A Summary of the Mie Scattering Calculation PhL 2.18.03

Units

We use cgs-Gaussian units throughout, as in "green Jackson". We also give occasional references to equations in this text.

Description of the Problem and its Parameters

Mie Scattering describes the scattering of an EM plane wave of wavelength λ (frequency ω) from a uniform, isotropic sphere of radius a. It is assumed that the incident plane wave is traveling in a medium characterized by uniform isotropic dimensionless parameters μ_o and ε_o, while the sphere is characterized by μ_i and ε_i , where subscripts mean outside and inside. It is convenient later to use the dimensionless parameters $n = \sqrt{\varepsilon\mu}$ and $g = n/\mu = \sqrt{\varepsilon/\mu}$, where n is the (possibly complex) index of refraction or "optical constant" , and where g is a parameter which arises when H fields are computed from E fields using the usual Maxwell curl equation, as will be seen below.

At "low" frequencies, for a conductor one has $n = \sqrt{\mu\, 4\pi i\, \sigma/\omega}$. For very large σ, the sphere becomes a "perfect" conductor and Mie Scattering replicates Jackson's Chapter 16 calculation of scattering from a perfectly conducting sphere. However, the main interest in Mie scattering is the case where μ=1 and ε>1 is some real number describing the sphere as a simple dielectric. There is plenty of complexity to be had in this simple case without worrying about complex index and other factors!

Typical values are $n_o = 1.00$ and $n_i = 1.33$ for a spherical water drop in the air, which causes rainbows and other effects, with a large range of radius values. For a floater in the eye, one might use $n_o = 1.33$ and $n_i = 1.37$ to model a bloated red blood cell in the vitreous from which the hemoglobin has escaped. In this case typical values of interest are $a = 4\ \mu$ and $\lambda = 0.5\ \mu$.

In the solution below, we will assume that the plane wave is traveling in the z direction, and has electric field linearly polarized in the x direction.

It is fascinating to realize that the exact solution of the Mie scattering problem includes all the complex rainbow, corona and glory effects that are normally explained by more intuitive ray-tracing techniques using Snell's Law to describe the behavior of rays when they touch the spherical surface. When we match boundary conditions later at the sphere surface, we are in effect incorporating Snell's Law in the most basic form into our solution, and so we expect these effects to be incorporated.

References

The original work was published by Mie in 1908 in a paper published in German,

Gustav Mie
Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen.
[Contributions to the optics of turbid media, especially colloidal metal suspensions.]
Annalen der Physik, Vierte Folge, Band 25, 1908, No. 3, S. 377-445.

English translation no. 79-21946, National Translation Center,
John Crerar Library, Chicago, IL 60616, USA

I have not seen this paper, and could not locate the English translation. Most current researchers refer to the following three secondary references, each of which is a monograph on EM scattering theory

M. Kerker, *The Scattering of Light and other Electromagnetic Radiation*, 1969.

Bohren & Huffman, *Absorption and Scattering of Light by Small Particles*, 1983, $53

van der Hulst , *Light Scattering by small particles*, 1981, Dover $15, Amazon $11.

Born and Wolf has a large section on the subject, and people mention Morse and Feshbach and Stratton as well. Unfortunately, except for Born and Wolf, even with Marriott Library I have not been able to see any of these references, and I have had to use tertiary web references such as Krugel's Chapter 2 on Interstellar Dust where he summarizes some of the Mie theory.

In 1909 Debye produced an alternative but related solution which more closely ties to the multi-bounce ray-tracing approach, which I have not investigated.

Both the Mie and Debye solutions are available in downloadable freeware GUI-based programs which do a very nice job plotting the differential cross section for arbitrary angle ranges and color mixes. Programs ("codes") for doing this were provided in FORTRAN in the Bohren & Huffman reference above, and since 1983 people have ported the code to C and have made modifications. My impression is that the GUI programs only compute Mie scattering in the far-field limit, which is of course usually what people are interested in, for example, with atmospheric particle or interstellar dust scattering.

Solution as an Integral

The problem of scattering from a dielectric sphere is more complex than one might at first think, unless the sphere radius is much smaller than λ. The complexity arises simply because any large stimulated radiating source has primary (pre-radiation) phase shifts between volume elements due to the incoming plane wave, and secondary (post-radiation) phase shifts between volume elements with respect to some observation point. Conceptually, this is not rocket science, and it is easy to write a an integral for the exact solution of the problem, namely, to use the general Green's Function solution for vector potential **A** due to a radiating current **J**, then we use for that current the dielectric polarization current

$$\mathbf{J} = -i\omega\mathbf{P} \qquad \text{with} \qquad \mathbf{P} = \mathbf{P_o}\ e^{ikz} \qquad \mathbf{P_o} = \chi\ \mathbf{E_o}$$

and we get this exact solution

$$\mathbf{A(r)} = -ik \int \frac{e^{ikR}}{R}\,\mathbf{P(r')}\,d^3\mathbf{r'} = -ik\mathbf{P_0}\int \frac{e^{ikR}}{R}\,\exp(ikz')\,d^3r' = -ik\mathbf{P_0}\,f(\mathbf{r},\mathbf{k})$$

where we have defined f as the "form factor". As usual, once you have **A**, you can compute **B** and then **E** in the usual manner. This solution is "exact" only in the (King) sense that we assume the radiated field within the source can be neglected relative to the incident field for purposes of stimulating the radiation.

In our applications, I believe this is well the case, although I should come up with a toy estimate to prove this is so. As noted, it is easy to write out this integral. If we put the z axis along the initial plane wave **k** vector, the result is

$$f(\mathbf{r}) = f(r,\theta,\phi) = \int \frac{e^{ikR}}{R} \exp(ikz')\, d^3r' = \int r'^2\, dr' \int d(\cos\theta') \int d\phi'$$

$$\frac{\exp[ik\sqrt{r^2 + r'^2 - 2\,r\,r'\,[\,\sin\theta\,\sin\theta'\cos(\phi-\phi') + \cos\theta\,\cos\theta'\,]}]}{\sqrt{r^2 + r'^2 - 2\,r\,r'\,[\,\sin\theta\,\sin\theta'\cos(\phi-\phi') + \cos\theta\,\cos\theta'\,]}} \exp(ikr'\cos\theta')$$

In fact, this is the general result for scattering from a dielectric object of any shape, where the integration extends over the volume of the object. Even for the simplest case of a sphere, the integration end points are simply stated, but the integral is simply not doable in closed form. One could do numerical triple-integration with probably the Cartesian version of the above formula.

It should be noted that one can do an approximate calculation for $\lambda << a$ to obtain an expression for f(r) shown above. Portis does such a calculation in his book, and all multipole orders are activated in his result. Although this solution is a great improvement over the E1 electric dipole solution which we might say is valid for $a <<< \lambda$, it has no validity whatsoever in the case that $\lambda > a$. In our applications, we always have $\lambda > \sim a$, and we certainly do not have $a << \lambda$, so the Portis type calculation is of no use, just as the electric dipole calculation is of no use. These approximations ARE of use when one is talking about nuclear, atomic or molecular radiators where a = 1A and λ = 5000A, say (atomic/optical numbers).

The Multipole Method of Solution

In this method, we obtain an exact solution of Maxwell's equations for our problem, although the result must be expressed as an infinite series of terms. In this exact solution, we do not need to make the King approximation mentioned above, this one is really exact, assuming an ideal dielectric. We obtain results for the **E** and **B** fields that are exact arbitrarily close to the dielectric sphere, and even in fact inside it.

We have detailed this calculation in another document, and here we want just to outline the approach and the solution.

The Plane Wave

We start by assuming that the incident plane wave is polarized in the $\hat{\mathbf{x}}$ direction, and we note that this direction corresponds to $\phi = 0$ in the spherical coordinates to be used. The first step involves expressing the $\hat{\mathbf{x}}$-polarized plane wave as a sum of multipole fields. I made use of Jackson's laborious calculation for helical polarizations (see 16.139) and made the appropriate linear combinations of his solutions for the $\hat{\mathbf{x}}$ polarization. The result can be written in the following compact form,

$$\mathbf{E_p} = \hat{\mathbf{x}}\, e^{ikz} = \Sigma_\ell\, i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\, \mathbf{M}^{(1)}_{o1\ell} - i\, \mathbf{N}^{(1)}_{e1\ell}\,]$$

$$\mathbf{B_p} = n\ \hat{\mathbf{y}}\ e^{ikz} = \ n\ \Sigma_\ell i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\ - \mathbf{M}^{(1)}_{e1\ell}\ - i\ \mathbf{N}^{(1)}_{o1\ell}\ ]$$

where, first of all, the p subscript on **E** and **B** stands for plane wave. The sum is for $\ell = 1$ to ∞, so that all partial waves are involved. We discuss the factor n below, it is the index of refraction or "optical constant" in the medium.

The vector functions **M** and **N** appearing inside the sums are these,

$$\mathbf{M_{o1\ell}} \equiv j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_{o1\ell} \qquad \text{where} \qquad \psi_{o1\ell} \equiv\ \ P_\ell^1(\cos\theta)\ \sin\phi$$
$$\mathbf{M_{e1\ell}} \equiv j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_{e1\ell} \qquad \text{where} \qquad \psi_{e1\ell} \equiv\ \ P_\ell^1(\cos\theta)\ \cos\phi$$

$$\mathbf{N_{o1\ell}} \equiv (1/k)\ \nabla \times \mathbf{M_{o1\ell}}\ = (1/k)\ \nabla \times [j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_{o1\ell}\ ]$$
$$\mathbf{N_{e1\ell}} \equiv (1/k)\ \nabla \times \mathbf{M_{e1\ell}}\ = (1/k)\ \nabla \times [j_\ell(kr)\ \ \mathbf{r} \times \nabla\psi_{e1\ell}\ ]$$

The superscript (1) on the **M** and **N** vectors indicate $j_\ell(kr)$ functions. These are appropriate for the plane wave expansion since they don't blow up at r=0.

The general forms of the **M** and **N** functions are quite familiar to anyone who has worked with the multipole expansion, see for example Jackson 16.47. However, the ψ functions appearing here are non-normalized linear combinations of the $Y_{\ell\,\pm1}$ spherical harmonics. That is, in each multipole order ℓ, only the components $m = \pm1$ make a contribution. The sum and difference combinations of $Y_{\ell m}$ used result in the obvious $\sin\phi$ and $\cos\phi$ for the azimuthal factors (since $m = \pm1$), and $\sin\phi$ is referred to as the "odd" function and $\cos\phi$ as the "even" function, and these labels o and e appear as subscripts the M and N functions. The other two subscripts are m and ℓ.

Why do only $m = \pm 1$ occur? One arm-waving explanation of this result is this: the multipole expansion of the plane wave itself ("orbital") has only m=0 terms as shown in Jackson 16.129, since it is obviously azimuthally symmetric. The photon is spin 1 and can add or subtract 1 unit to this m.

But we don't need to invoke quantum concepts like photons to get this result. We are really combining a spin ℓ object (orbital) with a spin 1 object which is the classical electric polarization vector, and these make $j = \ell+1$, ℓ, ℓ-1 in the usual manner, and it is really j that appears in the sums. We could I think do a very clean angular momentum analysis and show why this result happens using Clebsch-Gordon coefficients and all that stuff. In fact, this is partially done in the appendix of one of my stored PDF documents (which I have labeled "very good" but have not studied.) It should be noted that there is an ℓ=0 term in the expansion 16.129, but it gets knocked out of the field expansions because ℓ=0 cannot support $m = \pm1$.

It happens that the **M** and **N** vectors shown above satisfy the vector wave equation (with k^2) and have no divergence, so they are ideal objects to represent **E** and **B** fields where $\rho = 0$, which is everywhere in our Mie problem, except right at the boundary of the sphere where there is some surface charge running around due to the polarization difference between the two sides of the boundary.

Jackson makes use of the vector **X** as defined in 16.45 where $\mathbf{L} = -i\ \mathbf{r} \times \nabla$, so his **X** is just a multiple of our **M**, although it is true that we have a linear combination of $Y_{\ell m}$ instead of a fixed $Y_{\ell m}$.

In general, there are three "vector spherical harmonic" basis vectors **X**, **Y** and **Z**, where $\mathbf{X} \sim \mathbf{M}$ is the one we have talked about. **X** and **Z** are transverse to the sphere at radius r, **Y** is radial. It turns out that only $\mathbf{X} \sim \mathbf{M}$ and the linear combination of **Y** and **Z** which is **N** satisfy the vector wave equation, so that is why those combinations are what appears in the multipole expansion of **E** and **B**.

In passing, we want to note that there is a simple very useful rule for taking an **E** expansion formula and obtaining the corresponding **B** formula:

"To get B from E, for each term use the nature of, and -i times the coeff of, the cross term. "

To see what this means, just look at the plane wave example for E and B above.

Finally, we come to the factor of n in the equation above for B. This n is the index of refraction in the medium in which the plane wave is propagating. In the literature, this is assumed to be air or vacuum and one always sees n = 1 so there is no factor. However, this factor is very important because it reminds us of another rule we have to keep in mind.

That rule relates to Jackson's presentation of the multipole expansion in his Chapter 16. This is all done in vacuum and starts with Maxwell's equations. If we assume isotropic μ and ε instead of a vacuum, then all of Jackson's results need to be slightly modified at least in interpretation. Wherever there is a k factor, such as the (1/k) in the **N** functions, or such as inside $j_\ell(kr)$, one must think of it as $k = k_f\, n$, where $k_f = \omega/c$, the free-space value, and n is the index of the material. If at the same time as making these interpretative changes we also rescale the B field by factor n, defining B' = B/n, then all the μ and ε disappear from the Maxwell equations [see Jackson 7.1] which we start with in Jackson (written in terms of E and B'), and all his results stand, with the understanding that his B-fields are really B' fields. Thus, to get to the *actual* B field we must do B = nB' on his formulas. So, we add this to our general rule above and rewrite it as:

"To get B from E, for each term use the nature of, and -i times the coeff of, the cross term, and apply an overall factor of n to the final result for B. "

An easy way to derive this rule is to realize that $\mathbf{B} = -i/k_f\ \nabla \times \mathbf{E}$ where $k_f = \omega/c$, and to use the facts that $\nabla \times \mathbf{M} = k\mathbf{N}$ and $\nabla \times \mathbf{N} = k\mathbf{M}$, where here k is the specific value for the medium.

You will see this rule applied in all the expansions which follow.

Obtaining the above expression for the plane wave in partial waves is really the hardest part of the Mie scattering problem. From now on, things are relatively easy.

The Expansions

We assume the following forms for the plane wave, the scattered wave, and the internal wave. (The word "wave" really means the E and B fields). The term "internal" means inside the dielectric sphere. As noted in the overview, although we keep saying dielectric sphere, Mie scattering applies to the general case of complex n, which includes the conducting sphere.

We have then

$$\mathbf{E_p} = \Sigma_\ell K_\ell \left[\mathbf{M}^{(1)}_{o1\ell} - i\,\mathbf{N}^{(1)}_{e1\ell} \right] \qquad \text{// incident plane wave,} \quad K_\ell = i^\ell \frac{2\ell+1}{\ell(\ell+1)}$$

$$\mathbf{E_s} = \Sigma_\ell K_\ell \left[-b_\ell \mathbf{M}^{(3)}_{o1\ell} + i\,a_\ell\, \mathbf{N}^{(3)}_{e1\ell} \right] \qquad \text{// scattered field, (3) means } h^{(1)}_\ell(kr)$$

$$\mathbf{E_i} = \Sigma_\ell K_\ell \left[c_\ell\, \mathbf{M}^{(1)}_{o1\ell} - i\,d_\ell\, \mathbf{N}^{(1)}_{e1\ell} \right] \qquad \text{// internal field, (1) means } j_\ell(kr)$$

The **M** and **N** functions appearing in the scattered and internal field expansions are exactly as described above, *except* $j_\ell(kr)$ is replaced by $h^{(1)}_\ell(kr)$ since this spherical Bessel function has the right e^{+ikr}/r asymptotic behavior at large r for a scattered wave. The presence of $h^{(1)}_\ell(kr)$ is indicated by the (traditional I think) superscript (3) on the M and N functions for this wave.

Other than this choice of Bessel function, the scattered and internal fields are of the *most general form* allowed by Jackson's multipole expansion 16.47. Other values of m could be included in the scattered and internal field sums, but one would find that their presence would prohibit the upcoming boundary conditions from being met for those terms.

There are four coefficients a,b,c,d we need to find for each partial wave (ie, for each multipole expansion order ℓ), and these are defined with the traditional signs, phases and scales as shown above in order to make the solutions for these coefficients come out in the simplest form. Notice that we include the K_ℓ factor in the internal and scattered field expansions since it appears in the plane wave expansion.

For each of the above E expansions, there is a corresponding B field expansion obtainable by the rule quoted above. We get at once,

$$\mathbf{B_p} = n_o \Sigma_\ell K_\ell \left[- \mathbf{M}^{(1)}_{e1\ell} - i \mathbf{N}^{(1)}_{o1\ell} \right] \qquad K_\ell = i^\ell \frac{2\ell+1}{\ell(\ell+1)}$$

$$\mathbf{B_s} = n_o \Sigma_\ell K_\ell \left[a_\ell \mathbf{M}^{(3)}_{e1\ell} + i\, b_\ell \mathbf{N}^{(3)}_{o1\ell} \right]$$

$$\mathbf{B_i} = n_i \Sigma_\ell K_\ell \left[- d_\ell \mathbf{M}^{(1)}_{e1\ell} - i\, c_\ell \mathbf{N}^{(1)}_{o1\ell} \right] \qquad / \text{ i = inside, o = outside}$$

Notice now that we have to label the leading refraction indices as appropriate for inside or outside the scattering sphere. The various k's in the vector functions are also labeled in this way, we just don't see these labels yet in these compact formula notations.

Since we are going to apply a tangential magnetic boundary condition, and since we are allowing for linear magnetic media, we need to convert to **H** fields which don't see magnetization surface currents. In this conversion, we pick up extra the extra factor of μ since $H = B/\mu$. So,

$$\mathbf{H_p} = n_o/\mu_o \Sigma_\ell K_\ell \left[- \mathbf{M}^{(1)}_{e1\ell} - i \mathbf{N}^{(1)}_{o1\ell} \right] \qquad K_\ell = i^\ell \frac{2\ell+1}{\ell(\ell+1)}$$

$$\mathbf{H_s} = n_o/\mu_o \Sigma_\ell K_\ell \left[a_\ell \mathbf{M}^{(3)}_{e1\ell} + i\, b_\ell \mathbf{N}^{(3)}_{o1\ell} \right]$$

$$\mathbf{H_i} = n_i/\mu_i \Sigma_\ell K_\ell \left[- d_\ell \mathbf{M}^{(1)}_{e1\ell} - i\, c_\ell \mathbf{N}^{(1)}_{o1\ell} \right] \qquad / \text{ i = inside, o = outside}$$

In what follows, we define $g = (n/\mu) = \sqrt{\mu\varepsilon}/\mu = \sqrt{\varepsilon/\mu}$ for the leading factors in the H fields.

The Boundary Conditions

There are four boundary conditions which apply at the surface of the sphere, that is, at r = a:

r x **H** is continuous
r x **E** is continuous
r • **D** is continuous where $\mathbf{D} = \varepsilon\mathbf{E}$

$\mathbf{r} \bullet \mathbf{H}$ is continuous

These come from the usual loop and pillbox arguments, recalling that H sees only "free" surface current, and D sees only "free" surface charge, neither of which exist in this problem. Each of the $\mathbf{r} \times$ boundary conditions implies two constraint equations on our four coefficients, one to balance the **M** terms, and one to balance the **N** terms. Each $\mathbf{r} \bullet$ condition gives only one equation, since $\mathbf{r} \bullet \mathbf{M} = 0$. Thus, it would appear that we have 6 constraint equations on the 4 unknowns a,b,c,d, but it turns out that the two $\mathbf{r} \bullet$ conditions are not independent of the other four conditions, and add no knew information. So, the first two vector conditions are

$$\mathbf{r} \times \mathbf{E_i} = \mathbf{r} \times \mathbf{E_p} + \mathbf{r} \times \mathbf{E_s}$$
$$\mathbf{r} \times \mathbf{H_i} = \mathbf{r} \times \mathbf{H_p} + \mathbf{r} \times \mathbf{H_s}$$

When these are applied to the field expansions stated above, we find that

inside plane scattered

$$\mathbf{r} \times [\, c_\ell \, \mathbf{M}^{(1)}_{o1\ell} - i\, d_\ell \, \mathbf{N}^{(1)}_{e1\ell} \,] = \mathbf{r} \times [\, \mathbf{M}^{(1)}_{o1\ell} - i\, \mathbf{N}^{(1)}_{e1\ell} \,] + \mathbf{r} \times [\, - b_\ell \, \mathbf{M}^{(3)}_{o1\ell} + i\, a_\ell \, \mathbf{N}^{(3)}_{e1\ell} \,]$$

$$\mathbf{r} \times [\, - g_i \, d_\ell \, \mathbf{M}^{(1)}_{e1\ell} - i\, g_i \, c_\ell \, \mathbf{N}^{(1)}_{o1\ell} \,] = \mathbf{r} \times [\, - g_o \, \mathbf{M}^{(1)}_{e1\ell} - g_o \, i\, \mathbf{N}^{(1)}_{o1} \,] + \mathbf{r} \times [\, g_o \, a_\ell \, \mathbf{M}^{(3)}_{e1\ell} + i\, g_o \, b_\ell \, \mathbf{N}^{(3)}_{o1\ell} \,]$$

Since the **M** and **N** terms are independent in terms of vector spherical harmonic components, we separately balance the **M** and **M** coefficients. In order to do this, we need the following information,

$$\mathbf{r} \times \mathbf{M}^{(1)}_{o1\ell} = - f\, r^2 \, \nabla \psi_{o1\ell}$$
$$\mathbf{r} \times \mathbf{N}^{(1)}_{o1\ell} = - (1/k)(rf)' \, \mathbf{r} \times \nabla \psi_{o1\ell}$$

where f stands for the appropriate spherical Bessel function. A similar result holds for the two even M and N functions. Notice that the k here is not k(free) but is k in the specific medium, inside or outside the sphere. Think of these cross products in this manner,

$$\mathbf{r} \times \mathbf{M}^{(1)}_{o1\ell} = f \quad * \; - r^2 \, \nabla \psi_o$$
$$\mathbf{r} \times \mathbf{N}^{(1)}_{o1\ell} = (rf)' / n \quad * \; (1/k_f) \, \mathbf{r} \times \nabla \psi_o \qquad \text{where } k_f = \omega / c$$

The items on the right will be common factors that will cancel on both sides, so we need only worry about the items on the left in each term.

Balancing coefficients in the two boundary condition equations above then gives these four equations,

$$c_\ell \, j_i = 1 \, j_o - b_\ell \, h_o$$
$$- i\, d_\ell \, (r j_i)'/n_i = - i\, (r j_o)'/n_o + i\, a_\ell \, (r h_o)'/n_o$$
$$- g_i \, d_\ell \, j_i = - g_o \, j_o + g_o \, a_\ell \, h_o$$
$$- i\, g_i \, c_\ell \, (r j_i)'/ n_i = -i\, g_o \, (r j_o)'/ n_o + i\, g_o \, b_\ell \, (r h_o)'/ n_o$$

where we have used these new abbreviations

$$j_o = j_\ell(k_o r) \qquad j_i = j_\ell(k_i r) \qquad h_o = h^{(1)}_\ell(k_o r) \qquad o = \text{outside} \qquad i = \text{inside}$$

If we define the following ratios

$$n \equiv n_i/n_o \qquad \text{and} \qquad g \equiv g_i/g_o \qquad [\ \ g = n\,(\mu_o/\mu_i)\ \ ]$$

the four equations can be rewritten as,

$$\begin{aligned}
&c_\ell\, j_i \;=\; j_o \;-\; b_\ell\, h_o \\
-\;&d_\ell\,(r j_i)' \;=\; -\,n\,(r j_o)' + a_\ell\, n\,(r h_o)' \\
-\;&g\, d_\ell\, j_i \;=\; -\, j_o \;+\; a_\ell\, h_o \\
-\;&c_\ell\, g\;(r j_i)' \;=\; -n\,(r j_o)' + \;b_\ell\, n\,(r h_o)'
\end{aligned}$$

where it is always understood that r is set to a wherever it occurs. The notations like $(rj_i)'$ always lead to some ambiguity and clumsiness in that one might be confused as to what the differentiation is with respect to (r or x ?), and we hate to keep saying things like $\partial_r(r\, j_\ell(k_i r))|^{r=a}$ which is of course what is really meant. This is all avoided by replacing the Bessel functions with the Riccati-Bessel functions (see p 445 AS)

$$\psi(x_o) \equiv x_o\, j_\ell(x_o) \qquad x_o = k_o\, r \;= n_o\, k_f\; r \qquad k_f = \omega/c$$
$$\xi(x_o) \equiv x_o\, h^{(1)}_\ell(\, x_o) \qquad x_o = k_o\, r \;= n_o\, k_f\; r$$

which just involves a multiplication of each function by its argument. We emphasize that the argument is medium dependent, by showing the o = outside case explicitly. We shall also have need of this inside function,

$$\psi(x_i) \equiv x_i\, j_\ell(x_i) \qquad x_i = k_i\, r = n_i\, k_f\; r$$

Warning: Do not confuse this new use of the symbol ψ with the symbol ψ which appear inside the **M** and **N** functions. There is no connection whatsoever between them. We are using the traditional symbols here.

We now replace the regular Bessel functions with the Riccati versions using these substitutions and implied abbreviations,

$$\begin{aligned}
j_o &= \psi_o / (n_o x) & (r\, j_o)' &= \psi'_o \\
h_o &= \xi_o / (n_o x) & (r\, h_o)' &= \xi'_o \\
j_i &= \psi_i / (n_i x) & (r\, j_i)' &= \psi'_i
\end{aligned}$$

yielding these four equations,

$$\begin{aligned}
(1/n)\,\psi_i\, c_\ell \;&+\; \xi_o\, b_\ell \;&=\; \psi_o \\
(1/n)\,\psi'_i\, d_\ell \;&+\; \xi'_o\, a_\ell \;&=\; \psi'_o
\end{aligned}$$

$$(g/n)\psi_i\, d_\ell \quad + \quad \xi_o\, a_\ell \quad = \psi_o$$
$$(g/n)\, \psi'_i\, c_\ell \quad + \quad \xi'_o\, b_\ell \quad = \psi'_o$$

which we recognize as two equations for a and d, and two for b and c. The solutions are easily found,

$$a = \frac{g\,\psi_i\,\psi'_o - \psi_o\,\psi'_i}{g\,\psi_i\,\xi'_o - \xi_o\,\psi'_i}$$

$$b = \frac{\psi_i\,\psi'_o - g\,\psi_o\,\psi'_i}{\psi_i\,\xi'_o - g\,\xi_o\,\psi'_i}$$

$$c = n\,\frac{\psi_o\,\xi'_o - \xi_o\,\psi'_o}{\psi_i\,\xi'_o - g\,\xi_o\,\psi'_i} = \frac{n\,i}{\psi_i\,\xi'_o - g\,\xi_o\,\psi'_i}$$

$$d = n\,\frac{\psi_o\,\xi'_o - \xi_o\,\psi'_o}{g\psi_i\,\xi'_o - \xi_o\,\psi'_i} = \frac{n\,i}{g\psi_i\,\xi'_o - \xi_o\,\psi'_i}$$

where

$n = n_i / n_o$ = index inside / index outside $\qquad i = \sqrt{-1}$

$g = g_i / g_o = n_i / n_o * \mu_o / \mu_i$ $\qquad$ Note that $g = n$ when $\mu_o = \mu_i$

$\psi(x) = x\, j_\ell(x) \qquad \xi(x) = x\, h^{(1)}_\ell(x)$ $\qquad$ the Riccati-Bessel AS page 445

(just regular spherical Bessels mult by x)

o = outside, $x_o = n_o k_f\, a$

i = inside, $x_i = n_i k_f\, a$ $\qquad$ a = sphere radius, $k_f = \omega/c$

The simplification in the c and d coefficients results from the fact that the Wronskian $W(\psi,\xi) = i$.

Notice that the scattering coefficients a and b are functions only of ratio g, whereas the coefficients c and d which describe the fields inside the dielectric sphere are functions of both g and n.

So we now have a **complete solution** to the Mie scattering problem! We put these coefficients back into the field expansions given earlier, and we are done.

If we set $g = n = 1$, meaning the scattering object is only in our imagination, we find that $a = b = 0$ so there is no scattering, and we find that $c = d = 1$, so the internal field matches the plane wave.

If we set $g = n = \infty$, we recover the solution to the problem of scattering from a perfectly conducting sphere. We get this limit by assuming $n_i = \sqrt{\mu\, 4\pi i\, \sigma/\omega}$ where σ is the conductivity of the sphere. The c and d coefficients are 0 in this limit because the denominator Bessel functions blow up for large argument, meaning there are no fields inside the sphere. The scattering a and b coefficients however are quite finite and are given by

$$a = \frac{\psi'_o}{\xi'_o} = \frac{\partial_x [x\, j_\ell(x)]\,|^{x = xo}}{\partial_x [x\, h1_\ell(x)]\,|^{x = xo}} = (1/2)\ \left[\ 1\ + \frac{\partial_x [x\, h1^*_\ell(x)]\,|^{x = xo}}{\partial_x [x\, h1_\ell(x)]\,|^{x = xo}}\ \right]$$

$$b = \frac{\psi_o}{\xi_o} = \frac{j_\ell(x_o)}{h1_\ell(x_o)} = (1/2)\ \left[\ 1 + \frac{h1^*_\ell(x_o)}{h1_\ell(x_o)}\ \right]$$

where the ratios on the right are just phasors leading to the usual phase shift definitions as shown in Jackson. Notice the (1/2) factor Jackson has placed in front of the expansions shown in 16.141, which then matches the factors of (1/2) shown above.