Matrix Optics and Gaussian Beams PhL 12.29.02

1. I printed out a nice tutorial and went through this pleasant subject. The general idea is that in the paraxial limit and probably with some other assumptions, you can claim that the output position and slope of a "ray" are each linear functions of the input position and slope. Later I will come back and think about these assumptions, but I accept them for the moment, and this allows an optical component to be a "linear system" and allows them to be cascaded together by matrix multiplication. Also, I will go look up who first figured this out.

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}$$

This is the language of our tutorial, where α and α' are angles measured in radians, and are assumed to be small enough that $\sin\alpha = \alpha$, etc. In our tutorial, the prime symbol is used to distinguish a second plane of interest. Now, I think the following notation is much more commonly used:

$$\begin{bmatrix} x_2 \\ x'_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x'_1 \end{bmatrix}$$

where now 1 and 2 refer to the two sides of an optical component, and x' refers to the slope angle. We know from our tutorial what the matrix looks like for many different situations. For example, for a thin lens just going from one side to the other we get

$$\begin{bmatrix} x_2 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/f & f'/f \end{bmatrix} \begin{bmatrix} x_1 \\ x'_1 \end{bmatrix}$$

For a thin lens of index μ between n1 and n2, the formulas for f and f' are a bit messy, we have them on page 3 of our Chapter 6 tutorial. For n1 = n2 = 1 for air on both sides, we get $1/f = 1/f' = (\mu-1)(1/R1 - 1/R2)$ where R2 is the surface on the right which is a negative number for a double-convex lens.

2. Now, forget the meaning of R1 and R2 just presented (lens surface curvatures), we are going to now give these same symbols different meanings. Imagine an optical component A,B,C,D and we illuminate it from the left with a source which has radius of curvature R1 at the left plane of this optical system. We know that x1 = R1*x1' because x1' is assumed a very small angle. Assume that just to the right of the optical component, our wave has a radius of curvature R2. Then of course x2 = R2*x2'. If we write out the two equations above, we get

x2 = A x1 + B x1'
x2' = C x1 + D x1'

Divide the two equations to get

x2/x2' = [A x1/x1' + B] / [C x1/x1' + D]

or

$R_2 = [\ AR_1\ + B\ ] / [\ CR_1\ + D\ ]$

This shows that in general, a linear optical component changes the radius of curvature of an impinging wave as shown. The wave could be coming from a point source, or it could be coming from a Gaussian beam! In the case of a thin lens, this becomes

$R_2 = [\ 1R_1\ + 0\ ] / [\ (1/f)R_1\ + 1\ ] \qquad = R_1 / [\ (1/f)R_1\ + 1\ ]$

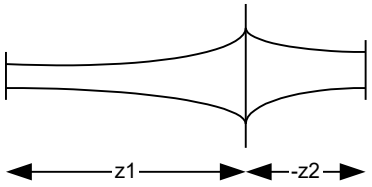
$1/R_2\ = 1/R_1 + 1/f$

I am familiar with this result from our phase analysis of a thin lens in Goodman, $\exp(-ik\rho^2/2R_1)$ is the phase of a point source R_1 to the left, and $\exp(-ik\rho^2/2f)$ is the quadratic phase added by the lens.

3. Now, a Gaussian beam coming from the left has a phase of $\exp(-ik\rho^2/2q(z))$ where $q(z) = z + iL_F$. This quantity q(z) is "sitting in the position" normally occupied by a radius of curvature. The real part of 1/q(z) is in fact exactly 1/R(z), the Gaussian beam local radius of curvature. Now the imaginary part of 1/q(z) causes the transverse Gaussian damping of the beam. If this imaginary part were small enough to be neglected (waist far to the left), we would identify q(z) with R(z) and we would at once claim that the following rule applies as this beam goes through an optical component:

$q_2 = [\ Aq_1\ + B\ ] / [\ Cq_1\ + D\ ]$

I think now the argument must go like this: since the first order system is completely described by A,B,C,D, then if q(z) is allowed to be continued off the real axis into the complex-z plane, the equation must still be true. I have not seen this argument made, however. Let's now assume this geometry for the moment:



Notice that z2 < 0, so - z2 is the distance on the right. Waists of the beams are shown left and right, and the middle vertical bar is our optical system. We can rewrite the above as:

$$z2 + iL2 = \frac{A(z1 + iL1) + B}{C(z1 + iL1) + D}$$

Taking the real part of this equation tells you z2, and the imaginary part tells you L2. In general, the waists need not have the same value at the optical component, unless it is say a thin lens. In this case we get this result

$$z2 + iL2 = \frac{z1 + iL1}{(1/f)(z1 + iL1) + 1}$$

If you equate the real and imaginary parts, it is still a mess, but here is what I got:

$$z_2 = \frac{z1(z1/f+1) + L1^2/f}{(z1/f+1)^2 + (L1/f)^2}$$

$$L_2 = \frac{L1(z1/f+1) - L1z1/f}{(z1/f+1)^2 + (L1/f)^2}$$

Now, suppose we take the special case of the left side beam having its waist *at* the thin lens, then z1 = 0 and these things simplify

$$z_2 = \frac{L1^2/f}{1+(L1/f)^2} = \frac{f}{1+(f/L1)^2}$$ // This appears in the waveoptics.pdf

$$L_2 = \frac{L1}{1+(L1/f)^2}$$

We can replace $W_2^2 = \lambda L_2/\pi$ and $W1_2^2 = \lambda L_1/\pi$ in the second equation to get

$$W_2 = \frac{W1}{\sqrt{1+(L1/f)^2}}$$ // This also appears in waveoptics.pdf

Divide the two equations above to get

$z_2/L_2 = f/L_1$

which is a pretty simple and exact result. I computed the waist on the right size $w_2(-z_2)$ and found that it is in fact exactly equal to $w_1(0) = W_1$, so the waists exactly match up, as the drawing suggests.

Finally, suppose our incoming Gaussian beam from the left is very parallel, meaning its divergence angle is very small, meaning that $L_1 >> \lambda$. Well, we can interpret L as the distance where the waist increases by sqrt(2). So if we assume this distance L1 is >> f, which is pretty reasonable, then we get these two results for our waisted hit on the lens:

$W_2 = W_1 * (f/L_1)$ // appears in the pdf
$L_2 = f^2 / L_1$
$z_2 = f$ // appears in the pdf

4. After a long battle, the above few equations are what I have been after. If you have a parallel beam coming in from somewhere that is gaussian with a very slow divergence angle (L1 is large), then you get on the right a Gaussian beam that waists down at a distance f from the lens, and the waist diameter at the focal point is the lens radius times (f/L1), which is a small but finite number. So this gives us the idea of a practical focal point that is not infinitely small. It gets so as L1 = infinity. Our waist is w(0) = a, the

radius of our lens (or pupil), so we know that $L1 = \pi a^2/\lambda$. The divergence of the beam is then $\lambda/(\pi a)$. So now we know quite a lot about our little situation.

Now maybe here is another way to estimate the value of L1. Suppose we made a flashlight with a light at the focal point and sent the beam out parallel through a lens or radius a. We know that we are going to have a diffraction divergence half-angle on that beam of theta = $(1.22/2)\ \lambda/a$. You can see this is similar to the angle $\lambda/(\pi/a)$, although the constants are not exactly the same.

Example: suppose in an eye we have pupil radius of a = 4 mm and focal length 1 cm, and $\lambda = .5\ \mu$. Then from the above we find that focal point waist $W2 = (f/\pi a) * \lambda = (5/6)\ \lambda \sim 0.5\ \mu$. This certainly is smaller than the size of a red blood cell which has a radius of 4 μ.