Scalar vs Vector; Jones Gaussian Beam derivation; waist as focal spot PhL 12.29.02

Scalar vs Vector. The BYU PDF file makes some good points about the entire scalar theory which I have been using. Although exp(ikx) is a solution to the vector wave equation, the form exp(ikr)/r is a solution to the scalar wave equation but NOT a solution to the vector equation. In order to have an E-field which solves the vector equation, you have to make some statement like this:

$E_\theta = \exp(ikr)/r$ other components 0 or $\mathbf{E} = (\exp(ikr)/r)\ \boldsymbol{\theta}\textbf{-hat}$

If you stuff this into the vector wave equation, this is what you get:

$(\nabla^2 + k^2)\ \mathbf{E} = (- (2/r^2)\ |E|)\mathbf{r}\textbf{-hat} - (\csc^2\theta/r^2)\ \mathbf{E} \neq 0$!

One problem of course occurs at the pole where $\theta = 0$ and the θ-hat direction becomes undefined. If you stay away from this area, then you can ignore both RHS terms for large r, and this must be the basis for using the field exp(ikr)/r as if it were a solution to the vector wave equation! [In Jackson, this is used as a solution of the scalar equation in static theory.]

The BYU paper does offer up some solutions which do solve the vector wave equation but which have some strange terms.

How large must r be? Large compared to the λ that is inside k.

Optics and exp(ikr)/r. So, all the stuff we do in optics with point sources and spherical wavefronts is subject to the above condition. The only real problem occurs when you have to deal with a "focal point" of a lens, because there the spherical radius gets infinitely small.

Gaussian Beam. Jones give a very nice derivation in full gory detail, but he leaves out one small statement, because he uses the scalar and not the vector wave equation. He should state that the general idea is that the electric field is a vector that is in something like the x direction and it is fixed. That is, it is in some transverse direction. Then we may regard his A_α as being E_x, say. We know that E_x satisfies the wave equation exactly, so that gives us a starting point. Of course as you move away from the center of the Gaussian beam, you are going to pick up a longitudinal component of the E field, some E_z so that E as a vector can stay perpendicular to the wavefronts. I suppose we could worry about this if we wanted. But that does not prevent E_x from being an exact solution of the wave equation, in theory. So these are the words he might have added.

Now here is the derivation in summary. You first write this assumed form:

$E_x(x,y,z) = f(x,y,z) \exp(ikz)$

You then assume that f varies slowly in the z direction relative to λ, and in that case, the wave equation for E_x becomes the equation boxed on page 24 for f. The next step is to find a nice solution to the paraxial wave equation for f. The general assumed form of f is shown as (13) where we have two functions P(z) and q(z) to work out. Jamming this form into the paraxial equation and screwing around eventually ends up with the final form on page 27.

Now I need to comment on this result. The beam is characterized by an overall length constant L_F which is related to the width at beam center w(0). That is, $L_F = \pi\ w(0)^2 / \lambda$. Note that L_F and w(0) move together, they are not independent. As you make the waist w(0) smaller, the divergence of the beam

increases. This angle is $\tan\theta = \lambda / (\pi\, w(0))$, so you can think of $\pi w(0)$ as a length which controls exactly this angle, a little triangle structure. As long as $w(0)$ is much larger than λ, the beam angle will always be very small.

The reason $w(z)$ is called the width is that it appears here: $\exp(-\rho^2 / w(z)^2)$. The beam really does have a Gaussian amplitude profile, and $w(z)$ really is the width of this thing at a given z. The formula for $w(z)^2$ is a simple quadratic as shown, and this is the plot you always see of the beam outline.

So, there is basically only one free parameter of a beam at some λ and that parameter is either the waist size, or the L_F length (sometimes called z_0), or it could be the opening angle.

Applied to the focal point. Now, imagine we bring in a gaussian beam from the left side of a lens so the waist is right at the lens. We know the lens is going to simply add some quadratic phase of the form $\exp(ik\rho^2 / 2f)$ where f is the focal length. This is going to cause a shift in one of the phase parameters of the incoming beam:

$$\exp(ik\rho^2 / 2f) * \exp(ik\rho^2 / 2R(z)) = \exp(ik\rho^2 / 2R'(z)) \qquad R'(z) = R(z) \,||\, f$$

[At this point, I did a bunch of algebra to match up the two beams, but then I realized this is not the standard way this problem is approached. I later continued this subject and got the answer I was after. Namely, that the beam width is related to the lens diameter, etc. See separate document for a continuation of this path.]