Scattering from Dielectric Objects without ignoring radiated field PhL 1.28.03

In the conventional approach, we start with this equation

$$\mathbf{A} = -ik \int \frac{e^{ikR}}{R} \mathbf{P(r')}\, d^3r'$$

and we know that $P(r') = \chi E(r')$, where E is the *total* electric field inside the medium. The coefficient χ already accounts for microscopic averaging and Jackson shows how you can relate it to γ, the χ factor for an individual atom or molecule in the material. So E is the total average macroscopic E field in the object of interest. Now we really have to say that $\mathbf{E} = \mathbf{E_0} + \mathbf{E_{rad}}$, where $\mathbf{E_0}$ is our incident plane way, and E_{rad} is the field we are trying to calculate, the one we will get from A, then B, then E. So we have then a bit of a problem. The usual assumption is that $\mathbf{E_{rad}}$ is much smaller than $\mathbf{E_0}$ so we can ignore it in the integral. When we do this, for example, in our sphere calculation, we get this result:

$$\mathbf{E} = k^2 \chi (4\pi a^3) \frac{e^{ikr}}{r} f(\theta) \ (\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}})$$

where $f(\theta)$ is a form factor which is on the order of unity. It is -1/3 at central and drops to zero etc. In our particular problem, we have $k = 2\pi/\lambda$ and $a = 4\ \mu$ and $\lambda = 1/2\ \mu$, so $ka = 16\pi$. Also, $(n-1) = 2\pi\chi$. So we can write the above as:

$$\mathbf{E_{rad}} = (ka)^2\, 4\pi\chi\, e^{ikr}\, (a/r)\ f(\theta)\ (\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}}) = (16\pi)^2\, 2\, (n-1)\, e^{ikr}\, (a/r)\ f(\theta)\ (\mathbf{E_0} - E_{0r}\, \hat{\mathbf{r}})$$

Unless n is exceedingly small, it is clear that our field is much larger than $\mathbf{E_0}$ at r = a, the surface of the sphere. But use of this equation is illegal there because in deriving the above, we assume r >> a in order to expand the exponential to first term. So we are not allowed to evaluate the above anywhere near the sphere, so we cannot know whether $\mathbf{E_{rad}} << \mathbf{E_0}$ inside the sphere!

What can we do about this?

(1) We can go back to (9.3) and try to do our calculation more correctly without assuming r >> d. But then we have J in the integral and we cannot do the parts trick to replace it with ρ. I thought I was doing this approach in the Appendix calculation below, but the parts trick does not work with the propagator in there, so cannot get P inside the integral.

(2) Looking at Jackson (6.27), why can't I associate J with the displacement current, set H = B, and get

$$\mathbf{J_{disp}} = (\varepsilon/4\pi)\, \partial_t\, \mathbf{E} = (-i\omega\varepsilon/4\pi)\, \mathbf{E}$$

The reason you cannot do this is that the J driving A does *not* include this term. The displacement current is just the $\partial_t\, \mathbf{E}$ in the curl B equation. Notice that the above also exists outside our scattering object!

(3) We can try to figure out what **J** is inside a dielectric. I know now what it is for a mechanical dipole, but don't know how to get to the macroscopic **J** in the dielectric material. I know for example that

$\mathbf{J_i(r)} = -2i\omega \mathbf{p}\, \delta(\mathbf{r} - \mathbf{x_i})$ = the current due to a very closely spaced mechanical dipole

I can put this into our equation for A to get

$$\mathbf{A(r)} = -2i\omega\, (\gamma\, \mathbf{E_i})\, \Sigma \frac{e^{ikRi}}{R_i} \quad \text{where } \mathbf{R_i} = |\, \mathbf{r} - \mathbf{x_i}|$$

where $\mathbf{E_i}$ is that internal macroscopic field that dipoles feel. In the world of page 116, we can replace E_i with $(4\pi/3)$ P and we then set $P = \chi E$ and then end up with

$$\mathbf{A(r)} = -2i\omega\, (\gamma(4\pi/3)\, \chi\, \mathbf{E})\, \Sigma \frac{e^{ikRi}}{R_i} \quad \text{where } \mathbf{R_i} = |\, \mathbf{r} - \mathbf{x_i}\,|$$

or something along these lines. The sum is due to the current delta functions. If we had a crystal, we might be able to actually do the sum, which is just over our scattering object. But in general, cannot do it, so cannot really know A, even in this toy model of nothing but dipoles. But it was a good shot.

(4) I can try the full **multipole** expansion approach and use boundary conditions at the dielectric sphere surface. I think this might be **Mie Scattering**. I have done some probing here, and things are mighty complicated. The claim is that Mie even ferrets out the rainbows and multiple paths somehow. If you use the Debye expansion you sort by multiple paths. It is pretty mysterious how a general solution can contain all this phenomenological stuff. Another point is that in the multipole expansion of a plane wave, you don't just have dipole, so higher moments are important in the general solution. I think it might be fun to take a look at this stuff. All I did was compute the dipole term in the radiation zone of a dielectric sphere. I am not even sure it is the leading term. The subject is of great interest because it includes familiar objects like rainbows, coronas, glories, and so on. You can even download programs to do the Mie scattering computation on your PC!

Here I thought I was computing A but I was using P inside even though cannot do parts on J to get it that way, so this entire calculation is meaningless.

I have done a bit on this and this is where I get:

$$\mathbf{A(r)} = -2\pi ik\chi\, \mathbf{E_0} \int dr'\, r'^2 \int_{-1}^{1} dx' \frac{e^{ikR}}{R}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r'}$ and where I selected the z' axis to point along the **r** direction, and where we compute that

$$R = \sqrt{r'^2 + z^2 - 2r'z\, x'} \qquad \text{where } x' = \cos\theta'.$$

This looks like one of those famous undoable integrals, but it is in fact quite doable at least in x'. Think first that

$$R = \sqrt{r'^2 + z^2 - 2r'z\,x'} \quad = \sqrt{a + b\,x'} \qquad\qquad a = r'^2 + z^2 \qquad b = -2r'z$$

Then set $y = \sqrt{a + b\,x'}$, $dy = (1/\sqrt{a + b\,x'})\,(b/2)dx'$ so that $dx'/\sqrt{a + b\,x'} = (2/b)dy$ which makes the integral quite trivial

$$\int_{-1}^{1} dx' \frac{e^{ikR}}{R} = (2/b) * \int_{\sqrt{a-b}}^{\sqrt{a+b}} dy\, e^{iky} = \frac{2}{ikb} * \{ \exp(ik\sqrt{a+b}) - \exp(ik\sqrt{a-b}) \}$$

$$= \frac{2}{ik(-2r'z)} * \{ \exp(ik[r'-z]) - \exp(ik[r'+z]) \}$$

and this leaves us with

$$\mathbf{A}(\mathbf{r}) = -2\pi ik\chi\, \mathbf{E_0} * \frac{2}{ik(-2z)} \int dr'\, r' \{ \exp(ik[r'-z]) - \exp(ik[r'+z]) \}$$

$$= -2\pi ik\chi\, \mathbf{E_0} * \frac{2}{ik(-2z)} [\, -2i\sin(kz)\,] * \int dr'\, r'\, e^{ikr'}$$

Maple tells us the inside integral from 0 to a is

$$-(1/k)^2 * \{ 1 + e^{ika} [\, ika - 1] \}$$

So final result is then

$$\mathbf{A}(\mathbf{r}) = -2\pi ik\chi\, \mathbf{E_0} * \frac{2}{ik(-2z)} [\, -2i\sin(kz)\,] * (-1)\,(1/k)^2 * \{ 1 + e^{ika} [\, ika - 1] \}$$

$$= (4\pi i\chi/k)\, \mathbf{E_0} \frac{\sin(kz)}{kz} \{ 1 + e^{ika} [\, ika - 1] \}$$

This is amazing! What have we done here? We started with Jackson (9.3)