Scattering from the dielectric sphere PhL 2.5.03
updated: 2.19.03

Assume ε_1 in the outer region, and ε_2 inside the sphere. Outer region = 1, inner = 2.

We start off as on page 569. We use 16.139 for the plane wave coming in (with the B' and k' mods), and we assume 16.141 for the scattered fields with these same mods. The new feature in this problem compared to the conducting sphere is that we have to model the fields inside the sphere. It seems to me that 16.131 is exactly what we want in this case: all modes must be finite at r=0 because there is nothing singular there. We don't expect to have infinite E or B fields at the origin!

Now here are the boundary conditions at the sphere surface:

1) $\varepsilon_1 \mathbf{E_1} \bullet \hat{\mathbf{r}} = \varepsilon_2 \mathbf{E_2} \bullet \hat{\mathbf{r}}$ // the normal field has a discontinuity due to pol charge

2) $\hat{\mathbf{r}} \times \mathbf{E_1} = \hat{\mathbf{r}} \times \mathbf{E_2}$ // the tangential field is same since no surface currents.

3) $\mathbf{B_1} \bullet \hat{\mathbf{r}} = \mathbf{B_2} \bullet \hat{\mathbf{r}}$ // the normal field is same since no magnetic charge on surface

4) $\hat{\mathbf{r}} \times \mathbf{B_1} = \hat{\mathbf{r}} \times \mathbf{B_2}$ // the tangential field is same since no surface currents.

Comments on conditions 2 and 4. These come from little contour loops which are done at the surface. We will find that things are continuous across the boundary as long as a little area loop of the curl X gives 0, where X = E or B depending on which condition you have. Since the loop is of super thin area, the only way to get something non-zero is to have some kind of infinite surface term like a surface current. We do expect to have such a thin surface charge of this nature. Looking at 6.112, we expect the surface charge to be $-\nabla \bullet P$. If we follow through on this with our pill box, we get condition 1. Now 6.112 also shows a volume *current* which is $d\mathbf{P}/dt$. What effect does this have? It tells us that $\nabla \times \mathbf{B} = 4\pi i \mu k \mathbf{P}$. This in turn tells us that $(B_1 - B_2) \bullet \hat{\mathbf{t}} = 4\pi i \mu k \int \mathbf{P} \bullet \mathbf{dA}$ where the integral is over the loop surface. Half the loop is in material 1, the other half in material 2. In either half, we have $P = \chi E$. Unless at least one of the tangential E fields is infinite at the surface, this integral goes to 0 as the surface shrinks, and we conclude that condition 4 is OK after all. Condition 2 is valid similarly provided the tangential B field is finite.

Comments on conditions 3 and 4. There are conditions on the true B field, but not on our special B' field that we need to use in the multipole theory in the presence of ε and μ ! Let's now replace 3 and 4 with corrected versions for the B' fields! (see separate document on this subject)

3) $\sqrt{\mu_1 \varepsilon_1}\, \mathbf{B'_1} \bullet \hat{\mathbf{r}} = \sqrt{\mu_2 \varepsilon_2}\, \mathbf{B'_2} \bullet \hat{\mathbf{r}}$

4) $\sqrt{\mu_1 \varepsilon_1}\, \hat{\mathbf{r}} \times \mathbf{B'_1} = \sqrt{\mu_2 \varepsilon_2}\, \hat{\mathbf{r}} \times \mathbf{B'_2}$.

So, we are going to have 4 "unknowns" in this problem: 2 coefficients in the scattered field equations, and 2 coefficients in the "inside field" equations. But we know that each curl condition above will yield

two separate conditions, so we would appear to have 6 conditions on our 4 unknowns. In the conducting sphere case we had 3 equations in 2 unknowns, but two of the equations were the same, maybe that will happen here as well.

Now let's write out the expansions:

$$\mathbf{E}_{inc} = (1/2)\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\; 2j_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1} \pm 2/k_1\,\nabla \times \{\, j_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{B'}_{inc} = (1/2)\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\mp 2i\, j_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1} - 2i/k_1\,\nabla \times \{\, j_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{E}_{sc} = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\;\alpha_{\pm}(\ell)\,h^{(1)}{}_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1} \pm \beta_{\pm}(\ell)/k_1\,\nabla \times \{\, h^{(1)}{}_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{B'}_{sc} = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\mp i\,\beta_{\pm}(\ell)\,h^{(1)}{}_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1} - i\alpha_{\pm}(\ell)/k_1\,\nabla \times \{\, h^{(1)}{}_{\ell}(k_1r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{E}_2 = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\;\alpha'_{\pm}(\ell)\,j_{\ell}(k_2r)\,\mathbf{X}_{\ell,\pm1} \pm \beta'_{\pm}(\ell)/k_2\,\nabla \times \{\, j_{\ell}(k_2r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{B'}_2 = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\mp i\,\beta'_{\pm}(\ell)\,j_{\ell}(k_2r)\,\mathbf{X}_{\ell,\pm1} - i\alpha'_{\pm}(\ell)/k_2\,\nabla \times \{\, j_{\ell}(k_2r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

Boundary Condition (1): $\quad \varepsilon_1\,(\mathbf{E}_{inc} + \mathbf{E}_{sc}) \bullet \hat{\mathbf{r}} = \varepsilon_2 \mathbf{E}_2 \bullet \hat{\mathbf{r}}$

We know that the first expansion term gives nothing, and in the second expansion term we get only the first term of 16.143. If we ignore common factors, we get,

$$\varepsilon_1\,\{\pm 2/k_1\,j_{\ell}(k_1r) \pm \beta_{\pm}(\ell)/k_1\,h^{(1)}{}_{\ell}(k_1r)\} = \varepsilon_2\,\{\pm \beta'_{\pm}(\ell)/k_2\,j_{\ell}(k_2r)\}$$

Boundary Condition (3): $\quad (\mathbf{B'}_{inc} + \mathbf{B'}_{sc}) \bullet \hat{\mathbf{r}} = \mathbf{B'}_2 \bullet \hat{\mathbf{r}}$

This is done in similar fashion, but with the B fields, and we get

$$-2i\sqrt{\mu_1\varepsilon_1}\,/k_1\,j_{\ell}(k_1r) - i\sqrt{\mu_1\varepsilon_1}\,\alpha_{\pm}(\ell)/k_1\,h^{(1)}{}_{\ell}(k_1r) = -i\alpha'_{\pm}(\ell)\sqrt{\mu_2\varepsilon_2}\,/k_2\,j_{\ell}(k_2r)$$

Boundary Condition (2): $\quad \hat{\mathbf{r}} \times (\mathbf{E}_{inc} + \mathbf{E}_{sc}) = \hat{\mathbf{r}} \times \mathbf{E}_2$

Here we pick up the two kinds of terms and we get two conditions, to wit,

$\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ terms: $\quad 2j_{\ell}(k_1r) + \alpha_{\pm}(\ell)\,h^{(1)}{}_{\ell}(k_1r) = \alpha'_{\pm}(\ell)\,j_{\ell}(k_2r)$

$-\mathbf{X}_{\ell m}/r$ terms: $\quad \pm 2/k_1\,\partial_r[r\,j_{\ell}(k_1r)] \pm \beta_{\pm}(\ell)/k_1\,\partial_r[r\,h^{(1)}{}_{\ell}(k_1r)] = \pm\,\beta'_{\pm}(\ell)/k_2\,\partial_r[r\,j_{\ell}(k_2r)]$

Boundary Condition (4): $\quad \hat{\mathbf{r}} \times (\mathbf{B'}_{inc} + \mathbf{B'}_{sc}) = \hat{\mathbf{r}} \times \mathbf{B'}_2$

Same idea as in the previous boundary condition:

$\hat{\mathbf{r}} \times \mathbf{X}_{\ell m}$ terms: $\quad \mp 2i\sqrt{\mu_1\varepsilon_1}\;\, j_{\ell}(k_1r) \mp i\sqrt{\mu_1\varepsilon_1}\,\beta_{\pm}(\ell)\,h^{(1)}{}_{\ell}(k_1r) = \mp i\sqrt{\mu_2\varepsilon_2}\,\beta'_{\pm}(\ell)\,j_{\ell}(k_2r)$

- $\mathbf{X_{\ell m}}$ /r terms: $- 2i\sqrt{\mu_1\varepsilon_1}\,/k_1\ \partial_r[r\, j_\ell(k_1 r)] - i\sqrt{\mu_1\varepsilon_1}\,\alpha_\pm(\ell)/k_1\ \partial_r[r\, h^{(1)}_\ell(k_1 r)] = - i\sqrt{\mu_2\varepsilon_2}\,\alpha'_\pm(\ell)/k_2\ \partial_r[r\, j_\ell(k_2 r)]$

Summary of Boundary Conditions:

(1) $\varepsilon_1 \{ \pm 2/k_1\, j_\ell(k_1 r) \pm \beta_\pm(\ell)/k_1\, h^{(1)}_\ell(k_1 r) \} = \varepsilon_2 \{ \pm \beta'_\pm(\ell)/k_2\, j_\ell(k_2 r) \}$

(2a) $2j_\ell(k_1 r) + \alpha_\pm(\ell)\, h^{(1)}_\ell(k_1 r) = \alpha'_\pm(\ell)\, j_\ell(k_2 r)$

(2b) $\pm 2/k_1\, \partial_r[r\, j_\ell(k_1 r)] \pm \beta_\pm(\ell)/k_1\, \partial_r[r\, h^{(1)}_\ell(k_1 r)] = \pm \beta'_\pm(\ell)/k_2\, \partial_r[r\, j_\ell(k_2 r)]$

(3) $- 2i\sqrt{\mu_1\varepsilon_1}\,/k_1\, j_\ell(k_1 r) - i\sqrt{\mu_1\varepsilon_1}\,\alpha_\pm(\ell)/k_1\, h^{(1)}_\ell(k_1 r) = - i\alpha'_\pm(\ell)\sqrt{\mu_2\varepsilon_2}\,/k_2\, j_\ell(k_2 r)$

(4a) $\mp 2i\sqrt{\mu_1\varepsilon_1}\ j_\ell(k_1 r) \mp i\sqrt{\mu_1\varepsilon_1}\,\beta_\pm(\ell)\, h^{(1)}_\ell(k_1 r) = \mp i\sqrt{\mu_2\varepsilon_2}\,\beta'_\pm(\ell)\, j_\ell(k_2 r)$

(4b) $- 2i\sqrt{\mu_1\varepsilon_1}\,/k_1\ \partial_r[r\, j_\ell(k_1 r)] - i\sqrt{\mu_1\varepsilon_1}\,\alpha_\pm(\ell)/k_1\ \partial_r[r\, h^{(1)}_\ell(k_1 r)] = - i\sqrt{\mu_2\varepsilon_2}\,\alpha'_\pm(\ell)/k_2\ \partial_r[r\, j_\ell(k_2 r)]$

We seem to have 6 conditions on our 4 coefficients, which is a little scary. I can see from the above that (1) and (4a) completely determine $\beta_\pm(\ell)$ and $\beta'_\pm(\ell)$. Similarly, it would seem that (2a) and (3) completely determine $\alpha_\pm(\ell)$ and $\alpha'_\pm(\ell)$. This means we have (2b) and (4b) left over, and these could mean we are overdetermined, but I presume they will in fact just be redundant.

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I am right now updating this document after having done the "second Mie scattering attempt". YES, two of the 6 equations above are redundant. I first did all this by hand with about 4 pages of manual calculations strewn with sign errors. Instead, let's plan to feed things into Maple for a quick and correct solution. I will use the four equations bolded above and copied here,

(2a) $2j_\ell(k_1 r) + \alpha_\pm(\ell)\, h^{(1)}_\ell(k_1 r) = \alpha'_\pm(\ell)\, j_\ell(k_2 r)$

(2b) $\pm 2/k_1\, \partial_r[r\, j_\ell(k_1 r)] \pm \beta_\pm(\ell)/k_1\, \partial_r[r\, h^{(1)}_\ell(k_1 r)] = \pm \beta'_\pm(\ell)/k_2\, \partial_r[r\, j_\ell(k_2 r)]$

(4a) $\mp 2i\sqrt{\mu_1\varepsilon_1}\ j_\ell(k_1 r) \mp i\sqrt{\mu_1\varepsilon_1}\,\beta_\pm(\ell)\, h^{(1)}_\ell(k_1 r) = \mp i\sqrt{\mu_2\varepsilon_2}\,\beta'_\pm(\ell)\, j_\ell(k_2 r)$

(4b) $- 2i\sqrt{\mu_1\varepsilon_1}\,/k_1\ \partial_r[r\, j_\ell(k_1 r)] - i\sqrt{\mu_1\varepsilon_1}\,\alpha_\pm(\ell)/k_1\ \partial_r[r\, h^{(1)}_\ell(k_1 r)] = - i\sqrt{\mu_2\varepsilon_2}\,\alpha'_\pm(\ell)/k_2\ \partial_r[r\, j_\ell(k_2 r)]$

Let's first get media factors all grouped as ratios, and simplify signs, cancel i's

(2a) $2j_\ell(k_1 r) + \alpha_\pm(\ell)\, h^{(1)}_\ell(k_1 r) = \alpha'_\pm(\ell)\, j_\ell(k_2 r)$

(2b) $+ 2\, \partial_r[r\, j_\ell(k_1 r)] + \beta_\pm(\ell)\partial_r[r\, h^{(1)}_\ell(k_1 r)] = + (k_1/k_2)\, \beta'_\pm(\ell)\, \partial_r[r\, j_\ell(k_2 r)]$

(4a) $\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}}\ \{ +2j_\ell(k_1 r) + \beta_\pm(\ell)\, h^{(1)}_\ell(k_1 r) \} = +\beta'_\pm(\ell)\, j_\ell(k_2 r)$

(4b) $\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\ \{ + 2\,\partial_r[r\, j_\ell(k_1 r)] + \alpha_\pm(\ell)\,\partial_r[r\, h^{(1)}_\ell(k_1 r)] \} = + (k_1/k_2)\,\alpha'_\pm(\ell)\,\partial_r[r\, j_\ell(k_2 r)]$

where in the last line we set $\mu_1 = \mu_2$. Now use $(k_1/k_2) = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$ to simplify one more time, and also set $n = k_2/k_1$ and recall that 2 = inside, so

(2a) $2 j_\ell(k_1 r) + \alpha_\pm(\ell)\, h^{(1)}_\ell(k_1 r) = \alpha'_\pm(\ell)\, j_\ell(k_2 r)$
(2b) $+ 2\,\partial_r[r\, j_\ell(k_1 r)] + \beta_\pm(\ell)\partial_r[r\, h^{(1)}_\ell(k_1 r)] = + (1/n)\,\beta'_\pm(\ell)\,\partial_r[r\, j_\ell(k_2 r)\,]$
(4a) $(1/n)\{ +2 j_\ell(k_1 r) + \beta_\pm(\ell)\, h^{(1)}_\ell(k_1 r)\ \} = +\beta'_\pm(\ell)\, j_\ell(k_2 r)$
(4b) $\{ + 2\,\partial_r[r\, j_\ell(k_1 r)] + \alpha_\pm(\ell)\,\partial_r[r\, h^{(1)}_\ell(k_1 r)] \} = +\ \alpha'_\pm(\ell)\,\partial_r[r\, j_\ell(k_2 r)]$

Now concerning each derivative, we can do for example this:

$\partial_r[r\, j_\ell(k_2 r)\,] = \partial_x [\, x\, j_\ell(x)\,]^{x = k2\,a} = \psi'_2$
$\partial_r[r\, h^{(1)}_\ell(k_1 r)] = \xi'_2$

and for each non-derivative function reference, we do like so,

$j_\ell(k_1 r) = \psi_1 / (k_1 r)$

where we are now converting to the Ricatti Bessel versions. This the gives,

(2a) $2\,\psi_1/k_1 + \alpha_\pm(\ell)\,\xi_1/k_1 = \alpha'_\pm(\ell)\psi_2/k_2$
(2b) $2\,\psi'_1 + \beta_\pm(\ell)\xi'_1 = + (1/n)\,\beta'_\pm(\ell)\psi'_2$
(4a) $2\,\psi_1/k_1 + \beta_\pm(\ell)\xi_1/k_1 = +n\,\beta'_\pm(\ell)\psi_2/k_2$
(4b) $2\,\psi'_1 + \alpha_\pm(\ell)\xi'_1 = +\ \alpha'_\pm(\ell)\psi'_2$

Now clear the factors of k, with $(1/n) = k_1 / k_2$

(2a) $2\,\psi_1 + \alpha_\pm(\ell)\,\xi_1 = (1/n)\,\alpha'_\pm(\ell)\psi_2$
(2b) $2\,\psi'_1 + \beta_\pm(\ell)\xi'_1 = + (1/n)\,\beta'_\pm(\ell)\psi'_2$
(4a) $2\,\psi_1 + \beta_\pm(\ell)\xi_1 = +\,\beta'_\pm(\ell)\psi_2$
(4b) $2\,\psi'_1 + \alpha_\pm(\ell)\xi'_1 = +\ \alpha'_\pm(\ell)\psi'_2$

Next, lets change notation so that 2 = inside = i, and 1 = outside = o,

(2a) $\psi_o + \alpha_\pm(\ell)/2\ \xi_o = (1/n)\,\alpha'_\pm(\ell)/2\psi_i$
(2b) $\psi'_o + \beta_\pm(\ell)/2\xi'_o = + (1/n)\,\beta'_\pm(\ell)/2\psi'_i$
(4a) $\psi_o + \beta_\pm(\ell)/2\xi_o = +\ \beta'_\pm(\ell)/2\psi_i$
(4b) $\psi'_o + \alpha_\pm(\ell)/2\xi'_o = +\ \alpha'_\pm(\ell)/2\psi'_i$

Now rewrite this in "standard form"

(2a) $(1/n)\,\alpha'_{\pm}(\ell)/2\psi_i \;-\; \alpha_{\pm}(\ell)/2\,\xi_o \;=\; \psi_o$

(2b) $(1/n)\,\beta'_{\pm}(\ell)/2\psi'_i \;-\; \beta_{\pm}(\ell)/2\,\xi'_o \;=\; \psi'_o$

(4a) $\beta'_{\pm}(\ell)/2\psi_i \;-\; \beta_{\pm}(\ell)/2\,\xi_o \;=\; \psi_o$

(4b) $\alpha'_{\pm}(\ell)/2\psi'_i \;-\; \alpha_{\pm}(\ell)/2\xi'_o \;=\; \psi'_o$

Now let's *compare* these to the results of our "second attempt" Mie calculation with $\mu_i = \mu_o$:

$$
\begin{aligned}
(1/n)\psi_i\, c_\ell \;+\; \xi_o\, b_\ell &= \psi_o \\
(1/n)\psi'_i\, d_\ell \;+\; \xi'_o\, a_\ell &= \psi'_o \\
\psi_i\, d_\ell \;+\; \xi_o\, a_\ell &= \psi_o \\
\psi'_i\, c_\ell \;+\; \xi'_o\, b_\ell &= \psi'_o
\end{aligned}
$$

By inspection, we have,

$$
\begin{aligned}
\alpha'_{\pm}(\ell)/2 &= c_\ell \\
\beta'_{\pm}(\ell)/2 &= d_\ell \\
-\,\alpha_{\pm}(\ell)/2 &= b_\ell \\
-\,\beta_{\pm}(\ell)/2 &= a_\ell
\end{aligned}
$$

and it should be taken note that the coefficients are *independent* of photon helicity.

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" The next trick is to come up with phase shifts, if that is possible. In Problem 16.12 Jackson hints that phase shifts are possible At some point, I could go look up the Mie and Debye 1908 paper. I would of course expect this subject to be summarized in many later works, but I have not found such a review on the web. I did want to do this for myself before reviewing someone else's results. Maybe Born and Wolf will have the results as Jackson hints. I wonder if there are advanced texts that are much heavier duty that even Jackson's text. "

I have left in that last old paragraph just for fun, we have now found the references, etc.

Conclusions

Let's go back and look at our expansions here,

$$\mathbf{E}_{inc} = (1/2)\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\;2j_\ell(k_1 r)\,\mathbf{X}_{\ell,\pm1} \;\pm 2/k_1\,\nabla \times \{\, j_\ell(k_1 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{E}_{sc} = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\;\alpha_{\pm}(\ell)\,h^{(1)}_\ell(k_1 r)\,\mathbf{X}_{\ell,\pm1} \;\pm \beta_{\pm}(\ell)/k_1\,\nabla \times \{\, h^{(1)}_\ell(k_1 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

$$\mathbf{E}_2 = (1/2)\,\Sigma\, i^{\ell}\sqrt{4\pi(2\ell+1)}\,[\;\alpha'_{\pm}(\ell)\,j_\ell(k_2 r)\,\mathbf{X}_{\ell,\pm1} \;\pm \beta'_{\pm}(\ell)/k_2\,\nabla \times \{\, j_\ell(k_2 r)\,\mathbf{X}_{\ell,\pm1}\}\;]$$

where

$$\mathbf{X}_{\ell,\pm1} = 1/\sqrt{\ell(\ell+1)}\;\;(-i)\;\mathbf{r} \times \nabla\, Y_{\ell,\pm1}$$

and

$$Y_{\ell,+1} = \sqrt{(2\ell+1)/4\pi\ell(\ell+1)}\ \ P_\ell^1(\cos\theta)\, e^{i\phi}$$
$$Y_{\ell,-1} = -\sqrt{(2\ell+1)/4\pi\ell(\ell+1)}\ \ P_\ell^1(\cos\theta)\, e^{-i\phi}$$

Let's define some M and N functions as follows

$$\mathbf{M_\pm} \equiv j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_\pm \qquad \text{where} \qquad \psi_\pm \equiv \pm P_\ell^1(\cos\theta)\, e^{\pm i\phi}$$

$$\mathbf{N_\pm} \equiv (1/k)\, \nabla \times \mathbf{M_\pm}$$

Then we can rewrite the above expansions as (assuming appropriate Bessel functions)

$$\mathbf{E}_{inc,\pm} = (-i) \Sigma K_\ell \left[\mathbf{M_\pm} \pm \mathbf{N_\pm} \right]$$

$$\mathbf{E}_{sc,\pm} = (-i) \Sigma K_\ell \left[\alpha_\pm(\ell)/2\ \mathbf{M_\pm} \pm \beta_\pm(\ell)/2\ \mathbf{N_\pm} \right]$$

$$\mathbf{E}_{2,\pm} = (-i) \Sigma K_\ell \left[\alpha'_\pm(\ell)/2\ \mathbf{M_\pm} \pm \beta'_\pm(\ell)/2\ \mathbf{N_\pm} \right]$$

These are the field expansions for circular polarization of the incident plane wave. Now let's define some half-the-sum and half-the-difference functions like so,

$$\mathbf{M_e} \equiv (1/2) \left[M_+ - M_- \right] = j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_e \qquad \text{where} \quad \psi_e \equiv P_\ell^1(\cos\theta)\cos\phi$$

$$\mathbf{M_o} \equiv (1/2i) \left[M_+ + M_- \right] = j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_o \qquad \text{where} \quad \psi_o \equiv P_\ell^1(\cos\theta)\sin\phi$$

We now use this linear combination

$$\mathbf{E} = (\mathbf{E}_+ + \mathbf{E}_-)/2 = \hat{\mathbf{x}}\ e^{ikz}$$

to find that (where we use our knowledge that $\alpha_\pm = \alpha$ is independent of helicity)

$$\mathbf{E}_{inc,} = (-i) \Sigma K_\ell \left[(i\ \mathbf{M_o}) + (\ \mathbf{N_e}) \right]$$

$$\mathbf{E}_{sc,} = (-i) \Sigma K_\ell \left[\alpha_\pm(\ell)/2\ (i\ \mathbf{M_o}) + \beta_\pm(\ell)/2\ (\ \mathbf{N_e}) \right]$$

$$\mathbf{E}_2 = (-i) \Sigma K_\ell \left[\alpha'_\pm(\ell)/2\ (i\ \mathbf{M_o}) + \beta'_\pm(\ell)/2(\ \mathbf{N_e}) \right]$$

which we rewrite as

$$\mathbf{E}_{inc,} = \Sigma K_\ell \left[\mathbf{M_o} - i\ \mathbf{N_e} \right]$$

$$\mathbf{E}_{sc,} = \Sigma K_\ell \left[\alpha_\pm(\ell)/2\ \mathbf{M_o} - i\ \beta_\pm(\ell)/2\ (\ \mathbf{N_e}) \right]$$

$$\mathbf{E}_2 = \Sigma K_\ell \left[\alpha'_\pm(\ell)/2\ \mathbf{M_o} - i\ \beta'_\pm(\ell)/2(\ \mathbf{N_e}) \right]$$

and now let's compare these to our "second Mie attempt" expansions

$$\mathbf{E_p} = \Sigma_\ell K_\ell \left[\mathbf{M}^{(1)}_{o1\ell} - i\,\mathbf{N}^{(1)}_{e1\ell} \right] \qquad K_\ell = i^\ell \frac{2\ell+1}{\ell(\ell+1)} \qquad \text{// incident plane wave}$$

$$\mathbf{E_s} = \Sigma_\ell K_\ell \left[- b_\ell \mathbf{M}^{(3)}_{o1\ell} + i\, a_\ell\, \mathbf{N}^{(3)}_{e1\ell} \right] \qquad \text{// scattered field, (3) means } h^{(1)}_\ell(kr)$$

$$\mathbf{E_i} = \Sigma_\ell K_\ell \left[c_\ell\, \mathbf{M}^{(1)}_{o1\ell} - i\, d_\ell \mathbf{N}^{(1)}_{e1\ell} \right] \qquad \text{// internal field, (1) means } j_\ell(kr)$$

$\mathbf{M_{o1\ell}} \equiv j_\ell(kr)\ \mathbf{r} \times \nabla\psi_{o1\ell}$ where $\psi_{o1\ell} \equiv P_\ell^1(\cos\theta)\sin\phi$

$\mathbf{M_{e1\ell}} \equiv j_\ell(kr)\ \mathbf{r} \times \nabla\psi_{e1\ell}$ where $\psi_{e1\ell} \equiv P_\ell^1(\cos\theta)\cos\phi$

and comparison tells us that

$$-\alpha_\pm(\ell)/2 = b_\ell \qquad -\beta_\pm(\ell)/2 = a_\ell$$
$$\alpha'_\pm(\ell)/2 = c_\ell \qquad \beta'_\pm(\ell)/2 = d_\ell$$

which exactly agrees with our previous conclusions (thank goodness).

Our conclusion is that the Mie coefficients for helical polarized scattering are the same, and we expect therefore that the differential cross section for any polarization linear or otherwise will be the same no matter how you polarize.