Radiation by a Dielectric and Metal Objects PhL 1.20.03 / 1.27.03
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1. Jackson and Electric Dipole Radiation

Look at Jackson page 179 in chapter 6. Here we see the vector field **A** being defined by $\mathbf{B} = \nabla \times \mathbf{A}$. This satisfies $\nabla \bullet \mathbf{B}=0$ for any function **A**. Jackson quickly shows on page 181 how you end up with $\Box^2 A_\mu = -4\pi J_\mu$ along with the condition that $\partial^\mu A_\mu = 0$ which you are free to do without affecting **B** and you thereby decouple the driving equations as shown. This is the Lorentz Gauge idea. The idea is to think of J_μ as driving your system, solve for A_μ, then you have the fields **E** and **B**. Jackson goes on to show how you take $\Box^2 A_\mu = -4\pi J_\mu$ and you think instead of $\Box^2 G(x;x') = -4\pi\delta^4(x-x')$ to get the Green's function (6.64), so that you can then solve for A in terms of an integral over J as in (6.65) or (6.66).

With this in hand, we jump off to Chapter 9 where (9.2) and (9.3) are what we have just shown. This is the vector field **A** as an integral of **J** times exp(ikR)/R similar to Kirchhoff theory, but here it is a volume integral, not a surface one . If we find **A**, then we get **B** and **E** as shown, assuming sine time. Note that $R = |\mathbf{r} - \mathbf{r}'|$, Jackson uses $|\mathbf{x} - \mathbf{x}'|$. The idea is that **x** is some far away observation point, whereas **x'** is a point in the localized volume of the radiator you are going to integrate over. For the exponential, we expand R only to its first term, and we just pull out the denominator, again, very similar to what we do in Kirchhoff. This gives (9.8) where we have $\exp(ik\hat{\mathbf{r}} \bullet \mathbf{r}')$ in the exponential [Jackson uses **n** for my $\hat{\mathbf{r}}$] . You can then power-series expand this exponential to get a factor $(k\hat{\mathbf{r}} \bullet \mathbf{r}')^n$ along with $J(\mathbf{r}')$ inside the

integral. The integer n is not the same as the angular momentum L in the full multipole expansion, and in fact here it is the n=0 term that will give the electric dipole radiation. In this entire Jackson section, we are now going to assume that $d << \lambda$ so the source of radiation is tiny compared to a wavelength. All three "zones" Jackson talks about have this assumption. In this case (which does not apply to floaters which are large compared to λ) we can use the n=0 term and have only electric dipole radiation!

With n=0, we get (9.13) on page 271. Since J is localized, we are allowed to replace it with $\mathbf{r}\,\nabla\bullet\mathbf{J}$ inside the integral, which is just parts integration, or see a standard identity. But $\partial^{\mu}A_{\mu}= 0$ says that $\nabla\bullet\mathbf{J} - i\omega\rho = 0$ [see page 178], and we have $\exp(-i\omega t)$ here , so you quickly get (9.16) with (9.17) which is the result we want.

Here is what this all says:

$$\mathbf{A} = -ik\,\frac{e^{ikr}}{r}\,\mathbf{p} \qquad \text{where} \quad \mathbf{p} = \int \mathbf{r}\,\rho(\mathbf{r})\,d^3r \;= \text{the electric dipole moment of radiator} \tag{1}$$

To get the **B field** you do $\nabla \times \mathbf{A} = \mathbf{p} \times \nabla(\frac{e^{ikr}}{r}) = \mathbf{p} \times \hat{\mathbf{r}}\,\partial_r(\frac{e^{ikr}}{r})$ which gives the result shown in (9.18), and if we take the large-r limit of (9.18) we get (9.19),

$$\mathbf{B} = k^2\,\frac{e^{ikr}}{r}\,\hat{\mathbf{r}} \times \mathbf{p} \qquad\qquad \mathbf{E} = k^2\,\frac{e^{ikr}}{r}\,(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} = \mathbf{B} \times \mathbf{n}$$

The direction of the **B** field is perp to both **p** and **r**. The cross product means we are picking up $\sin\theta$ where this is the polar angle measured down to the observation point from the north pole of the **p** vector. If we have an incident plane wave that is transversely polarized, this angle bears *no simple relationship* to the usual $\cos\theta$ angle we had in the Kirchhoff Som I formula! The **E** field has the same magnitude as the **B** field of course and lies in the plane of **r** and **p**, in the far field limit.

2. Dipole Radiation from a very small Dielectric Object

Now, let's imagine that the sources **J** and ρ in the above discussion are those of an atom in a material. The applied electric field causes a polarization of the electron cloud and nucleus system so that ρ is + on one side of the atom and - on the other end, at some instant, and current flows as this changes (but out of phase, as in a capacitor, see much later). Think of the atom as a 3D radiating antenna in this regard. We are treating the atom and field both classically, because the incoming flux is assumed large compared to $h\upsilon$. The radiated field is assumed weak compared to the incident field, a Kirchhoff-like assumption. Now imagine that we have a density n(atoms/cm^3) of atoms. Then we can define the macroscopic **polarization** as $\mathbf{P} = n\mathbf{p}$. Superposition says we add the fields (and potentials) of all the radiating atoms, so in (9.16) we just replace **p** with **P** V. We can only do this because our radiating object is tiny compared to λ, so all the radiators act in phase.

The result is then (9.16) and (9.19) [far field]where we replace **p** with $V\chi\mathbf{E_0}$. Thus, for example,

$$\mathbf{B} = k^2\,(\hat{\mathbf{r}} \times \mathbf{E_0})\,V\chi\,e^{ikr}/r \tag{2}$$

In Jackson, $\varepsilon = 1 + 4\pi\chi$. We know that $v = c/\sqrt{\mu\varepsilon}$ where μ and ε are "relative to 1", so that $n = \sqrt{\mu\varepsilon}$. In our case this means $n = \sqrt{\varepsilon} = \sqrt{1 + 4\pi\chi} \sim 1 + 2\pi\chi$. So $(n-1) = 2\pi\chi$ so we can rewrite the above as,

$$\mathbf{B} = k^2 (\hat{\mathbf{r}} \times \mathbf{E0})[(n-1)/2\pi] V e^{ikr}/r \qquad (3)$$

3. Thomson Scattering (just an aside)

If we do this for a single electron in free space (see Portis page 537), we solve the trivial ODE to find that $\mathbf{r} = e/(m\omega^2)\mathbf{E}$. Then we set $\mathbf{p} = -e\mathbf{r}$ (e>0) so $\mathbf{p} = -e^2/(m\omega^2)\mathbf{E}$ implying that $\chi = -e^2/(m\omega^2)$. Putting this into the above formula gives

$$\mathbf{B} = c^{-2} (\hat{\mathbf{r}} \times \mathbf{E_0}) (e^2/m) e^{ikr}/r \qquad (4)$$

which agrees with Portis p538 (7) when you remove the $\mu_0/4\pi$ to change from mks to gaussian. The cross-product angle here is between point r and polarization direction, not between r and the incident wave direction. This situation is called **Thomson Scattering**, and example of E1 dipole action. Notice that the ω^2 factor arising from dipole radiation is cancelled by the ω^2 factor in χ, so the result is independent of λ. If you integrate over all angles, you find a cross section $\sigma = (8\pi/3) r_0^2$ where $r_0 = e^2/m$ is the classical electron radius, and of course this cross section is independent of λ. (Compton, Klein-Nishina, plasma)

4. Dipole Radiation from larger Dielectric Objects

Jackson's equation (9.3) is still valid, and if x >> x' in magnitude (meaning r >>d) we can still get to the form (9.8). Let's look more closely at this exponential phase:

$$kR \approx k(r - \hat{\mathbf{r}} \bullet \mathbf{r}') = kr(1 - \hat{\mathbf{r}} \bullet \mathbf{r}' / r) \qquad (5)$$

Now, it is certainly true that the second term is much smaller than the first term because d << r. However, when you multiply by k, the real condition for ignoring the second term would be $kd << 2\pi$ or we could say $d/\lambda << 1$. Since we are NOT in this limit for a large source, we are not justified in ignoring this second term. So the bottom line is that we are blindly going to use Jackson's n=0 term only and we get the same result as above for the isolated atom. We can say that *we are only looking at* the E1 electric dipole part of the radiation. [Wrong! The first term 1 makes the dipole all by itself. The second term has contributions in all multipole orders, so we are "looking at" all these orders, but in the small d/λ limit.] At this point, I am not dead sure that the electric dipole dominates. However, Jackson considers this question in his multipole chapter and shows that the leading non-vanishing multipole always dominates in radiation from atomic systems, so E1 is going to be it for us since there is a non-vanishing dipole moment **p**. { But only true of $d/\lambda << 1$ } Note that all multipole terms have a "radiation field" with 1/r dependence, so you cannot argue by that alone that the higher moments are small. I think an incoming plane wave, although it can be expanded in multipole moments as per page 567 (various L, all M=0), is mainly E1 so is going to mainly stimulate E1 radiation from our radiator. { Wrong: it "stimulates" all partial waves! }

So our starting point then is (9.16) for the E1 radiation,

$$\mathbf{A(r)} = -ik\frac{e^{ikr}}{r}\mathbf{p} \qquad \text{where } \mathbf{p} = \int \mathbf{r}\,\rho(\mathbf{r})\,d^3r \text{ = the electric dipole moment of radiator} \qquad (1)$$

Notice that in this equation **p** is a constant, it is not p(r).

As before, we let **P** = n**p** [see Jackson p 118] and then we want to integrate over the source volume. We now replace **p** with **P** dV. Don't confuse this integration with the previous integration we did to get p in terms of ρ !!! This is a new integration. We are going to reintroduce the variable R to mean |**r** - **r'**| . The above formula applies only for a piece of the source right at the "center" of the source where r' = 0 by definition. The contribution of an *arbitrary* piece of the source will be the above formula with r replaced with R. We are just expressing the fact that the pieces of the source are at different locations and will have different distances R to the observation point. So, this gives us.

$$\mathbf{A} = -ik\int\frac{e^{ikR}}{R}\mathbf{P}\,d^3r' \qquad (6)$$

Now we expand R through its first term in the phase, but just treat the denominator as r, to get:

$$\mathbf{A(r)} = -ik\frac{e^{ikr}}{r}\int\mathbf{P(r')}\exp(-ik\,\hat{\mathbf{r}}\bullet\mathbf{r'})d^3r' \qquad (7)$$

Comparing this with the basic formula in terms of p, we conclude that we have

$$\mathbf{p_{eff}}(\hat{\mathbf{r}}) = \int\mathbf{P(r')}\exp(-ik\,\hat{\mathbf{r}}\bullet\mathbf{r'})d^3r' \qquad (8)$$

At this point, define $\mathbf{k} = k\,\hat{\mathbf{r}}$, meaning a vector pointing to the observation point. This is an unusual vector, easily confused with $\mathbf{r} = r\,\hat{\mathbf{r}}$, so pay attention to it. Also, we know that **P** and therefore $\mathbf{p_{eff}}$ points in the direction of our polarization, so let's call that direction $\hat{\mathbf{E}}_0$. So we get

$$\mathbf{p_{eff}}(\hat{\mathbf{r}}) = \int\mathbf{P(r')}\exp(-i\mathbf{k}\bullet\mathbf{r'})d^3r' = \{\int\mathbf{P(r')}\exp(-i\mathbf{k}\bullet\mathbf{r'})d^3r'\}\,\hat{\mathbf{E}}_0 = p_{eff}(\hat{\mathbf{r}})\,\hat{\mathbf{E}}_0 \qquad (9)$$

So our resulting E1 field will be Jackson (9.19) with **p** replaced by $\mathbf{p_{eff}}$. Whereas p was not a function of r, the magnitude of $\mathbf{p_{eff}}$ is a function of **r** due to the phase. That is, p_{eff} is a function of the *direction* $\hat{\mathbf{r}}$. So at this point we can write

$$\mathbf{A(r)} = -ik\frac{e^{ikr}}{r}\,\mathbf{p_{eff}}(\hat{\mathbf{r}}) = -ik\frac{e^{ikr}}{r}\,p_{eff}(\hat{\mathbf{r}})\,\hat{\mathbf{E}}_0 \qquad (10)$$

Now we use **P** = χ**E**, and for **E** we assume a plane wave coming in with wave vector $\mathbf{k_1}$.

$$\mathbf{E(r')} = \mathbf{E_0}\exp(i\,\mathbf{k_1}\bullet\mathbf{r'}) \qquad (11)$$

We then get

$$\mathbf{p_{eff}}(\hat{\mathbf{r}}) = \mathbf{E_0}\int \boldsymbol{\chi}(\mathbf{r'}) \exp(i\,\Delta\mathbf{k}\bullet\mathbf{r'})d^3r' \qquad \text{where } \Delta\mathbf{k} = \mathbf{k_1} - \mathbf{k},\quad \mathbf{k} = k\,\hat{\mathbf{r}} \tag{12}$$

If we look at the forward direction, and if we assume that χ is a constant, then we find:

$$\mathbf{p_{eff}}(\mathbf{r} = r\,\hat{\mathbf{k1}}) = \mathbf{E_0}\,\chi\,V$$

which is proportional to the number of atoms in our volume. [In a later document I show that this is in fact the E1 dipole only result]

The idea is this: due to cancellation of phases in a large source, the *effective* dipole moment of the source depends on where you view it from! It is strongest in the forward direction.

Now we need to do $\mathbf{B} = \nabla \mathrm{x} \mathbf{A}$. Let's pick the z axis to point in the same direction as k1. Then we can regard p_{eff} as $p_{eff}(\theta,\phi)$ relative to this axis. This gives:

$$\mathbf{B} = -ik\,\nabla \mathrm{x}\,\{\frac{e^{ikr}}{r}\,p_{eff}(\theta,\phi)\,\hat{\mathbf{E}}_0\} = ik\,\hat{\mathbf{E}}_0\ \mathbf{x}\,\nabla\{\frac{e^{ikr}}{r}\,p_{eff}(\theta,\phi)\} \tag{13}$$

Now,

$$\nabla\{\frac{e^{ikr}}{r}\,p_{eff}(\theta,\phi)\} = \frac{e^{ikr}}{r}\nabla\,p_{eff}(\theta,\phi) + \nabla(\frac{e^{ikr}}{r})\,p_{eff}(\theta,\phi) \tag{14}$$

But $\nabla\, p_{eff}(\theta,\phi)$ is of order 1/r (see spherical form of gradient), making the first term order $1/r^2$. We will neglect this compared to the leading term of our result. Now

$$\nabla(\frac{e^{ikr}}{r}) = \hat{\mathbf{r}}\,\partial_r(\frac{e^{ikr}}{r}) = \hat{\mathbf{r}}\,(\ ik\,e^{ikr}/r + e^{ikr}/r^2\ ) \tag{15}$$

and we drop this $1/r^2$ term as well to get a final result,

$$\mathbf{B} = ik\,\hat{\mathbf{E}}_0\ \mathbf{x}\,\hat{\mathbf{r}}\,(\ ik\,e^{ikr}/r)\,p_{eff}(\theta,\phi) = -k^2\,p_{eff}(\theta,\phi)\,\frac{e^{ikr}}{r}\,\hat{\mathbf{E}}_0\ \mathbf{x}\,\hat{\mathbf{r}} = -k^2\,p_{eff}(\theta,\phi)\,\frac{e^{ikr}}{r}\,\hat{\mathbf{E}}_0\ \mathbf{x}\,\hat{\mathbf{r}}$$

$$p_{eff}(\hat{\mathbf{r}}) = E_0\int \boldsymbol{\chi}(\mathbf{r'}) \exp(i\,\Delta\mathbf{k}\bullet\mathbf{r'})d^3r' \qquad \text{where } \Delta\mathbf{k} = \mathbf{k_1} - \mathbf{k}$$

One more rewrite,

$$\mathbf{B} = k^2\,f(\theta,\phi)\,\frac{e^{ikr}}{r}\,\hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{E_0} \tag{16}$$

$$f(\theta,\phi) = \int \boldsymbol{\chi}(\mathbf{r'}) \exp(i\,\Delta\mathbf{k}\bullet\mathbf{r'})d^3r' \qquad \text{where } \Delta\mathbf{k} = \mathbf{k_1} - \mathbf{k} \qquad \mathbf{k_1} = k\,\hat{\mathbf{z}}$$

where we are still allowing for a non-uniform χ in our dielectric scattering object.

In the case where χ is constant, we can rewrite one more time as,

$$\mathbf{B} = k^2 \chi V \frac{e^{ikr}}{r} g(\theta,\phi)\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{E_0} \qquad \mathbf{E} = k^2 \chi V \frac{e^{ikr}}{r} g(\theta,\phi)\ (\mathbf{E_0} - E_{0r}\ \hat{\mathbf{r}}\ ) \qquad (17)$$

$$\mathbf{k} \bullet \mathbf{k}_1 = k^2 (\hat{\mathbf{r}} \bullet \hat{\mathbf{z}}) = k^2 \cos\theta$$

$$g(\theta,\phi) = (1/V) \int \exp(i\, \mathbf{\Delta k} \bullet \mathbf{r'}) d^3r' \qquad \mathbf{\Delta k} = \mathbf{k_1} - \mathbf{k} \qquad \mathbf{k}_1 = k\, \hat{\mathbf{z}} \qquad \mathbf{k} = k\, \hat{\mathbf{r}} \qquad |\Delta k| = 2k\sin(\theta/2)$$

Notice that $|\Delta k|^2 = |\ \mathbf{k_1} - \mathbf{k}\ |^2 = 2k^2(1-\cos\theta) = 4k^2 \sin^2(\theta/2)$ giving the result above for $|\Delta k|$. The **E** field is a little more relevant in the electric dipole case. In general since there is no **J** outside the radiator we can say $\mathbf{E} = (1/ik)\ \nabla \times \mathbf{B} \approx -\ \hat{\mathbf{r}}\ \times \mathbf{B}$, where the second form results from keeping only the leading 1/r term, as in (9.19). But $-\ \hat{\mathbf{r}}\ \times \hat{\mathbf{r}}\ \times\ \mathbf{E_0} = (\mathbf{E_0} - E_{0r}\ \hat{\mathbf{r}}\ )$ where $E_{0r} = \mathbf{E_0} \bullet \hat{\mathbf{r}} =$ the projection of the incoming E vector onto the outgoing direction, a quantity that is normally small in the diffraction regime, but we shall keep it. Therefore we have this result for E

$$\mathbf{E} = k^2 \chi V \frac{e^{ikr}}{r} g(\theta,\phi)(\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}}\ )$$

which we now copy back into equation (17) to have everything in one place.

This $g(\theta,\phi)$ is a "form factor" and is normalized to be 1 in the forward direction. In that direction, you can see that this result for the **B** field in (17) is the same as in (2) above. The reason is this: think about what happens at some atom in the volume relative to a line perp to the z=0 plane which runs through the volume. A point to the left gets hit early by the plane wave, but the resulting radiation gets to the observer late by the exact same amount, so things offset exactly. It is as if the whole volume were jammed in a tiny space and was radiating as a unit. In any other direction, this argument cannot be made, and we have the form factor.

Spherical Coordinates

Consider now in more detail the integral

$$g(\theta,\phi) = (1/V) \int \exp(i\, \mathbf{\Delta k} \bullet \mathbf{r'}) d^3r'$$

Regardless of the shape of the object, we can choose to use spherical coordinates to attempt the d^3r' integration. In this case, we have <u>three possible choices</u> for positioning the integration z-axis: along **k,** along $\mathbf{k_1}$, or along $\mathbf{\Delta k}$.

<u>Spherical Coordinates with integration z axis along k_1</u>

If we align along the $\mathbf{k_1}$ axis, then the angles θ,ϕ in $g(\theta,\phi)$ will refer to the usual angles we think of in treating diffraction and scattering. In this situation, we would write,

$$\mathbf{k_1} = k(0,0,1)$$

$\mathbf{k} = k(S_\theta C_\phi, S_\theta S_\phi, C_\theta)$ $\quad\quad$ $\mathbf{r} = r(S_\theta C_\phi, S_\theta S_\phi, C_\theta)$ $\quad$ $\hat{\mathbf{r}} = (S_\theta C_\phi, S_\theta S_\phi, C_\theta)$

$\Delta\mathbf{k} = -k(S_\theta C_\phi, S_\theta S_\phi, C_\theta - 1)$

$\mathbf{r'} = r'(S_{\theta'} C_{\phi'}, S_{\theta'} S_{\phi'}, C_{\theta'})$

$\Delta\mathbf{k} \bullet \mathbf{r'} = -kr' (S_\theta S_{\theta'} C_{\phi-\phi'} + [C_\theta - 1] C_{\theta'})$

and then we could formally write the integral as

$$g(\theta,\phi) = (1/V) \int r'^2 \, dr' \int d(\cos\theta') \int d\phi' \exp(-ikr' \{ S_\theta S_{\theta'} C_{\phi-\phi'} + [C_\theta - 1] C_{\theta'} \}) \qquad (18)$$

Here you can see the appearance of θ and ϕ in the integrand.

If there happens to be cylindrical symmetry, then the ϕ' integration runs free 0 to 2π and can be shoved to the right to give,

$$g(\theta,\phi) = (1/V) \int r'^2 \, dr' \int d(\cos\theta') \exp(-ikr'[C_\theta - 1] C_{\theta'}) \int d\phi' \exp(-ikr' S_\theta S_{\theta'} C_{\phi-\phi'})$$

and we get the old Bessel result $2\pi J_0(kr' S_\theta S_{\theta'})$, we we end up with

$$g(\theta,\phi) = g(\theta) = (2\pi/V) \int r'^2 \, dr' \int d(\cos\theta') \exp(-ikr'[C_\theta - 1] C_{\theta'}) J_0(kr' S_\theta S_{\theta'}) \qquad (19)$$

but there is not much more we can do now! Even if the object has full spherical symmetry, it is not very clear what we should do next, both integrals look nasty (although they probably are doable). In the next section, we provide an alternate form for $g(\theta)$ that makes things a lot nicer.

<u>Spherical Coordinates with integration z axis along Δk</u>

Starting from the beginning, we have,

$$g = (1/V) \int \exp(i \, \Delta\mathbf{k} \bullet \mathbf{r'}) d^3r'$$

Now with this new choice of z-axis we we write

$\Delta\mathbf{k} = \Delta k(0,0,1)$

$\mathbf{r'} = r'(S_{\theta'} C_{\phi'}, S_{\theta'} S_{\phi'}, C_{\theta'})$

$\Delta\mathbf{k} \bullet \mathbf{r'} = \Delta k \, r' \, C_{\theta'}$

but what is Δk? In general we know that [we already did this above, so OK, we do it again...]

$$|\Delta\mathbf{k}|^2 = |\mathbf{k} - \mathbf{k_1}|^2 = k^2 + k^2 - 2 \, \mathbf{k} \bullet \mathbf{k_1} = 2k^2 (1 - \cos\theta) = 4 k^2 \sin^2(\theta/2) \qquad \text{so } \Delta k = 2 k \sin(\theta/2)$$

Notice that the dot product refers to the angle between $\mathbf{k}$ and $\mathbf{k_1}$ which is that between $\mathbf{r}$ and $\mathbf{k_1}$. This is *always* the angle we call θ in the "lab system". In our "integration system" the variable θ' is something completely different, referred to a different axis altogether, but that is fine, we don't care. We now have

$$g = (1/V)\int \exp(i\,\mathbf{\Delta k \bullet r'})d^3r' = (1/V)\int r'^2\,dr' \int d(\cos\theta') \int d\phi' \exp(i\,2\,k\,\sin(\theta/2)\,r'\,C_{\theta'})$$

No iff we have azimuthal symmetry, then $d\phi'$ integration yields 2π and we then have

$$g = g(\theta) = (1/V)\int \exp(i\,\mathbf{\Delta k \bullet r'})d^3r' = (2\pi/V)\int r'^2\,dr' \int d(\cos\theta')\,\exp(i\,2\,k\,\sin(\theta/2)\,r'\,C_{\theta'}) \qquad (20)$$

Compare this form to (19) above and you see that we have a great improvement in simplicity. There are no special functions present here, and we have the same two integrations.

Now suppose in addition we have full spherical symmetry of our object. Then we can do the $\cos\theta'$ integral which yields $2\,\sin(a)/a$ where $a = 2\,k\,\sin(\theta/2)\,r'$. This leaves us with:

$$g(\theta) = (4\pi/V)\int r'^2\,dr' \;\frac{\sin(2\,k\,\sin(\theta/2)\,r')}{2\,k\,\sin(\theta/2)\,r'} = (4\pi/V)\int r'^2\,dr' \;\frac{\sin(\Delta k\,r')}{\Delta k\,r'} \qquad (21)$$

and the remaining r' integral is a doable indefinite integral. We will do it for the sphere in a later section. One could also consider a spherical delta shell, or a spherical annular ring of some finite thickness.

The main point of having this section is that it provides an easy solution for spherically symmetric objects!

I see no reason at this time to develop the third case where the z axis is chosen along the **k** axis.

Cylindrical Coordinates Take #1

We are going to do a messy thing here. We are going to use both spherical and cylindrical coordinate at the same time, so we have to be careful. In the spherical system we put the z-axis in the k_1 direction, and we do the same for the cylindrical. Thus, both systems of coordinates have the same z variable. They also have the same ϕ variable, which is azimuthal around the z axis. Now let's write the k-stuff in terms of *spherical* coordinates, as above:

$$\mathbf{k_1} = k(0,0,1)$$
$$\mathbf{k} = k(S_\theta\,C_\phi,\, S_\theta S_\phi,\, C_\theta)$$
$$\mathbf{\Delta k} = -k(S_\theta\,C_\phi,\, S_\theta S_\phi,\, C_\theta - 1)$$

where θ and ϕ are the spherical angles relative to the z axis. Now, however, let's write r' in *cylindrical* coordinates:

$$\mathbf{r'} = (\rho' C_{\phi'},\, \rho' S_{\phi'},\, z')$$

When we say "let's write a vector in fred coordinates",we mean that our three components are always cartesian, so we know how to do dot products, but we are writing the cartesian *components* in terms of fred coordinates. We are allowed to write them in any coordinates we like, as long as each component is right. Now we take the dot product,

$$\mathbf{\Delta k \bullet r'} = -k\,[\, S_\theta C_\phi \rho' C_{\phi'} + S_\theta S_\phi \rho' S_{\phi'} + (C_\theta - 1) z'] = -k\,[\, S_\theta \rho' C_{\phi-\phi'} + (C_\theta - 1) z']$$

Now we will use cylindrical coordinates for the volume to get

$$g(\theta,\phi) = (1/V) \int \rho' d\rho' \int dz' \int d\phi' \; \exp[\, -i k (C_\theta - 1) z' \,] \exp(-ik S_\theta \rho' C_{\phi-\phi'}) \qquad (22)$$

Now, in the event that we have cylindrical symmetry of our object, we can move the ϕ' integration to the right and get the Bessel function $2\pi J_0(k\sin\theta\, \rho')$ so we get

$$g(\theta,\phi) = g(\theta) = (2\pi/V) \int \rho' d\rho' \int dz' \exp[\, -i k (C_\theta - 1) z' \,] J_0(k\sin\theta\, \rho') \qquad (23)$$

If the object is a cylinder or a cylindrical annulus etc, then we can move the dz' integration to the right to get

$$g(\theta,\phi) = g(\theta) = (2\pi/V) \int \rho' d\rho' J_0(k\sin\theta\, \rho') \;\; * \;\; \int dz' \exp[\, -i k (C_\theta - 1) z' \,]$$

We now have the product of two integrals each of which can easily be done. Finally, let's assume the object is a very thin annular disk in the z=0 plane. Then the dz' integral gives t and we get

$$g(\theta) = (2\pi t/V) \int_{\rho 1}^{\rho 2} \rho' d\rho' J_0(k\sin\theta\, \rho') = (2\pi t/V)\, \rho'^2 \frac{J_1(k\rho'\sin\theta)}{k\rho'\sin\theta} \Big|^{\rho 2}_{\rho 1}$$

At this point, we could consider a disk, a delta ring, or an annular ring. The disk result is this:

$$g(\theta) = (2\pi t/\pi a^2 t)\, \rho'^2 \frac{J_1(k\rho'\sin\theta)}{k\rho'\sin\theta} \Big|^{a}_{0} = (2\pi t/\pi a^2 t)\, a^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta} = 2 \frac{J_1(ka\sin\theta)}{ka\sin\theta} \qquad (24)$$

and we comment on this in a later section of this document.

Cylindrical Coordinates Take #2

Here instead of doing hybrid between spherical and cylindrical coordinates as we did in the last section, we now do everything in cylindricals. In this case we still put the cylindrical z axis long our k1 beam direction to get,

$\mathbf{k_1} = k(0,0,1) = (k/r)(0,0,r)$ where $r = \sqrt{z^2 + \rho^2}$ // select z along k1 as before

$\mathbf{k} = k\hat{\mathbf{r}} = (k/r)\,\mathbf{r} = (k/r)\,(\rho C_\phi, \rho S_\phi, z)$

$\mathbf{\Delta k} = \mathbf{k_1} - \mathbf{k} = -(k/r)\,(\rho C_\phi, \rho S_\phi, r - z)$

$\mathbf{r'} = (\rho' C_{\phi'}, \rho' S_{\phi'}, z')$

$\mathbf{\Delta k \bullet r'} = -(k/r)\,[\rho\rho' C_{\phi-\phi'} + z'(r - z)\,]$

Neither z nor ρ nor r is an integration variable, so we are happy to just use the symbol r where convenient. Then we get

$$g(\rho,z,\phi) = (1/V) \int \rho' d\rho' \int dz' \int d\phi' \ \exp(-i(k/r)\, z'(r - z) \exp[\ -i(k/r)\, \rho\rho' C_{\phi-\phi''}\ ]$$

Now iff cylindrical symmetry, the endpoints of the ϕ' integration are 0 to 2π, we move it to the right, and we get our Bessel function result $2\pi J_0(k/r)\, \rho\rho'$), with this final result

$$g(\rho,z,\phi) = g(\rho,z) = (2\pi/V) \int \rho' d\rho' \int dz' \exp[\ -i(k/r)\, z'(r - z)\ ]\ \ J_0(k\rho\rho'/r)$$

This is similar to our result in the last section. If the object has a uniform cross section, we can move the ρ' integration to the right and then we have decoupled the two integrations to get

$$g(\rho,z) = (2\pi/V) \int dz' \exp[\ -i(k/r)\, z'(r - z)\ ] \int \rho' d\rho' \ J_0(k\rho\rho'/r)$$

The connection between this and the last result is that $\rho/r = \sin\theta$ and $z/r = \cos\theta$. The results are of course then exactly the same.

The next step will be to compute form factors for various specific cases. All cases of scattering from dielectric objects have the form given above in (17), the only difference is in the form factor! Note that even *without* the form factor, we still get the $\sin\psi = \hat{\mathbf{r}} \times \hat{\mathbf{E}}_0$ angular dependence [this is the E1 dipole action] . If we just write out E_0 and r in their spherical forms, we find that

$\cos\psi = \sin\theta \cos(\phi - \phi')$ where we assume $\hat{\mathbf{E}}_0$ is at ϕ' relative to the x axis.

Thus we also know

$$\sin\psi = \text{sqrt}(\ 1 - \sin^2\theta \cos^2(\phi - \phi')\ )$$

A good problem: Use the above formula to compute scattering from a dielectric disk, then compare that to the Airy formula that arises from diffraction through a circular hole.

5. Quick note on Higher Moments. While running, I realized that you can take the n=1 term in Jackson's expansion in (9.9) and have an integral of x^i **J.** You do the same parts trick on this and get $(x^i)^2$ $\nabla\bullet\mathbf{J}$ as the integrand, and you then end up with a diagonal part of the "quadrupole" tensor integral over ρ. [I later see that Jackson actually does this in his book.] Two charges on a stick making a dipole have zero for this moment. In our problem here, we are stimulating the dielectric with a source that is primarily E1 (although I am sure all L values are in there, see (16.128) page 567). I am pretty sure the E1 is the main result, but will defer a more detailed study of this question for later. The main point is that the higher L moments are in those higher n terms in the series. [This subject has now been investigated more, see later document. All formulas in this paper which involve radiation from an extended dielectric source involve many multipoles, not just the E1 dipole, although the local "mechanism" for the radiation is the dipole polarization of the dielectric. Secondly, putting current = dP/dt is another way to start.]

6. Assumptions made in the dielectric case?

(1) We assumed that the total electric field E inside the dielectric is not significantly affected by the radiated E field. This is why we could assume that P is uniform inside the dielectric object. Otherwise P would have to follow the exact *total* E field including any strange directions it might have. So our assumption was really "weak dielectric" with small χ. This is the same idea as assuming cosine current in an antenna. I refer to this sometimes as the King Smiles assumption, since Prof. King always smiled about it.

(2) Did we assume anything about the SIZE of the dieletric object? Yes. The whole plan of calculation involved here uses an expansion of e^{ikR}. We expand as $R = r + \delta r + \delta^2 r + ...$ in the phase and we ignore higher terms like $\delta^2 r$ etc. This results in $e^{ikR} = e^{ikr} * e^{ik\delta r}$. The expansion $R = r + \delta r$ in the phase is only justified if $k\,\delta r << 2\pi$ over the entire volume of the source, meaning that $(d/\lambda) << 1$, because then we can ignore the *rest* of the series which we expect to be on this order or less. We might even go so far as to say $(d/\lambda) < 1/4$. Note that we do NOT expand $e^{ix} = 1 + ix$ anywhere, as I once thought. If your problem has $(d/\lambda) > 1$, then the phase expansion here is no good no matter how many terms you keep. You are out of the perturbation theory limit. You need to use another approach to the problem (such as floater).

(3) There is another assumption, but it has less impact. In (2) we wrote the condition for the e^{ikR} factor, and that condition is that $(d/\lambda) << 1$. But we also expand 1/R as 1/r and this implies that $\delta r/r$ is small, which means that $(d/r) << 1$ as well. So we need both $d << \lambda$ and we need $r >> d$. However, since this large r condition does not affect a phase, it just affects the general scaling and smooth shape of the result. But for very close-in work, yes, you will get significant error from violating this condition.

7. A rectangular thin dielectric plate.

Dimensions are 2a and 2b. The z integral gives thickness t. The result is this:

$$g = (1/[\,(2a)*(2b)*t]) \; [\,(2a)\,\sin(\Delta k_x a)/\,(\Delta k_x a)\,] \; [\,(2b)\,\sin(\Delta k_x b)/\,(\Delta k_x b)\,] \; [\,t]$$

So

$$g = [\,\sin(\Delta k_x a)/\,(\Delta k_x a)\,] \; [\,\sin(\Delta k_x b)/\,(\Delta k_x b)\,]$$

Since $\Delta\mathbf{k} = \mathbf{k_1} - \mathbf{k}$, if we put $\mathbf{k}_1 = k\,\hat{\mathbf{z}}$, then

$$\Delta k = (-kS_\theta\, C_\phi,\; -kS_\theta S_\phi,\; k - kC_\theta)$$

Let's also look at the angle ψ which appears as $\sin\psi$ in our cross product. Assume without loss of generality that $\hat{\mathbf{E}}_\mathbf{0} = \hat{\mathbf{y}}$. Then:

$$\hat{\mathbf{k}} \bullet \hat{\mathbf{y}} = \cos\psi = \sin\theta\,\sin\phi \quad => \quad \sin\psi = \mathrm{sqrt}(1 - S_\theta^2\, S_\phi^2)$$

OK, let's do try to be general and rotate our $\hat{\mathbf{E}}_0$ to some arbitrary location at angle ϕ' from the x axis. We know that the result will be:

$$\hat{\mathbf{k}} \bullet \hat{\mathbf{E}}_0 = \cos\psi = \sin\theta \cos(\phi-\phi') \qquad \Rightarrow \quad \sin\psi = \text{sqrt}(1 - S_\theta^2 C_{\phi-\phi'}^2)$$

Thus we get this result:

$$|\mathbf{B}| = k^2 \chi V \frac{e^{ikr}}{r} E_0 \sin\psi \, g(\theta,\phi)$$

$$= k^2 \chi V E_0 \frac{e^{ikr}}{r} \sqrt{1 - S_\theta^2 C_{\phi-\phi'}^2} \; [\sin(kaS_\theta C_\phi)/ (kaS_\theta C_\phi)] [\sin(kbS_\theta S_\phi)/ (kbS_\theta S_\phi)]$$

The energy flow is $(c/8\pi)|\mathbf{B}|^2$ from page 205 7.15, so

$$dP/dA = dP/(r^2 d\Omega) = (c/8\pi)|\mathbf{B}|^2$$
$$= (c/8\pi) k^4 V^2 \chi^2 E_0^2 (1/r^2) (1 - S_\theta^2 C_{\phi-\phi'}^2) [\sin(kaS_\theta C_\phi)/ (kaS_\theta C_\phi)]^2 [\sin(kbS_\theta S_\phi)/ (kbS_\theta S_\phi)]^2$$

or

$$dP/ d\Omega = (c/8\pi) k^4 \chi^2 E_0^2 (1 - S_\theta^2 C_{\phi-\phi'}^2) [\sin(kaS_\theta C_\phi)/ (kaS_\theta C_\phi)]^2 [\sin(kbS_\theta S_\phi)/ (kbS_\theta S_\phi)]^2$$

Now we can average this over ϕ' to average over all transverse polarizations from an incoherent source. This just means averaging the quantity $(1 - S_\theta^2 C_{\phi-\phi'}^2)$ and that yields $(1/2)[1 + C_\theta^2]$. So

$$dP/ d\Omega|_{\text{unpol}} = (c/8\pi) k^4 \chi^2 E_0^2 \frac{1 + C_\theta^2}{2} [\sin(kaS_\theta C_\phi)/ (kaS_\theta C_\phi)]^2 [\sin(kbS_\theta S_\phi)/ (kbS_\theta S_\phi)]^2$$

Let's go back to the form of Δk:

$$\Delta k = (-kS_\theta C_\phi, -kS_\theta S_\phi, k - kC_\theta)$$

Consider a point on an observation screen such that $x = r S_\theta C_\phi$, $y = r S_\theta S_\phi$, $z = r C_\theta$. Then we can say

$$\Delta k = (-kx/r -ky/r, k - kz/r)$$

$$g = [\sin(kax/r)/ (kax/r)] [\sin(kby/r)/ (kby/r)] \qquad (1 - S_\theta^2 C_{\phi-\phi'}^2) = [1 - (\hat{\mathbf{r}} \bullet \hat{\mathbf{E}}_0)^2]^2$$

And now the result takes a more congenial form

$$|\mathbf{B}| = k^2 \chi V \frac{e^{ikr}}{r} E_0 \sin\psi \, g(\theta,\phi)$$

$$= k^2 \chi V \frac{e^{ikr}}{r} E_0 [1 - \{(\hat{E}_{0y} y/r)^2 + (\hat{E}_{0x} x/r)^2\}] [\sin(kax/r)/ (kax/r)] [\sin(kby/r)/ (kby/r)]$$

The g factor does know know about the polarization direction. All the polarization dependence is in the first factor. Suppose we pick E_0 in the y direction and get [1 - $(y/r)^2$]. The more we climb up toward the pole of the E1 radiator in that direction, the weaker our signal gets. So what we have here is a traditional diffraction pattern but with a modulation factor (the dipole!) which reflects the polarization axis. The scalar diffraction theory as in Som I has a cosθ/R factor and some other internal quadrtic phase which might gives some of this factor maybe as [1 - $(\rho/r)^2$], say. { More on Som I below! }But in diffraction, we usually say x,y << r so we would not see this factor anyway, or it would be small. The other two factors are very clearly the standard Fraunhofer diffraction sinc functions.

Now, how are we supposed to compare this to the diffraction result? That result was for a rectangular aperture, this result is for a flat rectangular dielectric plate of the same size as the aperture. What is the Babinet/Airy situation here? We assumed a certain incoming field E_0 and from it we computed the radiated field. We know that the radiated E is the same size as the B, but the direction is tricky. Here is the direction of the radiated E field:

$$-\hat{\mathbf{E}} = \hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \hat{\mathbf{E}}_0 \quad \Rightarrow \quad \hat{\mathbf{E}} = \hat{\mathbf{E}}_0 - (\hat{\mathbf{r}} \bullet \hat{\mathbf{E}}_0)\,\hat{\mathbf{r}} = \hat{\mathbf{E}}_0 - (\hat{E}_{0x}\, x/r + \hat{E}_{0y}\, y/r)\,\hat{\mathbf{r}}$$

So let's cut to the chase here. For x,y << r, the radiated E field is in the same direction as the incident E field, no great surprise there. Then we make our Babinet/Airy statement:

$$\mathbf{E} = \mathbf{E}_0\, e^{ikz} + k^2 \chi\, V\, \mathbf{E}_0 \frac{e^{ikr}}{r} \; [\, \sin(kax/r)/(kax/r)\,]\,[\, \sin(kby/r)/(kby/r)\,]$$

This is superposition for a particular polarization state, but we have ignored the factor that depends on polarization now because it is small! Now we can expand r in the phase in the usual Fresnel manner to get:

$$\mathbf{E} = \mathbf{E}_0\, e^{ikz} \{ \quad 1 \quad + k^2 \chi\, A\, t\, (1/z) \exp(\, ik(x^2 + y^2)/2z)\, [\, \sin(kax/r)/(kax/r)\,]\,[\, \sin(kby/r)/(kby/r)\,] \; \}$$

where by the way V = (2a)(2b) t = A t = the volume of our thin dieletric rectangular plate.

As I compare this to the (4-28) Fraunhofer result on page 75 of Goodman, things are very close! There are three differences to think about :

(1) I have k^2 but Goodman has k -- this is a characteristic of dielectric vs metal, see proof below!
(2) My result is real, but his result has 1/j -- same cause as (1), see below
(2) I have the extra factor χ t in addition to the k business. (chacteristic of dielectric case)

8. General Radiation Theory Applied to a Flat Metallic Radiating Object

In the work above, we considered a dielectric object and based the study starting with **p** and **E1**. The E field induces P and that radiates to make our result. Suppose we have a rectangular plate of metal sitting at our little test site. If it were a perfect conductor, then J = σE would say J = 0 inside the metal. However, on the side surfaces we can have some n x B which can support surface current K. We know that this current will somehow cause a reflected wave on the left side that will cancel E at the left surface.

So, maybe we can compute K from the incoming plane wave B and then use that as J in (9.13) to compute the radiation! If this works, the result should be similar to the Babinet diffraction result.

The first stop on this trip is to look at the Snell picture on page 218 of Jackson. This shows the directions that all the E and B fields have at some instant in time. Notice that the three E fields are into the plane of paper in this picture. Now in the case of a mirror surface, which metal would be, there will be no refracted wave, and if we come in head-on, the reflected wave will have its E" exactly opposite the incoming E field, which causes zero E field at the metal surface at all times. In this case, the fields B and B" will be aligned, and we end up with 2B as the tangential field at the surface. It is this factor of 2 that we need to make this all work.

Next, from curl **B** = (4π/c) **J** applied to a little loop straddling the left metal surface, we conclude that there is a surface current on the left surface which will be K = (c/4π) B . The direction of K is perp to B at the surface. If you draw a picture with a B field tangent to the surface and n a unit vector to the left, then you can say that the direction of ∇ x B is n x B. The reason is that B=0 inside the metal, so the curl loop only gets a piece in the B direction on the left. If we draw a loop oriented to get a positive contribution to its integral B•dl, then we find that the current is in the n x B direction. Thus **K** = (c/4π) **n** x **B**. We can then define a conventional **J** by saying **J** = **K**δ(z) = (c/4π) **n** x $\mathbf{B_t}$ δ(z), where the left plate surface is at z=0. We have now taken note that it is the tangential B that drives this current. [Just as a surface charge shields a conductor's inside from a surface normal E field, so a surface current shields it from penetration by the plane wave]

Next we start with the completely general (9.3) and insert the above current and get this result:

$$\mathbf{A(r)} = (1/4\pi) \int \frac{e^{ikR}}{R} \; \mathbf{n} \times \mathbf{B_t}(r')\, dA'$$

This is not just E1, it is the exact result, and n is to the left still. To get our radiated B field, we just take the curl of this thing, and that goes in and acts on the G function and we get our usual result,

$$\nabla G \approx ik\, \hat{\mathbf{R}}\, G \qquad \text{// assumes } kR >> 1$$

so we then end up with

$$\mathbf{B}(r) = (ik/4\pi) \int \frac{e^{ikR}}{R} \; \hat{\mathbf{R}} \times (\mathbf{n} \times \mathbf{B_t}(r'))\, dA'$$

Now **n** x $\mathbf{B_t}$ is in the z=0 plane, and there is some angle ψ angle between this vector and $\hat{\mathbf{R}}$ **,** and so a factor of sinψ lurks in the integrand.

As our final step, let's <u>make n point to the right</u> instead of to the left, and set k = 2π/λ to get

$$\mathbf{B}(r) = (1/2i\lambda) \int \frac{e^{ikR}}{R} \; \hat{\mathbf{R}} \times (\mathbf{n} \times \mathbf{B_t}(r'))\, dA'$$

I think this result is exact for kR >> 1 where $\mathbf{B_t}$ is the exact field on the plate surface. In this sense, the result would apply to any size and shape plate. Using the A x B x C identity, we can rewrite the above as

$$\mathbf{B}(r) = (1/2i\lambda) \int \frac{e^{ikR}}{R} \; \{ (\hat{\mathbf{R}} \bullet \mathbf{B_t}(r'))\, \mathbf{n} - (\hat{\mathbf{R}} \bullet \mathbf{n})\, \mathbf{B_t}(r') \}\, dA'$$

As elsewhere, we are inclined to ignore the first term because it is of order (x-x')/z compared to the second term, and this is small in the usual diffraction geometry. Also, we set $(\hat{\mathbf{R}} \bullet \mathbf{n}) = \cos\theta$ to get:

$$\mathbf{B}(r) = -\ (1/2i\lambda) \int \mathbf{B}_t(r')\ \frac{e^{ikR}}{R}\ \cos\theta\ dA'$$

Now, suppose the plate is large compared to λ so behaves as a real mirror. Then as discussed above in our "first step", we can say that $\mathbf{B}_t = 2\ \mathbf{B}_0$ where $\mathbf{B}_0$ is the plane-wave B field that *would* be there if there were no plate present! So we get the final result:

$$\mathbf{B}(r) = -\ (1/i\lambda) \int \mathbf{B}_0(r')\ \frac{e^{ikR}}{R}\ \cos\theta\ dA'$$

which is the famous Sommerfeld I formula, except we have an overall minus sign! We want this minus sign, as explained in the next section!

So what have we done here? We have started with the radiation formula and we have applied it to a metal plate of size large compared to λ, and we have observed it from far away so that z >> d, and we have in doing so exactly replicated the Sommerfeld diffraction formula that would apply to a complementary aperture in a metal screen, apart from an overal minus sign.

9. Comparison between radiation from dieletric, radiation from metal, and diffraction from a hole.

The diffraction formulas involve dG/dn which introduces a factor of 1/jλ into the integral formulas. Apart from this factor, you install your source scalar field and do the integral against exp(ikr)/r. I have derived the vector version of this from Jackson in a separate document, and of course we get the same result.

In the radiation formula world, we start with Jackson (9.3) for the vector potential A. This formula is driven by the current J and has only a 1/c factor in cgs-esu. Then B = curlA as we have done above. For the dipole in the dielectric case, replacing J with iωρ adds one factor if ik, really from $\partial\rho/\partial t$ { Or think of this as coming from dP/dt.} . But then curlA does another derivative on exp(ikr) and makes a second ik, so we end up with a real factor k^2 in our result. The idea is that going from J to ρ causes 90 degrees of phase shift since a time derivative, and going A to B makes another 90 degree shift since diffing exp(ikr) in space.

Think about what is happening in the dielectric radiation case. You have a stimulating E field which directly drives the dipole moment p and the charge density ρ which creates local p with no phase shift. As shown in (9.14), this creates an effective current which is 90 degrees out of phase with E, $\mathbf{J} = -i\omega\ \mathbf{r'}\ \rho$, and we get our first factor of ik as noted above. Then the curl in B = curl A makes a second factor of ik and we end up with k^2 in the radiation formula.

In the case of radiating from a metallic object, as just done in the previous section, the electric field drives a current *directly* with no phase shift (the surface current K). Thus, we only pick up the one factor of ik when we do B = curl A.

In the case of radiating from an aperture, we can imagine that E directly drives E in the aperture with no phase shift, and this is similar to the metal case above, and we only get the one ik factor.

Now here is a conclusion to make on my favorite subject called Babinet/Airy. We have shown that "radiation" from a metallic plate is given by the exact same formula as diffraction through the complementary hole but with a minus sign. In the radiation case, we know that at the screen we really have to add $E_0 + E_{radiation}$ to get the total field. In the Babinet case, we know that the result is $E_0 - E_{diffraction}$. Thus, we do expect a sign difference in the two results!!!

So finally after many weeks I am now more or less confirming the Babinet/Airy idea.

Another comment seems in order here. For radiation from a flat metal object, we do get the exact same result as from diffraction through the complementary aperture apart from a minus sign, and this confirms our Babinet/Airy approach. But for a dielectric object, we HAVE NO diffraction formula because we don't really have a diffraction theory where the screen is a dielectric. We could make one I suppose along the lines of Jackson's derivation. So there is nothing directly to compare to. But there are similarities which we will now ferret out.

10. Redo the Dielectric Object Case

Let's start with (6) above which we repeat here, { think of J = dP/dt }

$$\mathbf{A} = -ik \int \frac{e^{ikR}}{R} \mathbf{P}\, d^3r' \qquad (6)$$

Apply the curl to both sides, and use $\nabla G \approx ik\, \hat{\mathbf{R}}\, G$ to get

$$\mathbf{B} = k^2 \int \frac{e^{ikR}}{R} \hat{\mathbf{R}} \times \mathbf{P}\, d^3r'$$

Now make the obvious approximations to rewrite this as

$$\mathbf{B} = k^2 \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times \int e^{ik(R-r)}\, \mathbf{P}\, d^3r'$$

Finally, set $\mathbf{P(r')} = \chi \mathbf{E_0} \exp(i\mathbf{k_1} \bullet \mathbf{r'})$, use $ik(R - r) \approx ik(-\hat{\mathbf{r}} \bullet \mathbf{r'}) = -i\, \mathbf{k} \bullet \mathbf{r'}$ and assume constant χ to get

$$\mathbf{B} = k^2 \frac{e^{ikr}}{r} \chi \; \hat{\mathbf{r}} \times \mathbf{E_0} \int \exp(i[\mathbf{k_1} - \mathbf{k}] \bullet \mathbf{r'})\, d^3r'$$

Then define the form-factor in this way

$$g(\mathbf{k}) = (1/V) \int \exp(i[\mathbf{k_1} - \mathbf{k}] \bullet \mathbf{r'})\, d^3r'$$

and we replicate our result (17),

$$\mathbf{B} = k^2 \frac{e^{ikr}}{r} \chi \; (1/V) \; \hat{\mathbf{r}} \times \mathbf{E_0}\, g(\mathbf{k}) \qquad \mathbf{E} = k^2 \chi V \frac{e^{ikr}}{r} \; g(\mathbf{k}) \; (\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}})$$

Now this exact same form-factor is going to arise *whenever* we have an integral of this form,

$$\int \frac{e^{ikR}}{R}\,\mathbf{P}\,\mathbf{d}^3\mathbf{r}'$$

in which quantity P come from an incident plane wave and so has $\exp(ik_1 \bullet \mathbf{r}')$ dependence. For example, above we got this formula for scattering from a metallic plate,

$$\mathbf{B}(r) = -(1/i\lambda) \int \mathbf{B_0}(r') \frac{e^{ikR}}{R} \cos\theta \; dA'$$

in the presence of a plane wave. Based on the above, we rewrite this as

$$\mathbf{B}(r) = -(1/i\lambda) \frac{e^{ikr}}{r} \mathbf{B_0} \int \exp(i[\mathbf{k_1} - \mathbf{k}] \bullet \mathbf{r}')\, \mathbf{d}^3\mathbf{r}' \cos\theta' \; dA'$$

The only catch here is that we are doing an area integral, and we have an extra cosθ factor sitting in there, so we are NOT going to get the exact same form factor here. But this is a nice way to write a diffraction integral when a plane wave comes in at an oblique angle.

On the other hand, if the expo moves fast relative to cosθ, and since you know it moves most slowly in the forward direction, you can use the stationary phase idea to pull cosθ out of the integral and just evaluate it in the forward direction cosθ = 1. In this case, we really DO get the standard g form factor, although it is integrated only over an *area*. Thus, we expect diffration to give similar results to radiation theory in the case of thin objects! We found above that this is the case for the thin rectangular dielectric where we get a product of sinc functions, and we will see this below where we get the Airy result for a thin disk. The idea of cos = 1 is associated with the Fraunhofer limit.

Notice that we get this extra cosθ factor in the metal case because we had to start with n x B inside the integral, whereas with a dielectric we start with just P without the n x operator. The cosine then arises as we show above when B = curl A is done.

11. Comments on Portis's Dielectric Scattering Derivation

This discussion seems to begin on page 540, but it really begins on page 537 with a discussion of Thomson scattering, the Thomson formula being P (7). Portis is writing in (4) and (6) formulas for the A and B field of a dipole in terms of retarded time, whereas in the above we have always assumed sine time, so we have need for t-R/c factors. In effect, we have already dealt with all that, which is where the expo phase factor comes from in Jackson 9.3. You can of course not have the expo and have retarded time inside J in (9.3), and that seems to be the approach of Portis. Eventually the expo must "out" and we see it finally in his (26). Portis's notation just stinks here! He has r1 being the integration coordinate, and **k** being the incident k-vector, but $k\hat{\mathbf{r}}$ is the outgoing k-vector which I would normally write write as **k,** so (26) looks very confusing. Portis writes his final result as (25) without pulling the obvious expo stuff out of the integrand, and with r instead of r1 in χ. He never writes down the final result in a clean form!

Then suddenly in (27) he writes the same form-factor with some strange constants added outside, and then (28) is the final result in terms of this weirdly scaled form factor. This is awesomely confusing, just at a point where he had a chance to be clear about something. No reason is given for the scaling factors chosen. No reason is given in (29) for why the forward direction gives the "effective number of free charges". We don't even have free charges in the problem we are in the middle of working on! It's as if he pasted this section on f in from somewhere else. { But I now now that the forward direction gives the pure dipole result, which might be proportional then to the number of dipole objects }

So I think I am justified in having been immensely confused by Portis's presentation, and my derivation seems much more solid and clear.

However, to Portis's credit, he does come up with **Babinet for dielectrics** n page 544 which I have been seeking. He does not have a diffraction theory for computing the aperture, but he does know out to compute the dielectric disk shown on the right with P down. He presumably knows how to calculate the wall of dielectric, but he does not do that because he knows that it only affects "the forward direction", and away from this direction he can then equate the hole with the disk. I would guess that the wall is going to reflect some of the incident plane wave, and has the net effect of reducing the plane wave amplitude on the right by some snell formula. So this plan does not allow a comparison of diffraction to radiation, but it represents a reverse use of Babinet thinking to get the field of a hole in a dielectric! He goes on to do the disk which I want to do but have not yet done {done below}. I notice that he gets the Airy result.

Overall, this Chapter of Portis is interesting. A grab bag of problem solutions. Thomson, dielectric sphere, dielectric disk and then hole, then a square plate and hole, then multi-slits, and then a tapered aperture idea. Then suddenly on page 553 the theme changes to holograms. Then on 557 we are into crystal diffraction, and then page 561 we are off into Rayleigh scattering, and then page 567 is Cerenkov radiation from tachyons, and then later sublight particles in a medium. This is his second last chapter, so I guess he decided to just throw in all this stuff. The last chapter then is on lasers. Then as if that is not enough, we have appendices A through K covering quantum mechanics, vectors, complex integration, you name it! Very ambitious.

12. Scattering from a Dielectric Disk

From (17) we recall the general solution for any shape,

$$\mathbf{B} = k^2 \chi V \frac{e^{ikr}}{r} g(\theta,\phi)\ \hat{\mathbf{r}}\ \mathbf{x}\ \mathbf{E_0} \qquad (17)$$

$$g(\theta,\phi) = (1/V) \int \exp(i\,\mathbf{\Delta k \bullet r'}) d^3r' \qquad \text{where } \mathbf{\Delta k = k_1 - k} \qquad \mathbf{k}_1 = k\,\hat{\mathbf{z}}$$

and in cylindrical symmetry we have

$$g(\theta,\phi) = g(\theta) = (1/V) \int \rho'\, d\rho' \int d\phi' \int dz\ \exp(-ik\, S_\theta\, \rho' \cos\phi') \exp(ik[1 - C_\theta]z')$$

$$= \frac{1}{\pi a^2 t} \int_0^a \rho d\rho \int_0^{2\pi} d\phi \int_0^t dz \exp(i\, \Delta k_x \rho \sin\phi) \exp(i \Delta k_z\, z)$$

We assume a super-thin disk so the z integral gives t which then cancels the t in the front factor. The ϕ integral is then our old standard

$$\int_0^{2\pi} d\phi \exp(i\,\Delta k_\rho \rho \cos\phi) = 2\pi J_0(\Delta k_\rho \rho)$$

and then the ρ integration is another old standard:

$$\int_0^{a} \rho d\rho\, J_0(\Delta k_\rho \rho) = a^2 \frac{J_1(\Delta k_\rho a)}{(\Delta k_\rho a)}$$

so the complete result is then

$$g = 2\frac{J_1(\Delta k_\rho a)}{(\Delta k_\rho a)}$$

As noted in the general case, we have $\Delta k_\rho = -kS_\theta$, so result is

$$g = 2\frac{J_1(ka\sin\theta)}{(ka\sin\theta)}$$

and our complete answer is then

$$\mathbf{E} = k^2 \chi V \frac{e^{ikr}}{r} g(\theta,\phi)(\mathbf{E_0} - E_{0r}\,\hat{\mathbf{r}})$$

$$= k^2 \chi (\pi a^2 t) \frac{e^{ikr}}{r} 2\frac{J_1(ka\sin\theta)}{(ka\sin\theta)} (\mathbf{E_0} - E_{0r}\,\hat{\mathbf{r}})$$

In general the second term is small being the order of ρ/r in the diffraction limit, in which case we would claim a Fraunhofer result, which is independent of polarization direction,

$$\mathbf{E} = k^2 \chi\ (\pi a^2 t)\ \frac{e^{ikr}}{r} 2\frac{J_1(ka\sin\theta)}{(ka\sin\theta)}\ \mathbf{E_0}$$

This is the result I have long wondered about! You get exactly the Airy pattern $\sin\theta = \rho/z$ where now we have changed ρ to be ρ on the observation screen. We can also replace χ with the index to get:

$$\mathbf{E} = k^2 [\ (n-1)/2\pi\ ]\ (\pi a^2 t)\ \frac{e^{ikr}}{r} 2\frac{J_1(ka\rho/z)}{(ka\rho/z)}\ \mathbf{E_0}$$

So this would be our best model so far for a floater. But we have more yet to come!

13. **Scattering from a dielectric sphere**

Start from the general formula from (17) above,

$$g(\theta,\phi) = (1/V) \int \exp(i\, \mathbf{\Delta k \bullet r'}) d^3r' \qquad \text{where } \mathbf{\Delta k = k_1 - k} \qquad \mathbf{k_1} = k\, \hat{\mathbf{z}}$$

But we have already noted that with spherical symmetry we have, with V = that of a sphere,

$$g = \frac{4\pi}{(4/3)\pi a^3} \int_0^a r'^2\, dr' \frac{\sin(\Delta k\, r')}{(\Delta k\, r')}$$

This r' integral can be done as indefinite with Maple help and we get

$$g = \frac{4\pi}{(4/3)\pi a^3}\ a^3 \left[\frac{\cos(\Delta k\, a)}{(\Delta k\, a)^2} - \frac{\sin(\Delta k\, a)}{(\Delta k\, a)^3} \right] = 3 \left[\frac{\cos(\Delta k\, a)}{(\Delta k\, a)^2} - \frac{\sin(\Delta k\, a)}{(\Delta k\, a)^3} \right]$$

where $\Delta k = 2k \sin(\theta/2)$. So here is the scattered electric field from a dielectric sphere:

$$\mathbf{E} = k^2 \chi (4\pi a^3) \frac{e^{ikr}}{r} \left[\frac{\cos(2ka \sin(\theta/2))}{(2ka \sin(\theta/2))^2} - \frac{\sin(2ka \sin(\theta/2))}{(2ka \sin(\theta/2))^3} \right] (\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}})$$

where the 3 from the volume cancels the 3 from the g factor.

14. Scattering from a dielectric spherical shell of radius a and thickness t

In this case we get

$$g = \frac{4\pi}{4\pi a^2 t} \int_a^{a+t} r'^2\, dr' \frac{\sin(\Delta k\, r')}{(\Delta k\, r')} = \frac{4\pi}{4\pi a^2 t} * t * a^2 * \frac{\sin(\Delta ka)}{(\Delta ka)} = \frac{\sin(\Delta ka)}{(\Delta ka)}$$

and then

$$\mathbf{E} = k^2 \chi (4\pi a^2 t) \frac{e^{ikr}}{r} \frac{\sin(2ka \sin(\theta/2))}{(2ka \sin(\theta/2))} (\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}})$$

This might apply to a floater that has nothing but the membrane left!

15. What about absorbing objects and floater models?

On page 120 Jackson puts an atomic electron in a harmonic oscillator with a restoring force and comes up with a crude formula for χ in that case which we could extend to the macroscopic χ and of course we then get a real value for χ. We want to add a damping term and get complex χ, but he does not do that. Portis does similar stuff on page 78, making a simple model for the atom. Portis actually does a much fancier job with some good toy models. Portis touches on the subject of absorption on page 546, but his point there is to show that, if you assume field is absorbed within some distance d inside an object, then for flat objects the z-integral only gets evaluated at one endpoint in (45), and you pick up a factor of ki, and then you replicate the diffraction formulas from your radiation theory. So he is trying to show that opaque flat objects do the same as metal flat objects for diffraction. In this case, all the radiation is done by the first thin layer of the flat object that the incident wave hits.

I think we could extend this idea to a spherical absorbing object. Perhaps only some outer shell is then active, and your result looks more like the dielectric shell result and less like the dielectric sphere result. If floater blood cells were filled with aborbing stuff, this might be the right model. If they are instead filled with water with a thin **membrane**, then we still expect the thin-shell result. The question might then be to figure out the index of that membrane which is made of some lipid stuff. Campbell page 156 says the membrane is about 7.5 nm thick, or 75 A, much smaller than λ = 5000 A. The phospholipid molecule is shown on page 74. All I can say is that this membrane material probably has an index that differs somewhat from water, but which way I don't know since we think water is excluded from the inside of the membrane. You can learn on this subject by searching on "scattering from cells". I found a PDF with this interstesting table,

TABLE I
INDEX OF REFRACTION VALUES USED IN THE FDTD SIMULATION

Cell Component	Index of Refraction	Reference
extracellular fluid	1.35	[28]
cytoplasm	1.37	[29]
nucleus	1.39	[29]
mitochondria	1.42	[30]
melanin	1.7	[31]

The authors note "There have been few published reports of the index of refraction of cells and organelles, and the values listed in Table I were taken from the literature and used in these simulations." (www.nmr.mgh.harvard.edu/~adunn/papers/dunn_fdtd3d.pdf .) The paper also shows some experimentally measured diffraction patterns! So I could just use that data and ignore the theory, another approach. In this paper, the cell is modeled as a composite object with a scattering of organelles and a nucleus, etc. The membrane is not even considered or mentioned. Perhaps it is so thin that diffraction from it would be tiny compared to diffraction from the other parts. In particular, the fluid inside the cell shows up at index 1.37 above. However, in a floater stuff may have diffused out leaving only the membrane. I would have to learn more about what is in a red blood cell (no nucleus I recall), and what is in a hydrated on in the vitreous. All fascinating stuff. For the moment though I keep this document focussed on the simple calculations of dielectric scattering. The tough part is saying something about the amplitude of the scattering, but I think we can make some reasonable assumptions on this! Continuing this elsewhere!