direction* determined by fx and fy. If you write the plane wave in the usual manner exp (i k•r), you find that the unit vector k is given by k = λfx, λfy, and don't care for the third component. We only care for the moment about r with z=0. You can therefore think of the quantities λfx and λfy as direction cosines describing the direction which this plane wave is traveling, where you just assume that the third direction cosine is sqrt (1 - the squares of these two). So this third component of the direction is forced by the other two. Thus, for example, if you think of small values for fx and fy, their direction cosines are small, and the third one will be big, and you will have a plane wave traveling mostly to the right. Now, in the Fourier integration over the projected components, your variables fx and fy will go over an infinite range. However, when they are large enough that the third direction cosine is forced imaginary, THOSE components will be doing damping in the z direction, and we don't include them in the integration!

So this is the first big new idea here: interpret the Fourier decomposition as saying you are adding up plane waves in many different *directions*, and the amplitude of each of these waves is A(fx,fy). However, for our wave optics, we will only integrate over fx and fy that give no damping of the wave in z.

The second big idea is shown on page 57. He writes out the Fourier expansion of U(x,y,z), notes that it still applies the wave equation (don't forget that now after all this time). That in turn implies that the Fourier component A(fx,fy; z) satisfies a certain simple differential equation in z, the solution of which is given in 3-66. This is a major result, it is saying that we KNOW how the Fourier transform propagates in the z direction! You simply add a phase factor exp(ikγz) to move ahead z, where γ is that third direction cosine. Notice that he has to scale the transforms frequency arguments in a funny way to make this work:

$$A(\alpha/\lambda, \beta/\lambda; z) = A(\alpha/\lambda, \beta/\lambda; 0) * \exp[ik\gamma z] \qquad \text{where } \gamma = \text{sqrt}(1 - \alpha^2 - \beta^2)$$

Now we can jump to the bottom of page 60 where we get Goodman's filter interpolation. You can interpret "free space" as a filter. It is a low pass filter in both directions, bounded by the circle limit. The transfer function of the filter is the phase shown, so it is a dispersive filter. The bandwidth is related to λ which controls the circle radius. He also notes that this assumption gives the correct result for diffraction patterns! That is, it fully agrees with the Sommerfeld approach.

Case (b): We are back on page 102 with an image transparency distance d to the left of our thin lens, text on page 104. Now finally I understand 5-17 which shows how you propagate your Fourier transform to the right by a distance d. Next, you let this thing pass through the lens to the focal plane, as we did in Case 1. Equation 5-18 does this part, which we just quote from Case 1. Now we combine the two pieces together to get the full effect from the plane d to the left, over to the focal plane. The result is now seen to be a slight generalization of our Case 1 result. If we set d=0, we duplicate case 1. But if we set d = f, then we kill the leading phase and we get an exact Fourier transform. So when you go from a focal plane on one side to that on the other, you get an exact Fourier transform!

Before leaving this case, Goodman mentions the idea of vignetting with regard to the figure on page 5.6. If you are doing a case 2 trace, you need to look backwards from any given point on the resulting

focal plane screen and you see that a finite lens diameter puts a restriction on that input plane, so you can include this in your integration as shown in 5-20 with a pupil function which does exactly this thing.

Case (c) is next, and I think this has a very special interest to me! Think of the transparency here as a floater sitting a distance d to the *left* of the focal point! His analysis here I think is just what I would have done if I had to do it all myself. There are several items to take note of: (1) since the rays are coning down on the right, you get an amplitude constant gain by factor f/d, just conservation of energy. (2) the diameter of the cone intersecting the transparency is of course now smaller than the lens, so you need to add a pupil function expressing this fact. Again, the scaling is f/d. (3) Now, most importantly, since this transparency is being hit by a spherical wavefront, not a plane one, you have a phase shift across the transparency from the incident beam. This phase shift is the exponential factor shown in 5-21. When we put our transparency to the left of the lens, this same phase shift factor was created by the lens itself with a distance f in the phase formula. Here we have this same phase but with distance d, since that is the curvature of the sphere where it intersects the image input plane. Just as in case 1, this extra phase cancels the phase in the Fresnel formula, and we end up with a pretty much straight Fourier transform result, as shown in 5-22. Goodman has included the scaled pupil function just mentioned. Apart from this function and a slightly different overall phase AND the overall scale factor f/d, this case gives the same result as Case 1: the focal plane displays the Fourier Transform of your transparency. By moving the transparency, you change the overall gain on the result.

So, we now have these three little "building blocks". For each one, the basic idea was to figure out what was happening phase-wise on the input plane, then we use Fresnel to map it to where we want.

[In all three cases above, called (a), (b) and (c), we are computing the field on the right-side focal plane for a transparency inserted at three different places to the left. In all three cases, the result is a direct Fourier transform (Fraunhofer integral), even though distances are short. There is an overall multiplying phase in all three cases, which only vanishes in case (b) if you put your film at the left focal plane. So the idea seems to be that at the right focal plane of a lens, you get the "Fourier Transform" of an image no matter where you put that image plane on the left, times a quadratic phase factor. Thus, you have a machine that can compute 2D Fourier transforms very quickly!]

Image Plane Idea. Now with the geometry on page 108, we are going to try to do a non-blurry image. We would like the impulse response to be a delta function, apart from some magnification factor M, so we exactly duplicate our source object. The impulse response describes what you see at (u,v) in your image due to a point spherical source at point (ε,eta) on the source plane. To do this, we do the three steps we already know about. First, given our unit point source, we handle the propagation on the left side of the lens by applying a quadratic phase due to the spherical wave hitting the flat lens plane -- the radius there is z1. Second, we go through the lens and this generates another phase. Finally, we propagate distance z2 to the desired imagine plane. To do this propagation, instead of using the fancy Fourier thing we used in Case 2, we just use the Fresnel original formula which describes what happens when you move z distance down the line. These three effects are combined into 5-28. We first assume at this point the lens-makers formula to kill off the messy quadratic phase factor inside the integral. We then make reasonable hand-waving arguments to kill off the phases outside the integral. We then end up with result 5-33. Were it not for the pupil function, this would integrate to our desired double delta function, and we would have a perfect image. As it is, we have to instead regard the transfer function not as just this delta, but as the transform of the pupil function as shown, and this is just the Fraunhofer diffraction pattern of the pupil function. So, in this approach we have in effect derived both the magnification factor and the lens formula, without doing any geometric optics ray tracing, so to speak.

This brings us to page 112 of Goodman. He goes on to rescale the impulse response h we just worked with to get it into a form where we get a simple convolution situation to account for diffraction. You first imagine $U_g(u,v)$ to be an idealized geometric image, with proper scaling and magnification going through an optical system, let's say one lens. Then you can say the true image $U_i(u,v)$ is a convolution of the geometric one with the pupil transfer function, and this is a simple way to account for diffraction effects.

His next section develops a nice little operator formalism to handle our little "building blocks" when we do wave optics.

The rest of the book goes on to investigate all kinds of interesting topics, including wet gates and LCD displays and holograms and so on. I have to admit, this is the book I like, and the one I should buy:

Introduction to Fourier Optics, Joseph W. Goodman, 2nd Ed, 1996. $120.00 at Amazon!

I need for sure to copy critical pages of the above book before I return it!

Status. I think I have not read enough optics to start working on my own personal problem. I think I know how to "propagate" any image through an optical system.

So let's try it. Suppose we have a disk between the lens and focal point. I now know that this disk will be illuminated by spherically coning down waves, and I know how to write the phase over the disk surface. I know how this image looks at the focal plane, we have a Fourier Transform of the disk there, since this is Case 3 above. So at the focal plane, we are going to have the Fourier Transform of the disk, which is the Airy function.