Doing the Disk Integral. Recall from earlier notes that, according to Babinet, the intensity at z behind a disk of radius a in a plane wave is given by this formula (in the Fraunhofer approximation that $B << \pi$)

$$|E(r)|^2 = 1 - 2\sin(Ar^2)q(r) + q(r)^2$$

where

$$q(r) = B * 2J_1(Cr)/(Cr) \qquad B = (\pi a^2)/(\lambda z) \qquad C = ka/z \qquad A = k/2z \qquad k = 2\pi/\lambda$$

and we know the general shape of this function. I plotted it in Excel and it is very ugly but shows a "bright spot" in the middle at r=0. As r gets larger, $|E(r)|^2$ oscillates faster and faster. I tried doing numerical integration of this thing in Maple and had problems doing so because of the oscillation. But then I started thinking about doing a definite integral from r = 0 to ∞. After many erroneous attempts, I was able to do the above integral exactly using definite integrals of Bessel functions shown on GR pages 692 and 755. The integral you want is over area: we want to integrate the above intensity over a large circular area behind the disk. We approximate the last two terms by making that area infinite for them only. The necessary integrals are these:

$$\int_0^\infty [J_1(Cr)]^2\, dr/r = 1/2$$

$$\int_0^\infty \sin(Ar^2)J_1(Cr)\, dr = (1/4A)\cos(B/2)j_0(B/2) = (1/4A)\sin(B)/B$$

Here then is the exact result:

$$\int_0^\infty |E(r)|^2 dA = \int_0^\infty 1 \bullet dA - (\pi a)^2 \left[2\frac{\sin B}{B} - 1\right] \qquad \text{where } dA = 2\pi r dr. \qquad B = \frac{\pi a^2}{\lambda z}$$

This is a stunningly simple result for this ugly definite integral! Then when we actually apply the Fraunhofer limit, it becomes even simpler:

$$\int_0^\infty |E(r)|^2 dA = \int_0^\infty 1 \bullet dA - (\pi a)^2 \qquad \text{where } dA = 2\pi r dr. \qquad B = \frac{\pi a^2}{\lambda z}$$

Note that the second AND third terms of $|E(r)|^2$ contribute important amounts to this result. When starting, you might have ignored the third term since it is order B^2 but you would then get a dramatically wrong result! You would conclude that the "rest" part was $-2\pi a^2$ so your answer would be off by 100%! It must be that the second term oscillates so fast that its contribution to the integral is cut down to match the size of the third term's contribution.

How do we interpret this amazingly simple result? Let's integrate over a disk of radius R that is perhaps 10 times the radius of the Airy pattern, so we "pick up" all the action. Then what we get is this:

energy hitting large circle of radius R centered behind the disk $= \pi R^2 - \pi a^2$

with unit plane wave intensity. You interpret this as saying that the disk absorbed or back-scattered an energy = πa^2. Making this statement from the start would have been a hell of an easier way to "do" the integral, but it is nice to see that everything is consistent! The result is the same as would be obtained in geometric optics with no diffraction. It is fascinating to see that, although Fraunhofer can spread out the shadow of the disk to something much larger than πa^2 , he cannot cause violation of energy conservation.