

Gauge Invariance and Construction of the Fermion-Boson Vertex in QED3

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Preface

The study of physics at the level of the fundamental constituents of the universe has represented a challenge for the human mind. The classification of such objects by their properties is very helpful for understanding the interactions among them. One of these properties, their mass, remains as one of the most intriguing, since, up to date, there is no theory that can explain its origin. The manner in which we understand the interactions at the fundamental level is by the exchange of mediators in the Standard Model (SM) of Particle Physics. All the processes are then described in terms of Feynman diagrams, and in order to compare with the experimental results, one must calculate the amplitudes for a given event, adding the contributions of all the possible ways in which the event can take place. It is true that this model satisfactorily reproduces the dynamics of the particles through the symmetries of the mathematical objects that describe such particles, but it assumes that they are massless. On the other hand, in the different high energy laboratories around the world, masses for the fundamental constituents of the universe have been measured, in some cases very accurately. This reflects the fact of the breaking of the symmetry of the SM. In order to conciliate this discrepancy between theory and experiment, the quest for a self consistent theoretical model and its experimental corroboration that explains the origin of the masses of the fundamental blocks of the universe has begun.

The most famous as well as popular of these models is the so-called Higgs mechanism, which explains that the mass of the particles emerges by their interaction with the Higgs boson, a phenomenon that involves the Spontaneous Breaking of the Chiral Symmetry in the SM. The search for this boson in particle accelerators in the US and in Europe has impulsed the technological development, due to the technical difficulties and low budgets to produce and detect it; and the theoretical advance, which offers simpler channels towards its discovery. At the same time, it has promoted the increase of the number of students and scientists dedicated to this branch of physics. In spite of the efficiency on which the SM describes the phenomenology of the fundamental particles, there are still some problems.

The Higgs boson has not been discovered experimentally yet. The SM assumes that this boson is fundamental, but it does not exist a fundamental scalar in nature, at least, it has not been observed yet. On the other hand, when we try to extend the SM to include some other forces by constructing a Grand

Unification Theory, we face the hierarchy problem, and in order to solve it, one must perform an unnatural fine tuning of the Higgs parameters. Supersymmetry or SUSY takes care of this problem at the expense of doubling the spectrum of particles. Fundamental scalars are still there and SUSY keeps on eluding us.

As an alternative, models have been proposed in which the existence of a scalar particle is no longer necessary for the symmetry breaking, since it is given in the presence of condensates. Some of these models are Technicolor and its Extended versions, Top Condensate and Top Color. Predictions from these models are obtained by non perturbative calculations, that even in the simplest cases are very hard to perform and they are based in severely incorrect assumptions. For example, in the Top Condensate Models in the context of the Schwinger-Dyson Equations (SDE)-which we shortly will talk about-, the QCD corrections are only calculated in Landau gauge. If one repeats the same exercise in other gauges, one finds that the top quark mass depends upon the gauge, which is physically senseless. Trying to solve this situation in complicated theories like QCD is a formidable problem. In this thesis we discuss similar topics in a simple model, QED in 2+1 dimensions.

We decided to study the Dynamical Breaking of Chiral Symmetry in the context of Schwinger-Dyson equations. These equations are an infinite set of integral relations among the Green's functions of a Quantum Field Theory, and they provide the analytic structure that such functions must have. Their implementation is extended to the High Energy Physics as well as Nuclear Physics, therefore, the solutions for these equations are of interest for a wide sector of physicists around the world.

Solutions to the SDE are obtained by truncating the infinite tower of integral equations, and the main task for the solution of these equations should be to incorporate all of the gauge identities of the theory in order to calculate gauge parameter independent physical observables. To pursue this objective, analytical studies, up to where the complexity of the theory allows, are performed. Such studies are either based on the use of ansätze for the Greens function or in their spectral representation. There are also successful attempts to describe the dynamics of the fundamental particles by discretising the space-time, i. e., assuming a lattice where the dynamics takes place. Then, the different processes are calculated with the unavoidable use of computers, which the more powerful, the more useful in this context.

Even though there have been attempts to solve the DSE in QCD and in QED in a four dimensional space-time, such studies face a difficulty inherent to four-dimensional theories: Ultraviolet divergences. Added to this, the initial assumptions to truncate the infinite tower of SDE have not allowed to clearly identify the sources of the gauge dependence of physical observables, nor the role which play the gauge invariance constraints in the restoration of the gauge independence of these quantities, like the Ward Identities for QED or the Slavnov-Taylor Identities for QCD, or some others, like the Landau-Khalatnikov-Fradkin transformations in QED, which describe the precise manner in which the Green's function vary under a gauge transformation.

Therefore, we decided to focus our research in Quantum Electrodynamics on

a plane, or QED3, where, besides of enjoying of the benefits which theories of phenomena occurring on planar surfaces offer, we can identify exactly the role of the initial assumptions made to solve the SDE and the gauge invariance constraints, since QED3 lacks of ultraviolet divergences.

The special features of the electromagnetic dynamics on a plane are studied in Chapter 1, while the derivation of the SDE as well as the Ward-Green-Takahashi Identities (WGTI) and the Landau-Khalatnikov-Fradkin (LKF) transformations is carried out in Chapter 2. For a better understanding of the phenomenon of the Dynamical Generation of Fermion Masses, in Chapter 3 we solve the SDE for the Fermion Propagator with the only assumption that the fermions interact among them in the simplest known way, that is, assuming that the vertex of the interaction is only the bare one. This allows us to identify the sources of the gauge dependence for the two relevant physical observables, the Euclidean mass and the Chiral Condensate. This scenario also allows us to study the role of the WGTI in the restauration of the gauge independence of the above mentioned physical observables. The conclusion that the imposition of the WGTI is a necessary, but not sufficient condition to guarantee the gauge independence of the physical observables is translated as the necessity for the incorporation of other gauge invariance constraints. The LKF transformations are the next ingredient to consider, but their implementation is not as simple as the WGTI. In Chapter 4 we describe the manner to implement these transformations as a requirement that the solutions for the SDE must fulfill. Being then only left with the initial assumption about the vertex, in Chapter 5 we advocated our attention to remove it by constructing the vertex for the electromagnetic interaction making use of Perturbation Theory as a guide. The main reason for this is that in Perturbation Theory, gauge identities (WGTI and LKF transformations) and the gauge independence of physical observables are satisfied order by order. Then if our initial assumption for the fermion-boson vertex reduces to its Feynman expansion in the weak coupling regime, we stand our best chance to obtain the correct gauge behavior in the nonperturbative regime. Our construction is the most ambitious of its kind, because, besides considering the gauge invariance constraints of QED3, it deals with massive fermions in the interaction. Our vertex provides the first insight of the nonperturbative interaction, because when we write an effective Ward identity for the transverse part of the vertex (i.e., a relation between the transverse vertex and the fermion propagator), we are left with an interaction explicitly independent of the electromagnetic coupling. Consequences of this construction as well as the possible paths for the implementation of this vertex in similar studies are discussed in Chapter 6, where besides we offer our conclusions. We keep the hope that the technique we developed can be applied in future to alternative models to the SM which help us to discover the origin of masses for the fundamental particles, where more reliable calculations are required with improved vertex ansätze.

From this work there have been published the following papers :

1. *Constructing the fermion-boson vertex in three-dimensional QED.* A. Bashir y A. Raya. Phys. Rev. **D64** 105001 (2001).
2. *Gauge dependence of mass and the condensate in chirally asymmetric phase of QED3.* A. Bashir, A. Huet y A. Raya. Phys. Rev. **D66** 025029 (2002).
3. *Landau-Khalatnikov-Fradkin transformation and the fermion propagator in Quantum Electrodynamics.* A.Bashir y A. Raya. Phys. Rev. **D66** 105005 (2002).

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Chapter 1

Quantum Electrodynamics on a Plane

1.1 Introduction

Even though we live in a four-dimensional space-time –three of its dimensions being spatial and one temporal–, many theories are also interesting when they are formulated in other dimensions. Phenomena like the Dynamical Breaking of Chiral Symmetry in Quantum Field Theories are more easily understood when such theories restrict their dynamics to less than four dimensions. Three-dimensional models are very useful to understand phenomena that happen on planar surfaces, where precisely one counts with two spatial dimensions and one temporal. They are also helpful to study the high temperature behavior of four-dimensional theories, since these theories have as infinite-temperature limit their three-dimensional counterparts [1, 2]. In such cases, a three-dimensional theory is interesting not only as an academic model, but because they have a direct physical interpretation. QED3 is one of such models. This is the subject of our interest in this thesis and we start from an introduction to it in this chapter.

In QED3, the phenomenon of Dynamical Mass Generation has been studied as an alternative mechanism to that of Higgs in order to explain how fundamental particles acquire such property (an excellent review can be found in [3]). Although the Higgs mechanism offers a solution to this question, it is partial in the sense that it relates particles to their masses, but it says nothing about the value of these masses. Besides, if the Higgs boson is detected in experiments at LEP and/or Tevatron, it can take a while before determining whether it is fundamental or not.

The theory we study has a coupling e^2 with dimensions of mass, which allows the dynamically generated mass to be related to this coupling, and it is no longer necessary to introduce by hand a cut-off mass scale for this model, since e^2 defines this scale. Even more, this theory is superrenormalizable, and lacks

of ultraviolet divergences, one of the most serious obstacles to study physical phenomena in four-dimensional theories. The order of divergence of QED3 is [4] :

$$w = 3 - f - \frac{1}{2}b - \frac{1}{2}n , \quad (1.1)$$

where f is the number of external fermions in a diagram, b the number of external photons and n the perturbative order of the diagram. It can be shown that the Green's functions become finite in the ultraviolet region. However, it does not mean that the theory is free from divergences. In our case, the problem is that there are infrared divergent integrals, at least in the massless theory. That is why one is forced to start from certain assumptions that remove these divergences to solve the Schwinger-Dyson Equations (SDE).

Therefore, we have a very interesting model, from which we can learn pretty much about the Dynamical Generation of Masses and the analytic structure of the Fermion Propagator in general [5], but which is mathematically more easily treatable than in four dimensions. Results we obtained can serve as a guide to more complicated theories, like Grand Unification Theories.

This theory, besides, possesses a direct physical relevance. QED3 with a dynamically generated mass has been proposed as a model for two-dimensional superconductivity, specially for the discovery of superconducting quasiplanar oxides at high temperature, like La_2CuO_4 y YBa_2Cu_6 [6]. There are other models that describe bidimensional superconductivity at high temperature, like anyon models, but they are problematic, because they lead to parity violation, which is not observed in nature.

There are also lattice studies on the structure of the Fermion and Photon Propagators which allow us to understand the Dynamical Generation of Masses. In this kind of studies, the search for the relevant critical exponents is more easily carried out than in analytical studies in the continuum. Also, there are studies of QED3 at finite temperature, due to the potentially predictive power of this model to explain early universe phenomena, and it is also considered as the first step towards the comprehension of the hadronic structure, since in its unquenched version, i. e., considering vacuum polarization effects, the theory exhibits confinement [7]. This can be seen in a heuristical manner by considering the classical potential¹

$$\begin{aligned} V(x) &\equiv \int_{-\infty}^{\infty} dx_0 \int \frac{d^3q}{(2\pi)^3} e^{i(q \cdot x + q_0 x_0)} e^2 \Delta^T(q) \\ &= \int \frac{d^2q}{(2\pi)^2} e^{iq \cdot x} e^2 \Delta^T(q) , \end{aligned} \quad (1.2)$$

where

$$\Delta^T(q) = \frac{1}{q^2[1 + \Pi(q)]} \quad (1.3)$$

¹Although the arguments of Green's functions are the momenta squared, in order to simplify the notation we will consider that $G = G(p)$.

is related to the transverse part of the photon propagator as :

$$\Delta_{\mu\nu}^T(q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \frac{\Delta^T(q)}{q^2}. \quad (1.4)$$

Setting $\Pi(q) = 0$, one obtains

$$V(x) = \frac{e^2}{2\pi} \ln e^2 x, \quad (1.5)$$

which is a logarithmically confinement potential. e^2 is obviously the electromagnetic coupling.

In this chapter we will show the dynamics of the electromagnetically interacting objects on a plane. We start by explicitly showing the three-dimensional equivalent to the Lorentz Group. Next we study the Dirac equation explicitly, using 2×2 and 4×4 representations for the Dirac matrices. Then, we study the chiral symmetry, that we will be considering throughout the next chapters, and the discrete symmetries of QED3. In the end, we shall discuss why we prefer to study QED3 in its representation 4×4 in this thesis.

1.2 Lorentz Group

Lorentz Group can be represented by 3×3 matrices, and there are 9 parameters to classify them. However, the relation

$$\Lambda_\alpha^\mu g_{\mu\nu} \Lambda_\beta^\nu = g_{\alpha\beta} \quad (1.6)$$

fixes 6 of these parameters, leaving only three, corresponding to two boosts, each along the corresponding axis, and one rotation.

Explicit representation for the boosts are :

$$\begin{aligned} B_1(\gamma) &= \begin{pmatrix} \gamma & \gamma v & 0 \\ \gamma v & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ B_2(\gamma) &= \begin{pmatrix} \gamma & 0 & \gamma v \\ 0 & 1 & 0 \\ \gamma v & 0 & \gamma \end{pmatrix}. \end{aligned} \quad (1.7)$$

Obviously $\gamma = \sqrt{1 - v^2}$. To obtain the infinitesimal generators, we perform the parametrization

$$\gamma = \cosh \psi. \quad (1.8)$$

Then,

$$\begin{aligned} B_1(\psi) &= \begin{pmatrix} \cosh \psi & \sinh \psi & 0 \\ \sinh \psi & \cosh \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ B_2(\psi) &= \begin{pmatrix} \cosh \psi & 0 & \sinh \psi \\ 0 & 1 & 0 \\ \sinh \psi & 0 & \cosh \psi \end{pmatrix}. \end{aligned} \quad (1.9)$$

These infinitesimal generators satisfy :

$$K_i = -i \left. \frac{dB_i}{d\psi} \right|_{\psi=0}, \quad (1.10)$$

which explicitly correspond to the matrices

$$\begin{aligned} K_1 &= \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \end{aligned} \quad (1.11)$$

A rotation on the plane is described by the matrix

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}. \quad (1.12)$$

This matrix has the infinitesimal generator

$$J = -i \left. \frac{dR}{d\theta} \right|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \quad (1.13)$$

Now, these three generators are in one-to-one correspondence with the parameters of the Lorentz Group. They obey the following commutation relations :

$$\begin{aligned} [K_1, K_2] &= iJ \\ [J, K_1] &= -iK_2 \\ [J, K_2] &= +iK_1. \end{aligned} \quad (1.14)$$

As it is well known (see, for instance, [8]), pure Lorentz transformations do not form a group. Let us observe that J is a Hermitian matrix, but neither K_1 nor K_2 are. However, with the definitions

$$A_1 = iK_1, \quad A_2 = -iK_2, \quad A_3 = J, \quad (1.15)$$

relations (1.14) can be re-written as :

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad (1.16)$$

where

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{else.} \end{cases}$$

Therefore, we conclude that the Lorentz Group on a plane is isomorphic to the $SU(2)$ group. Owing to the reduced number of degrees of freedom on a plane, Dirac equation can also be studied in its matrix representation of dimensions less than 4. We discuss this matter in the next section.

1.3 Dirac Equation and the Lagrangian

In order to solve the Dirac Equation, let us recall that Paul Adrien Maurice's idea (see, for instance [9]) was precisely to write down a Hamiltonian with linear dependence on ∂_0 , and consequently on ∂_i , in such a way that its most general form was

$$H\psi = (\alpha \cdot P + \beta m)\psi, \quad (1.17)$$

with the requirement that it must fulfill the relativistic relation

$$H^2\psi = (P^2 + m^2)\psi. \quad (1.18)$$

Dirac found that α and β are no numbers, but matrices with the following characteristics :

- α_1, α_2 y α_3 commute with each other.
- $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$.

Dirac equation can be written then as :

$$(i\partial - m)\psi = 0, \quad (1.19)$$

with $\partial = \gamma^\mu \partial_\mu$, where

$$\gamma^\mu \equiv (\beta, \beta\alpha). \quad (1.20)$$

These matrices satisfy the Clifford's algebra :

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (1.21)$$

Once the representation for the γ matrices has been chosen, one can proceed to solve the Dirac equation.

1.3.1 2×2 Representation

In QED3, only three γ matrices are necessary to write down the Dirac Equation. The problem is that these matrices should be treated in a special manner when the number of dimensions of the space-time is other than four. In the three-dimensional case, a basis for the Clifford's algebra is given by the monomials [10] :

$$1, \gamma^0, \gamma^1, \gamma^2, \gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^1\gamma^2, \gamma^0\gamma^1\gamma^2. \quad (1.22)$$

If Γ_i represents any element of this set of monomials, it follows that the matrix $\gamma = \gamma^0\gamma^1\gamma^2$ satisfies

$$[\gamma, \Gamma_i] = 0, \quad (1.23)$$

Then, by Schur's lemma,

$$\gamma = cI, \quad (1.24)$$

where c is a constant. We also have that

$$(\gamma)^2 = -1, \quad (1.25)$$

therefore,

$$c = \pm i . \quad (1.26)$$

We can see that we have two inequivalent representations for the Clifford's algebra, depending upon the choice of the two signs for γ . In the case of 2×2 matrices, Pauli's σ matrices represent the Dirac matrices. We start with the metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (1.27)$$

and proceed to identify [4] :

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2 . \quad (1.28)$$

The (anti-)commutation relations among these matrices are then :

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \quad (1.29)$$

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\alpha} \gamma_\alpha . \quad (1.30)$$

These relations imply

$$\gamma_\mu \gamma^\mu = 3 , \quad (1.31)$$

$$\gamma_\mu \not{p} \gamma^\mu = -\not{p} , \quad (1.32)$$

$$\epsilon_{\mu\nu\alpha} \epsilon^{\beta\gamma\alpha} = \delta_\mu^\beta \delta_\nu^\gamma - \delta_\mu^\gamma \delta_\nu^\beta , \quad (1.33)$$

$$\epsilon^{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} = 3 , \quad (1.34)$$

and we have the traceology :

$$Tr [\gamma^\mu] = 0 , \quad (1.35)$$

$$Tr [\gamma^\mu \gamma^\nu] = 2g^{\mu\nu} , \quad (1.36)$$

$$Tr [\gamma^\mu \gamma^\nu \gamma^\rho] = -2i\epsilon^{\mu\nu\rho} , \quad (1.37)$$

$$Tr [\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 2(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) . \quad (1.38)$$

Let us notice that the Lorentz indices run from 0 to 2. In our case

$$\gamma^0 = \beta = \sigma_3, \quad \gamma^1 = \beta\alpha_1 = i\sigma_1, \quad \gamma^2 = \beta\alpha_2 = i\sigma_2. \quad (1.39)$$

To solve the equation, let us proceed in the standard way, that is, let us consider first the particle at rest. The corresponding equation reads

$$\begin{aligned} [i\gamma^0 \partial_0 - m]\psi(t) &= 0 , \\ \left[i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_0 - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \psi_A(t) \\ \psi_B(t) \end{pmatrix} &= 0 , \end{aligned}$$

or,

$$\begin{pmatrix} \frac{\partial \psi_A(t)}{\partial t} \\ -\frac{\partial \psi_B(t)}{\partial t} \end{pmatrix} = -im \begin{pmatrix} \psi_A(t) \\ \psi_B(t) \end{pmatrix} . \quad (1.40)$$

So, we find the solutions :

$$\psi_A(t) = e^{-imt}\psi_A(0), \quad \psi_B(t) = e^{imt}\psi_B(0), \quad (1.41)$$

such that the wavefunction can be written as :

$$\psi(t) = \begin{pmatrix} e^{-imt}\psi_A(0) \\ e^{imt}\psi_B(0) \end{pmatrix}, \quad (1.42)$$

with

$$\psi(0) = \begin{pmatrix} \psi_A(0) \\ \psi_B(0) \end{pmatrix}. \quad (1.43)$$

If we look for independent solutions of the form

$$\psi_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (1.44)$$

we find that the solutions for a particle at rest are :

$$\begin{aligned} \psi_1(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-imt}, & E > 0, \\ \psi_2(t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{imt}, & E < 0. \end{aligned} \quad (1.45)$$

For a moving particle, we look for solutions of the form

$$\psi(x) = e^{-ix \cdot p} u(p). \quad (1.46)$$

Let us observe that

$$\partial_\mu \psi(x) = \partial_\mu [e^{-ix \cdot p}] u(p) = -ip_\mu e^{-ix \cdot p} u(p) = -ip_\mu \psi(x), \quad (1.47)$$

therefore, we can write the Dirac equation as follows :

$$(\gamma^\mu p_\mu - m)u(p) = 0. \quad (1.48)$$

Since

$$\begin{aligned} \gamma^\mu p_\mu &= \gamma^0 p_0 - \vec{\gamma} \cdot \vec{p} \\ &= \begin{pmatrix} E & -(p^2 + ip^1) \\ p^2 - ip^1 & -E \end{pmatrix}, \end{aligned} \quad (1.49)$$

in matrix form we have

$$\begin{pmatrix} E - m & -(p^2 + ip^1) \\ p^2 - ip^1 & -E - m \end{pmatrix} \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} = 0, \quad (1.50)$$

which leads us to the following system of equations for the spinor componets :

$$\begin{aligned} (E - m)u_A(p) - (p^2 + ip^1)u_B(p) &= 0, \\ (p^2 - ip^1)u_A(p) - (E + m)u_B(p) &= 0. \end{aligned} \quad (1.51)$$

To solve this system, we choose $u_A(p) = 1$, such that

$$\begin{aligned} u_B(p) &= \left[\frac{p^2 - ip^1}{E + m} \right], \\ \Rightarrow \psi_1(x) &= \left(\frac{1}{\frac{p^2 - ip^1}{E + m}} \right) e^{-ip \cdot x}. \end{aligned} \quad (1.52)$$

This is the positive energy solution. If now we choose $u_B(p) = 1$,

$$\begin{aligned} u_A(p) &= \left[\frac{p^2 + ip^1}{E - m} \right], \\ \Rightarrow \psi_2(x) &= \left(\frac{\frac{p^2 + ip^1}{E - m}}{1} \right) e^{-ip \cdot x}, \end{aligned} \quad (1.53)$$

which is the negative energy counterpart.

1.3.2 4×4 Representation

In this case, we choose the Weyl or chiral representation for the γ matrices [9] :

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \quad (1.54)$$

To solve the Dirac equation, once again we start with the equation for the particle at rest, which reads

$$\begin{aligned} \left(\begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix} \partial_0 - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \psi_A(t) \\ \psi_B(t) \end{pmatrix} &= 0, \\ \begin{pmatrix} i\sigma_3 \frac{\partial}{\partial t} - mc & 0 \\ 0 & -i\sigma_3 \frac{\partial}{\partial t} - mc \end{pmatrix} \begin{pmatrix} \psi_A(t) \\ \psi_B(t) \end{pmatrix} &= 0, \end{aligned} \quad (1.55)$$

and that leads us to the following system of equations for the spinors :

$$\begin{aligned} i\sigma_3 \frac{\partial \psi_A}{\partial t} &= m\psi_A(t) \\ i\sigma_3 \frac{\partial \psi_B}{\partial t} &= -m\psi_B(t). \end{aligned} \quad (1.56)$$

If we decompose the spinors into their components in the form :

$$\psi_A(t) = \begin{pmatrix} \psi_{A_1}(t) \\ \psi_{A_2}(t) \end{pmatrix}, \quad \psi_B(t) = \begin{pmatrix} \psi_{B_1}(t) \\ \psi_{B_2}(t) \end{pmatrix}, \quad (1.57)$$

we obtain now the following system of equations :

$$i \frac{\partial \psi_{A_1}(t)}{\partial t} = -m\psi_{A_1}(t), \quad (1.58)$$

$$\begin{aligned}
i \frac{\partial \psi_{A_2}(t)}{\partial t} &= m \psi_{A_2}(t) , \\
i \frac{\partial \psi_{B_1}(t)}{\partial t} &= m \psi_{B_1}(t) , \\
i \frac{\partial \psi_{B_2}(t)}{\partial t} &= -m \psi_{B_2}(t) ,
\end{aligned}$$

whose solutions are

$$\begin{aligned}
\psi_{A_1}(t) &= e^{-imt} \psi_{A_1}(0) , \\
\psi_{A_2}(t) &= e^{imt} \psi_{A_2}(0) , \\
\psi_{B_1}(t) &= e^{imt} \psi_{B_1}(0) , \\
\psi_{B_2}(t) &= e^{-imt} \psi_{B_2}(0) .
\end{aligned} \tag{1.59}$$

Then, the wavefunction $\psi(t)$ can be written as

$$\psi(t) = \begin{pmatrix} e^{-imt} \psi_{A_1}(0) \\ e^{imt} \psi_{A_2}(0) \\ e^{imt} \psi_{B_1}(0) \\ e^{-imt} \psi_{B_2}(0) \end{pmatrix} . \tag{1.60}$$

We can, as we did before, choose independent solutions such that the particle at rest can be described as

$$\begin{aligned}
\psi_1(t) &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} , & \psi_2(t) &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{imt} , \\
\psi_3(t) &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imt} , & \psi_4(t) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-imt} .
\end{aligned} \tag{1.61}$$

When the particle is moving, the Dirac equation is expressed as :

$$\begin{pmatrix} \sigma_3 - i\sigma^1 p^1 - i\sigma_2 p^2 - m & 0 \\ 0 & -\sigma_3 + i\sigma^1 p^1 + i\sigma_2 p^2 - m \end{pmatrix} \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} = 0 , \tag{1.62}$$

from where we obtain the system of equations

$$\begin{aligned}
[\sigma_3 - i\sigma^1 p^1 - i\sigma_2 p^2 - m] u_A(p) &= 0 , \\
[-\sigma_3 + i\sigma^1 p^1 + i\sigma_2 p^2 - m] u_B(p) &= 0 .
\end{aligned} \tag{1.63}$$

Once more, if we decompose the spinors into their components

$$u_A(p) = \begin{pmatrix} u_{A_1}(p) \\ u_{A_2}(p) \end{pmatrix} , \quad u_B(p) = \begin{pmatrix} u_{B_1}(p) \\ u_{B_2}(p) \end{pmatrix} , \tag{1.64}$$

we arrive to the system of equations :

$$\begin{aligned}
(E - m)u_{A_1}(p) - (p^2 + ip^1)u_{A_2}(p) &= 0, \\
(p^2 - ip^1)u_{A_1}(p) - (E + m)u_{A_2}(p) &= 0, \\
-(E + m)u_{B_1}(p) - (p^2 + ip^1)u_{B_2}(p) &= 0, \\
-(p^2 - ip^1)u_{B_1}(p) - (E - m)u_{B_2}(p) &= 0.
\end{aligned} \tag{1.65}$$

To solve such system, we take advantage of the consequences of choosing a particular normalization for the spinors components, as displayed below :

- $u_{A_1}(p) = 1 \Rightarrow u_{A_2}(p) = \frac{p^2 - ip^1}{E + m}$,
- $u_{A_2}(p) = 1 \Rightarrow u_{A_1}(p) = \frac{p^2 + ip^1}{E - m}$,
- $u_{B_1}(p) = 1 \Rightarrow u_{B_2}(p) = \frac{p^2 - ip^1}{E - m}$,
- $u_{B_2}(p) = 1 \Rightarrow u_{B_1}(p) = \frac{p^2 + ip^1}{E + m}$.

Therefore, we have the independent solutions

$$\begin{aligned}
u_1(p) &= \begin{pmatrix} 1 \\ \frac{p^2 - ip^1}{E + m} \\ 0 \\ 0 \end{pmatrix}, & u_2(p) &= \begin{pmatrix} \frac{p^2 - ip^1}{E + m} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \\
u_3(p) &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{p^2 - ip^1}{E - m} \end{pmatrix}, & u_4(p) &= \begin{pmatrix} 0 \\ 0 \\ \frac{p^2 + ip^1}{E + m} \\ 1 \end{pmatrix}.
\end{aligned} \tag{1.66}$$

Since the solutions are decoupled, we can merge the corresponding solutions for positive and negative energy as

$$\begin{aligned}
u_P(p) &= \begin{pmatrix} 1 \\ \frac{p^2 - ip^1}{E + m} \\ \frac{p^2 + ip^1}{E + m} \\ 1 \end{pmatrix} \\
u_N(p) &= \begin{pmatrix} \frac{p^2 + ip^1}{E - m} \\ 1 \\ 1 \\ \frac{p^2 - ip^1}{E - m} \end{pmatrix},
\end{aligned} \tag{1.67}$$

where we understand

$$u_P(p) = \begin{pmatrix} u_P^1 \\ u_P^2 \end{pmatrix}, \quad u_N(p) = \begin{pmatrix} u_N^1 \\ u_N^2 \end{pmatrix}. \tag{1.68}$$

In this way we represent the solutions to the Dirac equation.

1.4 Discrete Symmetries

We are now in position to discuss the discrete symmetries of Dirac equation. For that purpose, we will follow the work of [13], where we use only the 2×2 representation for the Dirac matrices. We should first introduce two Dirac fields ψ and ϕ which satisfy, respectively,

$$\begin{aligned} (i\gamma^\alpha \partial_\alpha - \gamma \partial_2 - m)\psi(x) &= 0, \\ (-i\gamma^\alpha \partial_\alpha + \gamma \partial_2 - m)\phi(x) &= 0, \end{aligned} \quad (1.69)$$

where the index α runs from 0 to 1 and $\gamma = \gamma^0 \gamma^1 \gamma^2$. In order to study the charge conjugation operation C , we must write the Dirac equation when the field ψ interacts with an external magnetic field :

$$\{i\gamma^\alpha \partial_\alpha - \gamma \partial_2 - e(\gamma^\alpha A_\alpha + i\gamma A_2) - m\}\psi(x) = 0. \quad (1.70)$$

It follows that, under charge conjugation,

$$\{i\gamma^\alpha \partial_\alpha - \gamma \partial_2 + e(\gamma^\alpha A_\alpha + i\gamma A_2) - m\}\psi^C(x) = 0. \quad (1.71)$$

On the other hand, we have

$$\{i\gamma^\alpha \partial_\alpha - \gamma \partial_2 + e(\gamma^\alpha A_\alpha + i\gamma A_2) - m\}C\bar{\psi}^T(x) = 0, \quad (1.72)$$

where C is the charge conjugation matrix and T denotes the matrix transpose operation. Comparing these equations, we obtain that

$$\psi^C(x) = C\bar{\psi}^T(x), \quad (1.73)$$

which is the usual charge conjugation operation.

Parity P in three dimensions corresponds to the inversion of one axis, say x , because the inversion of both the axes can be obtained from a rotation of π of the plane. Dirac equation modified by a Parity transformation then reads :

$$(i\gamma^0 \partial_0 - i\gamma^1 \partial_1 - \gamma \partial_2 - m)\psi^P(t, -x, y) = 0. \quad (1.74)$$

For the field ϕ , we rewrite the corresponding Dirac equation without performing the Parity operation as follows :

$$(i\gamma^0 \partial_0 + i\gamma^1 \partial_1 + \gamma \partial_2 - m)\phi(t, x, y) = 0. \quad (1.75)$$

Multiplying by $1 = \gamma^0 \gamma^0$ those terms with γ^1 and γ , our equation then reads :

$$\begin{aligned} (i\gamma^0 \partial_0 + i\gamma^1 \gamma^0 \gamma^0 \partial_1 + \gamma \gamma^0 \gamma^0 \partial_2 - m)\phi(t, x, y) &= 0, \\ (i\gamma^0 \partial_0 - i\gamma^0 \gamma^1 \gamma^0 \partial_1 - \gamma^0 \gamma \gamma^0 \partial_2 - m)\phi(t, x, y) &= 0, \\ (i\gamma^0 \gamma^0 \partial_0 - i\gamma^0 \gamma^1 \partial_1 - \gamma^0 \gamma \partial_2 - m\gamma^0)\gamma^0 \phi(t, x, y) &= 0, \\ (i\gamma^0 \partial_0 - i\gamma^1 \partial_1 - \gamma \partial_2 - m)\gamma^0 \phi(t, x, y) &= 0, \end{aligned} \quad (1.76)$$

$$(1.77)$$

such that, under a comparison,

$$\psi^P(t, -x, y) = \gamma^0 \phi(t, x, y) . \quad (1.78)$$

It is convenient at this stage to take a closed look at the mass terms of QED3. If we write the four-dimensional spinors as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \quad (1.79)$$

a Parity transformation acts on them in the following way :

$$\psi_1 \rightarrow \sigma_1 \psi_2 , \quad \psi_2 \rightarrow \sigma_1 \psi_1 . \quad (1.80)$$

Obviously, the ordinary mass term is invariant under this transformation,

$$m \bar{\psi} \psi = m (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) , \quad (1.81)$$

since Pauli matrices are unitary and hermitian. The other mass term

$$\tilde{m} \bar{\psi} \frac{1}{2} [\gamma^3, \gamma^5] \psi = \tilde{m} (\psi_1^\dagger \sigma_3 \psi_1 + \psi_2^\dagger \sigma_3 \psi_2) , \quad (1.82)$$

is invariant under chiral transformations (1.89), but not under Parity, (1.78).

For the Time Reversal τ , we will need to relate somehow the fields ψ and ϕ . Dirac equation under a Time Reversal operation for the field ψ reads :

$$(-i\gamma^0 \partial_0 + i\gamma^1 \partial_1 - \gamma \partial_2 - m) \psi^\tau(-t, \vec{x}) = 0 . \quad (1.83)$$

Using the same reasoning as before, the field ϕ satisfies :

$$\begin{aligned} (i\gamma^0 \partial_0 + i\gamma^1 \partial_1 - \gamma \partial_2 - m) C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (-i\gamma \gamma^0 \partial_0 - i\gamma \gamma^1 \partial_1 - \gamma \partial_2 - m) C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (i\gamma \gamma^0 \partial_0 + i\gamma \gamma^1 \partial_1 - \gamma \partial_2 - m) C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (i\gamma \gamma^0 \partial_0 + i\gamma \gamma^1 \partial_1 - \gamma \gamma \partial_2 + m \gamma) \gamma C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (-i\gamma^0 \partial_0 - i\gamma^1 \partial_1 + \gamma \partial_2 - m) \gamma C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (-i\gamma^0 \partial_0 - i\gamma^1 \gamma^0 \gamma^0 \partial_1 + \gamma \gamma^0 \gamma^0 \partial_2 - m) \gamma C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (-i\gamma^0 \partial_0 + i\gamma^0 \gamma^1 \gamma^0 \partial_1 - \gamma^0 \gamma \gamma^0 \partial_2 - m) \gamma C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (-i\gamma^0 \gamma^0 \partial_0 + i\gamma^0 \gamma^1 \partial_1 - \gamma^0 \gamma \partial_2 - m \gamma^0) \gamma^0 \gamma C \bar{\phi}^T(t, \vec{x}) &= 0 , \\ (-i\gamma^0 \partial_0 + i\gamma^1 \partial_1 - \gamma \partial_2 - m) \gamma^0 \gamma C \bar{\phi}^T(t, \vec{x}) &= 0 , \end{aligned} \quad (1.84)$$

from where we conclude that

$$\psi^\tau(-t, \vec{x}) = \gamma^0 \gamma C \bar{\phi}^T(t, \vec{x}) . \quad (1.85)$$

We see that if we require the massive spin-1/2 fields to be invariant under C , P and τ respectively, we need to introduce the field ϕ which satisfies Dirac equation with the second representation for γ . It is worth to mention that if

originally the lagrangian does not include such field, as in our case, the usual mass term breaks P and τ , but retains $CP\tau$.

Finally, if we consider the transformation

$$\phi \rightarrow \phi' = i\gamma\phi, \quad (1.86)$$

we observe that ϕ' satisfies the same equation as ϕ , except for the sign for the mass term. The boost along the y axis and the rotation around it for the field ϕ correspond to the same transformations for the field ψ , but in the opposite direction, due to the sign of γ . On the other hand, ϕ' behaves as ψ under Lorentz transformations, although with the opposite sign for the mass term. This corresponds to the fact that both the representations of the Clifford's algebra are the same (under transformations of γ) as the representation of the Lorentz Group. Therefore, we can conclude that if we require invariance under C , P and τ , there must exist two fields with the opposite sign for their mass terms, and under C , P and τ , such fields are interchanged.

The invariant lagrangian under discrete transformations is given by

$$\mathcal{L} = i\bar{\psi}(\gamma^\alpha\partial_\alpha + i\gamma\partial_2)\psi + m\bar{\psi}\psi + i\bar{\phi}'(\gamma^\alpha\partial_\alpha + i\gamma\partial_2)\phi' - m\bar{\phi}'\phi'. \quad (1.87)$$

1.5 Chiral Symmetry

As we mentioned before, we can choose the 2×2 representation for the γ^μ matrices as the Pauli matrices, and we then use two-dimensional spinors. However, there is no other 2×2 matrix which anticommutes with the σ matrices, therefore, the massive theory does not possess a greater symmetry than the massless one [11]. This is an obstacle to define chirality. That is why we use four-dimensional spinors and also those γ^μ matrices from the four-dimensional space-time. The massless theory in this case is invariant under two chiral-like transformations :

$$\psi \rightarrow e^{i\alpha\gamma^3}\psi, \quad (1.88)$$

$$\psi \rightarrow e^{i\alpha\gamma^5}\psi. \quad (1.89)$$

and, therefore, the lagrangian is invariant under a global $U(2)$ symmetry with the generators

$$1, \gamma^3, \gamma^5, [\gamma^3, \gamma^5]. \quad (1.90)$$

This symmetry, however, is broken with a mass term of the form $m\bar{\psi}\psi$. Also, a dynamically generated mass will break this symmetry, as in four-dimensions. In QED3 there is another possible mass term which is invariant under Chiral transformations, but not under Parity (as we saw in the last section). Such term has the form

$$\frac{1}{2}\tilde{m}\bar{\psi}[\gamma^3, \gamma^5]\psi.$$

We are only considering the typical mass term, since it has been shown, by analyzing the effective potential, that solutions to the SDE with this mass term are energetically preferred [11, 12]. Another reason to do so is the conservation of parity in QED3. As a result, the Lagrangian that we shall be considering in this thesis is the one we are familiar with in QED4, i.e.,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \quad (1.91)$$

where the notations carry the usual meaning.

These are the main features of Quantum Electrodynamics on a plane. To study the phenomenon of Dynamical Mass Generation, the one we are concerning with, we must firstly study the gauge structure of QED3, in particular we must know the Schwinger-Dyson equations in this context, and also we must explore two of the consequences of gauge covariance of the theory: the Ward-Green-Takahashi Identities, which relate Green's functions among them, and the Landau-Khalatnikov-Fradkin transformations of these functions, which realize their gauge behavior under a variation of gauge. As the Lagrangian does not change its form, the derivation of the Schwinger-Dyson equations etc. does not really differ from the one in 3 spatial dimensions. We take up these derivations in the next chapter.

Chapter 2

Schwinger-Dyson Equations and Gauge Invariance Constraints

2.1 Introduction

As in any other theory, the electromagnetic dynamics can be obtained, in arbitrary dimensions, from the lagrangian and its corresponding action

$$S = \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) . \quad (2.1)$$

The equation of motion for the field ϕ is obtained after imposing the staticity condition on the action

$$\delta S = 0 . \quad (2.2)$$

From this condition, we obtain the Euler-Lagrange equations for this field

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0 . \quad (2.3)$$

In this chapter we start from the QED action. Functional derivatives of this action lead us to the Schwinger-Dyson Equations (SDE) for the Green's functions. We also repeat the derivation of the Ward-Green-Takahashi Identity (WGTI) and the Landau-Khalatnikov-Fradkin (LKF) transformations, two of the gauge identities of QED necessary in order to ensure that solutions to the SDE in the study of the Dynamical Generation of Masses with a Non Perturbative Vertex reproduce gauge parameter independent physical observables, as they must be.

2.2 Electromagnetic Action

Electromagnetic quantization can be obtained either from the canonical formulation or by means of the Feynman's Path Integral. We prefer the later scheme, since the integral formulation allows us to neatly obtain the SDE. Richard's idea was that in order to know the transition amplitude between two quantum states, we must sum over all possible histories in which such transition can take place, i. e.,

$$U(t_f, x_f \leftarrow t_i, x_i) = \int \mathcal{D}x \mathcal{D}p e^{iS}, \quad (2.4)$$

where $\mathcal{D}x$ and $\mathcal{D}p$ are the measures.

We saw before that the electromagnetic dynamics is obtained from Dirac equation (1.19)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

This equation can be obtained from the free lagrangian density

$$\mathcal{L}_{free} = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi. \quad (2.5)$$

We must consider the interaction between fermions and the electromagnetic field, given by the term $e\gamma^\mu A_\mu$. To make sure that the lagrangian density being invariant under global gauge transformations, we must replace the ordinary derivatives by covarian derivatives by using the minimal substitution principle :

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu. \quad (2.6)$$

In this way, the lagrangian density is expressed as :

$$\mathcal{L}_{int} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \quad (2.7)$$

We still need to consider the interaction of the magnetic field with itself. This is obtained from the term

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (2.8)$$

Gathering terms, QED action is

$$S[\bar{\psi}, \psi, A_\mu] = \int d^d x \left[\sum_{f=1}^N \bar{\psi}^f (i\gamma^\mu D_\mu^f - m^f) \psi^f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (2.9)$$

where f is a flavor label and N is the number of different fermion flavors. Although we are considering bare quantities, the results we obtain are also valid for renormalized quantities, provided this process is performed properly. As for the Green's functions, its bareness will be shown explicitly in order to distinguish them from the corresponding complete functions.

Let us turn our attention to the generating functional

$$Z[\bar{\eta}, \eta, J_\mu] = \int d\mu(\bar{\psi}, \psi, A) e^{(iS[\bar{\psi}, \psi, A_\mu] + i \int d^d x [\sum_f (\bar{\psi}^f \eta^f + \bar{\eta}^f \psi^f) + A_\mu J^\mu])}, \quad (2.10)$$

where $\bar{\eta}^f$, η^f and J_μ are the sources for the fermions, antifermions and photons, respectively, and where it is defined, as in [3]

$$d\mu(\bar{\psi}, \psi, A) = \Pi_f \mathcal{D}\bar{\psi}^f \mathcal{D}\psi^f \Pi_\mu \mathcal{D}A_\mu . \quad (2.11)$$

To complete the operational definition of QED, let us note that the action is invariant under the abelian local transformations

$$\begin{aligned} \psi(x) &\rightarrow \psi^\lambda(x) = e^{-ie\lambda(x)} \psi , \\ \bar{\psi}(x) &\rightarrow \bar{\psi}^\lambda(x) = e^{ie\lambda(x)} \bar{\psi} \\ A_\mu(x) &\rightarrow A_\mu^\lambda(x) = A_\mu(x) - \partial_\mu \lambda(x) \end{aligned} \quad (2.12)$$

where $\lambda(x)$ is an arbitrary scalar function and, for the time being, we have suppressed flavor labels. Under such circumstances, the generating functional is senseless, since for every single fields configuration $\{\bar{\psi}(x), \psi(x), A_\mu(x)\}$, due to the gauge invariance, there exists an infinite number of related configurations $\{\bar{\psi}^\lambda(x), \psi^\lambda(x), A_\mu^\lambda(x)\}$, which have the same action

$$S[\bar{\psi}, \psi, A_\mu] = S[\bar{\psi}^\lambda, \psi^\lambda, A_\mu^\lambda] . \quad (2.13)$$

The Grassman integration over $\bar{\psi}$ and ψ yield the same result, independently of $\lambda(x)$, since the corresponding Jacobian is unity. Then, there is a divergence in the functional integration over the field A_μ . The correct definition of the measure must ensure that the integration over the gauge field is extended only to inequivalent configurations under gauge transformations.

This problem can be solved by introducing the Fadeev-Popov determinant. The net effect of this procedure in QED is simply to introduce a gauge fixing term in the action. A commonly used choice for this term is

$$S[\bar{\psi}, \psi, A_\mu] \rightarrow S_\xi[\bar{\psi}, \psi, A_\mu] = S[\bar{\psi}, \psi, A_\mu] - \frac{1}{2\xi} \int d^d x (\partial_\mu A^\mu)^2 , \quad (2.14)$$

where ξ is the gauge fixing parameter.

2.3 Schwinger-Dyson Equations (SDE)

It is known, from some time ago, that from the field equations for a Quantum Field Theory, it can be derived a system of coupled integral equations which relates the Green's functions of such theory among them. This infinite tower of equations is known as the Schwinger-Dyson Equations [14]. We will use the integral functionals formulation to derive the SDE following the works on [3] and [15].

We start from eq. (2.14) and the generating functional (2.10). Let us take into account that the fermionic fields $\{\bar{\psi}, \psi\}$ and their sources $\{\bar{\eta}, \eta\}$ are elements of the Grassman algebra, that is, all of these field anticommute among

themselves; and that A_μ and its source J_μ are c -numbers. We also take the standard notations and conventions, where

$$\not{A} = A_\mu \gamma^\mu = g^{\mu\nu} A_\mu \gamma_\nu, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \text{etc.}$$

Let us note that for an electron, the physical charge must be $e^{phys} = -e$, where, by definition, $e = |e|$ is the magnitude of the charge of the electron.

2.3.1 SDE for the Photon Propagator

Let us consider the generating functional (2.10). The generating functional for connected Green's functions $\mathcal{G}[\bar{\eta}, \eta, J_\mu]$ is given by

$$Z[\bar{\eta}, \eta, J_\mu] = e^{\mathcal{G}[\bar{\eta}, \eta, J_\mu]}. \quad (2.15)$$

To obtain the SDE corresponding to the Photon Propagator we simply use the fact that the functional integral of a total functional derivative vanishes with the appropriate boundary conditions. For example,

$$\begin{aligned} 0 &= \int d\mu(\bar{\psi}, \psi, J_\mu) \frac{\delta}{\delta A_\mu(x)} e^{\{i(S_\xi[\bar{\psi}, \psi, A_\mu] + \int d^d x [\bar{\psi}^f \eta^f + \bar{\eta}^f \psi^f + A_\mu J^\mu])\}} \\ &= \int d\mu(\bar{\psi}, \psi, J_\mu) \left\{ \frac{\delta S_\xi}{\delta A_\mu(x)} + J^\mu(x) \right\} \\ &\quad e^{\{i(S_\xi[\bar{\psi}, \psi, A_\mu] + \int d^d x [\bar{\psi}^f \eta^f + \bar{\eta}^f \psi^f + A_\mu J^\mu])\}} \\ &= \left\{ \frac{\delta S_\xi}{\delta A_\mu(x)} \left[-\frac{\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta J^\mu} \right] + J^\mu(x) \right\} Z[\bar{\eta}, \eta, J_\mu]. \end{aligned} \quad (2.16)$$

Differentiating the action (2.14), we immediately obtain

$$\frac{\delta S_\xi}{\delta A_\mu(x)} = \left[\partial_\rho \partial^\rho g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu + \sum_f e^f \bar{\psi}^f \gamma^\mu \psi^f, \quad (2.17)$$

from where it follows that, after we divide by Z , we can write eq. (2.17) as :

$$\begin{aligned} &\left[\partial_\rho \partial^\rho g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] \frac{\delta \mathcal{G}}{i\delta J^\nu(x)} + \\ &\quad \sum_f e^f \left(\frac{\delta \mathcal{G}}{\delta \eta^f(x)} \gamma^\mu \frac{\delta \mathcal{G}}{\delta \bar{\eta}^f(x)} + \frac{\delta}{\delta \eta^f(x)} \left[\gamma^\mu \frac{\delta \mathcal{G}}{\delta \bar{\eta}^f(x)} \right] \right) = -J^\mu(x). \end{aligned} \quad (2.18)$$

This equation represents a compact form of the non perturbative equivalent to the Maxwell's equations. This is useful for us to obtain an expression for the photon vacuum polarization. Now we can take the Legendre transformation and introduce the generating functional for one-particle irreducible (1PI) Green's functions, $\Gamma[\bar{\psi}, \psi, A_\mu]$:

$$\mathcal{G}[\bar{\eta}, \eta, A_\mu] \equiv i\Gamma[\bar{\psi}, \psi, A_\mu] + i \int d^d x [\bar{\psi}^f \eta^f + \bar{\eta}^f \psi^f + A_\mu J^\mu]. \quad (2.19)$$

From the Grassman integration, it follows that $Z[\bar{\psi}, \psi, A_\mu]$ and consequently $\mathcal{G}[\bar{\eta}, \eta, J_\mu]$ depend only on even powers of $\bar{\eta}$ and η , which in turn implies that setting $\bar{\eta} = \eta = 0$ after taking the derivative of $\mathcal{G}$ (or Z), we will have nonvanishing results only for the same number of derivatives with respect to $\bar{\eta}$ and η . Similarly, in the absence of derivatives with respect to the fermionic fields, it can be seen that only an even number of derivatives of Z and $\mathcal{G}$ with respect to J_μ survive when we set $J = 0$. From the last expression, we have

$$\begin{aligned} A_\mu(x) &= \frac{\delta \mathcal{G}}{i \delta J^\mu(x)}, \quad \psi^f(x) = \frac{\delta \mathcal{G}}{i \delta \bar{\eta}^f(x)}, \quad \bar{\psi}^f(x) = -\frac{\delta \mathcal{G}}{i \delta \eta^f(x)}, \\ J_\mu(x) &= -\frac{\delta \Gamma}{\delta A^\mu(x)}, \quad \eta^f(x) = -\frac{\delta \Gamma}{\delta \bar{\psi}^f(x)}, \quad \bar{\eta}^f(x) = \frac{\delta \Gamma}{\delta \psi^f(x)}. \end{aligned} \quad (2.20)$$

From here we obtain expressions for $\bar{\psi}$, ψ and A_μ in terms of $\bar{\eta}$, η and J_μ and viceversa; for example,

$$\bar{\psi}_\alpha^f(x) = \bar{\psi}_\alpha^f[\bar{\eta}, \eta, J_\mu] = i \frac{\delta \mathcal{G}[\bar{\eta}, \eta, J_\mu]}{\delta \eta_\alpha^f},$$

with the spinorial indices explicitly shown. It is now easy to see that, setting $J = 0$ after we differentiate Γ , we will have nonvanishing results only when we have the same number of derivatives of $\bar{\psi}$ and ψ , in analogy with the case of $\mathcal{G}$. Making use of the expressions (2.21), let us take a look at the following term :

$$\begin{aligned} i \int d^d z \frac{\delta^2 \mathcal{G}}{\delta \eta_\alpha^f(x) \delta \bar{\eta}_\gamma^h(z)} \frac{\delta^2 \Gamma}{\delta \psi_\gamma^h(z) \delta \bar{\psi}_\beta^g(y)} \Big|_{\eta=\bar{\eta}=\psi=\bar{\psi}=0} \\ = \int d^d z \frac{\delta \psi_\gamma^h(z)}{\delta \eta_\alpha^f(x)} \frac{\delta \eta_\beta^g(y)}{\delta \bar{\psi}_\gamma^h(z)} \Big|_{\eta=\bar{\eta}=\psi=\bar{\psi}=0} \\ = \frac{\delta \eta_\beta^g(y)}{\delta \eta_\alpha^f(x)} \Big|_{\psi=\bar{\psi}=0} = \delta_{\alpha\beta} \delta_{fg} \delta^d(x-y). \end{aligned} \quad (2.21)$$

Therefore, when the fermionic sources $(\bar{\eta}, \eta)$ are null, we can write eq. (2.18) as :

$$\begin{aligned} \frac{\delta \Gamma}{\delta A^\mu(x)} \Big|_{\psi=\bar{\psi}=0} &= \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] A^\nu(x) \\ &\quad - i \sum_f e^f \text{Tr}[\gamma_\mu S_F^f(x, x, [A_\mu])], \end{aligned} \quad (2.22)$$

where we have identified the term

$$S_F^f(x, y, [A_\mu]) = i \frac{\delta \mathcal{G}}{\delta \eta^f(y) \delta \bar{\eta}^f(x)} = -i \frac{\delta \mathcal{G}}{\delta \eta(x) \delta \eta^f(y)} \quad (2.23)$$

as the Fermion Propagator of flavor f in an external magnetic field A_μ . One of the consequences of eq. (2.21) is that the inverse of this Green's function is

given by :

$$S_F^f(x, y, [A_\mu])^{-1} = \frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(y)} \Big|_{\psi=\bar{\psi}=0} . \quad (2.24)$$

Obviously, the complete Green's function for the fermion $S_F(x, y)$ is obtained after setting $A_\mu = 0$ in eq. (2.24).

To obtain the corresponding SDE for the photon polarization tensor, we only need to act with $\delta/\delta A_\nu(y)$ on eq. (2.22) and set $J_\mu(x) = 0$. Let us see :

$$\begin{aligned} & \frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\nu(y)} \Big|_{A_\mu=\psi=\bar{\psi}=0} \\ &= \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] \delta^d(x-y) \\ & - i \sum_f e^f \text{Tr} \left[\gamma_\mu \frac{\delta}{\delta A_\nu(y)} \left(\frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(x)} \Big|_{\psi=\bar{\psi}=0} \right)^{-1} \right] . \end{aligned} \quad (2.25)$$

The right hand side of this equation can be interpreted in a better way by observing that :

$$\begin{aligned} & \frac{\delta}{\delta A_\nu(y)} \left(\frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(x)} \Big|_{\psi=\bar{\psi}=0} \right)^{-1} \\ &= - \int d^d u \, d^d w \left(\frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(w)} \Big|_{\psi=\bar{\psi}=0} \right)^{-1} \\ & \times \frac{\delta}{\delta A_\nu(y)} \frac{\delta^2 \Gamma}{\delta \psi^f(u) \delta \bar{\psi}^f(w)} \left(\frac{\delta^2 \Gamma}{\delta \psi^f(w) \delta \bar{\psi}^f(x)} \Big|_{\psi=\bar{\psi}=0} \right)^{-1} , \end{aligned} \quad (2.26)$$

an analogous result to

$$\begin{aligned} \frac{d}{dx} [A(x) A^{-1}(x) = I] &= 0 = \frac{dA(x)}{dx} A^{-1}(x) + A(x) \frac{dA^{-1}(x)}{dx} \\ &\Rightarrow \frac{dA^{-1}(x)}{dx} = -A^{-1}(x) \frac{dA(x)}{dx} A^{-1}(x) , \end{aligned} \quad (2.27)$$

which hold for finite-dimensional matrices. Equation (2.26) involves the fermion-boson Vertex of [1-PI]

$$e^f \Gamma_\mu^f(x; y, z) = \frac{\delta}{\delta A^\mu(x)} \frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(y)} \Big|_{0=A_\mu=\psi=\bar{\psi}} , \quad (2.28)$$

which should not be confused with the generating functional Γ . Similarly to the fermionic case, the second derivative of Γ with respect to A_μ generates the inverse of the photon propagator $(\Delta^{-1})^{\mu\nu}(x, y)$ (left hand side of eq. (2.25)).

Therefore, from eqs. (2.22), (2.25) and (2.26), we obtain the SDE for the inverse of the photon propagator :

$$\begin{aligned} (\Delta^{-1})^{\mu\nu}(x, y) &= \left. \frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\nu(y)} \right|_{A_\mu = \psi = \bar{\psi} = 0} \\ &= \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] \delta^d(x - y) + \Pi_{\mu\nu}(x, y) , \end{aligned} \quad (2.29)$$

where we have identified the photon polarization tensor $\Pi_{\mu\nu}$:

$$\Pi_{\mu\nu}(x, y) = i \sum_f (e^f)^2 \int d^d z_1 d^d z_2 \text{Tr}[\gamma_\mu S_F^f(x, z_1) \Gamma_\nu^f(y; z_1, z_2) S_F^f(z_2, x)] . \quad (2.30)$$

Making use of the traslational invariance, we can write the photon propagator in momentum space

$$\Delta_{\mu\nu}(q) = \frac{-g_{\mu\nu} + (q_\mu q_\nu / (q^2 + i\epsilon))}{q^2 + i\epsilon} \frac{1}{1 + \Pi(q)} - \xi \frac{q_\mu q_\nu}{(q^2 + i\epsilon)^2} , \quad (2.31)$$

where, as usual, we define the scalar polarization $\Pi(q)$ as :

$$\Pi_{\mu\nu}(q) \equiv (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi(q) .$$

Let us note that $\Pi(q)$ is independent of the gauge parametre ξ in QED, as a result of current conservation. At the lowest order in Perturbation Theory, we have that $\Pi(q) = 0$, as well as

$$\Gamma_\nu^f(y; z_1, z_2) = \gamma_\nu \delta^d(y - z_1) \delta^d(y - z_2) \quad \text{y} \quad (i \not{\partial} - m^f) S_F^f(x, y) = \delta^d(x - y) . \quad (2.32)$$

Once we factorized e^f , there is no explicit flavor dependence for the proper vertex. We have seen that from the second derivative of the generating functional $\Gamma[\bar{\psi}, \psi, A_\mu]$ we obtain the photon and fermion propagators, and from the third one, the proper vertex of the fermion-boson interaction. Generally, higher order derivatives of $\Gamma[\bar{\psi}, \psi, A_\mu]$ yield the corresponding proper Green's functions, where the number and type of derivatives yield the number and type of legs in the proper Green's functions. The SDE for the Photon Propagator is shown in Diagram (1)¹.

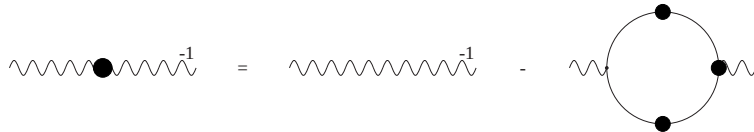


Diagram (1) : SDE for the Photon Propagator.

¹Diagrams were generated with AXODRAW [16]

The representation of the SDE in momentum space is immediately obtained by taking the Fourier transform of the expression in coordinate space, or, more easily, using the usual Feynman rules for the diagrams based in the lowest order perturbative contribution to the non perturbative quantities. For example, for the photon polarization tensor we obtain

$$i\Pi_{\mu\nu}(q) = (-1) \sum_f (e^f)^2 \int \frac{d^d k}{(2\pi)^d} \text{Tr}[(i\gamma_\mu)(iS_F^f(k))(i\Gamma_\nu^f(k, k+q))(iS_F^f(k+q))] , \quad (2.33)$$

where the factor (-1) arises from the fermion loop as usual.

2.3.2 SDE for the Fermion Propagator

Following a similar procedure, we can derive the integral equation for the Fermion Propagator starting from

$$\begin{aligned} 0 &= \int d\mu(\bar{\psi}, \psi, A) \frac{\delta}{\delta\bar{\psi}(x)} e^{\{i(S_\xi[\bar{\psi}, \psi, A_\mu] + \int d^d x [\bar{\psi}^f \eta^f + \bar{\eta}^f \psi^f + A_\mu J^\mu])\}} \quad (2.34) \\ &= \left\{ \frac{\delta S_\xi}{\delta\bar{\psi}(x)} \left[-\frac{\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta J} \right] + \eta^f(x) \right\} Z[\bar{\eta}, \eta, J_\mu] \\ &= \left[\eta^f(x) + \left(i\cancel{\partial} - m^f + e^f \gamma^\mu \frac{\delta}{i\delta J^\mu(x)} \right) \frac{\delta}{i\delta\bar{\eta}(x)} \right] Z[\bar{\eta}, \eta, J_\mu] . \quad (2.35) \end{aligned}$$

The last line of this expresion in the functional non perturbative equivalent to the Dirac equation.

As before, we act with $\delta/\delta\eta^f(y)$ on this expresion to obtain

$$\begin{aligned} &\delta^d(x-y) Z[\bar{\eta}, \eta, J_\mu] \Big|_{\eta=\bar{\eta}=0} \\ &- \left(i\cancel{\partial} - m^f + e^f \gamma^\mu \frac{\delta}{i\delta J^\mu(x)} \right) Z[\bar{\eta}, \eta, J_\mu] \Big|_{\eta=\bar{\eta}=0} S_F^f(x, y; [A_\mu]) = 0 , \quad (2.36) \end{aligned}$$

with obvious notation. Now, using eqs. (2.15) and (2.21), we can rewrite :

$$\begin{aligned} &\delta^d(x-y) \\ &- \left(i\cancel{\partial} - m^f + e^f \mathcal{A}(x; [J]) + e^f \gamma^\mu \frac{\delta}{i\delta J^\mu(x)} \right) S_F^f(x, y; [A_\mu]) = 0 , \quad (2.37) \end{aligned}$$

which defines the non perturbative connected two-points Green's function.

The electromagnetic potential vanishes in the absence of an external source, that is, $A_\mu(x; [J=0]) = 0$, in such a way that it is only written to exhibit the content of the remaining functional derivation for eq. (2.37), which can be done exploiting the identity (2.26) :

$$\frac{\delta}{i\delta J^\mu(x)} S_F^f(x, y; [A_\mu])$$

$$\begin{aligned}
&= \int d^d z \frac{\delta A_\nu(z)}{i\delta J^\mu(x)} \frac{\delta}{\delta A_\nu(z)} \left(\frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(y)} \Big|_{\psi=\bar{\psi}=0} \right)^{-1} \\
&= -e^f \int d^d z d^d u d^d w \frac{\delta A_\nu(z)}{i\delta J^\mu(x)} S_F^f(x, u) \Gamma_\nu(u, w; z) S_F^f(w, y) \\
&= -e^f \int d^d z d^d u d^d w i\Delta_{\mu\nu}(x, z) S_F^f(x, u) \Gamma_\nu(u, w; z) S_F^f(w, y) , \quad (2.38)
\end{aligned}$$

where in the last line we take $J = 0$. From here, in the absence of external sources, eq. (2.37) is equivalent to the expression :

$$\begin{aligned}
\delta^d(x - y) &- (i\partial - m^f) S_F^f(x, y) \\
&= -i(e^f)^2 \int d^d z d^d u d^d w \Delta^{\mu\nu}(x, z) \gamma_\mu S_F^f(x, u) \Gamma_\nu(u, w; z) S_F^f(w, y) . \quad (2.39)
\end{aligned}$$

It is usual to write the SDE corresponding to the inverse Fermion Propagator. Therefore, multiplying by $S_F^{f-1}(y, y')$, integrating with respect to y and relabeling $y' = y$:

$$\begin{aligned}
S_F^{f-1}(x, y) &- (i\partial - m^f) \delta^d(x - y) \\
&= -i(e^f)^2 \int d^d z d^d u \Delta^{\mu\nu}(x, z) \gamma_\mu S_F^f(x, u) \Gamma_\nu(u, y; z) . \quad (2.40)
\end{aligned}$$

The photon propagator couples eqs. (2.40) and (2.30). In this way, one observes that the equations for the two-points functions couple to each other, and both depend on the three-points Green's function $\Gamma^{f\mu}$. This is the first indication of a general rule which says that the SDE for an n -points function is coupled to others of the same order or lower orders, and to functions of order $(n + 1)$.

Diagram (2) shows the SDE corresponding to the Fermion Propagator

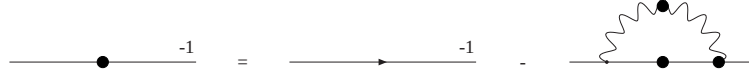


Diagram (2) : SDE for the Fermion Propagator.

2.3.3 SDE for the Fermion-Boson Vertex

The corresponding equation for the three-point vertex can be obtained in a similar fashion. For completeness, we present it in momentum space, where it is written more concisely :

$$\begin{aligned}
i\Gamma_\mu^f(p', p) &= i\gamma_\mu + \sum_g \int \frac{d^d l}{(2\pi)^d} (iS_F^g(p' + l)) \\
&\times (i\Gamma_\mu^g(p' + l, p + l))(iS_F^g(p + l)) K^{gf}(p + l, p' + l, l) . \quad (2.41)
\end{aligned}$$

K is the fermion-antifermion scattering kernel. The diagrammatic representation of this equation is shown in Diagram (3) :

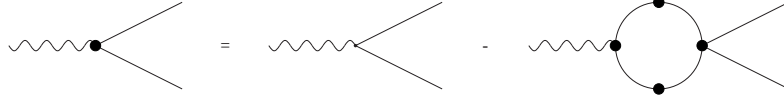


Diagram (3) : SDE for the Vertex.

Clearly, Γ_μ couples to the two-point function for the fermion S_F , and to the fermion-antifermion scattering amplitude M , a four-point function, which again illustrates the general rule.

To solve the SDE for the Fermion Propagator, the simplest approximation for the Vertex that has been used is the so-called rainbow or ladder approximation, which consists in approximate M by iterating the lowest order perturbative contribution to the kernel K , along with the substitution of the fermionic propagators by their bare counterparts, $S_F^{0f}(p) = 1/[\not{p} - m^f]$. This and other approximation will be discussed below, avoiding flavor labels.

2.3.4 Solving the SDE

Quenched Approximation

In massless QED3 in the quenched approximation [7, 17, 18], which corresponds to neglect fermion-loop contributions to the vacuum polarization, that is, to take

$$\Pi(q) = 0 \quad (2.42)$$

in the Photon Propagator, we face infrared divergences with the ordinary Perturbation Theory. A commonly used remedy for this situation is to soften the infrared behavior of the Photon Propagator by including fermion-loop contributions to the vacuum polarization. At the lowest order for a fermion of mass m , this contribution to the Polarization Scalar is :

$$\Pi(k) = \frac{\alpha}{k^2} \left[2m + \frac{k^2 - m^2}{k} \arcsin \left(\frac{k}{\sqrt{k^2 + 4m^2}} \right) \right]. \quad (2.43)$$

In a theory with N massless fermions, the Polarization Scalar is then :

$$\Pi(k) = \frac{\tilde{\alpha}}{k}, \quad (2.44)$$

where

$$\tilde{\alpha} = \frac{Ne^2}{8},$$

such that the photon propagator behaves as $1/q$ for $q^2 \rightarrow 0$, that is, the infrared divergence has been softened without altering the ultraviolet properties of the propagator.

The quenched approximation in the SDE for the Fermion Propagator corresponds to Diagram (4) :

$$\text{Fermion line with a dot} \stackrel{-1}{=} \text{Fermion line with an arrow} \stackrel{-1}{=} \text{Fermion line with a self-energy loop} - \text{Fermion line with a vertex correction}$$

Diagram (4) : SDE for the Fermion Propagator (quenched approximation)

and at the one-loop level in the ordinary Perturbation Theory there is no distinction between the quenched and the unquenched approximations.

Rainbow or Ladder Approximation

In the study of the Dynamical Generation of Masses, one commonly used approximation is to set [3]

$$\Gamma^\mu(k, p) = \gamma^\mu, \quad (2.45)$$

which is known as the rainbow or ladder approximation. If this approximation is added to the quenched one, the SDE for the Fermion Propagator decouples for the corresponding equations for the Photon Propagator and for the Vertex. The following Diagram :

$$\text{Fermion line with a dot} \stackrel{-1}{=} \text{Fermion line with an arrow} \stackrel{-1}{=} \text{Fermion line with a self-energy loop} - \text{Fermion line with a vertex correction}$$

Diagram (5) : SDE for the Fermion Propagator (quenched approximation and bare vertex)

describes this conjunction of approximations.

Beyond the Rainbow Approximation

One of the problems with the rainbow approximation is the violation of gauge covariance, particularly of the Ward-Green-Takahashi Identity. The correct form for the fermion-boson vertex is crucial to restore the gauge covariance of the SDE and should be such that the above mentioned identity is fulfilled, among other requirements. Ball and Chiu [19] have studied the structure of this vertex and have proposed their now famous ansatz, which we will discuss afterwards. The restoration of the gauge covariance for the physical observables is one of the main motivations for the construction of the fermion-boson vertex, which we will carry out in Chapter 5.

1/N Expansion

Ordinary Perturbation Theory in terms of the coupling seems to break down due to the infrared divergences of the Green's functions. These divergences can be avoided making use of another expansion parameter. We can take N fermion flavors and expand in $1/N$ [5]. It has been shown that massless QED3 is finite order by order in this approximation, and besides, the infrared behavior of the Photon Propagator softenes, as we pointed out before. Therefore, we have a theory with N massless fermions and we take the large- N limit, keeping Ne^2 fixed in the weak coupling regime. If we write

$$e^2 = \frac{8}{N}, \quad (2.46)$$

the new perturbative expansion for the Photon Propagator is shown in Diagram (6) :

$$\text{wavy line with dot} = \text{wavy line} + N \text{ wavy line with fermion loop} + O(1/N)$$

Diagram (6) : SDE for the Fermion Propagator ($1/N$ expansion).

and after its evaluation, it yield a Polarization Scalar for the Photon

$$\Pi(q) = q, \quad (2.47)$$

which leads to a $1/q$ behavior for the Photon Propagator in the infrared domain.

The SDE for the Fermion Propagator in this scheme is given by

$$S_F(p)^{-1} = \not{p} - \frac{8i}{N} \int \frac{d^3k}{(2\pi)^3} \gamma^\mu S_F(p) \Gamma^\nu(p, k) \Delta_{\mu\nu}(p - k). \quad (2.48)$$

There exists a controversy on whether in the study of this equation, see for example Pennington *et. al.* [20] and Atkinson *et. al.* [21], DCSB takes place for arbitrary number of flavours or there exists a critical number of such flavours separating the chirally symmetric and asymmetric phases of unquenched QED. However, we do not take up these matters in this thesis.

Gauge Technique

There exist a scheme to solve the SDE which differs substantially in the method with the previously mentioned studies : The Gauge Technique [25, 26, 27, 28, 29, 30, 31, 32, 33]. This scheme, based in Minkowski space, assumes that the elements of the SDE (Propagators and Vertices) have a spectral representation, in term of which the SDE are reformulated and directly solved. For example, it assumes that there exists a spectral function ρ_ψ such that the Fermion

Propagator can be written as :

$$S_F(p) = \int_{-\infty}^{\infty} d\omega \frac{\rho_\psi(\omega)}{\not{p} - \omega}, \quad (2.49)$$

and the fermion-boson Vertex has a similar form. In fact, an ansatz in this scheme is

$$S_F(p) \Gamma_\mu^{GT}(p, q) S_F(q) = \int_{-\infty}^{\infty} d\omega \rho_\psi(\omega) \frac{1}{\not{p} - \omega} \gamma_\mu \frac{1}{\not{q} - \omega}, \quad (2.50)$$

Inserting these expressions into the SDE, we obtain a linear equation for the spectral density. This is an important feature of the Gauge Technique : It reduces the SDE to linear equations. Another advantage is that the Ward-Green-Takahashi identity is automatically taken into account.

2.4 Ward-Green-Takahashi Identity

One of the consequences of gauge covariance is that Green's functions obey certain identities which relate one of these functions to the others. These are called Ward-Geen-Takahashi identities (WGTI), [34, 35, 36], and they come out from the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. They play a crucial role in the proof for the renormalizability of the theory. One of them, simply known as the WGTI, relates the [1-PI] Vertex to the propagators, and it has been widely implemented in SDE studies based either on the Gauge Technique, for instance, [25, 26, 27, 28, 29, 30, 32, 33], and on making an ansatz for the fermion-boson vertex, [19, 37, 38, 39, 40, 41, 42, 43].

We derive this identity following the textbook [8], starting from the generating functional (2.10) with the action (2.14), which includes the gauge fixing term. Let us recall that without such term (and the source terms), the lagrangian corresponding to this action is gauge invariant. This makes Z to be infinite and spoils the search for the photon propagator. In order to find a finite propagator, we are forced to introduce a gauge fixing term (and a ghost term, which in the abelian case, we can absorb into the normalization). This means that the lagrangian related to the action (2.14) is no longer gauge invariant. The physical consequences of the theory, expressed in terms of Green's functions, should not depend upon the gauge, in such a way that Z *must be* gauge invariant. This is a nontrivial requirement, and leads us to a differential equation for Z , which we will find below.

Let us take the transformations (2.13) to be infinitesimal, that is,

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \lambda(x) \\ \psi &\rightarrow \psi - ie\lambda(x)\psi \\ \bar{\psi} &\rightarrow \bar{\psi} + ie\lambda(x)\bar{\psi}. \end{aligned} \quad (2.51)$$

Under these transformations, neither the gauge fixing term, nor the source terms are gauge invariant, in such a way that the integrand of Z acquires a factor

$$e^{\left\{ i \int dx \left[-\frac{1}{\xi} (\partial^\mu A_\mu) \partial_\rho \partial^\rho \lambda + J^\mu \partial_\mu \lambda + ie\lambda (\bar{\eta}\psi - \bar{\psi}\eta) \right] \right\}}, \quad (2.52)$$

which, being λ infinitesimal, can be rewritten as

$$1 + i \int dx \left[-\frac{1}{\xi} (\partial^\mu A_\mu) \partial_\rho \partial^\rho + \partial^\mu J_\mu \lambda - ie(\bar{\eta}\psi - \bar{\psi}\eta) \right] \lambda, \quad (2.53)$$

where we have integrated by parts to remove the derivative operator from λ . Gauge invariance of Z implies that the operator (2.53), when acting on Z , is merely the identity. Since λ is an arbitrary function, this implies that

$$\left[-\frac{1}{\xi} \partial_\rho \partial^\rho (\partial^\mu A_\mu) + \partial^\mu J_\mu \lambda - ie(\bar{\eta}\psi - \bar{\psi}\eta) \right] Z = 0. \quad (2.54)$$

Substituting the fields by derivatives with respect to their sources,

$$\psi \rightarrow \frac{1}{i} \frac{\delta}{\delta \bar{\eta}}, \quad \bar{\psi} \rightarrow \frac{1}{i} \frac{\delta}{\delta \eta}, \quad A_\mu \rightarrow \frac{1}{i} \frac{\delta}{\delta J^\mu}, \quad (2.55)$$

we find the following functional differential equation for Z :

$$\left[\frac{i}{\xi} \partial_\rho \partial^\rho \partial^\mu \frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - e \left(\bar{\eta} \frac{\delta}{\delta \bar{\eta}} - \eta \frac{\delta}{\delta \eta} \right) \right] Z[\bar{\eta}, \eta, J_\mu] = 0. \quad (2.56)$$

Taking the transformation (2.15), the last expression can be written as an equation for $\mathcal{G}$:

$$\frac{i}{\xi} \partial_\rho \partial^\rho \partial^\mu \frac{\delta \mathcal{G}}{\delta J^\mu} - \partial^\mu J_\mu - e \left(\bar{\eta} \frac{\delta \mathcal{G}}{\delta \bar{\eta}} - \eta \frac{\delta \mathcal{G}}{\delta \eta} \right) = 0, \quad (2.57)$$

where $\mathcal{G} = \mathcal{G}[\bar{\eta}, \eta, J_\mu]$. Finally, let us turn this expression into an equation for the vertex function Γ , given by the transformation (2.19). Making use of the identities (2.21), eq. (2.57) becomes :

$$-\frac{1}{\xi} \partial_\rho \partial^\rho \partial^\mu A_\mu(x) + \partial_\mu \frac{\delta \Gamma}{\delta A_\mu(x)} - ie\psi \frac{\delta \Gamma}{\delta \psi(x)} + ie\bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}(x)} = 0. \quad (2.58)$$

Now, taking the functional derivative with respect to $\bar{\psi}(x_1)$ and $\psi(y_1)$, and setting $\bar{\psi} = \psi = A = 0$, the first term vanishes, and therefore,

$$\begin{aligned} -\partial_x^\mu \frac{\delta^3 \Gamma}{\delta \bar{\psi}(x_1) \delta \psi(y_1) \delta A^\mu(x)} \Big|_{\bar{\psi}=\psi=A=0} &= ie\delta(x-x_1) \frac{\delta^2 \Gamma}{\delta \bar{\psi}(x_1) \delta \psi(y_1)} \Big|_{\psi=\bar{\psi}=0} \\ &\quad - ie\delta(x-y_1) \frac{\delta^2 \Gamma}{\delta \bar{\psi}(x_1) \delta \psi(y_1)} \Big|_{\psi=\bar{\psi}=0}. \end{aligned} \quad (2.59)$$

The left hand side of this equation is the derivative of the [1-PI] fermion-boson vertex (2.28), and the next two terms are the inverses of the exact Fermion Propagators (2.23). The content of eq. (2.59) becomes clear if we express it in momentum space. For such purpose, we define the proper vertex function $\Gamma_\mu(k, p, q)$ as

$$\begin{aligned} \int dx dx_1 dy_1 e^{i(qx_1 - ky_1 - px)} \frac{\delta^3 \Gamma}{\delta \bar{\psi}(x_1) \delta \psi(y_1) \delta A^\mu(x)} \Big|_{\bar{\psi}=\psi=A=0} \\ = ie(2\pi)^4 \delta(q - k - p) \Gamma_\mu(k, p, q). \end{aligned} \quad (2.60)$$

On the other hand, we define the Fermion Propagator in momentum space as :

$$\int dx_1 dy_1 e^{i(qx_1 - ky_1)} \left. \frac{\delta^2 \Gamma}{\delta \bar{\psi}(x_1) \delta \psi(y_1)} \right|_{\psi=\bar{\psi}=0} = (2\pi)^4 \delta(q-p) i S_F^{-1}(p) . \quad (2.61)$$

Therefore, multiplying eq. (2.59) by $e^{i(qx_1 - ky_1 - px)}$ and integrating over x , x_1 and y_1 , we have

$$q^\mu \Gamma_\mu(k, p, q) = S_F^{-1}(k) - S_F^{-1}(p) . \quad (2.62)$$

or, in the limit $k \rightarrow p$,

$$\frac{\partial S_F^{-1}}{\partial p^\mu} = \Gamma_\mu(p, p) . \quad (2.63)$$

WGTI is one of the requirements for the restoration of the gauge covariance of the physical observables that have been employed in SDE studies. Its implementation is pretty much simple, and it has been widely used.

There exist also a WGTI for the Photon Propagator, which is given by the expression :

$$q_\mu \Pi^{\mu\nu}(q) = 0 . \quad (2.64)$$

This expression is useful, because it allows us to define the Polarization Scalar in the traditional way. The fact that $\Pi^{\mu\nu}$ is transverse, leads us to the masslessness for the photon. We no longer take into account this identity.

2.5 Landau-Khalatnikov-Fradkin Transformations

In a gauge field theory, Green's functions transform in a specific manner under a variation of gauge. In Quantum Electrodynamics, and in honor to Lev Davidovich and his colleagues who first obtained them, these transformations carry the name of Landau-Khalatnikov-Fradkin (LKF) transformations, [44, 45, 46]. These were also derived by Johnson and Zumino through functional methods,[47, 48]². LKF transformations are non perturbative in nature, and therefore, they have the potential to play an important role to address the problems of gauge invariance which plague the strong coupling SDE studies. In general, the rules governing these transformation are far from simple. The fact that they better describe their essence in coordinate space, make them even more complex. As a result, these transformations have played a less significant and practical role than desired in SDE studies.

We display its derivation below, following the work of Zumino [48]. We start by noticing that in Landau gauge, the Photon Propagator can be written as

$$\Delta_{\mu,\nu}(x;0) = \left[g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right] \Delta_c(x) , \quad (2.65)$$

²Fukuda, Kubo and Yokoyama have looked for a possible formalism where renormalization constants of the wave function are in fact gauge invariant [49]

where $\Delta_c(x)$ is the so-called Feynman function

$$\Delta_c(x) = -\frac{\delta(x)}{\partial^2} . \quad (2.66)$$

In an arbitrary covariant gauge, the Photon Propagator can be parametrized by an arbitrary function Δ_d in the form :

$$\Delta_{\mu\nu}(x; \Delta_d) = \Delta_{\mu\nu}(x; 0) + \partial_\mu \partial_\nu \Delta_d(x) . \quad (2.67)$$

Let us recall now that the generating functional is expressed as

$$Z[\bar{\eta}, \eta, J_\mu] = \langle 0 | T e^{i \int dx (\bar{\eta} \psi + \bar{\psi} \eta + A_\mu J^\mu)} | 0 \rangle . \quad (2.68)$$

Just to obtain the LKF transformations, for the moment we are not assuming that

$$\partial_\mu J^\mu = 0 . \quad (2.69)$$

Now, under a gauge transformation,

$$Z_\lambda[\bar{\eta}, \eta, J_\mu] = Z_0 [\bar{\eta} e^{ie\lambda}, \eta e^{-ie\lambda}, J^\mu] e^{ie \int dx J_\mu \partial^\mu \lambda} , \quad (2.70)$$

which can be written in a differential form with the expression

$$i \frac{\delta Z}{\delta \lambda} = \left(\partial^\mu J_\mu + e \eta \frac{\delta}{\delta \eta} - e \bar{\eta} \frac{\delta}{\delta \bar{\eta}} \right) Z . \quad (2.71)$$

With these definitions, it is easy to verify that the generating functional satisfies the following set of differential equations :

$$\left\{ \partial^\sigma \left(\partial_\mu \frac{\delta}{i \delta J^\sigma} - \partial_\sigma \frac{\delta}{i \delta J^\mu} \right) + (\delta_\mu^\sigma + a_\mu \partial^\sigma) \left(i e \frac{\delta}{i \delta \eta} \gamma_\sigma \frac{\delta}{i \delta \bar{\eta}} - J_\sigma \right) \right\} Z = 0 \quad (2.72)$$

$$\left\{ \left[\gamma^\mu \left(i e \frac{\delta}{i \delta J^\mu} \right) + m \right] \frac{\delta}{i \delta \bar{\eta}} - \eta \right\} Z = 0 \quad (2.73)$$

$$\left\{ -\frac{\delta}{i \delta \eta} \left[-\gamma^\mu \left(\partial_\mu + i e \frac{\delta}{i \delta J^\mu} \right) + m \right] - \bar{\eta} \right\} Z = 0 \quad (2.74)$$

$$\left(a^\mu \frac{\delta}{i \delta J^\mu} + \lambda \right) Z = 0 . \quad (2.75)$$

The vector operator a_μ is introduced firstly for the sake of consistency of the notation. It satisfies

$$\partial^\mu a_\mu = -1 , \quad (2.76)$$

and a convenient choice for it defines the different gauges as well. For instance,

$$a_\mu = \partial_\mu (-\partial^2 - i\epsilon)^{-1} \quad (2.77)$$

defines the Landau gauge, and for a Lorentz frame, characterized by a time-like unitary vector n_μ ,

$$a_\mu = \frac{\partial_\mu + n_\mu (n \cdot \partial)}{\partial^2 + (n \cdot \partial)^2} \quad (2.78)$$

corresponds to the Coulomb gauge. Turning our attention back to the generating functional, if an object $\mathcal{F}$, constructed as a functional derivative of the generating functional, is gauge invariant in the sense that it does not change with the choice of λ , then we have that

$$\frac{\delta \mathcal{F}}{\delta \lambda} = 0 . \quad (2.79)$$

In particular, if we take the Fermion Propagator as

$$S_F(x, y) = \frac{1}{iZ} \frac{\delta^2 Z}{\delta \eta(y) \delta \bar{\eta}(x)} , \quad (2.80)$$

we obtain from eq. (2.70) or (2.71), after setting $\eta = \bar{\eta} = 0$,

$$Z_\lambda[0, 0, J^\mu] S_{F\lambda}(x, y) = e^{[ie(\lambda(x) - \lambda(y)) + i \int \partial_\mu J^\mu \lambda]} Z_0[0, 0, J^\mu] S_{F0}(x, y) , \quad (2.81)$$

or,

$$i \frac{\delta}{\delta \lambda(z)} [Z[0, 0, J^\mu] S_F(x, y)] = [\partial_\mu J^\mu(z) - e\delta(x - z) + e\delta(y - z)] Z[0, 0, J^\mu] S_F(x, y) . \quad (2.82)$$

To obtain a convenient expression for the generating functional which allows us to deduce the LKF transformations, let us consider first the Fermion Propagator in an external magnetic field B_μ given by

$$\tilde{S}_F[x, y; B_\mu] \equiv \tilde{S}_F(B) = \frac{\delta(x - y)}{\gamma^\mu (\partial_\mu - ieB_\mu) + m} . \quad (2.83)$$

We write then the vacuum Polarization Scalar, in obvious notation, as :

$$\Pi(B) = e^{-Tr(\ln \tilde{S}_F(B) \tilde{S}_F^{-1}(0))} . \quad (2.84)$$

By direct verification we have the identity

$$\left(\frac{\delta}{\delta B_\mu} - e \frac{\delta}{i\delta \eta} \gamma^\mu \frac{\delta}{i\delta \bar{\eta}} \right) [e^{i\eta \tilde{S}_F(B) \eta} \Pi(B)] = 0 . \quad (2.85)$$

The generating functional can then be written in the following way :

$$Z[\bar{\eta}, \eta, J_\mu] = e^{i\bar{\eta} \tilde{S}_F \left[\frac{\delta}{i\delta J_\mu} \right] \eta} \Pi \left[\frac{\delta}{i\delta J_\mu} \right] e^{\left[\frac{i}{2} (J_\mu + a_\mu \partial^\rho J_\rho) \Delta_c (J^\mu + a^\mu \partial^\rho J_\rho) - i \partial^\rho J_\rho \lambda - \frac{i}{2} \partial^\rho J_\rho \Delta_d \partial^\sigma J_\sigma \right]} . \quad (2.86)$$

This expression satisfies eqs. (2.72) to (2.74) automatically, and eq.(2.75) is satisfied when we take $\Delta_d = 0$. We can verify directly this sentence, except in

the case of eq. (2.72), where we should observe firstly that, on acting with the operator

$$\partial^\sigma \left(\partial_\mu \frac{\delta}{i\delta J^\sigma} - \partial_\sigma \frac{\delta}{i\delta J^\mu} \right)$$

in the last exponential of (2.86), we obtain a factor

$$J_\mu + a_\mu \partial^\rho J_\rho . \quad (2.87)$$

Those factors with λ and Δ_d do not contribute. We must now move this factor to the left of the terms of (2.86) which contain derivatives with respect to J_μ . The net effect of this operation is to replace

$$J_\mu \rightarrow J_\mu - ie \frac{\delta}{i\delta\eta} \gamma_\mu \frac{\delta}{i\delta\bar{\eta}} , \quad (2.88)$$

by virtue of (2.85). This verifies eq. (2.72). We also take advantage of

$$\Pi \left[\frac{\delta}{i\delta J_\mu} \right] \partial^\rho J_\rho = \partial^\rho J_\rho \Pi \left[\frac{\delta}{i\delta J_\mu} \right] \quad (2.89)$$

and

$$\begin{aligned} \tilde{S}_F \left[\frac{\delta}{i\delta J_\mu}; x, y \right] \partial^\rho J_\rho(z) &= \\ [\partial^\rho J_\rho(z) - e\delta(x-z) + e\delta(y-z)] \tilde{S}_F \left[\frac{\delta}{i\delta J_\mu}; x, y \right] , \end{aligned} \quad (2.90)$$

to find the relation

$$Z[0, 0, J^\mu] S_F[x, y; J^\mu] = \tilde{S}_F \left[\frac{\delta}{i\delta J_\mu}; x, y \right] Z[0, 0, J^\mu] , \quad (2.91)$$

which involves the propagator (2.80). This expression is very useful to obtain in a direct way the LKF transformation for the Fermion Propagator. We start from considering an infinitesimal transformation of the function Δ_d . Such variation induces a transformation on the generating functional given by

$$\delta Z = \frac{i}{2} \iint \frac{\delta}{\delta\lambda} (\delta\Delta_d) \frac{\delta}{\delta\lambda} Z . \quad (2.92)$$

Now, setting $\eta = \bar{\eta} = 0$, the induced change in the generating functional without fermionic sources is

$$\delta Z[0, 0, J^\mu] = -\frac{i}{2} \iint \partial^\mu J_\mu (\delta\Delta_d) \partial^\rho J_\rho Z[0, 0, J^\mu] , \quad (2.93)$$

or, in finite form

$$Z'[0, 0, J^\mu] = e^{-\frac{i}{2} \iint \partial^\mu J_\mu (\delta\Delta_d) \partial^\rho J_\rho} Z[0, 0, J^\mu] . \quad (2.94)$$

Therefore, for the Fermion Propagator we have

$$\delta(Z[0, 0, J^\mu]S_F) = \frac{i}{2} \iint \frac{\delta}{\delta\lambda}(\delta\Delta_d) \frac{\delta}{\delta\lambda}(Z[0, 0, J^\mu]S_F) , \quad (2.95)$$

which, along with eq. (2.82), implies

$$\begin{aligned} S'_F(x, y; J^\mu) &= e^{ie^2[\delta\Delta_D(x-y)-\delta\Delta_d(0)]+ie\int[\delta\Delta_d(x-z)-\delta\Delta_d(y-z)]\partial^\rho J_\rho(z)dz} \\ &\quad S_F(x, y; J^\mu) . \end{aligned} \quad (2.96)$$

For $J = 0$ we find

$$S'_F(x, y; 0) = e^{ie^2[\delta\Delta_d(x-y)-\delta\Delta_d(0)]} S_F(x, y; 0) \quad (2.97)$$

Then, for a finite change of Δ_d , the transformation law for the Fermion Propagator reads :

$$S_F(x; \xi) = S_F(x; 0) e^{-i[\Delta_d(0)-\Delta_d(x)]} . \quad (2.98)$$

For the Vertex, we have

$$\begin{aligned} B_\mu(z; x, y|\Delta) &= B_\mu(z; x, y|0) e^{-i[\Delta_d(0)-\Delta_d(x-y)]} \\ &\quad + S_F(x-y; 0) e^{-i[\Delta_d(0)-\Delta_d(x-y)]} \frac{\partial}{\partial z_\mu} [\Delta_d(x-z) - \Delta_d(z-y)] , \end{aligned} \quad (2.99)$$

where B_μ is the non-amputated vertex, defined in momentum space in terms of the amputated vertex Γ_μ as :

$$B_\mu(k, p) = S_F(k) \Gamma_\nu(k, p) S_F(p) \Delta_{\mu\nu}(q) . \quad (2.100)$$

The transformation rule for the partially amputated Vertex

$$\Lambda_\mu(k, p) = S_F(k) \Gamma_\mu(k, p) S_F(p) \quad (2.101)$$

follows from eqs. (2.67), (2.98) and (2.99), and it is simply given by :

$$\Lambda_\mu(z; x, y|\Delta_d) = \Lambda_\mu(z; x, y|0) e^{-i[\Delta_d(0)-\Delta_d(x-y)]} . \quad (2.102)$$

In the usual covariant way for gauge fixing, the Photon Propagator takes the form :

$$\Delta_{\mu\nu}(q, \xi) = \frac{1}{q^2[1 + \Pi(q)]} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \xi \frac{q_\mu q_\nu}{q^4} , \quad (2.103)$$

which is obtained by taking Δ_d in eq. (2.67) as :

$$\Delta_d(x) = -i\xi e^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{e^{-iq \cdot x}}{q^4} , \quad (2.104)$$

where e^2 is the dimensionless electromagnetic coupling, and μ is the 't Hooft mass scale in Dimensional Regularization.

These LKF transformations are ruled by very complex laws. Being written in coordinate space adds to their complexity, and they have been less used in the context of SDE, as compared to the WGTI, particularly in the study of the phenomenon of Dynamical Mass Generation.

To have a better understanding of the role that either the WGTI and the LKF transformations play in the restoration of gauge independence for the physical observables, it is necessary to first know the phenomenon we are dealing with, and those assumptions that simplify the most the SDE, to our knowledge, to make use of the bare vertex. This is the scenario we will develop in the next chapter.

Chapter 3

Gauge Dependence of Physical Observables

In a gauge theory, at the level of physical observables, gauge symmetry reflects as the fact that they be independent of the gauge parameter. Perturbation theory respects these requirements and besides the Ward-Green-Takahashi identities (WGTI) and the Landau-Khalatnikov-Fradkin (LKF) transformations remain valid at every level of approximation. However, this has not been achieved in general in the non perturbative study of gauge field theories through Schwinger-Dyson equations (SDE) carried out so far although significant progress has been made. The gauge technique of Salam, Delbourgo and later collaborators, [25, 26, 27, 28, 29, 30, 31], was developed to incorporate the constraint imposed by WGTI. However, as pointed out in [33], gauge technique can become completely reliable only after incorporating transverse Green functions with correct analytic and gauge-covariance properties. Another method widely used to explore the non-perturbative structure of the SDE is to make an *ansatz* for the full fermion-boson vertex and then study the gauge dependence of the physical observables related to the phenomenon of Dynamical Chiral Symmetry Breaking (DCSB). This method has been quite popular in four dimensional Quantum Electrodynamics (QED). For example, the vertex *ansatz* proposed by Curtis and Pennington, [37], has been extensively used to study the gauge dependence of the fermion propagator and the dynamical generation of fermion mass in Quenched QED, e.g., [38, 39, 40, 41]. Later on, in the work of Bashir and Pennington, [42, 43], an improved vertex which achieves complete gauge independence of the critical coupling above which mass is dynamically generated is proposed. These methods use the cut-off regularization to study the gauge dependence of the physical observables. As the cut-off method in general does not respect gauge symmetry, a criticism of these works has been raised recently, [50, 51, 52]. They suggest dimensional regularization scheme to study the chirally asymmetric phase of QED so that the possible gauge dependence coming from the inappropriate regulator could be filtered out.

Three dimensional Quantum Electrodynamics (QED3) provides us with a neat laboratory to study DCSB as it is ultraviolet well-behaved and hence the source of gauge non-invariance finds its roots only in the simplifying assumptions employed and *not* in the choice of the regulator. Burden and Roberts, [7], studied the gauge dependence of the chiral condensate in quenched QED3 and proposed a vertex which appreciably reduces this gauge dependence in the range $0 - 1$ of the covariant gauge parameter ξ . Unfortunately, the choice of their vertex does not transform correctly under the operation of charge conjugation. Moreover, the selected range of values for ξ is very narrow, close to the vicinity of the Landau gauge. In this chapter, we undertake the calculation of the Euclidean mass of the fermion (referred to as *mass* from now onwards) and the condensate for a wide range of values of ξ in the bare vertex approximation in the following schemes :

- Setting $F(p) = 1$.
- Including the equation for $F(p)$, and
- Making a partial use of the WGTI.

We have not achieved a complete gauge independence neither for the mass nor for the chiral condensate, which is a sing for the necessity of the construction and use of the full vertex.

3.1 The Fermion Propagator

In quenched QED3, the SDE for the fermion propagator in the Minkowski space can be written as :

$$S_F^{-1}(p) = S_F^{0-1}(p) - ie^2 \int \frac{d^3k}{(2\pi)^3} \Gamma^\nu(k, p) S_F(k) \gamma^\mu \Delta_{\mu\nu}^0(q), \quad (3.1)$$

where $q = k - p$, e is the electromagnetic coupling, $\Gamma^\nu(k, p)$ is the full fermion-photon vertex, $S_F^0(p)$ and $\Delta_{\mu\nu}^0(q)$ are the bare fermion and photon propagators defined as

$$S_F^0(p) = 1/\not{p}, \quad \Delta_{\mu\nu}^0(q) = -\frac{g_{\mu\nu}}{q^2} + (1 - \xi) \frac{q_\mu q_\nu}{q^4}, \quad (3.2)$$

and $S_F(p)$ is the full fermion propagator, which we prefer to write in the following most general form :

$$S_F(p) = \frac{F(p)}{\not{p} - \mathcal{M}(p)}. \quad (3.3)$$

$F(p)$ is referred to as the wavefunction renormalization and $\mathcal{M}(p)$ as the mass function and ξ is the usual covariant gauge parameter.

Eq. (3.1) is a matrix equation. It consists of two independent equations, which can be decoupled by taking its trace after multiplying it with 1 and $\not{p}$,

respectively. Making use of Eqs. (3.2,3.3) and replacing the full vertex by its bare counterpart, these equations can be written as :

$$\begin{aligned}\frac{1}{F(p)} &= 1 + \frac{\alpha}{2\pi^2 p^2} \int d^3k \frac{F(k)}{k^2 + \mathcal{M}^2(k)} \frac{1}{q^4} \\ &\quad [-2(k \cdot p)^2 + (2 - \xi)(k^2 + p^2)k \cdot p - 2(1 - \xi)k^2 p^2] , \\ \frac{\mathcal{M}(p)}{F(p)} &= \frac{\alpha(2 + \xi)}{2\pi^2} \int d^3k \frac{F(k) \mathcal{M}(k)}{k^2 + \mathcal{M}^2(k)} \frac{1}{q^2} ,\end{aligned}\quad (3.4)$$

where $\alpha = e^2/(4\pi)$ as usual. Carrying out angular integration after the Wick rotation to the Euclidean space, the above equations acquire the form :

$$\frac{1}{F(p)} = 1 - \frac{\alpha\xi}{\pi p^2} \int_0^\infty dk \frac{k^2 F(k)}{k^2 + \mathcal{M}^2(k)} \left[1 - \frac{k^2 + p^2}{2kp} \ln \left| \frac{k+p}{k-p} \right| \right] , \quad (3.5)$$

$$\frac{\mathcal{M}(p)}{F(p)} = \frac{\alpha(\xi + 2)}{\pi p} \int_0^\infty dk \frac{k F(k) \mathcal{M}(k)}{k^2 + \mathcal{M}^2(k)} \ln \left| \frac{k+p}{k-p} \right| . \quad (3.6)$$

A trivial solution to Eq. (3.6) is $\mathcal{M}(p) = 0$, which corresponds to the usual perturbative solution. We are interested in a non-trivial solution by solving Eqs. (3.5) and (3.6) simultaneously. Such a solution for $\mathcal{M}(p)$ is related to the mass m and the chiral condensate $\langle \bar{\psi}\psi \rangle$. Assuming a simple analytic continuation from Minkowski to the Euclidean space, neglecting the rotation of the integration contour, we define $m = \mathcal{M}(m)$. It is true that this is not the physical mass for the fermion, and we do not expect it to be exactly gauge invariant. However, since $m \sim \mathcal{M}(0)$, we can consider it as an effective mass. At most we can expect this Euclidean mass to be approximately gauge invariant, in the sense that it is close to the physical mass [40]. On the other hand, in reference [7], Burden and Roberts demonstrated that the standard definition of the fermion condensate (Eq. (3.9) in [7]) in terms of an integral over the mass function is in excellent numerical agreement with the prediction of the operator product expansion [53] which allows us to write $\langle \bar{\psi}\psi \rangle = 4p^2 \mathcal{M}(p)/(2 + \xi)$ (in units of e^4) in the limit when $p^2 \rightarrow \infty$. Such an expansion is valid only for values of the gauge parameter in the range $[0, 1]$. As expected, we find that in this limit, $\mathcal{M}(p)$ falls as $1/p^2$ so that the condensate does not depend upon the momentum variable p . We shall study the gauge dependence of the mass and the condensate in the next section.

3.2 Effect of the Wavefunction Renormalization

In studying DCSB, it has been a common practice to make the approximation $F(p) = 1$ so that we only have to solve Eq. (3.6). The justification for this approximation stems from the fact that perturbatively $F(p) = 1 + \mathcal{O}(\alpha\xi/\pi)$. If α is small and we are sufficiently close to the Landau gauge, one would naturally expect that $F(p) \approx 1$. Although, it has been quite customary to employ this approximation, there exist several works which include both the equations. We

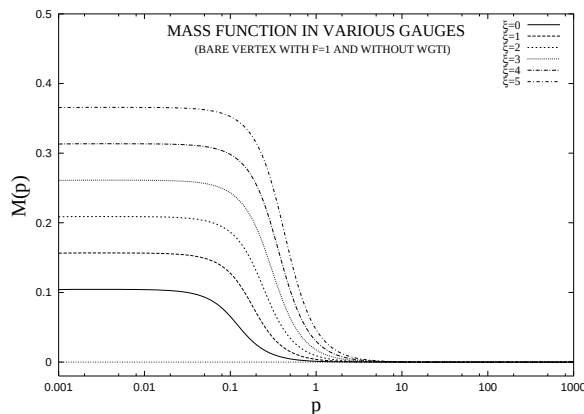


Figure 3.1: Mass Function $\mathcal{M}(p)$ in the $F(p) = 1$ approximation.

study the effects of neglecting the wavefunction renormalization quantitatively. Fig. (3.1) depicts the mass function $\mathcal{M}(p)$ for $F(p) = 1$ in various gauges. As expected, the mass function is roughly a constant for low values of p and falls as $1/p^2$ for large values of p . The integration region chosen is from 10^{-3} to 10^3 and we select 26 points per decade. The mass probes low momentum region of this graph, whereas, the condensate is extracted from its asymptotic behaviour. Obviously, the mass seems to vary in more or less equally spaced steps with the variation of the gauge parameter. In order to obtain a quantitative value of the mass, we select neighbouring points p_a and p_b ($p_a > p_b$), such that $\mathcal{M}(p_a) < p_a$ and $\mathcal{M}(p_b) > p_b$. We then approximate the mass by the following relation :

$$m = \frac{\mathcal{M}(p_b) - \mathcal{M}(p_a)}{p_b - p_a}(m - p_a) + \mathcal{M}(p_a) . \quad (3.7)$$

As for the condensate, the figure does not distinguish between the results for various gauges. Therefore, we have to look at the numbers explicitly. Table (1) shows the value of the condensate for ξ ranging from 1 – 5. Momentum p is displayed in units of e^2 and the condensate in units of $10^{-3}e^4$. The point $p = 1000$ was chosen to calculate the condensate. This number seems sufficiently large as the $1/p^2$ behaviour seems to set in much earlier ($p \approx 300$), as noted also in [7]. In Figs. (3.2) and (3.3) we display the gauge dependence of the chiral condensate and the mass for $F(p) = 1$ in a wide range of values of the gauge parameter. The condensate varies heavily with the change of gauge, roughly twice per unit change in the value of ξ . Gauge dependence of the mass is not too different either.

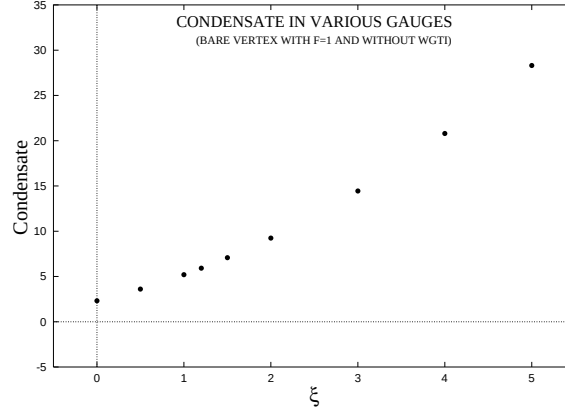


Figure 3.2: Condensate $\langle \bar{\psi}\psi \rangle$ in the $F(p) = 1$ approximation.

Repeating the exercise by taking both the equations, namely Eqs. (3.5) and (3.6), into account, we see similar qualitative behaviour of the mass function. It is roughly a constant for low values of p and falls as $1/p^2$ for large values of p , Fig. (3.4). The large p behaviour is also evident from the entries in Table (2). As for the wavefunction renormalization, it also is constant for small values of p . As p becomes large it goes to 1, Fig. (3.5). In Table (2) we also give a comparison with the work of Burden and Roberts, [7]. As mentioned earlier, they restrict themselves to the close vicinity of the Landau gauge, where our results are in excellent agreement. We investigate the gauge dependence of the condensate as well as the mass far beyond the Landau gauge. A graphical description can be found in Figs. (3.6) and (3.7). The following points are important to note :

- The wavefunction renormalization plays an extremely important role in restoring the gauge invariance of the chiral condensate as well as the mass of the fermion. Although the qualitative behaviour of the mass function in various regimes of momenta remains largely unchanged, whether or not we employ the approximation $F(p) = 1$, quantitative dependence of the physical observables mentioned above on the covariant gauge parameter ξ reduces a great deal by including the wavefunction renormalization.
- As we move away from the Landau gauge towards large positive values of ξ , the gauge dependence of the condensate as well as the mass keeps dimini-

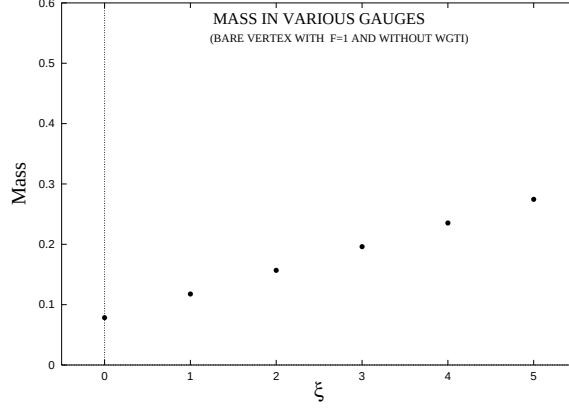


Figure 3.3: Mass in the $F(p) = 1$ approximation.

shing, without resorting to any sophisticated *ansatze* for the fermion-boson interaction.

3.3 Effect of the Ward-Green-Takahashi Identity

The bare photon propagator which appears in Eq. (3.1) can be split up in longitudinal and transverse parts as follows :

$$\Delta_{\mu\nu}^0(q) = \Delta_{\mu\nu}^{0T}(q) - \xi \frac{q_\mu q_\nu}{q^4}, \quad (3.8)$$

where

$$\Delta_{\mu\nu}^{0T}(q) = -g_{\mu\nu}/q^2 + q_\mu q_\nu/q^4.$$

Employing this decomposition, we can rewrite Eq. (3.1) as :

$$\begin{aligned} S_F^{-1}(p) &= S_F^{0-1}(p) - ie^2 \int \frac{d^3k}{(2\pi)^3} \Gamma^\nu(k, p) S_F(k) \gamma^\mu \Delta_{\mu\nu}^{0T}(q) \\ &\quad + ie^2 \xi \int \frac{d^3k}{(2\pi)^3} \Gamma^\nu(k, p) S_F(k) \gamma^\mu \frac{q_\mu q_\nu}{q^4}. \end{aligned} \quad (3.9)$$

It is well known that the use of the WGTI, in the equivalent of the last term of Eq. (3.9) in QED4, filters out a spurious term which is an artifact of using the gauge dependent cut-off regulator. Therefore, one is naturally motivated to use this decomposition in dimensions other than four. Now employing the bare vertex ansatz $\Gamma^\mu(k, p) = \gamma^\mu$, multiplying Eq. (3.9) by 1 and $\not{p}$ respectively and

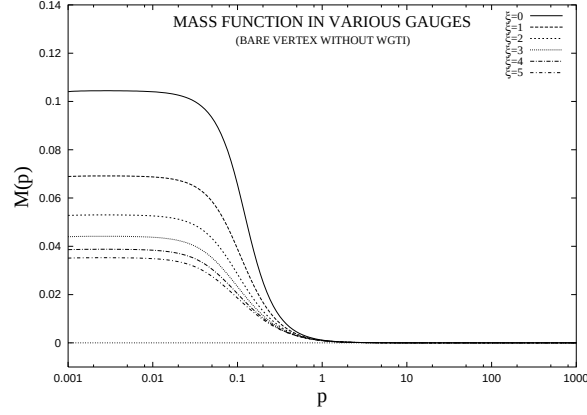


Figure 3.4: Mass Function including the equation for the Wavefunction Renormalization.

Wick-rotating to the Euclidean space, we obtain the following equations :

$$\frac{1}{F(p)} = 1 + \frac{\alpha}{2\pi^2 p^2} \int d^3 k \frac{F(k)}{k^2 + \mathcal{M}^2(k)} \frac{1}{q^4} \left[2(q \cdot p)(q \cdot k) + \frac{\xi}{F(p)} [p^2(q \cdot k) + \mathcal{M}(k)\mathcal{M}(p)(q \cdot p)] \right], \quad (3.10)$$

$$\frac{\mathcal{M}(p)}{F(p)} = \frac{\alpha}{2\pi^2} \int d^3 k \frac{F(k)}{k^2 + \mathcal{M}^2(k)} \frac{1}{q^2} \left[2\mathcal{M}(k) - \frac{\xi}{q^2} \frac{1}{F(p)} [\mathcal{M}(k)(p \cdot q) - \mathcal{M}(p)(k \cdot q)] \right]. \quad (3.11)$$

On carrying out angular integration,

$$\begin{aligned} \frac{1}{F(p)} = & 1 + \frac{\alpha\xi}{\pi p^2} \int_0^\infty dk \frac{k^2 F(k)/F(p)}{k^2 + \mathcal{M}^2(k)} \\ & \left[\frac{p^2}{k^2 - p^2} + \frac{p}{2k} \ln \left| \frac{k+p}{k-p} \right| \right. \\ & \left. + \mathcal{M}(k)\mathcal{M}(p) \left\{ \frac{1}{k^2 - p^2} - \frac{1}{2kp} \ln \left| \frac{k+p}{k-p} \right| \right\} \right], \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{\mathcal{M}(p)}{F(p)} = & \frac{\alpha}{\pi} \int_0^\infty dk \frac{k^2 F(k)}{k^2 + \mathcal{M}^2(k)} \left[\frac{2\mathcal{M}(k)}{kp} \ln \left| \frac{k+p}{k-p} \right| \right. \\ & \left. - \frac{\xi}{F(p)} \left\{ \frac{\mathcal{M}(k) - \mathcal{M}(p)}{k^2 - p^2} - \frac{\mathcal{M}(k) + \mathcal{M}(p)}{2kp} \ln \left| \frac{k+p}{k-p} \right| \right\} \right]. \end{aligned} \quad (3.13)$$

As the terms of the type $1/(k^2 - p^2)$ are harder to deal with numerically, we

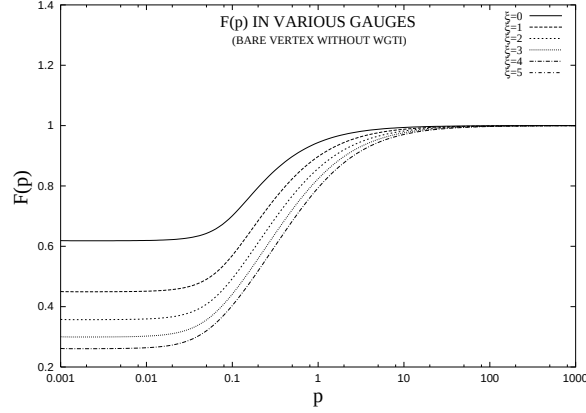


Figure 3.5: Wavefunction Renormalization.

use the approximation $F(p) = 1$ to analyze the effect of the WGTI. Under this simplification, we only have to solve

$$\mathcal{M}(p) = \frac{\alpha}{\pi} \int_0^\infty dk \frac{k^2}{k^2 + \mathcal{M}^2(k)} \left[\frac{2\mathcal{M}(k)}{kp} \ln \left| \frac{k+p}{k-p} \right| - \xi \left\{ \frac{\mathcal{M}(k) - \mathcal{M}(p)}{k^2 - p^2} - \frac{\mathcal{M}(k) + \mathcal{M}(p)}{2kp} \ln \left| \frac{k+p}{k-p} \right| \right\} \right]. \quad (3.14)$$

The mass function obtained on solving Eq. (3.14) is depicted in Fig. (3.8), which, along with Table (3), reveals that its qualitative behaviour remains unchanged both for small and large values of p . In Figs. (3.9) and (3.10), we compare the gauge dependence of the condensate and the mass with and without the usage of the WGTI. We find that the gauge dependence of these quantities seems to increase by incorporating the said identity. Similar behaviour was observed in QED4 by Gusynin *et. al.* [50] in studying the gauge dependence of the critical coupling above which mass is generated. They carried out a numerical analysis of the criticism raised by Dong *et. al.* [18] on the work of Atkinson *et. al.* [54] who did not employ the WGTI as suggested in Eq. (3.9), resulting in the appearance of a spurious cut-off dependent term¹. Gusynin *et. al.* found that if one employs the WGTI, the critical coupling is more steeply gauge dependent. We find similar behaviour for the mass and the condensate in QED3 in the approximation $F(p) = 1$ in this section.

¹To trace back the origin of this error, consult the footnote on page 7680 of the reference [42]

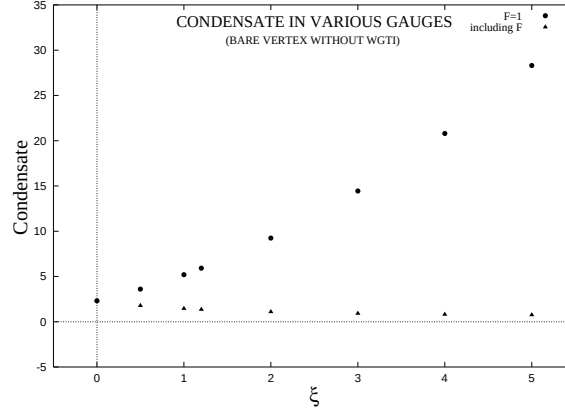


Figure 3.6: Condensate: Comparison between the cases with and without the usage of $F(p) = 1$.

3.4 Dimensional Regularization Method

In this section we compare our numerical results with those obtained by employing the dimensional regularization scheme, [50, 51, 52]. For simplicity, we restrict ourselves only to the Landau gauge without incorporating the WGTI. In this case, the equation for the mass function acquires the following form in Euclidean space in arbitrary dimensions :

$$\mathcal{M}(p) = 4\pi\alpha(d-1) \int \frac{d^d k}{(2\pi)^d} \frac{\mathcal{M}(k)}{k^2 + \mathcal{M}^2(k)} \frac{1}{q^2}, \quad (3.15)$$

where α is a dimensionful coupling except in four dimensions. We define $d = 4 - 2\epsilon$ and

$$\alpha = \alpha_d \mu^{2\epsilon}, \quad (3.16)$$

α_d being dimensionless. We now use the volume element $d^d k = k^{d-1} dk d\Omega_d$, where $d\Omega_d$ is the d -dimensional solid angle defined as

$$d\Omega_d = \prod_{l=1}^{d-1} \sin^{d-1-l} \theta_l d\theta_l.$$

The angle θ_{d-1} varies from 0 to 2π , whereas all other angles vary from 0 to π . Choosing θ_1 to be the angle between k and p , we can easily carry out the remaining angular integrations to arrive at :

$$\mathcal{M}(p) = \frac{2(d-1)\alpha}{(4\pi)^{\frac{d-1}{2}} \Gamma(\frac{d-1}{2})} \int_0^\infty dk^2 \frac{k^{d-2} \mathcal{M}(p)}{k^2 + \mathcal{M}^2(p)} \int_0^\pi d\theta_1 \frac{\sin^{d-2} \theta_1}{q^2}. \quad (3.17)$$

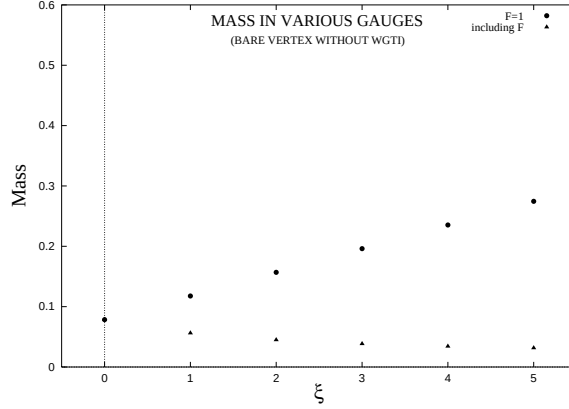


Figure 3.7: Mass: Comparison between the cases with and without the usage of $F(p) = 1$.

Using the standard formula, [55],

$$\int_0^\pi dx \frac{\sin^{2\sigma-1} x}{[1 + 2a \cos x + a^2]^\lambda} = B(\sigma, 1/2) {}_2F_1(\lambda, \lambda - \sigma + 1/2, \mu + 1/2; a^2) \quad |a| < 1,$$

integration over θ_1 yields :

$$\begin{aligned} \mathcal{M}(p) &= \frac{(3-2\epsilon)\alpha}{(4\pi)^{1-\epsilon}\Gamma(2-\epsilon)} \int_0^\infty dk^2 \frac{(k^2)^{1-\epsilon} \mathcal{M}(k)}{k^2 + \mathcal{M}^2(k)} \\ &\left[\frac{1}{k^2} {}_2F_1\left(1, \epsilon; 2-\epsilon; \frac{p^2}{k^2}\right) \theta(k^2 - p^2) + \frac{1}{p^2} {}_2F_1\left(1, \epsilon; 2-\epsilon; \frac{k^2}{p^2}\right) \theta(k^2 - p^2) \right]. \end{aligned} \quad (3.18)$$

This equation was studied in detail in [50] in four dimensions, taking ϵ to be a small positive number. The factor $(k^2)^{-\epsilon}$ in the numerator regulates the otherwise divergent behaviour of the integrand for large momenta. As noted in [50], the hypergeometric function does not play any role in regularization and hence can simply be replaced by $F(1, 0; 2, z) = 1$. In case of three dimensions, the hypergeometric function develops a pole for $k^2 = p^2$, as is obvious from the following identity :

$${}_2F_1(1, \epsilon; 2-\epsilon; 1) = \frac{1-\epsilon}{1-2\epsilon}. \quad (3.19)$$

As was pointed out earlier, such terms are hard to deal with numerically. Due to increasing computational time and memory, we go only up to $\epsilon = 0.48$, starting from $\epsilon = 0.4$. To obtain satisfying results, we need to use increasingly more points per decade as we approach closer to $\epsilon = 0.5$. For instance, we use

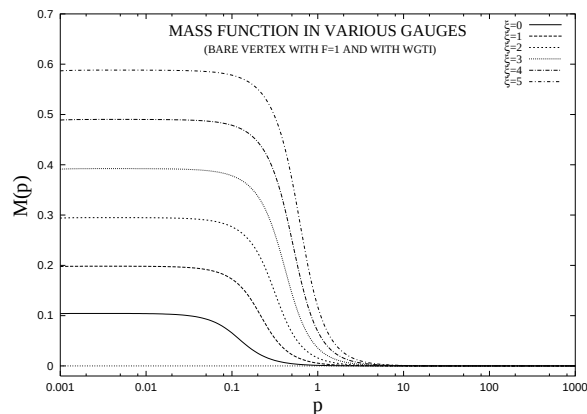


Figure 3.8: Mass Function including the effect of the WGTI for $F(p) = 1$.

100 points per decade for $\epsilon = 0.46$. The problems of ever increasing computational time and memory limited us to use 140 points per decade for the case of $\epsilon = 0.48$. Despite this large number, we believe that the corresponding result falls short of the desired accuracy. As a result, there is a slight rise at the end of the flat region of the mass function, and the final descent begins rather late, Fig. (3.11). The problematic pole for $\epsilon = 0.5$ in Eq. (3.19) corresponds to the relatively well-controlled singularity in the following expression

$${}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{1}{2z} \ln \frac{1+z}{1-z} \quad (3.20)$$

for $z \rightarrow 1$. A comparison between the mass function obtained from techniques based upon the dimensional regularization scheme and the one computed in Section 3.3 is also depicted in Fig (3.11). Taking the numerical limitation for $\epsilon = 0.48$ into account, we note that as ϵ approaches the value of 0.5, we get closer and closer to the result obtained previously, where we work in 3-dimensions to start with.

3.5 Towards the Full Vertex

WGTI with the bare vertex is valid only if bare fermion propagators are involved. When we use full propagators, physical observables calculated from the corresponding SDE exhibit gauge dependence. Temptation emerges then to move towards the full vertex, and the first requirement we must think of is that it should satisfy the WGTI. Hence we face the construction of the Ball-Chiu vertex detailed displayed below. The full vertex satisfies WGTI

$$q_\mu \Gamma^\mu(k, p) = S_F^{-1}(k) - S_F^{-1}(p) . \quad (3.21)$$

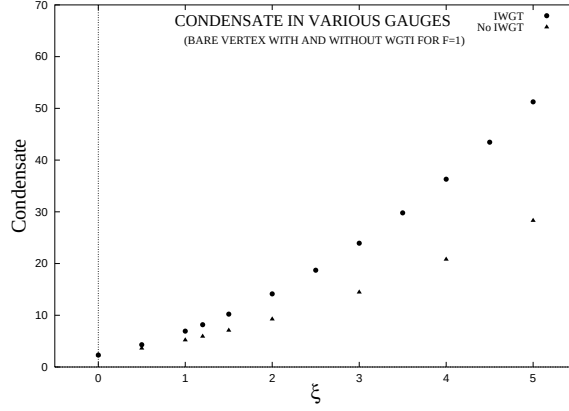


Figure 3.9: Effect of the WGTI on the gauge dependence of the condensate for $F(p) = 1$.

This relation allows us to decompose the complete vertex into its longitudinal $\Gamma_L^\mu(k, p)$ and transverse $\Gamma_T^\mu(k, p)$ parts :

$$\Gamma^\mu(k, p) = \Gamma_L^\mu(k, p) + \Gamma_T^\mu(k, p) , \quad (3.22)$$

where the transverse part satisfies

$$q_\mu \Gamma_T^\mu(k, p) = 0 \quad y \quad \Gamma_T^\mu(p, p) = 0 , \quad (3.23)$$

and therefore, remains undetermined by the WGTI. Following the work of Ball and Chiu, we define the longitudinal part of the vertex in terms of the Fermion Propagator. In the limit $k \rightarrow p$, WGTI is written as :

$$\Gamma^\mu(p, p) = \frac{\partial}{\partial p_\mu} S_F^{-1}(p) . \quad (3.24)$$

Substituting the expresion for the full Fermion Propagator, we observe that

$$\begin{aligned} \Gamma^\mu(p, p) &= \frac{\partial}{\partial p_\mu} S_F^{-1}(p) \\ &= \frac{\partial}{\partial p_\mu} \frac{\not{p} - \mathcal{M}(p)}{F(p)} \\ &= \frac{\gamma^\mu}{F(p)} + 2p^\mu \not{p} \frac{\partial}{\partial p^2} - 2p^\mu \frac{\partial}{\partial p^2} \frac{\mathcal{M}(p)}{F(p)} . \end{aligned} \quad (3.25)$$

After we make the substitutions

$$\frac{1}{F(p)} \rightarrow \frac{1}{2} \left[\frac{1}{F(k)} + \frac{1}{F(p)} \right] ,$$

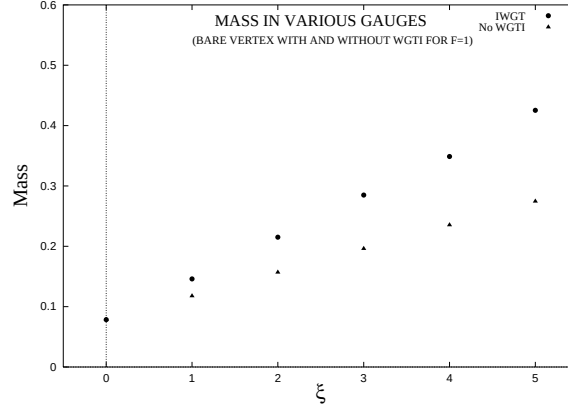


Figure 3.10: Effect of the WGTI on the gauge dependence of the mass for $F(p) = 1$.

$$\begin{aligned}
 p^\mu &\rightarrow \frac{1}{2}(k^\mu + p^\mu), \\
 \not{p} &\rightarrow \frac{1}{2}(\not{k} + \not{p}), \\
 \frac{\partial}{\partial p^2} \frac{1}{F(p)} &\rightarrow \frac{1}{k^2 - p^2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right], \\
 \frac{\partial}{\partial p^2} \frac{\mathcal{M}(p)}{F(p)} &\rightarrow \frac{1}{k^2 - p^2} \left[\frac{\mathcal{M}(k)}{F(k)} - \frac{\mathcal{M}(p)}{F(p)} \right],
 \end{aligned} \tag{3.26}$$

we can define the longitudinal or Ball-Chiu (BC) vertex as :

$$\begin{aligned}
 \Gamma_{BC}^\mu &= \frac{\gamma^\mu}{2} \left[\frac{1}{F(k)} + \frac{1}{F(p)} \right] + \frac{1}{2} \frac{(\not{k} + \not{p})(k + p)^\mu}{(k^2 - p^2)} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \\
 &+ \frac{(k + p)^\mu}{(k^2 - p^2)} \left[\frac{\mathcal{M}(k)}{F(k)} - \frac{\mathcal{M}(p)}{F(p)} \right].
 \end{aligned} \tag{3.27}$$

Without loss of generality, the transverse vertex can be expressed as :

$$\Gamma_T^\mu(k, p) = \sum_{i=1}^8 \tau_i(k^2, p^2, q^2) T_i^\mu(k, p), \tag{3.28}$$

with the appropriate $\{T^\mu\}$ basis. Functions τ_1, τ_4, τ_5 and τ_7 are proportional to the mass m , and therefore in massless studies they do not appear. During the last few years, a programme has been started towards the construction of vertex ansätze which impose constraints on the transverse part of the vertex. Some of the most famous attempts are discussed below.

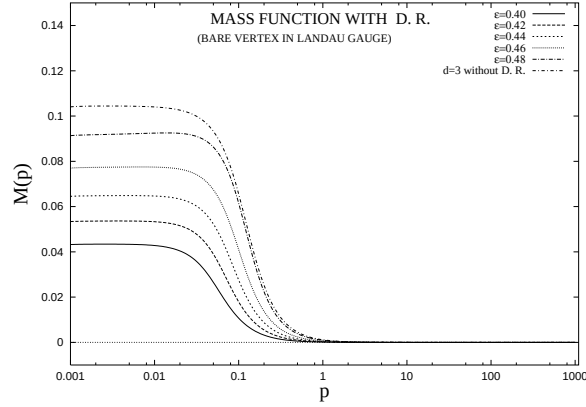


Figure 3.11: Mass Function for various values of ϵ . Notice that $\epsilon = 0.5$ corresponds to 3 dimensions. The result for $\epsilon = 0.48$ could not be achieved with the desired accuracy due to the computing time and memory increase.

3.5.1 Curtis-Pennington Vertex

In four dimensions, the question of why cannot we set all of the τ 's as zero was answered by Curtis and Pennington, arguing to Perturbation Theory and the multiplicative renormalizability properties of the Fermion Propagator.

Their vertex (CP)

$$\Gamma_{CP}^\mu = \Gamma_{BC}^\mu + \frac{1}{2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \frac{\gamma^\mu(k^2 - p^2) - (k + p)^\mu \not{q}}{d(k, p)}, \quad (3.29)$$

where

$$d(k, p) = \frac{(k^2 - p^2)^2 + [\mathcal{M}^2(k) + \mathcal{M}^2(p)]^2}{k^2 + p^2}, \quad (3.30)$$

comes out from the following assumptions :

- Transverse vertex must agree with perturbative results at the one-loop level in the relevant kinematic regime $k \gg p$.
- Fermion Propagator obtained from this vertex should be multiplicatively renormalizable.
- This Propagator should transform correctly under its LKF transformation.
- Vertex should have correct charge conjugation properties.
- Transverse vertex should vanish in Landau gauge.
- Transverse Vertex should not depend upon the angle between fermion momenta.

- The only contributing coefficient is τ_6 .
- Transverse vertex as well as the longitudinal one are written without explicit dependence on the gauge parameter ξ ².

Transverse vertex is written in such a way that, in the massless case, their

$$\tau_6 = \frac{1}{2} \frac{k^2 + p^2}{(k^2 - p^2)^2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \quad (3.31)$$

is antisymmetric under the interchange of k and p .

This vertex, however, does not lead to a completely gauge independent critical coupling. In the massless case, it exhibits a kinematic singularity when $k^2 \rightarrow p^2$. In comparison with Perturbation Theory at one-loop, CP vertex agrees with perturbative results only in the kinematic regime above mentioned. Multiplicative renormalizability for the Fermion Propagator is achieved only in the leading term of the power law for the wavefunction renormalization. Finally, the LKF transformation of this propagator is valid only up to the leading logarithmic term.

3.5.2 Burden-Roberts Vertex

Burden and Roberts have parametrized a slight modification to the BC vertex in the following way [7] :

$$\begin{aligned} \Gamma_{BR}^\mu = & \left[a \frac{1}{F(k)} + (1-a) \frac{1}{F(p)} \right] \gamma^\mu \\ & + \frac{(k+p)^\mu ((1-a) \not{k} - a \not{p})}{k^2 - p^2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \\ & - \frac{(k+p)^\mu}{k^2 - p^2} \left[\frac{\mathcal{M}(k)}{F(k)} - \frac{\mathcal{M}(p)}{F(p)} \right], \end{aligned} \quad (3.32)$$

where

$$a = \frac{1}{2} + \delta. \quad (3.33)$$

With the assumptions that the vertex should satisfy the WGTI and that it should be free from kinematical singularities to all orders in Perturbation Theory, their parametrization allows to optimize δ such that the most gauge independence of the chiral condensate is achieved. They report $\delta = 0.03$ as the best value. However, we observe that the BR vertex can be written as :

$$\begin{aligned} \Gamma_{BR}^\mu &= \Gamma_{BC}^\mu + \delta \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] [k^2 - p^2] \{ \gamma^\mu (p^2 - k^2) + (k+p)^\mu \not{q} \} \\ &= \Gamma_{BC}^\mu + \delta \tau_6 T_6^\mu, \end{aligned} \quad (3.34)$$

²This, as we will show, is not possible.

with

$$\tau_6 = (k^2 - p^2) \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right]. \quad (3.35)$$

Charge conjugation symmetry for the vertex requires τ_6 to be antisymmetric under the interchange of k and p , but the BR vertex is symmetric. Besides, for $\delta = 0.03$ this vertex does not agree with perturbative results, and therefore we neglect it as a good choice.

3.5.3 Dong-Munczek-Roberts Vertex

Dong *et. al.* [18] have proposed a vertex (DMR) that satisfies WGTI and ensures gauge covariance of the Fermion Propagator in the massless case by means of the so-called Transversality Condition :

$$\int \frac{d^d k}{(2\pi)^d} \Delta_{\mu\nu}^T(p-k) \gamma^\mu S_F(k) \Gamma^\nu(k,p) = 0. \quad (3.36)$$

DMR vertex is constructed under the assumption that the coefficients of the transverse part are independent of the angle between fermion momenta. In it, the following constraints are set on the transverse coefficients :

$$\begin{aligned} f^i &= 0 \quad \text{for } i \neq 3, 8 \\ f^3 &= \frac{1}{2} \left(\frac{d}{2} - 1 \right) f^8 \\ f^8 &= \frac{1}{\frac{d}{2} - 1} \frac{(d-1)I_3}{I_1 - I_3}, \end{aligned} \quad (3.37)$$

d is the number of space-time dimensions and

$$\begin{aligned} I_1 &= k^2 p^2 \mathcal{I}_1 \\ I_2 &= \frac{1}{2} ((k^2 + p^2) \mathcal{I}_1 - 1) \\ I_3 &= \frac{1}{2} [k^2 + p^2] I_2, \end{aligned} \quad (3.38)$$

where they define

$$\mathcal{I}_n = \int d\Omega_d \frac{1}{(k-p)^{2n}}, \quad (3.39)$$

being the solid angle

$$\int d\Omega_d \equiv \frac{1}{\mathcal{N}} \left[\int_0^\pi d\theta_2 \sin^{d-2} \theta_2 \int_0^\pi d\theta_3 \sin^{d-3} \theta_3 \dots \int_0^{2\pi} d\theta_{d-1} \right], \quad (3.40)$$

and

$$\mathcal{N} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (3.41)$$

Functions f^i are related to the τ_i in the form :

$$\tau_i = \frac{1}{k^2 - p^2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] f^i . \quad (3.42)$$

In this construction it is shown that the effect of including only τ_6 as in the CP vertex can also be reproduced by considering τ_3 and τ_8 . This vertex, however, exhibit logarithmic kinematic singularities. It has the correct charge conjugation properties and leads to a multiplicatively renormalizable Fermion Propagator.

3.5.4 Burden-Tjiang Vertex

Using a similar reasoning, Burden and Tjiang [56] have deconstructed a one-parameter ansätze family for massless QED3. Constraints on the transverse coefficients in the BT vertex are the following :

$$\begin{aligned} \bar{f} &= -2(1 + \beta) \frac{I(k, p)}{J(k, p)} \\ f^3 &= -\beta \frac{I(k, p)}{J(k, p)} \\ f^6 &= 0 , \end{aligned} \quad (3.43)$$

where

$$\begin{aligned} I(k, p) &= \frac{(k^2 + p^2)^2}{16kp} \ln \left(\frac{(k + p)^2}{(k - p)^2} \right) - \frac{1}{4}(k^2 + p^2) \\ J(k, p) &= \frac{(k^2 - p^2)^2}{16kp} \ln \left(\frac{(k + p)^2}{(k - p)^2} \right) - \frac{1}{4}(k^2 + p^2) , \end{aligned} \quad (3.44)$$

They have used the relation (3.42) and they define

$$\bar{\tau} = \tau_8 + (k^2 + p^2)\tau_2 . \quad (3.45)$$

Choosing $\beta = 1$ leads to the DMR vertex. This deconstruction is based upon the assumptions that the transverse vertex vanishes in Landau gauge and takes $\tau_6 = 0$. Transversality Condition is the strongest argument to ensure the gauge covariance of the Fermion Propagator. A possible drawback mentioned by the authors is that for any value of β , for $k^2 = p^2$ but $k_\mu \neq p_\mu$, the BT vertex exhibits a logarithmic singularity.

A criticism to this vertex has been exposed by Bashir *et. al.* [57]. They assure that τ_6 cannot be taken as zero and that the transverse coefficients should depend upon the angle between fermion momenta. One important observation is that the parameter β , introduced explicitly gauge parameter independent by the authors, should have such dependence as pointed out by the corresponding perturbative calculation. Finally, in this work they point out that the Transversality Condition is not valid beyond the one-loop level.

3.5.5 Bashir-Pennington Vertex

With the assumptions that the transverse vertex vanishes in Landau gauge and that it has no dependence upon the angle between fermion momenta, Bashir and Pennington [42]. have constructed a vertex that satisfies :

- WGTI,
- that the Fermion Propagator is multiplicatively renormalizable,
- that with the correct choice of the functions that define this vertex, it agrees with Perturbation Theory in the weak coupling regime is warranted,
- offers a strictly gauge independent critical coupling.

BP Vertex imposes integral and differential constraints on the functions that define its transverse part. For the massless case, Kızılersü, Bashir and Pennington [58] removed the initial assumptions, including all the corrections in the exponent of the power law for the wavefunction renormalization. This is done by introducing new functions and constraints on them.

Continuing with the programme, we should start an ambitious quest towards the construction of a vertex for the fermion-boson interaction in massive QED3, taking into account all the nice features of the above mentioned vertices, but getting rid of their drawbacks. It is clear that the inclusion of the WGTI is common in all the vertex ansätze. This is due to the fact that its implementation is straightforward. Demanding correct gauge behavior of Green's functions is a nontrivial requirement. We will show this in the next chapter, where we study the LKF transformation for the Fermion Propagator at the tree level.

Appendix: Tables

Below we display the Tables that contain the numeric results of our calculation :

ξ	p	$\frac{4}{2+\xi} p^2 M(p)$	ξ	p	$\frac{4}{2+\xi} p^2 M(p)$
0.0	1000	2.31109	2.5	1000	11.6992
	642.233	2.31117		642.233	11.7001
	316.228	2.31119		316.228	11.7003
0.5	1000	3.61103	3.0	1000	14.4432
	642.233	3.61119		642.233	14.4444
	316.228	3.61124		316.228	14.4448
1.0	1000	5.19982	3.5	1000	17.4761
	642.233	5.20009		642.233	17.4777
	316.228	5.20017		316.228	17.4782
1.2	1000	5.91622	4.0	1000	20.7977
	642.233	5.91654		642.233	20.7998
	316.228	5.91664		316.228	20.8005
1.5	1000	7.07745	4.5	1000	24.4081
	642.233	7.07787		642.233	24.4108
	316.228	7.07800		316.228	24.4117
2.0	1000	9.24390	5.0	1000	28.3073
	642.233	9.24452		642.233	28.3106
	316.228	9.24473		316.228	28.3117

TABLE 1. Condensate in Various Gauges for $F(p) = 1$

ξ	p	$\frac{4}{2+\xi} p^2 M(p)$	$\frac{4}{2+\xi} p^2 M(p)$
		BHR	BR
0.0	1000	2.31109	2.316
	642.233	2.31117	
	316.228	2.31119	
0.5	1000	1.77309	1.775
	642.233	1.77313	
	316.228	1.77306	
1.0	1000	1.44791	1.447
	642.233	1.44793	
	316.228	1.44780	
1.2	1000	1.35288	1.352
	642.233	1.35288	
	316.228	1.35274	
1.5	1000	1.23591	
	642.233	1.23591	
	316.228	1.23574	
2.0	1000	1.09014	
	642.233	1.09012	
	316.228	1.08992	

ξ	p	$\frac{4}{2+\xi} p^2 M(p)$	$\frac{4}{2+\xi} p^2 M(p)$
		BHR	BR
2.5	1000	0.98597	
	642.233	0.98594	
	316.228	0.98571	
3.0	1000	0.90941	
	642.233	0.90938	
	316.228	0.90912	
3.5	1000	0.85201	
	642.233	0.85197	
	316.228	0.85169	
4.0	1000	0.80839	
	642.233	0.80833	
	316.228	0.80803	
4.5	1000	0.77497	
	642.233	0.77491	
	316.228	0.77458	
5.0	1000	0.74933	
	642.233	0.74927	
	316.228	0.74891	

TABLE 2. Condensate In Various Gauges Including $F(p)$

ξ	p	$\frac{4}{2+\xi} p^2 M(p)$	ξ	p	$\frac{4}{2+\xi} p^2 M(p)$
0.0	1000	2.31109	2.5	1000	18.7057
	642.233	2.31117		642.233	18.7085
	316.228	2.31119		316.228	18.7136
0.5	1000	4.30196	3.0	1000	23.9226
	642.233	4.30221		642.233	23.9268
	316.228	4.30248		316.228	23.9345
1.0	1000	6.93473	3.5	1000	29.7865
	642.233	6.93529		642.233	29.7924
	316.228	6.93609		316.228	29.8037
1.2	1000	8.16826	4.0	1000	36.2975
	642.233	8.16900		642.233	36.3056
	316.228	8.17011		316.228	36.3212
1.5	1000	10.2122	4.5	1000	43.4556
	642.233	10.2133		642.233	43.4663
	316.228	10.2150		316.228	43.4872
2.0	1000	14.1356	5.0	1000	51.2608
	642.233	14.1375		642.233	51.2746
	316.228	14.1406		316.228	51.3020

TABLE 3. Condensate in Various Gauges for $F(p) = 1$ using WGTI

Chapter 4

Landau-Khalatnikov-Fradkin Transformations and the Fermion Propagator

4.1 Introduction

Looking for the gauge independence of the physical observables, we found that the Ward-Green-Takahashi Identity (WGTI) is a necessary condition for it, but not sufficient. We see, therefore, the need for the incorporation of other gauge invariance constraints for this purpose. We incorporate the Landau-Khalatnikov-Fradkin (LKF) transformations, [44, 45, 46, 47, 48], which tell us the manner in which Green's functions change under a variation of gauge. Rules governing these transformations are far from simple. In counterpart, WGTI [34, 35, 36], are more simple, and therefore, they have been widely implemented. We can make larger this set of identities by transforming also the gauge parameter ξ [59, 60], and arrive to the Nielsen Identities (NI). One advantage of these identities over the conventional Ward identities is that $\partial/\partial\xi$ becomes part of the new relations involving Green's functions. This fact was exploited in [61, 62] to prove the gauge independence of some quantities related to two-point Green's functions at the one-loop level and to all orders in Perturbation Theory, respectively. Since it is a difficult task to establish the gauge independence of the physical observables in the Schwinger-Dyson Equations (SDE) study, NI can play a significant role in addressing this issue, along with the WGTI and the LKF transformations. However, in this chapter, we have focused only in the later.

The LKF transformation for the three-point vertex is complicated and hampers direct extraction of analytical restrictions on its structure. Burden and

Roberts, [7], carried out a numerical analysis to compare the self-consistency of various *ansatze* for the vertex, [19, 37, 63], by means of its LKF transformation. In addition to these numerical constraints, indirect analytical insight can be obtained on the nonperturbative structure of the vertex by demanding correct gauge covariance properties of the fermion propagator. In the context of gauge technique, examples are [64, 65, 66]. Concerning the works based upon choosing a vertex *ansatze*, references [18, 37, 42, 43, 56, 58] employ this idea ¹. However, all the work in the later category has been carried out for massless QED3 and QED4. The masslessness of the fermions implies that the fermion propagator can be written only in terms of one function, the so called wavefunction renormalization, $F(p)$. In order to apply the LKF transform, one needs to know a Green function at least in one particular gauge. This is a formidable task. However, one can rely on approximations based on perturbation theory. It is customary to take $F(p) = 1$ in the Landau gauge, an approximation justified by one loop calculation of the massless fermion propagator in arbitrary dimensions, see for example, [67]. The LKF transformation then implies a power law for $F(p)$ in QED4 and a simple trigonometric function in QED3. To improve upon these results, one can take two paths: (i) incorporate the information contained in higher orders of perturbation theory and (ii) study the massive theory. As pointed out in [58], in QED4, the power law structure of the wavefunction renormalization remains intact by increasing order of approximation in perturbation theory although the exponent of course gets contribution from next to leading logarithms and so on. In [58], constraint was obtained on the 3-point vertex by considering a power law where the exponent of this power law was not restricted only to the one loop fermion propagator. In QED3, the two loop fermion propagator was evaluated in [22, 57, 68], where it was explicitly shown that the approximation $F(p) = 1$ is only valid upto one loop, thus violating the *transversality condition* advocated in [56]. The result found there was used in [23] to find the improved LKF transform.

In the present Chapter, we calculate the LKF transformed fermion propagator in massive QED3 and QED4 ². We start with the simplest input which corresponds to the lowest order of perturbation theory, i.e., $S(p) = 1/i\not{p} - m$ in the Landau gauge. On LKF transforming, we find the fermion propagator in an arbitrary covariant gauge. In the case of QED3, we obtain the result in terms of basic functions of momenta. In QED4, the final expression is in the form of hypergeometric functions. Coupling α enters as parameter of this transcendental function. A comparison with perturbation theory needs the expansion of the hypergeometric function in terms of its parameters. We use the technique developed by Moch *et. al.*, [69], for the said expansion. We compare our results with the one loop expansion of the fermion propagator in QED4 and QED3, [70, 71], and find perfect agreement upto terms independent of the gauge parameter at one loop, a difference permitted by the structure of the LKF transformations. We believe that the incorporation of LKF transformations, along with WGTI,

¹A criticism of the vertex construction in [56] was raised in [57].

²In the context of gauge technique, gauge covariance of the spectral functions in QED was studied in [64, 65, 66].

in the SDE can play a key role in addressing the problems of gauge invariance. For example, in the study of the SDE of the fermion propagator, only those assumptions should be permissible which keep intact the correct behaviour of the Green functions under the LKF transformations, in addition to ensuring that the WGTI is satisfied. It makes it vital to explore how two and three point Green functions transform in a gauge covariant fashion. In this Chapter, we consider only a two point function, namely, the fermion propagator.

4.2 Fermion Propagator and the LKF Transformation

We start by expanding out the fermion propagator, in momentum and coordinate spaces respectively, in its most general form as follows :

$$S_F(p; \xi) = A(p; \xi) + i \frac{B(p; \xi)}{\not{p}} \equiv \frac{F(p; \xi)}{i \not{p} - \mathcal{M}(p; \xi)} , \quad (4.1)$$

$$S_F(x; \xi) = \not{x} X(x; \xi) + Y(x; \xi) , \quad (4.2)$$

where we explicitly write the gauge in which we specify the Fermion Propagator. Motivated from the lowest order perturbation theory, we take

$$F(p; 0) = 1 \quad \text{and} \quad \mathcal{M}(p; 0) = m . \quad (4.3)$$

Perturbation theory also reveals that this result continues to hold true to one loop order for the wavefunction renormalization. Eqs. (4.1,4.2) are related to each other through the following Fourier transforms :

$$S_F(p; \xi) = \int d^d x e^{ip \cdot x} S_F(x; \xi) \quad (4.4)$$

$$S_F(x; \xi) = \int \frac{d^d p}{(2\pi)^d} e^{-ip \cdot x} S_F(p; \xi) , \quad (4.5)$$

where d is the dimension of space-time. Let us recall that the LKF transformation relating the coordinate space fermion propagator in Landau gauge to the one in an arbitrary covariant gauge reads (2.98) :

$$S_F(x; \xi) = S_F(x; 0) e^{-i[\Delta_d(0) - \Delta_d(x)]} ,$$

and, also that (2.104)

$$\Delta_d(x) = -i\xi e^2 \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{e^{-ip \cdot x}}{p^4} .$$

Taking ψ to be the angle between x and p , we can write

$$d^d p = dp p^{d-1} \sin^{d-2} \psi d\psi \Omega_{d-2} ,$$

where $\Omega_{d-2} = 2 \pi^{(d-1)/2} / \Gamma((d-1)/2)$. Hence

$$\Delta_d(x) = -i\xi e^2 \mu^{4-d} f(d) \int_0^\infty dp p^{d-5} \int_0^\pi d\psi \sin^{d-2} \psi e^{-ipx \cos \psi} , \quad (4.6)$$

where $f(d) = \Omega_{d-2} / (2\pi)^d$. Performing angular and radial integrations, we arrive at the following equation

$$\Delta_d(x) = -\frac{i\xi e^2}{16(\pi)^{d/2}} (\mu x)^{4-d} \Gamma\left(\frac{d}{2} - 2\right) . \quad (4.7)$$

With these tools at hand, the procedure now is as follows :

- Start with the lowest order fermion propagator and Fourier transform it to coordinate space.
- Apply the LKF transformation law.
- Fourier transform the result back to momentum space.

4.3 Three Dimensional Case

Employing Eqs. (4.1,4.2,4.3,4.5), the lowest order three dimensional fermion propagator in Landau gauge in the position space is given by

$$\begin{aligned} X(x; 0) &= -\frac{e^{-mx}(1+mx)}{4\pi x^3} , \\ Y(x; 0) &= -\frac{me^{-mx}}{4\pi x} . \end{aligned} \quad (4.8)$$

Once in the coordinate space, we can apply the LKF transformation law using expression (4.7) explicitly in three dimensions :

$$\Delta_3(x) = -\frac{i\alpha\xi x}{2} , \quad (4.9)$$

where $\alpha = e^2/4\pi$. The fermion propagator in an arbitrary gauge is then

$$S_F(x; \xi) = S_F(x; 0) e^{-(\alpha\xi/2)x} . \quad (4.10)$$

For Fourier transforming back to momentum space, we use

$$\begin{aligned} A(p; \xi) &= -\frac{F(p; \xi)\mathcal{M}(p; \xi)}{p^2 + \mathcal{M}^2(p; \xi)} = \int d^3x e^{ip \cdot x} Y(x; \xi) \\ iB(p; \xi) &= -\frac{ip^2 F(p; \xi)}{p^2 + \mathcal{M}^2(p; \xi)} = \int d^3x p \cdot x e^{ip \cdot x} X(x; \xi) . \end{aligned} \quad (4.11)$$

Performing the angular integration, we get

$$A(p; \xi) = -\frac{m}{p} \int_0^\infty dx \sin px e^{-(m+\alpha\xi/2)x}, \quad (4.12)$$

$$B(p; \xi) = \frac{1}{p} \int_0^\infty \frac{dx}{x^2} (1+mx) [px \cos px - \sin px] e^{-(m+\alpha\xi/2)x}, \quad (4.13)$$

and the radial integration then yields

$$A(p; \xi) = -\frac{4m}{4p^2 + (2m + \alpha\xi)^2} \quad (4.14)$$

$$B(p; \xi) = -\frac{4p^2 + \alpha\xi(2m + \alpha\xi)}{4p^2 + (2m + \alpha\xi)^2} + \frac{\alpha\xi}{2p} \arctan [2p/(2m + \alpha\xi)]. \quad (4.15)$$

We can now arrive at the following expressions for the wavefunction renormalization and the mass function, respectively :

$$F(p; \xi) = -\frac{\alpha\xi}{2p} \arctan [2p/(2m + \alpha\xi)] + \frac{2p(4p^2 + \alpha^2\xi^2)}{\phi(p; \xi)} - \frac{\alpha\xi(4p^2 + \alpha\xi(2m + \alpha\xi)) \arctan [2p/(2m + \alpha\xi)]}{\phi(p; \xi)}, \quad (4.16)$$

$$\mathcal{M}(p; \xi) = \frac{8p^3 m}{\phi(p; \xi)}, \quad (4.17)$$

where

$$\phi(p; \xi) = 2p(4p^2 + \alpha\xi(2m + \alpha\xi)) - \alpha\xi(4p^2 + (2m + \alpha\xi)^2) \arctan [2p/(2m + \alpha\xi)]. \quad (4.18)$$

In the massless limit, we immediately recuperate the well-known results :

$$F_{\text{nm}}(p; \xi) = 1 - \frac{\alpha\xi}{2p} \arctan \frac{2p}{\alpha\xi}, \quad (4.19)$$

$$\mathcal{M}_{\text{nm}}(p; \xi) = 0.$$

In the weak coupling, we can expand out Eqs.(4.16,4.17) in powers of α . To $\mathcal{O}(\alpha)$, we find

$$F(p; \xi) = 1 + \frac{\alpha\xi}{2p^3} [(m^2 - p^2) \arctan [p/m] - mp], \quad (4.20)$$

$$\mathcal{M}(p; \xi) = m \left[1 + \frac{\alpha\xi}{2p^3} \{ (m^2 + p^2) \arctan [p/m] - mp \} \right]. \quad (4.21)$$

Let us compare these results with the ones obtained in [71] :

$$F_{1\text{-loop}}(p; \xi) = 1 + \frac{\alpha\xi}{2p^3} [(m^2 - p^2) \arctan [p/m] - mp], \quad (4.22)$$

$$\mathcal{M}_{1\text{-loop}}(p; \xi) = m \left[1 + \frac{\alpha}{2p^3} \{ [\xi(m^2 + p^2) + 4p^2] \arctan [p/m] - \xi mp \} \right]. \quad (4.23)$$

We of course only expect the results to be in agreement upto a term proportional to α , as allowed by the structure of the LKF transformations. There is no such term in Eq. (4.22). Therefore, the agreement is exact. Eq. (4.21) and Eq. (4.23) become identical only after we subtract out the non-vanishing term in the Landau gauge from Eq. (4.23) to write out the *subtracted* mass function at one loop as :

$$\mathcal{M}_{1\text{-loop}}^S(p; \xi) = m \left[1 + \frac{\alpha\xi}{2p^3} \{ (m^2 + p^2) \arctan[p/m] - mp \} \right] . \quad (4.24)$$

One can numerically check that without the above mentioned subtraction, Eqs. (4.21,4.23) approach the same value only in the large momentum regime.

4.4 Four Dimensional Case

Employing Eqs. (4.1,4.2,4.3,4.5), the lowest order four dimensional fermion propagator in coordinate space is given by

$$X(x; 0) = -\frac{m^2}{4\pi^2 x^2} K_2(mx) , \quad (4.25)$$

$$Y(x; 0) = -\frac{m^2}{4\pi^2 x} K_1(mx) , \quad (4.26)$$

where K_1 and K_2 are Bessel functions of the second kind. In order to apply the LKF transformation in four dimensions, we expand Eq. (4.7) around $d = 4 - \epsilon$ and use the following identities

$$\begin{aligned} \Gamma\left(-\frac{\epsilon}{2}\right) &= -\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) , \\ x^\epsilon &= 1 + \epsilon \ln x + \mathcal{O}(\epsilon^2) , \end{aligned}$$

to obtain

$$\Delta_4(x) = i \frac{\xi e^2}{16\pi^{2-\epsilon/2}} \left[\frac{2}{\epsilon} + \gamma + 2 \ln \mu x + \mathcal{O}(\epsilon) \right] . \quad (4.27)$$

Note that we cannot write a similar expression for $\Delta_4(0)$ because of the presence of the term proportional to $\ln x$. Therefore, we introduce a cut-off scale x_{min} . Now

$$\Delta_4(x_{min}) - \Delta_4(x) = -i \ln \left(\frac{x^2}{x_{min}^2} \right)^\nu , \quad (4.28)$$

where $\nu = \alpha\xi/4\pi$. Hence

$$S_F(x; \xi) = S_F(x; 0) \left(\frac{x^2}{x_{min}^2} \right)^{-\nu} . \quad (4.29)$$

For Fourier transforming back to momentum space we use the following expressions

$$A(p; \xi) = -\frac{F(p; \xi) \mathcal{M}(p; \xi)}{p^2 + \mathcal{M}^2(p; \xi)} = \int d^4x e^{ip \cdot x} Y(x; \xi) \quad (4.30)$$

$$iB(p; \xi) = -\frac{ip^2 F(p; \xi)}{p^2 + \mathcal{M}^2(p; \xi)} = \int d^4x \, p \cdot x \, e^{ip \cdot x} X(x; \xi) .$$

On carrying out angular integration, we obtain :

$$A(p; \xi) = -\frac{m^2}{p} x_{min}^{2\nu} \int_0^\infty dx x^{-2\nu+1} K_1(mx) J_1(px) , \quad (4.31)$$

$$B(p; \xi) = -m^2 \int_0^\infty dx x^{-2\nu+1} K_2(mx) J_2(px) . \quad (4.32)$$

The radial integration then yields :

$$A(p; \xi) = -\frac{1}{m} \left(\frac{m^2}{\Lambda^2} \right)^\nu \Gamma(1-\nu) \Gamma(2-\nu) {}_2F_1 \left(1-\nu, 2-\nu; 2; -\frac{p^2}{m^2} \right) , \quad (4.33)$$

$$B(p; \xi) = -\frac{p^2}{2m^2} \left(\frac{m^2}{\Lambda^2} \right)^\nu \Gamma(1-\nu) \Gamma(3-\nu) {}_2F_1 \left(1-\nu, 3-\nu; 3; -\frac{p^2}{m^2} \right) , \quad (4.34)$$

where we have identified $2/x_{min} \rightarrow \Lambda$. The above equations imply

$$\begin{aligned} F(p; \xi) = & \frac{\Gamma(1-\nu)}{2m^2 \Gamma(3-\nu) {}_2F_1(1-\nu, 3-\nu; 3; -p^2/m^2)} \left(\frac{m^2}{\Lambda^2} \right)^\nu \\ & \left[4m^2 \Gamma^2(2-\nu) {}_2F_1^2 \left(1-\nu, 2-\nu; 2; -\frac{p^2}{m^2} \right) \right. \\ & \left. + p^2 \Gamma^2(3-\nu) {}_2F_1^2 \left(1-\nu, 3-\nu; 3; -\frac{p^2}{m^2} \right) \right] , \quad (4.35) \end{aligned}$$

$$\mathcal{M}(p; \xi) = \frac{2m {}_2F_1(1-\nu, 2-\nu; 2; -p^2/m^2)}{(2-\nu) {}_2F_1(1-\nu, 3-\nu; 3; -p^2/m^2)} . \quad (4.36)$$

Eqs. (4.35,4.36) constitute the LKF transformation of Eqs. (4.3). We shall now see that although Eqs. (4.3) correspond to the lowest order propagator, their LKF transformation, Eqs. (4.35,4.36), is nonperturbative in nature and contains information of higher orders.

4.4.1 Case $\alpha = 0$

Let us switch off the coupling and put $\alpha = 0$ which implies $\nu = 0$. Now using the identity

$${}_2F_1(1, 2; 2; -p^2/m^2) = {}_2F_1(1, 3; 3; -p^2/m^2) = (1 + p^2/m^2)^{-1} , \quad (4.37)$$

it is easy to see that

$$F(p; \xi) = 1 \quad \text{and} \quad \mathcal{M}(p; \xi) = m , \quad (4.38)$$

which coincides with the lowest order perturbative result as expected.

4.4.2 Case $m \gg p$

In the limit $m \gg p$, the hypergeometric functions in Eqs. (4.35,4.36) can be easily expanded in powers of p^2/m^2 , using the identity

$${}_2F_1\left(\alpha, \beta; \gamma; -\frac{p^2}{m^2}\right) = 1 - \frac{\alpha\beta}{\gamma} \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^2}{m^2}\right)^2. \quad (4.39)$$

Retaining only $\mathcal{O}(p^2/m^2)$ terms, we arrive at :

$$\begin{aligned} F(p; \xi) &= \frac{\Gamma(1-\nu)\Gamma(2-\nu)}{(1-\nu/2)} \left(\frac{m^2}{\Lambda^2}\right)^\nu \\ &\quad \left[1 + \frac{2\nu}{3} \left(1 - \frac{5\nu}{8}\right) \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^2}{m^2}\right)^2\right], \\ \mathcal{M}(p; \xi) &= \frac{m}{(1-\nu/2)} \left[1 + \frac{\nu}{6}(1-\nu) \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^2}{m^2}\right)^2\right]. \end{aligned} \quad (4.40)$$

Now carrying out an expansion in α and substituting $\nu = \alpha\xi/4\pi$, we get the following $\mathcal{O}(\alpha)$ expressions :

$$F(p; \xi) = 1 + \frac{\alpha\xi}{4\pi} \left[2\gamma - \frac{1}{2} + \frac{2p^2}{3m^2} + \ln \frac{m^2}{\Lambda^2}\right], \quad (4.41)$$

$$\mathcal{M}(p; \xi) = m \left\{1 + \frac{\alpha\xi}{8\pi} \left[1 + \frac{p^2}{3m^2}\right]\right\}. \quad (4.42)$$

Let us now compare these expressions against the one-loop perturbative evaluation of the massive fermion propagator, see e.g., [70] :

$$F_{1\text{-loop}}(p; \xi) = 1 - \frac{\alpha\xi}{4\pi} \left[C\mu^\epsilon + \left(1 - \frac{m^2}{p^2}\right)(1-L)\right], \quad (4.43)$$

$$\mathcal{M}_{1\text{-loop}}(p; \xi) = m + \frac{\alpha m}{\pi} \left[\left(1 + \frac{\xi}{4}\right) + \frac{3}{4}(C\mu^\epsilon - L) + \frac{\xi}{4} \frac{m^2}{p^2}(1-L)\right], \quad (4.44)$$

where

$$\begin{aligned} L &= \left(1 + \frac{m^2}{p^2}\right) \ln \left(1 + \frac{p^2}{m^2}\right), \\ C &= -\frac{2}{\epsilon} - \gamma - \ln \pi - \ln \left(\frac{m^2}{\mu^2}\right). \end{aligned}$$

Knowing the fermion propagator even in one particular gauge is a prohibitively difficult task. Therefore, Eqs. (4.3) have to be viewed only as an approximation. For the wavefunction renormalization $F(p; 0)$, this approximation is valid upto one loop order, whereas, for the mass function, it is true only to the lowest order. Therefore we cannot expect the LKF transform of Eqs. (4.3) to yield

correctly each term in the perturbative expansion of the fermion propagator. However, it should correctly reproduce all those terms at every order of expansion which vanish in the Landau gauge at $\mathcal{O}(\alpha)$ and beyond. Therefore, we expect Eq. (4.43) to be exactly reproduced and Eq. (4.44) to be reproduced upto the terms which vanish in the Landau gauge at $\mathcal{O}(\alpha)$. After subtracting these terms, the resulting *subtracted* mass function is :

$$\mathcal{M}_{1-\text{loop}}^S(p; \xi) = m + \frac{\alpha\xi m}{4\pi} \left[1 + \frac{m^2}{p^2}(1-L) \right] . \quad (4.45)$$

In the limit $m \rightarrow \infty$, the wavefunction renormalization acquires the form

$$F(p; \xi)_{1-\text{loop}} = 1 + \frac{\alpha\xi}{4\pi} \left[-C\mu^\epsilon - \frac{1}{2} + \frac{2p^2}{3m^2} \right] , \quad (4.46)$$

while the *subtracted* mass function is

$$\mathcal{M}_{1-\text{loop}}^S(p; \xi) = m \left\{ 1 + \frac{\alpha\xi}{8\pi} \left[1 + \frac{p^2}{3m^2} \right] \right\} . \quad (4.47)$$

The last two expressions are in perfect agreement with Eqs. (4.41,4.42) after we make the identification :

$$-C\mu^\epsilon \rightarrow 2\gamma + \ln \frac{m^2}{\Lambda^2} . \quad (4.48)$$

4.4.3 Case of Weak Coupling

The case $m \gg p$ is relatively easier to handle as we merely have to expand ${}_2F_1(\beta, \gamma; \delta; x)$ in powers of x and retain only the leading terms. If we want to obtain a series in powers of the coupling alone, we need the expansion of the hypergeometric functions in terms of its parameters β and γ . We follow the technique developed in [69]. One of the mathematical objects we shall use for such an expansion are the Z -sums defined as :

$$Z(n; m_1, \dots, m_k; x_1, \dots, x_k) = \sum_{n \geq i_1 > i_2 > \dots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \dots \frac{x_k^{i_k}}{i_k^{m_k}} . \quad (4.49)$$

For $x_1 = \dots = x_k = 1$ the definition reduces to the Euler-Zagier sums, [72, 73] :

$$Z(n; m_1, \dots, m_k; 1, \dots, 1) = Z_{m_1, \dots, m_k}(n) . \quad (4.50)$$

Euler-Zagier sums can be used in the expansion of Gamma functions. For positive integers n we have [69]:

$$\Gamma(n + \epsilon) = \Gamma(1 + \epsilon)\Gamma(n) \left[1 + \epsilon Z_1(n-1) + \dots + \epsilon^{n-1} Z_{11\dots 1}(n-1) \right] . \quad (4.51)$$

The first sum $Z_1(n-1)$, e.g., is just the $(n-1)$ -th harmonic number, H_{n-1} , of order 1 :

$$Z_1(n-1) = \sum_{i=1}^{n-1} \frac{1}{i} \equiv H_{n-1} . \quad (4.52)$$

With these definitions in hand, we proceed to expand a hypergeometric function, ${}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x)$, as an example, assuming $|x| < 1$:

$$\begin{aligned} {}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) &= 1 + \frac{\Gamma(2)}{\Gamma(1 + \varepsilon)\Gamma(2 + \varepsilon)} \\ &\quad \sum_{n=1}^{\infty} \frac{\Gamma(1 + \varepsilon + n)\Gamma(2 + \varepsilon + n)}{\Gamma(2 + n)} \frac{x^n}{n!} \end{aligned} \quad (4.53)$$

$$\begin{aligned} &= 1 + \frac{1}{(1 + \varepsilon)\Gamma^2(1 + \varepsilon)} \\ &\quad \sum_{n=1}^{\infty} \frac{(1 + \varepsilon + n)(\varepsilon + n)^2\Gamma^2(\varepsilon + n)}{\Gamma(2 + n)} \frac{x^n}{n!} . \end{aligned} \quad (4.54)$$

Employing Eq. (4.51), we can expand the last expression in powers of ε to any desired order of approximation. We shall be interested only in terms upto $\mathcal{O}(\alpha)$.

$$\begin{aligned} {}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) &= 1 + \sum_{n=1}^{\infty} x^n - \varepsilon \sum_{n=1}^{\infty} x^n + \varepsilon \sum_{n=1}^{\infty} \frac{2 + 3n}{n(n + 1)} x^n \\ &\quad + 2\varepsilon \sum_{n=1}^{\infty} H_{n-1} x^n . \end{aligned} \quad (4.55)$$

Performing the summations, we obtain

$${}_2F_1(1 + \varepsilon, 2 + \varepsilon; 2; x) = \frac{1}{1 - x} \left[1 - \varepsilon \left\{ 1 + \frac{1 + x}{x} \ln(1 - x) \right\} \right] . \quad (4.56)$$

Similarly,

$$\begin{aligned} {}_2F_1(1 + \varepsilon, 3 + \varepsilon; 3; x) &= \frac{1}{1 - x} \\ &\quad - \varepsilon \left\{ \frac{1}{x} + \frac{3}{2} \frac{1}{1 - x} + \left(\frac{1 + x}{x^2} + \frac{2}{1 - x} \right) \ln(1 - x) \right\} . \end{aligned} \quad (4.57)$$

Substituting back into Eqs. (4.35,4.36) and identifying $\varepsilon = -\nu$, we obtain

$$\begin{aligned} F(p; \xi) &= 1 - \frac{\alpha\xi}{4\pi} \left[-2\gamma - \ln \frac{m^2}{\Lambda^2} + \left(1 - \frac{m^2}{p^2} \right) (1 - L) \right] , \quad (4.58) \\ \mathcal{M}(p; \xi) &= m + \frac{\alpha\xi m}{4\pi} \left[1 + \frac{m^2}{p^2} (1 - L) \right] , \end{aligned}$$

which matches exactly onto the one loop result of Eqs. (4.43,4.45) after the same identification as before, i.e., (4.48). Therefore, we have seen that the LKF transformation of the bare propagator contains important information of higher orders in perturbation theory.

LKF transformations, being non perturbative in nature, tell us the non perturbative way in which the Fermion Propagator [74] and the Vertex transform

under a gauge variation. Most of the works on SDE violate these transformations, with the exception of few of them which include the corresponding transformation to the Fermion Propagator at one-loop level. Since in Perturbation Theory all of the gauge invariance constraints are valid order by order, we can exploit this fact to construct a vertex ansatz such that in its perturbative expansion, it automatically satisfies, along with its associated Fermion Propagator, their corresponding LKF transformation, as we will do in the next Chapter.

Appendix

Most of the integrals involved in this Chapter are listed below for a quick reference [55, 75] :

$$\int_0^\pi d\psi \sin^{d-2} \psi \cos \psi e^{-ipx \cos \psi} = -i\sqrt{\pi} \left(\frac{px}{2}\right)^{1-\frac{d}{2}} \Gamma\left(\frac{d-1}{2}\right) J_{\frac{d}{2}}(px), \quad (4.59)$$

$$\int_0^\infty x^{d/2-1} J_{d/2}(ax) = \frac{\Gamma(d/2)}{2^{1-d/2} a^{d/2}}. \quad (4.60)$$

For the three dimensional case, the needed integrals are :

$$\int_0^\pi d\theta \sin \theta e^{-ipx \cos \theta} = \frac{2 \sin px}{px}, \quad (4.61)$$

$$\int_0^\pi d\theta \cos \theta \sin \theta e^{-ipx \cos \theta} = 2i \left[\frac{\cos px}{px} - \frac{\sin px}{(px)^2} \right], \quad (4.62)$$

$$\int_0^\infty dp \frac{p^3}{(p^2 + m^2)} \left[\frac{\cos px}{px} - \frac{\sin px}{(px)^2} \right] = -\frac{\pi}{2} \frac{(1 + mx)}{x^2} e^{-mx}, \quad (4.63)$$

$$\int_0^\infty dp \frac{p \sin px}{(p^2 + m^2)} = \frac{\pi}{2} e^{-mx}, \quad (4.64)$$

$$\frac{1}{p} \int_0^\infty \frac{dx}{x^2} e^{-ax} [px \cos px - \sin px] = -1 + \frac{a}{p} \arctan \frac{p}{a}, \quad (4.65)$$

$$\frac{1}{p} \int_0^\infty \frac{dx}{x} e^{-ax} [px \cos px - \sin px] = \frac{a}{a^2 + p^2} - \frac{1}{p} \arctan \frac{p}{a}, \quad (4.66)$$

$$\int_0^\infty dx \sin px e^{-(m+\alpha\xi/2)x} = \frac{p}{(m+\alpha\xi/2)^2 + p^2}. \quad (4.67)$$

For the four dimensional case, we used the following integrals in particular :

$$\int_0^\pi d\theta \sin^2 \theta e^{-ipx \cos \theta} = \frac{\pi}{px} J_1(px), \quad (4.68)$$

$$\int_0^\infty dp \frac{p^{\nu+1} J_\nu(px)}{(p^2 + m^2)^{\mu+1}} = \frac{m^{\nu-\mu} x^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(mx) , \quad (4.69)$$

$$\begin{aligned} \int_0^\infty dx x^{-\lambda} K_\mu(ax) J_\nu(bx) &= \frac{a^{\lambda-\nu-1} b^\nu}{2^{\lambda+1} \Gamma(1+\nu)} \Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right) \\ &\quad \Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right) \\ &\quad \times {}_2F_1\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{b^2}{a^2}\right) . \end{aligned} \quad (4.70)$$

Some of the series used in our calculation are as follows :

$$\sum_{n=1}^\infty H_{n-1} x^n = -\frac{x \ln(1-x)}{1-x} , \quad (4.71)$$

$$\sum_{n=1}^\infty \frac{n+1}{n(n+2)} x^n = -\frac{2+x}{4x} - \frac{(1+x^2) \ln(1-x)}{2x^2} , \quad (4.72)$$

$$\sum_{n=1}^\infty \frac{1}{(n+1)(n+2)} x^n = \frac{2-x}{2x} + \frac{(1-x) \ln(1-x)}{x^2} . \quad (4.73)$$

Chapter 5

Constructing the Vertex

5.1 Introduction

In studies on the Dynamical Breaking of Chiral Symmetry (DBCS) in QED3 in the quenched and unquenched approximations [7, 11, 12, 17, 18, 50, 56, 76, 77, 78, 79, 80, 81, 82], we look for gauge independent physical observables. For that purpose, we have seen previously the necessity of including the Ward-Green-Takahashi Identity (WGTI) as well as the Landau-Khalatnikov-Fradkin (LKF) transformations. We know that, in Perturbation Theory, these gauge invariance constraints are satisfied order by order, thus, the vertex should be modified in every order of approximation. This fact has been exploited by, for example, [22, 23, 57, 68]. In this Chapter we carry out the construction of the non perturbative Vertex based on its perturbative counterpart, following the work [83].

Making use of the WGTI, that relates the Vertex to the Fermion Propagator, one part of the Vertex, called longitudinal, can be expressed in terms of the said propagator [19]. We perform the evaluation of the Fermion Propagator at the one-loop level and hence we determine the longitudinal vertex at the same order. We also calculate the Full vertex at one-loop, and a mere subtraction of the longitudinal part yields the transverse part, which is not fixed by the WGTI. According to the choice of Ball and Chiu, later modified by Kızılersü *et. al.* [70], the transverse vertex can be expressed in terms of 8 independent spin structures. Vertex should be free from kinematical singularities. Ball and Chiu choose the basis in such a way that the coefficient of every element of this basis is independently free from kinematical singularities in Feynman gauge. It was later shown by Kızılersü *et. al.* [70] that a similar calculation to the one of Ball and Chiu in an arbitrary covariant gauge does not have the same nice feature. Consequently, they proposed a modified basis whose coefficients are free from kinematical singularities in an arbitrary covariant gauge. The calculation in the present Chapter confirms that all the vectors of the modified basis also retain this feature for massive QED3. The final result for the transverse vertex is

written in terms of basic functions of the momenta in a form suitable for its extension to the non-perturbative domain, following the ideas of Curtis and Pennington [37].

Using perturbative constraints as a guide, we carry out a construction of the non-perturbative vertex, which has no explicit dependence on the coupling α . This vertex has an explicit dependence on the gauge parameter ξ . For practical purposes of the numerical study of DCSB, we also construct an effective vertex which shifts the angular dependence from the unknown Fermion Propagator functions to the known basic functions, without changing its perturbative properties at the one-loop level. We believe that this vertex should lead to a more realistic study of the dynamically generated masses through the corresponding SDEs.

5.2 Longitudinal and Transverse Vertex to One-Loop

5.2.1 The Fermion Propagator

One-loop Fermion Propagator can be obtained by evaluating the graph in Diagram (7).

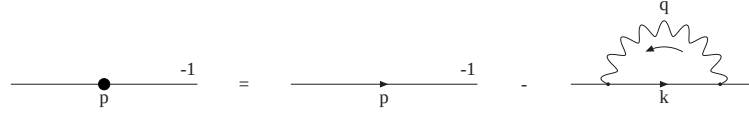


Diagram 7 : One-loop Correction to the Fermion Propagator.

This graph corresponds to the following equation :

$$iS_F(p)^{-1} = iS_F^0{}^{-1}(p) + e^2 \int \frac{d^3k}{(2\pi)^3} \gamma^\mu S_F^0(k) \gamma^\nu \Delta_{\mu\nu}^0(q), \quad (5.1)$$

where $q = k - p$. The bare Fermion and Photon Propagators are, respectively :

$$\begin{aligned} S_F^0(p) &= \frac{1}{\not{p} - m}, \\ \Delta_{\mu\nu}^0(q) &= -[q^2 g_{\mu\nu} + (\xi - 1)q_\mu q_\nu] / q^4, \end{aligned} \quad (5.2)$$

where m is the bare mass of the fermion and $S_F(p)$ represents the full propagator, defined in eq. (3.3).

Taking the trace of Eq. (5.1), having multiplied it with $\not{p}$ and with 1 respectively, one can obtain two independent equations. On simplifying, these equations can be written as :

$$\frac{1}{F(p)} = 1 + \frac{i4\pi\alpha\xi}{p^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{q^4(k^2 - m^2)} [(k^2 + p^2)k \cdot p - 2k^2 p^2] \quad (5.3)$$

$$\frac{\mathcal{M}(p)}{F(p)} = m - i4\pi\alpha(\xi + 2) \int \frac{d^3k}{(2\pi)^3} \frac{m}{q^2(k^2 - m^2)}, \quad (5.4)$$

On Wick rotating to the Euclidean space and carrying out angular and radial integrations, we arrive at :

$$\begin{aligned} \frac{1}{F(p)} &= 1 - \frac{\alpha\xi}{2p^2} [m - (m^2 + p^2) I(p)] , \\ \frac{\mathcal{M}(p)}{F(p)} &= m [1 + \alpha(\xi + 2) I(p)] , \end{aligned} \quad (5.5)$$

where we have used the simplifying notation $I(p^2) = (1/\sqrt{-p^2}) \arctan \sqrt{-p^2/m^2}$. Equations (3.3) and (5.5) form the complete Fermion Propagator at one loop.

5.2.2 Longitudinal Vertex to One Loop

We take the longitudinal part of the Vertex as the one that satisfies the WGTI, i. e., the Ball-Chiu Vertex, eq. (3.27). On substituting eq. (5.5) in the said expression, we obtain :

$$\begin{aligned} \Gamma_L^\mu (= \Gamma_{BC}^\mu) &= \left[1 + \frac{\alpha\xi}{4} \sigma_1 \right] \gamma^\mu + \frac{\alpha\xi}{4} \sigma_2 [k^\mu \not{k} + p^\mu \not{p} + k^\mu \not{p} + p^\mu \not{k}] \\ &+ \alpha(\xi + 2) \sigma_3 [k^\mu + p^\mu] , \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} \sigma_1 &= \frac{m^2 + k^2}{k^2} I(k) + \frac{m^2 + p^2}{p^2} I(p) - m \frac{k^2 + p^2}{k^2 p^2} , \\ \sigma_2 &= \frac{1}{k^2 - p^2} \left[\frac{m^2 + k^2}{k^2} I(k) - \frac{m^2 + p^2}{p^2} I(p) + m \frac{k^2 - p^2}{k^2 p^2} \right] , \\ \sigma_3 &= m [I(k) - I(p)] . \end{aligned} \quad (5.7)$$

Eqs. (5.6) and (5.7) give the longitudinal part of the fermion-photon vertex to one loop for the massive QED3.

5.2.3 Full Vertex to One Loop

Full Vertex can be obtained from Diagram (8) :

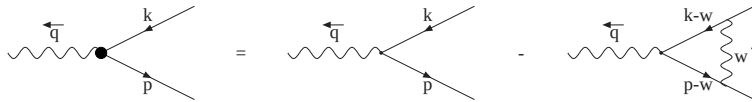


Diagram 8 : One-loop Correction to the Vertex.

and can be expressed as :

$$\Gamma^\mu(k, p) = \gamma^\mu + \Lambda^\mu . \quad (5.8)$$

Using the Feynman rules, Λ^μ to $O(\alpha)$ is simply given by :

$$-ie\Lambda^\mu = \int_M \frac{d^3w}{(2\pi)^3} (-ie\gamma^\alpha) iS_F^0(p-w) (-ie\gamma^\mu) iS_F^0(k-w) (-ie\gamma^\beta) i\Delta_{\alpha\beta}^0(w) , \quad (5.9)$$

where the loop integration is to be performed in Minkowski space (in Euclidean space, these definitions are modified by the correspondig factors of i). Λ^μ can be expressed as :

$$\begin{aligned} \Lambda^\mu = & -\frac{i\alpha}{2\pi^2} \left\{ [\gamma^\alpha \not{p} \gamma^\mu \not{k} \gamma_\alpha + m(4k^\mu + 4p^\mu - \not{p} \gamma^\mu - \gamma^\mu \not{k}) - m^2 \gamma^\mu] J^{(0)} \right. \\ & - [\gamma^\alpha \not{p} \gamma^\mu \gamma^\nu \gamma_\alpha + \gamma^\alpha \gamma^\nu \gamma^\mu \not{k} \gamma_\alpha + 6mg^{\mu\nu}] J_\nu^{(1)} + \gamma^\alpha \gamma^\nu \gamma^\mu \gamma^\lambda \gamma_\alpha J_{\nu\lambda}^{(2)} \\ & + (\xi - 1) \left[\gamma^\mu K^{(0)} - [\gamma^\nu \not{p} \gamma^\mu + \gamma^\mu \not{k} \gamma^\nu + 2mg^{\mu\nu}] J_\nu^{(1)} \right. \\ & \left. \left. + [\gamma^\nu \not{p} \gamma^\mu \not{k} \gamma^\lambda + m(\gamma^\nu \not{p} \gamma^\mu \gamma^\lambda + \gamma^\nu \gamma^\mu \not{k} \gamma^\lambda) + m^2 \gamma^\nu \gamma^\mu \gamma^\lambda] I_{\nu\lambda}^{(2)} \right] \right\} , \end{aligned} \quad (5.10)$$

where the integrals $K^{(0)}$, $J^{(0)}$, $J_\mu^{(1)}$, $J_{\mu\nu}^{(2)}$, $I^{(0)}$, $I_\mu^{(1)}$ and $I_{\mu\nu}^{(2)}$ are :

$$\begin{aligned} K^{(0)} &= \int_M d^3w \frac{1}{[(p-w)^2 - m^2][(k-w)^2 - m^2]} \\ J^{(0)} &= \int_M d^3w \frac{1}{w^2 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\ J_\mu^{(1)} &= \int_M d^3w \frac{w_\mu}{w^2 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\ J_{\mu\nu}^{(2)} &= \int_M d^3w \frac{w_\mu w_\nu}{w^2 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\ I^{(0)} &= \int_M d^3w \frac{1}{w^4 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\ I_\mu^{(1)} &= \int_M d^3w \frac{w_\mu}{w^4 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\ I_{\mu\nu}^{(2)} &= \int_M d^3w \frac{w_\mu w_\nu}{w^4 [(p-w)^2 - m^2][(k-w)^2 - m^2]} . \end{aligned} \quad (5.11)$$

We evaluate these integrals following the techniques developed in [19, 57, 68, 70]. The results are tabulated below, employing the notation $\Delta^2 = (k \cdot p)^2 - k^2 p^2$ and $X_0 = (2/i\pi^2)X^{(0)}$ for $X = I, J, K$.

The Scalar Integral $J^{(0)}$

In arbitrary dimensions,

$$J^{(0)} = \int \frac{d^d w}{[(k-w)^2 - m^2][(p-w)^2 - m^2]w^2} . \quad (5.12)$$

Employing Feynman parametrization,

$$\begin{aligned} J^{(0)} &= \Gamma(3) \int d^d w \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \\ &\times \frac{\delta(\alpha_s - 1)}{[\alpha_1[(k-w)^2 - m^2] + \alpha_2[(p-w)^2 - m^2] + \alpha_3 w^2]^3} , \end{aligned} \quad (5.13)$$

with

$$\alpha_s = \alpha_1 + \alpha_2 + \alpha_3 . \quad (5.14)$$

We define D as

$$\begin{aligned} D &= \alpha_1[(k-w)^2 - m^2] + \alpha_2[(p-w)^2 - m^2] + \alpha_3 w^2 \\ &= \alpha_s w^2 - 2(\alpha_1 k \cdot w + \alpha_2 p \cdot w) + \alpha_1(k^2 - m^2) + \alpha_2(p^2 - m^2) \\ &= \alpha_s \left[w^2 - \frac{2(\alpha_1 k + \alpha_2 p) \cdot w}{\alpha_s} + \frac{(\alpha_1 k + \alpha_2 p)^2}{\alpha_s^2} \right. \\ &\quad \left. - \frac{(\alpha_1 k + \alpha_2 p)^2}{\alpha_s^2} + \frac{\alpha_1(k^2 - m^2) + \alpha_2(p^2 - m^2)}{\alpha_s} \right] \\ &= \alpha_s \left[\left(w - \frac{\alpha_1 k + \alpha_2 p}{\alpha_s} \right)^2 + \right. \\ &\quad \left. \frac{1}{\alpha_s^2} \left\{ \alpha_s [\alpha_1(k^2 - m^2) + \alpha_2(p^2 - m^2)] - (\alpha_1 k + \alpha_2 p)^2 \right\} \right] . \end{aligned} \quad (5.15)$$

Let

$$w - \frac{\alpha_1 k + \alpha_2 p}{\alpha_s} \rightarrow w . \quad (5.16)$$

Then,

$$\begin{aligned} J^{(0)} &= 2 \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \frac{\delta(\alpha_s - 1)}{\alpha_s^3} \\ &\times \int \frac{d^d w}{[w^2 + \alpha_s^{-2} \{ \alpha_s [\alpha_1(k^2 - m^2) + \alpha_2(p^2 - m^2)] - (\alpha_1 k + \alpha_2 p)^2 \}]^3} . \end{aligned} \quad (5.17)$$

Making use of the formulas

$$\begin{aligned} \int \frac{d^d w}{w^n} &= 0 \\ \int \frac{d^d w}{(w^2 - s)^n} &= (-1)^n i \pi^{\frac{d}{2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} s^{\frac{d}{2} - n} , \end{aligned} \quad (5.18)$$

we can write

$$J^{(0)} = -i\pi^{\frac{d}{2}}\Gamma\left(3 - \frac{d}{2}\right) \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \quad (5.19)$$

$$\times \frac{\delta(\alpha_s - 1)}{\alpha_s^{d-3} \{-\alpha_s[\alpha_1(k^2 - m^2) + \alpha_2(p^2 - m^2)] + (\alpha_1 k + \alpha_2 p)^2\}^{3-\frac{d}{2}}}.$$

At this point, we remind the Cheng-Wo Theorem [84] :

Cheng-Wu Theorem 1 *If*

$$I = \int_0^1 \prod_{i=1}^n d\alpha_i \delta\left(1 - \sum_{i=1}^n \alpha_i\right) F(\alpha), \quad (5.20)$$

then we can write

$$I = \int_0^\infty \prod' d\alpha_i \int_0^1 \prod'' d\alpha_i \delta\left(1 - \sum'' \alpha_i\right) F(\alpha), \quad (5.21)$$

where the set of α has been split into two nonempty sets

$$\{\alpha'\} = \{\alpha_1, \dots, \alpha_k\}, \quad \{\alpha''\} = \{\alpha_{k+1}, \dots, \alpha_n\}. \quad (5.22)$$

Making use of this theorem,

$$J^{(0)} = -i\pi^{\frac{d}{2}}\Gamma\left(3 - \frac{d}{2}\right) \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_2 \int_0^1 d\alpha_1 \quad (5.23)$$

$$\times \frac{\delta(\alpha_1 - 1)}{\alpha_s^{d-3} \{-\alpha_s[\alpha_1(k^2 - m^2) + \alpha_2(p^2 - m^2)] + (\alpha_1 k + \alpha_2 p)^2\}^{3-\frac{d}{2}}}.$$

After a trivial integration over α_1 , we obtain

$$D = (1 + \alpha_2 + \alpha_3)^{d-3} [m^2(1 + \alpha_2 + \alpha_3)(1 + \alpha_2) - k^2(1 + \alpha_2 + \alpha_3 - 1) - p^2(1 + \alpha_2 + \alpha_3)\alpha_2 + \alpha_2^2 p^2 + 2\alpha_2 k \cdot p]^{3-\frac{d}{2}}. \quad (5.24)$$

Using $-2k \cdot p = q^2 - k^2 - p^2$,

$$D = (1 + \alpha_2 + \alpha_3)^{d-3} [m^2(1 + \alpha_2 + \alpha_3)(1 + \alpha_2) - \alpha_3 k^2 - \alpha_2 \alpha_3 p^2 - \alpha_2 q^2]^{3-\frac{d}{2}}. \quad (5.25)$$

Therefore,

$$J^{(0)} = -i\pi^{\frac{d}{2}}\Gamma\left(3 - \frac{d}{2}\right) \int_0^\infty d\alpha_3 \int_0^\infty \frac{d\alpha_2}{(1 + \alpha_2 + \alpha_3)^{d-3}} \quad (5.26)$$

$$\times \frac{1}{[m^2(1 + \alpha_2 + \alpha_3)(1 + \alpha_2) - \alpha_3 k^2 - \alpha_2 \alpha_3 p^2 - \alpha_2 q^2]^{3-\frac{d}{2}}}.$$

In our three-dimensional case

$$\begin{aligned}
J^{(0)} &= -i\pi^{\frac{3}{2}} \frac{1}{2} \pi^{\frac{1}{2}} \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_2 \\
&\quad \times \frac{1}{[m^2(1+\alpha_2+\alpha_3)(1+\alpha_2) - \alpha_3 k^2 - \alpha_2 \alpha_3 p^2 - \alpha_2 q^2]^{\frac{3}{2}}} \\
&= \frac{i\pi^2}{2} J_0, \tag{5.27}
\end{aligned}$$

where

$$\begin{aligned}
J_0 &= - \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_2 \\
&\quad \times \frac{1}{[m^2(1+\alpha_2+\alpha_3)(1+\alpha_2) - \alpha_3 k^2 - \alpha_2 \alpha_3 p^2 - \alpha_2 q^2]^{\frac{3}{2}}}. \tag{5.28}
\end{aligned}$$

Being simpler the integration over α_3 first, we take

$$\begin{aligned}
D &= m^2(1+\alpha_2+\alpha_3)(1+\alpha_2) - \alpha_3 k^2 - \alpha_2 \alpha_3 p^2 - \alpha_2 q^2 \\
&= \alpha_3 [m^2(1+\alpha_2) - k^2 - \alpha_2 p^2] + m^2(1+\alpha_2)^2 - \alpha_2 q^2 \\
&= \alpha_3 [\alpha_2(m^2 - p^2) + (m^2 - k^2)] + m^2(1+\alpha_2)^2 - \alpha_2 q^2. \tag{5.29}
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_0 &= - \int_0^\infty d\alpha_2 \int_0^\infty \frac{d\alpha_3}{[\alpha_3 [\alpha_2(m^2 - p^2) + (m^2 - k^2)] + m^2(1+\alpha_2)^2 - \alpha_2 q^2]^{\frac{3}{2}}} \\
&= 2 \int_0^\infty d\alpha_2 \frac{1}{[\alpha_2(m^2 - p^2) + (m^2 - k^2)]} \\
&\quad \times \frac{1}{\{\alpha_3 [\alpha_2(m^2 - p^2) + (m^2 - k^2)] + m^2(1+\alpha_2)^2 - \alpha_2 q^2\}^{\frac{1}{2}}} \Big|_{\alpha_3=0}^{\alpha_3=\infty} \\
&= -2 \int_0^\infty d\alpha_2 \frac{1}{[\alpha_2(m^2 - p^2) + (m^2 - k^2)]} \frac{1}{[m^2(1+\alpha_2)^2 - \alpha_2 q^2]^{\frac{1}{2}}} \\
&= -\frac{2}{(m^2 - p^2)} \int_0^\infty d\alpha_2 \frac{1}{\left[\alpha_2 + \frac{m^2 - k^2}{m^2 - p^2}\right]} \frac{1}{[m^2(1+\alpha_2)^2 - \alpha_2 q^2]^{\frac{1}{2}}}. \tag{5.30}
\end{aligned}$$

Let

$$\alpha_2 + \frac{m^2 - k^2}{m^2 - p^2} = z \quad \Rightarrow \quad d\alpha_2 = dz. \tag{5.31}$$

Integration limits are transformed in the following way :

$$\alpha_2 = 0 \Rightarrow z = \frac{m^2 - k^2}{m^2 - p^2} \quad \alpha_2 \rightarrow \infty \Rightarrow z \rightarrow \infty. \tag{5.32}$$

Now we rewrite the integrand as :

$$m^2(1+\alpha_2)^2 - \alpha_2 q^2 = m^2 \left(1 + z - \frac{m^2 - k^2}{m^2 - p^2}\right)^2 - \left[z - \frac{m^2 - k^2}{m^2 - p^2}\right] q^2$$

$$\begin{aligned}
&= m^2 + \left[z^2 + \frac{(k^2 - p^2)^2}{(m^2 - p^2)^2} + 2z \frac{k^2 - p^2}{m^2 - p^2} \right] \\
&\quad - zq^2 + q^2 \frac{m^2 - k^2}{m^2 - p^2} \\
&= m^2 z^2 + z \left[\frac{2m^2(k^2 - p^2) - q^2(m^2 - p^2)}{m^2 - p^2} \right] \\
&\quad + \frac{m^2(k^2 - p^2)^2 + q^2(m^2 - k^2)(m^2 - p^2)}{(m^2 - p^2)^2} \\
&= cz^2 + bz + a \equiv R, \tag{5.33}
\end{aligned}$$

where

$$\begin{aligned}
c &= m^2, \\
b &= \frac{2m^2(k^2 - p^2) - q^2(m^2 - p^2)}{m^2 - p^2}, \\
a &= \frac{m^2(k^2 - p^2)^2 + q^2(m^2 - k^2)(m^2 - p^2)}{(m^2 - p^2)^2} = \frac{\chi}{(m^2 - p^2)^2}. \tag{5.34}
\end{aligned}$$

Consequently,

$$J_0 = -\frac{2}{m^2 - p^2} \int_{\frac{m^2 - k^2}{m^2 - p^2}}^{\infty} \frac{dz}{z} \frac{1}{\sqrt{R}} = \frac{2}{(m^2 - p^2)} \frac{1}{\sqrt{-a}} \arctan \frac{2a + bz}{2\sqrt{-a}\sqrt{R}} \Big|_{\frac{m^2 - k^2}{m^2 - p^2}}^{\infty}. \tag{5.35}$$

We consider

1. Evaluation at infinity

$$\begin{aligned}
\lim_{z \rightarrow \infty} \frac{2a + bz}{2\sqrt{-a}\sqrt{R}} &= \frac{b}{2\sqrt{-a}\sqrt{c}} \\
&= \frac{m^2(k^2 - p^2)[2m^2 - k^2 - p^2] + \chi}{2m\sqrt{-\chi}(m^2 - k^2)}. \tag{5.36}
\end{aligned}$$

2. Evaluation at the lower limit

$$\frac{2a + bz}{2\sqrt{-a}\sqrt{R}} \Big|_{z = \frac{m^2 - k^2}{m^2 - p^2}} = \frac{-\chi + m^2(k^2 - p^2)(2m^2 - k^2 - p^2)}{2m\sqrt{-\chi}(m^2 - p^2)}, \tag{5.37}$$

and then,

$$\begin{aligned}
J_0 &= \frac{-2}{\sqrt{-\chi}} \left\{ \arctan \left[\underbrace{\frac{m^2(k^2 - p^2)(2m^2 - k^2 - p^2) + \chi}{2m\sqrt{-\chi}(m^2 - k^2)}}_a \right] \right. \\
&\quad \left. - \arctan \left[\underbrace{\frac{-\chi + m^2(k^2 - p^2)(2m^2 - k^2 - p^2)}{2m\sqrt{-\chi}(m^2 - p^2)}}_b \right] \right\}. \tag{5.38}
\end{aligned}$$

Since we defined

$$I(y) = \frac{1}{\sqrt{-y^2}} \arctan \sqrt{\frac{-y^2}{m^2}}, \quad (5.39)$$

we can write

$$I(\eta_i^2 \chi) = \frac{1}{\sqrt{-\eta_i^2 \chi}} \arctan \sqrt{\frac{-\eta_i^2 \chi}{m^2}} = \frac{1}{\eta_i \sqrt{-\chi}} \arctan \frac{\eta_i}{m} \sqrt{-\chi}. \quad (5.40)$$

Rewriting then eq. (5.38) with the identifications

$$\begin{aligned} a &= \frac{m^2(k^2 - p^2)(2m^2 - k^2 - p^2) + \chi}{2m\sqrt{-\chi}(m^2 - k^2)} \\ &= \frac{\sqrt{-\chi}}{2m} \left[- \left\{ \frac{m^2(k^2 - p^2)(2m^2 - k^2 - p^2) + \chi}{2\chi(m^2 - k^2)} \right\} \right] \\ &\equiv \frac{\sqrt{-\chi}}{2m} \eta_1, \end{aligned} \quad (5.41)$$

where

$$\eta_1 = - \left\{ \frac{m^2(k^2 - p^2)(2m^2 - k^2 - p^2) + \chi}{2\chi(m^2 - k^2)} \right\}. \quad (5.42)$$

In a similar way

$$\begin{aligned} b &= \frac{-\chi + m^2(k^2 - p^2)(2m^2 - k^2 - p^2)}{2m\sqrt{-\chi}(m^2 - p^2)} \\ &= \frac{\sqrt{-\chi}}{2m} \left[\frac{\chi - m^2(k^2 - p^2)(2m^2 - k^2 - p^2)}{\chi(m^2 - p^2)} \right] \\ &\equiv \frac{\sqrt{-\chi}}{2m} \eta_2, \end{aligned} \quad (5.43)$$

where

$$\eta_2 = \left[\frac{\chi - m^2(k^2 - p^2)(2m^2 - k^2 - p^2)}{\chi(m^2 - p^2)} \right]. \quad (5.44)$$

Finally

$$J_0 = \left[-\eta_1 I \left(\frac{\eta_1 \sqrt{\chi}}{2} \right) + \eta_2 I \left(\frac{\eta_2 \sqrt{\chi}}{2} \right) \right]. \quad (5.45)$$

The Scalar Integral $K^{(0)}$

We have,

$$K^{(0)} = \int \frac{d^3 w}{[(p-w)^2 - m^2][(k-w)^2 - m^2]}. \quad (5.46)$$

Employing Feynman parametrization :

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}. \quad (5.47)$$

We take

$$\begin{aligned} a &= [(p-w)^2 - m^2] , \\ b &= [(k-w)^2 - m^2] . \end{aligned} \quad (5.48)$$

We define now

$$\begin{aligned} D &= z[(p-w)^2 - m^2] + (1-z)[(k-w)^2 - m^2] \\ &= z(p^2 - 2p \cdot w) + (1-z)(k^2 - 2k \cdot w) + w^2 - m^2 . \end{aligned} \quad (5.49)$$

We perform now the following change of variable

$$w \rightarrow w' = w - k(1-z) . \quad (5.50)$$

Then, we obtain

$$D = zp^2 - 2zp \cdot w' - 2p \cdot kz(1-z) + k^2z(1-z) + w'^2 - m^2 . \quad (5.51)$$

A second change of variable

$$w' \rightarrow w = w' - zp , \quad (5.52)$$

implies

$$\begin{aligned} D &= w^2 + (k-p)^2z(1-z) - m^2 \\ &= w^2 + q^2z(1-z) - m^2 . \end{aligned} \quad (5.53)$$

Therefore,

$$K^{(0)} = \int_0^1 dz \int d^3w \frac{1}{[w^2 + q^2z(1-z) - m^2]^2} . \quad (5.54)$$

Using eqs. (5.18), we arrive at :

$$\begin{aligned} K^{(0)} &= \int_0^1 dz (-1)^2 i\pi^{\frac{3}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(2)} [m^2 - q^2z(1-z)]^{-\frac{1}{2}} \\ &= i\pi^2 \int_0^1 dz [m^2 - q^2z(1-z)]^{-\frac{1}{2}} \\ &= \frac{2i\pi^2}{\sqrt{-q^2}} \arctan \sqrt{\frac{-q^2}{4m^2}} \\ &= i\pi^2 I\left(\frac{q}{2}\right) , \end{aligned} \quad (5.55)$$

which is our final expression.

The Tensor Integral $J_\mu^{(1)}$

We have

$$J_\mu^{(1)} = \int d^3w \frac{w_\mu}{w^2 [(p-w)^2 - m^2] [(k-w)^2 - m^2]} . \quad (5.56)$$

We write this integral in its most general form as :

$$J_\mu^{(1)} = \frac{i\pi^2}{2} \{k_\mu J_A(k, p) + p_\mu J_B(k, p)\} . \quad (5.57)$$

Contracting with k^μ and p^μ , we obtain the following system of equations :

$$\begin{aligned} k^\mu J_\mu^{(1)} &= \frac{i\pi^2}{2} [k^2 J_A + (k \cdot p) J_B] \\ p^\mu J_\mu^{(1)} &= \frac{i\pi^2}{2} [(k \cdot p) J_A + p^2 J_B] . \end{aligned} \quad (5.58)$$

On the other hand,

$$\begin{aligned} p^\mu J_\mu^{(1)} &= \frac{(p^2 - m^2)}{2} \int d^3w \frac{1}{w^2 [(p-w)^2 - m^2] [(k-w)^2 - m^2]} \\ &- \frac{1}{2} \int d^3w \frac{1}{[(k-w)^2 - m^2] w^2} \\ &+ \frac{1}{2} \int d^3w \frac{1}{[(p-w)^2 - m^2] [(k-w)^2 - m^2]} \\ &= \frac{(p^2 - m^2)}{2} J^{(0)} + \frac{1}{2} K^{(0)} - \frac{1}{2} I(0, 1, 1) . \end{aligned} \quad (5.59)$$

$I(0, 1, 1)$ will be defined as a Master Integral when we calculate $I^{(0)}$, and we use the relation :

$$p \cdot w = \frac{1}{2} (p^2 + w^2 - (p-w)^2 - m^2 + m^2) . \quad (5.60)$$

In a similar way, we also have that :

$$k^\mu J_\mu^{(1)} = \frac{(k^2 - m^2)}{2} J^{(0)} + \frac{1}{2} K^{(0)} - \frac{1}{2} I(1, 0, 1) . \quad (5.61)$$

We can now solve the system of equations (5.58). On solving, we find :

$$\begin{aligned} J_A(k, p) &= -\frac{2}{\Delta^2} \left\{ [p^2(k^2 - k \cdot p) - m^2(p^2 - k \cdot p)] \frac{J_0}{4} + k \cdot p I(k) \right. \\ &\quad \left. - p^2 I(p) + \frac{1}{2} (p^2 - k \cdot p) I(q/2) \right\} , \\ J_B(k, p) &= J_A(p, k) . \end{aligned} \quad (5.62)$$

Equations (5.57) and (5.62) form the final answer.

The Tensor Integral $J_{\mu\nu}^{(2)}$

We have

$$J_{\mu\nu}^{(2)} = \int d^3w \frac{w_\mu w_\nu}{w^2 [(p-w)^2 - m^2] [(k-w)^2 - m^2]} . \quad (5.63)$$

We write this integral in its most general form as :

$$\begin{aligned} J_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{g_{\mu\nu}}{3} K_0 + \left(k_\mu k_\nu - g_{\mu\nu} \frac{k^2}{3} \right) J_C + \left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu} \frac{2(k \cdot p)}{3} \right) J_D \right. \\ &\quad \left. + \left(p_\mu p_\nu - g_{\mu\nu} \frac{p^2}{3} \right) J_E \right\} . \end{aligned} \quad (5.64)$$

Contracting with p^μ ,

$$\begin{aligned} p^\mu J_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{p_\nu}{3} K_0 + \left((k \cdot p) k_\nu - p_\nu \frac{k^2}{3} \right) J_C \right. \\ &\quad \left. + \left(p^2 k_\nu + (k \cdot p) p_\nu - p_\nu \frac{2(k \cdot p)}{3} \right) J_D + \left(p^2 p_\nu - p_\nu \frac{p^2}{3} \right) J_E \right\} . \end{aligned} \quad (5.65)$$

Performing the remaining contractions we obtain :

$$p^\nu p^\mu J_{\mu\nu}^{(2)} = \frac{i\pi^3}{2} \left\{ \frac{p^2}{3} K_0 + \left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) J_C + \frac{4}{3} p^2 (k \cdot p) J_D + \frac{2}{3} p^4 J_E \right\} . \quad (5.66)$$

Other possible contractions are :

$$\begin{aligned} k^\nu p^\mu J_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{(k \cdot p)}{3} K_0 + \frac{2}{3} k^2 (k \cdot p) J_C + \left(k^2 p^2 + \frac{(k \cdot p)^2}{3} \right) J_D \right. \\ &\quad \left. + \frac{2}{3} p^2 (k \cdot p) J_E \right\} \end{aligned} \quad (5.67)$$

and

$$\begin{aligned} k^\mu J_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{k_\nu}{3} K_0 + \frac{2}{3} k^2 k_\nu J_C + \left((k \cdot p) k_\nu + k^2 p_\nu - \frac{2}{3} k_\nu (k \cdot p) \right) J_D \right. \\ &\quad \left. + \left((k \cdot p) p_\nu - k_\nu \frac{p^2}{3} \right) J_E \right\} , \end{aligned} \quad (5.68)$$

wich, after a second contraction, yield :

$$k^\nu k^\mu J_{\mu\nu}^{(2)} = \frac{i\pi^3}{2} \left\{ \frac{k^2}{3} K_0 + \frac{2}{3} k^4 J_C + \frac{4}{3} k^2 (k \cdot p) J_D + \left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) J_E \right\} \quad (5.69)$$

and

$$\begin{aligned} p^\nu k^\mu J_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{(k \cdot p)}{3} K_0 + \frac{2}{3} k^2 (k \cdot p) J_C + \left(k^2 p^2 + \frac{1}{3} (k \cdot p)^2 \right) J_D \right. \\ &\quad \left. + \frac{2}{3} p^2 (k \cdot p) J_E \right\} . \end{aligned} \quad (5.70)$$

On the other hand,

$$\begin{aligned}
p^\mu J_{\mu\nu}^{(2)} &= \int d^3w \frac{p \cdot w w_\nu}{w^2[(p-w)^2 - m^2][(k-w)^2 - m^2]} \\
&= \frac{(p^2 - m^2)}{2} \int d^3w \frac{w_\nu}{w^2[(p-w)^2 - m^2][(k-w)^2 - m^2]} \\
&+ \int d^3w \frac{w_\nu}{[(p-w)^2 - m^2][(k-w)^2 - m^2]} \\
&- \frac{1}{2} \int d^3w \frac{w_\nu}{w^2[(k-w)^2 - m^2]} \\
&= \frac{(p^2 - m^2)}{2} J_\nu^{(1)} + \frac{1}{2} A_\nu(k, p) - \frac{1}{2} E_\nu(k) .
\end{aligned} \tag{5.71}$$

Similarly

$$k^\mu J_{\mu\nu}^{(2)} = \frac{(k^2 - m^2)}{2} J_\nu^{(1)} + \frac{1}{2} A_\nu(k, p) - \frac{1}{2} E_\nu(p) . \tag{5.72}$$

The new integrals needed are $E_\nu(p)$ and $A_\nu(k, p)$. Before proceeding with the calculation of $J_{\mu\nu}^{(2)}$, let us pay attention to them. First,

$$E_\nu(p) = \int d^3w \frac{w_\nu}{w^2[(p-w)^2 - m^2]} . \tag{5.73}$$

Using Feynman Parametrization,

$$E_\nu(p) = \int_0^1 dx \int d^3w \frac{w_\nu}{[x[(p-w)^2 - m^2] + (1-x)w^2]^2} . \tag{5.74}$$

Let

$$\begin{aligned}
D &= x[(p-w)^2 - m^2] + (1-x)w^2 \\
&= w^2 + xp^2 - 2xp \cdot w - xm^2 .
\end{aligned} \tag{5.75}$$

We take now

$$w = w' + xp , \tag{5.76}$$

such that

$$D = w'^2 + p^2 x(1-x) - m^2 x , \tag{5.77}$$

and, consequently,

$$\begin{aligned}
E_\nu(p) &= \int_0^1 dx \int d^3w' \frac{w'_\nu + p_\nu x}{[w'^2 + p^2 x(1-x) - m^2 x]^2} \\
&= p_\nu \int_0^1 dx x (-1)^2 i\pi^{\frac{3}{2}} \frac{\Gamma(2 - \frac{3}{2})}{\Gamma(2)} [-p^2 x(1-x) + m^2 x]^{\frac{3}{2}-2} \\
&= i\pi^2 p_\nu \int_0^1 dx x^{\frac{1}{2}} [-p^2(1-x) + m^2]^{-\frac{1}{2}} \\
&= \frac{i\pi^2 p_\nu}{p^2} \left[m - \frac{(m^2 - p^2)}{\sqrt{-p^2}} \arctan \sqrt{\frac{-p^2}{m^2}} \right] \\
&\stackrel{k \leftrightarrow p}{=} E_\nu(k) .
\end{aligned} \tag{5.78}$$

The other integral we need to calculate is

$$A_\nu(k, p) = \int d^3w \frac{w_\nu}{[(p-w)^2 - m^2][(k-w)^2 - m^2]} . \quad (5.79)$$

After Feynman parametrization, we write :

$$A_\nu(k, p) = \int_0^1 dz \int d^3w \frac{w_\nu}{\{z[(p-w)^2 - m^2] + (1-z)[(k-w)^2 - m^2]\}^2} . \quad (5.80)$$

We define

$$D = z[(p-w)^2 - m^2] + (1-z)[(k-w)^2 - m^2] . \quad (5.81)$$

After the change of variable

$$w' = w - zp - (1-z)k , \quad (5.82)$$

we have

$$D = w'^2 + q^2 z(1-z) - m^2 , \quad (5.83)$$

and, therefore,

$$A_\nu(k, p) = \int_0^1 dz \int d^3w \frac{w'_\nu + p_\nu z + k_\nu(1-z)}{[w'^2 + q^2 z(1-z) - m^2]^2} . \quad (5.84)$$

On simplifying,

$$\begin{aligned} A_\nu(k, p) &= \int_0^1 dz i\pi^2 \frac{p_\nu z + k_\nu(1-z)}{\sqrt{m^2 - q^2 z(1-z)}} \\ &= i\pi^2 (k+p)_\nu \int_0^1 dz \frac{z}{\sqrt{m^2 - q^2 z(1-z)}} \\ &= \frac{i\pi^2 (k+p)_\nu}{\sqrt{-q^2}} \arctan \sqrt{\frac{-q^2}{4m^2}} \equiv \frac{1}{2} K^{(0)}(k+p)_\nu . \end{aligned} \quad (5.85)$$

Let us turn back to the claculation of $J_{\mu\nu}^{(2)}$. Second contraction yields :

$$\begin{aligned} p^\nu p^\mu J_{\mu\nu}^{(2)} &= \frac{(p^2 - m^2)}{2} \frac{i\pi^3}{2} [(k \cdot p) J_A + p^2 J_B] + \frac{1}{2} p^\nu A_\nu - \frac{p^\nu}{2} E_\nu(k) \\ k^\nu p^\mu J_{\mu\nu}^{(2)} &= \frac{(p^2 - m^2)}{2} \frac{i\pi^3}{2} [k^2 J_A + (k \cdot p) J_B] + \frac{1}{2} k^\nu A_\nu - \frac{k^\nu}{2} E_\nu(k) \\ k^\nu k^\mu J_{\mu\nu}^{(2)} &= \frac{(k^2 - m^2)}{2} \frac{i\pi^3}{2} [k^2 J_A + (k \cdot p) J_B] + \frac{1}{2} k^\nu A_\nu - \frac{k^\nu}{2} E_\nu(p) \\ p^\nu k^\mu J_{\mu\nu}^{(2)} &= \frac{(k^2 - m^2)}{2} \frac{i\pi^3}{2} [(k \cdot p) J_A + p^2 J_B] + \frac{1}{2} p^\nu A_\nu - \frac{p^\nu}{2} E_\nu(p) . \end{aligned} \quad (5.86)$$

We arrive then to the following system of equations :

$$\left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) J_C + \frac{4}{3} p^2 (k \cdot p) J_D + \frac{2}{3} p^4 J_E = a$$

$$\begin{aligned}
\frac{2}{3}k^4 J_C + \frac{4}{3}k^2(k \cdot p)J_D + \left((k \cdot p)^2 - \frac{k^2 p^2}{3}\right) J_E &= b \\
\frac{2}{3}k^2(k \cdot p)J_C + \left(k^2 p^2 + \frac{(k \cdot p)^2}{3}\right) J_D + \frac{2}{3}p^2(k \cdot p)J_E &= c, \quad (5.87)
\end{aligned}$$

where

$$\begin{aligned}
a &= \frac{2}{i\pi^3} p^\mu p^\nu J_{\mu\nu}^{(2)} - \frac{p^2}{3} K_0 \\
b &= \frac{2}{i\pi^3} k^\mu k^\nu J_{\mu\nu}^{(2)} - \frac{k^2}{3} K_0 \\
c &= \frac{1}{i\pi^3} [k^\nu p^\mu J_{\mu\nu}^{(2)} + p^\nu k^\mu J_{\mu\nu}^{(2)} - \frac{(k \cdot p)}{3} K_0]. \quad (5.88)
\end{aligned}$$

Solutions to this system are :

$$\begin{aligned}
J_C(k, p) &= \frac{1}{\Delta^2} \left\{ [p^2(k \cdot p - 2k^2) - m^2(k \cdot p - 2p^2)] \frac{J_A}{2} - p^2(p^2 - m^2) \frac{J_B}{2} \right. \\
&\quad \left. + \frac{k \cdot p}{k^2} (m^2 - k^2) I(k) + \frac{1}{2} (k \cdot p + p^2) I(q/2) - m \frac{k \cdot p}{k^2} \right\}, \\
J_D(k, p) &= \frac{1}{2\Delta^2} \left\{ [k^2(3k \cdot p - p^2) - m^2(3k \cdot p - k^2)] \frac{J_A}{2} \right. \\
&\quad \left. + [p^2(3k \cdot p - k^2) - m^2(3k \cdot p - p^2)] \frac{J_B}{2} \right. \\
&\quad \left. - (m^2 - k^2)I(k) - (m^2 - p^2)I(p) - \frac{1}{2}(k + p)^2 I(q/2) + 2m \right\}, \\
J_E(k, p) &= J_C(p, k), \quad (5.89)
\end{aligned}$$

which completes or calculation of $J_{\mu\nu}^{(2)}$.

The Scalar Integral $I^{(0)}$

We have

$$I^{(0)} = \int d^3w \frac{1}{w^4 [(p-w)^2 - m^2][(k-w)^2 - m^2]}. \quad (5.90)$$

In general, let us define the family of integrals :

$$I(\nu_1, \nu_2, \nu_3) = \int d^d w \frac{1}{[(p-w)^2 - m^2]^{\nu_1} [(k-w)^2 - m^2]^{\nu_2} [w^2]^{\nu_3}}. \quad (5.91)$$

We use the following Integration by Parts (IBP) identity :

$$\int d^d w \frac{\partial}{\partial w_\mu} \left[\frac{(q_i - w)_\mu}{[(p-w)^2 - m^2]^{\nu_1} [(k-w)^2 - m^2]^{\nu_2} [w^2]^{\nu_3}} \right] = 0. \quad (5.92)$$

After the differentiation we obtain :

$$\begin{aligned} & \int d^d w (q_i - m) \left[\frac{2\nu_1(p-w)^\mu}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{2\nu_2(k-w)^\mu}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} - \frac{2\nu_2 w^\mu}{A^{\nu_1} B^{\nu_2}} C^{\nu_3+1} \right] \\ & + \int d^d w \frac{(-d)}{A^{\nu_1} B^{\nu_2} C^{\nu_3}} = 0 , \end{aligned} \quad (5.93)$$

since

$$\frac{\partial}{\partial w_\mu} w_\mu = g_\mu^\mu = d . \quad (5.94)$$

We have defined

$$A = (p-w)^2 - m^2 , \quad B = (k-w)^2 - m^2 , \quad C = w^2 . \quad (5.95)$$

Therefore,

$$\begin{aligned} dI(\nu_1, \nu_2, \nu_3) &= \int d^d w (q_i - w) \\ & \left[\frac{2\nu_1(p-w)^\mu}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{2\nu_2(k-w)^\mu}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} - \frac{2\nu_2 w^\mu}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right] . \end{aligned} \quad (5.96)$$

Assuming that q_i is defined such that $q_1 = p$, $q_2 = k$ and $q_3 = 0$. Then, for $i = 0$:

$$\begin{aligned} dI(\nu_1, \nu_2, \nu_3) &= \int d^d w (p-w) \\ & \left[\frac{2\nu_1(p-w)^\mu}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{2\nu_2(k-w)^\mu}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} - \frac{2\nu_2 w^\mu}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right] . \end{aligned} \quad (5.97)$$

Let us observe the following products :

1.

$$\begin{aligned} 2(p-w) \cdot (k-w) &= (p-w)^2 + (k-w)^2 - (p-k)^2 - m^2 - m^2 + 2m^2 \\ &= A + B - [(p-k)^2 - 2m^2] . \end{aligned} \quad (5.98)$$

2.

$$2(p-w) \cdot (-w) = (p-w)^2 - m^2 + w^2 - p^2 + m^2 = A + C - p^2 + m^2 . \quad (5.99)$$

3.

$$(p-w)^2 = (p-w)^2 - m^2 + m^2 = A + m^2 . \quad (5.100)$$

Substituting these products into eq. (5.97), we obtain :

$$\begin{aligned}
dI(\nu_1, \nu_2, \nu_3) &= \int d^d w \left[\frac{2\nu_1[A + m^2]}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{\nu_2[A + B - [(p-k)^2 - 2m^2]]}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} \right. \\
&\quad \left. + \frac{\nu_3[A + C - (p^2 - m^2)]}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right] \\
&= \int d^d w \left[\frac{2\nu_1 + \nu_2 + \nu_3}{A^{\nu_1} B^{\nu_2} C^{\nu_3}} + \frac{\nu_2}{A^{\nu_1-1} B^{\nu_2+1} C^{\nu_3}} + \frac{2\nu_1 m^2}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} \right. \\
&\quad \left. - \frac{\nu_2[(p-k)^2 - 2m^2]}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} + \frac{\nu_3}{A^{\nu_1-1} B^{\nu_2} C^{\nu_3+1}} - \frac{\nu_3(p^2 - m^2)}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right] \\
&= (2\nu_1 + \nu_2 + \nu_3)I(\nu_1, \nu_2, \nu_3) + \nu_2 I(\nu_1 - 1, \nu_2 + 2, \nu_3) \\
&\quad + 2m^2 \nu_1 I(\nu_1 + 1, \nu_2, \nu_3) - \nu_2 [(p-k)^2 - 2m^2] I(\nu_1, \nu_2 + 1, \nu_3) \\
&\quad + \nu_3 I(\nu_1 - 1, \nu_2, \nu_3 + 1) - \nu_3 (p^2 - m^2) I(\nu_1, \nu_2, \nu_3 + 1) \quad (5.101)
\end{aligned}$$

Rearranging the terms in such a way that those integrals with $\nu_1 + \nu_2 + \nu_3$ as the sum of their arguments are on the right hand side, while those with $\nu_1 + \nu_2 + \nu_3 + 1$ on the left hand side, we get :

$$\begin{aligned}
&-2\nu_1 m^2 I(\nu_1 + 1, \nu_2, \nu_3) + \nu_2 [(p-k)^2 - 2m^2] I(\nu_1, \nu_2 + 1, \nu_3) \\
&+ \nu_3 (p^2 - m^2) I(\nu_1, \nu_2, \nu_3 + 1) = (2\nu_1 + \nu_2 + \nu_3 - d) I(\nu_1, \nu_2, \nu_3) \\
&+ \nu_2 I(\nu_1 - 1, \nu_2 + 1, \nu_3) + \nu_3 I(\nu_1 - 1, \nu_2, \nu_3 + 1) . \quad (5.102)
\end{aligned}$$

For $i = 2$ we have :

$$\begin{aligned}
dI(\nu_1, \nu_2, \nu_3) &= \int d^d w (k - w)_\mu \\
&\quad \left[\frac{2\nu_1 (p - w)^\mu}{A^{\nu_1+1} B^{\nu_2} C^{\nu_3}} + \frac{2\nu_2 (k - w)^\mu}{A^{\nu_1} B^{\nu_2+1} C^{\nu_3}} - \frac{2\nu_3 w^\mu}{A^{\nu_1} B^{\nu_2} C^{\nu_3+1}} \right] \\
&= (\nu_1 + 2\nu_2 + \nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2 - 1, \nu_3) \\
&\quad + \nu_3 I(\nu_1, \nu_2 - 1, \nu_3 + 1) \\
&\quad - [(p-k)^2 - 2m^2] \nu_1 I(\nu_1 + 1, \nu_2, \nu_3) \\
&\quad + 2\nu_2 m^2 I(\nu_1, \nu_2 + 1, \nu_3) \\
&\quad - \nu_3 (k^2 - m^2) I(\nu_1, \nu_2, \nu_3 + 1) \quad (5.103)
\end{aligned}$$

or

$$\begin{aligned}
&\nu_1 [(p-k)^2 - 2m^2] I(\nu_1 + 1, \nu_2, \nu_3) - 2\nu_2 m^2 I(\nu_1, \nu_2 + 1, \nu_3) \\
&+ \nu_3 (k^2 - m^2) I(\nu_1, \nu_2, \nu_3 + 1) = (\nu_2 + 2\nu_2 + \nu_3 - d) I(\nu_1, \nu_2, \nu_3) \\
&+ \nu_1 I(\nu_1 + 1, \nu_2 - 1, \nu_3) + \nu_3 I(\nu_1, \nu_2, \nu_3 + 1) , \quad (5.104)
\end{aligned}$$

and for $i = 3$:

$$dI(\nu_1, \nu_2, \nu_3) = \int d^d w (-w_\mu)$$

$$\begin{aligned}
& \left[\frac{2\nu_1(p-w)^\mu}{A^{\nu_1+1}B^{\nu_2}C^{\nu_3}} + \frac{2\nu_2(k-w)^\mu}{A^{\nu_1}B^{\nu_2+1}C^{\nu_3}} - \frac{2\nu_3w^\mu}{A^{\nu_1}B^{\nu_2}C^{\nu_3+1}} \right] \\
&= (\nu_1 + \nu_2 + 2\nu_3)I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2, \nu_3 - 1) \\
&\quad + \nu_2 I(\nu_1 + \nu_2 + 1, \nu_3 - 1) \\
&\quad - \nu_1(p^2 - m^2)I(\nu_1 + 1, \nu_2, \nu_3) \\
&\quad - \nu_2(k^2 - m^2)I(\nu_1, \nu_2 + 1, \nu_3)
\end{aligned} \tag{5.105}$$

or

$$\begin{aligned}
& \nu_1(p^2 - m^2)I(\nu_1 + 1, \nu_2, \nu_3) + \nu_2(k^2 - m^2)I(\nu_1, \nu_2 + 1, \nu_3) = \\
& (\nu_1 + \nu_2 + 2\nu_3 - d)I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2, \nu_3 - 1) \\
& + \nu_2 I(\nu_1, \nu_2 + 1, \nu_3 - 1) .
\end{aligned} \tag{5.106}$$

Therefore, we need to solve the system of equations :

$$\begin{aligned}
& -2\nu_1 m^2 I(\nu_1 + 1, \nu_2, \nu_3) + \nu_2(q^2 - 2m^2)I(\nu_1 + \nu_2 + 1, \nu_3) \\
& \quad + \nu_3(p^2 - m^2)I(\nu_1, \nu_2, \nu_3 + 1) = a \\
& \nu_1(q^2 - 2m^2)I(\nu_1 + 1, \nu_2, \nu_3) - 2\nu_2 m^2 I(\nu_1, \nu_2 + 1, \nu_3) \\
& \quad + \nu_3(k^2 - m^2)I(\nu_1, \nu_2, \nu_3 + 1) = b \\
& \nu_1(p^2 - m^2)I(\nu_1 + 1, \nu_2, \nu_3) + \nu_2(k^2 - m^2)I(\nu_1, \nu_2 + 1, \nu_3) \\
& \quad + 0I(\nu_1, \nu_2, \nu_3 + 1) = c ,
\end{aligned} \tag{5.107}$$

where

$$\begin{aligned}
a &= (2\nu_1 + \nu_2 + \nu_3 - d)I(\nu_1 + \nu_2 + \nu_3) + \nu_2 I(\nu_1 - 1, \nu_2 + 1, \nu_3) \\
&\quad + \nu_3 I(\nu_1 - 1, \nu_2, \nu_3 + 1) \\
b &= (\nu_1 + 2\nu_2 + \nu_3 - d)I(\nu_1 + \nu_2 + \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2 - 1, \nu_3) \\
&\quad + \nu_3 I(\nu_1, \nu_2 - 1, \nu_3 + 1) \\
c &= (\nu_1 + \nu_2 + 2\nu_3 - d)I(\nu_1 + \nu_2 + \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2, \nu_3 - 1) \\
&\quad + \nu_2 I(\nu_1, \nu_2 + 1, \nu_3 - 1) .
\end{aligned} \tag{5.108}$$

In obvious notation :

$$\begin{pmatrix} -2m^2\nu_1 & \nu_2(q^2 - 2m^2) & \nu_3(p^2 - m^2) \\ \nu_1(q^2 - 2m^2) & -2m^2\nu_2 & \nu_3(k^2 - m^2) \\ \nu_1(p^2 - m^2) & \nu_2(k^2 - m^2) & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} , \tag{5.109}$$

which formally we can write as :

$$SI = A . \tag{5.110}$$

Then,

$$|S| = 2\nu_1\nu_2\nu_3[m^2(k^2 - m^2)^2 + m^2(p^2 - m^2)^2 + (k^2 - m^2)(p^2 - m^2)(q^2 - 2m^2)] , \tag{5.111}$$

and besides

$$\begin{aligned}
|S| I_3 &= \nu_1 \nu_2 [(q^2 - 2m^2)(k^2 - m^2) + 2m^2(p^2 - m^2)]a \\
&\quad + \nu_1 \nu_2 [(q^2 - 2m^2)(p^2 - m^2) + 2m^2(k^2 - m^2)]b \\
&\quad + \nu_1 \nu_2 [4m^4 - (q^2 - 2m^2)^2]c .
\end{aligned} \tag{5.112}$$

For $\nu_1 = \nu_2 = \nu_3 = 1$,

$$\begin{aligned}
a &= I(1, 1, 1) + I(0, 2, 1) + I(0, 1, 2) \\
b &= I(1, 1, 1) + I(2, 0, 1) + I(1, 0, 2) \\
c &= I(1, 1, 1) + I(2, 1, 0) + I(1, 2, 0)
\end{aligned} \tag{5.113}$$

with $I^{(0)} = I(1, 1, 2)$ and $J^{(0)} = I(1, 1, 1)$. So, we need to calculate the integrals $I(0, 2, 1) \stackrel{k \leftrightarrow p}{=} I(2, 0, 1)$, $I(0, 1, 2) \stackrel{k \leftrightarrow p}{=} I(1, 0, 2)$ and $I(2, 1, 0) \stackrel{k \leftrightarrow p}{=} I(1, 2, 0)$. For that purpose, we define the integral

$$I_{pn}(k, p, m_1, m_2) = \int d^d w \frac{1}{[(k-w)^2 - m_1^2]^p [(p-w)^2 - m_2^2]^n} . \tag{5.114}$$

Using Feynman parametrization, we know that

$$\frac{1}{A^n B^p} = \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)} \int_0^1 dx x^{n-1} (1-x)^{p-1} \frac{1}{[xA + (1-x)B]^{n+p}} . \tag{5.115}$$

Let

$$\begin{aligned}
A &= [(p-w)^2 - m^2] , \\
B &= [(k-w)^2 - m^2] ,
\end{aligned} \tag{5.116}$$

and

$$\begin{aligned}
D &= xA + (1-x)B \\
&= x[p^2 + w^2 - 2p \cdot w - m_2^2] + (1-x)[k^2 + w^2 - 2k \cdot w - m_1^2] \\
&= w^2 - 2w \cdot [px + k(1-x)] + p^2 x + k^2(1-x) - m_2^2 x - m_1^2(1-x) .
\end{aligned} \tag{5.117}$$

We make the change of variable

$$w' = w - [px + k(1-x)] , \tag{5.118}$$

such that

$$w^2 - 2w \cdot [px + k(1-x)] = w'^2 - [px + k(1-x)]^2 . \tag{5.119}$$

In this way, D is rewritten as :

$$\begin{aligned}
D &= w'^2 - [px + k(1-x)]^2 + p^2 x + k^2(1-x) - m_2^2 x - m_1^2(1-x) \\
&= w'^2 + p^2 x(1-x) + k^2 x(1-x) - 2k \cdot px(1-x) - m_2^2 x - m_1^2(1-x) \\
&= w'^2 + q^2 x(1-x) - m_2^2 x - m_1^2(1-x) .
\end{aligned} \tag{5.120}$$

Therefore,

$$\begin{aligned}
I_{pn}(k, p, m_1, m_2) &= \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)} \int_0^1 dx x^{n-1} (1-x)^{p-1} \int d^d w' \frac{1}{D^{n+p}} \\
&= \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(p)} \int_0^1 dx x^{n-1} (1-x)^{p-1} (-1)^{n+p} i \pi^{d/2} \frac{\Gamma(n+p-d/2)}{\Gamma(n+p)} s^{\frac{d}{2}-n-p} \\
&= i(-1)^{n+p} \pi^{d/2} \frac{\Gamma(n+p-d/2)}{\Gamma(n)\Gamma(p)} \int_0^1 dx x^{n-1} (1-x)^{p-1} s^{\frac{d}{2}-n-p}, \quad (5.121)
\end{aligned}$$

with

$$s = -q^2 x(1-x) + m_2^2 x + m_1^2 (1-x). \quad (5.122)$$

For the integrals of our interest, we help ourselves with

$$\begin{aligned}
I_{11}(k, p, m_1, m_2) &= \int \frac{d^d w}{[(p-w)^2 - m_2^2][(k-w)^2 - m_1^2]} \\
&= i \pi^{\frac{d}{2}} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx s^{\frac{d}{2}-2} \\
&= i \pi^2 \int_0^1 dx s^{-\frac{1}{2}}, \quad (5.123)
\end{aligned}$$

where, in the last line, we set $d = 3$. So

$$I(2, 1, 0) = \frac{1}{2m_1} \frac{\partial}{\partial m_1} I_{11}(k, p, m_1, m_2) \Big|_{m_1=m_2=m}. \quad (5.124)$$

Now,

$$\frac{\partial}{\partial m_1} s^{-\frac{1}{2}} = -m_1 (1-x) s^{-\frac{3}{2}}, \quad (5.125)$$

and therefore,

$$\begin{aligned}
I(2, 1, 0) &= -\frac{i\pi^2}{2} \int_0^1 dx \frac{(1-x)}{[-q^2 x(1-x) + m^2]^{\frac{3}{2}}} \\
&= \frac{-i\pi^2}{m[4m^2 - q^2]} \\
&\stackrel{k \leftrightarrow p}{=} I(1, 2, 0). \quad (5.126)
\end{aligned}$$

On the other hand

$$\begin{aligned}
I(2, 0, 1) &= \frac{1}{2m_1} \frac{\partial}{\partial m_1} I_{11}(k, p, m_1, m_2) \Big|_{k=m_1=0, m_2=m} \\
&= -\frac{i\pi^2}{2} \int_0^1 dx \frac{x}{[-p^2 x(1-x) + m^2 x]^{\frac{3}{2}}} \\
&= -\frac{i\pi^2}{m(m^2 - p^2)} \\
&\stackrel{k \leftrightarrow p}{=} I(0, 2, 1). \quad (5.127)
\end{aligned}$$

To solve $I(1, 0, 2)$, we take the integral

$$I^\nu = \int d^d w \frac{w^\nu}{w^4[(p-w)^2 - m^2]} . \quad (5.128)$$

Then,

$$\begin{aligned} p_\nu I^\nu &= \frac{1}{2}(p^2 - m^2) \int d^d w \frac{1}{w^4[(p-w)^2 - m^2]} \\ &\quad + \frac{1}{2} \int d^d w \frac{1}{w^2[(p-w)^2 - m^2]} - \frac{1}{2} \int \frac{d^d w}{w^4} \\ &= \frac{1}{2}(p^2 - m^2)I(1, 0, 2) + \frac{1}{2}I(1, 0, 1) , \end{aligned} \quad (5.129)$$

since the last term vanishes in the dimensional regularization scheme. It is easy to see that

$$\begin{aligned} I(1, 0, 1) &= I_{11}(k, p, m_1, m_2) \Big|_{k=m_1=0, m_2=m} \\ &= i\pi^2 \int_0^1 [-p^2 x(1-x) + m^2 x]^{-\frac{1}{2}} \\ &= \frac{2i\pi^2}{\sqrt{-p^2}} \arctan \sqrt{\frac{-p^2}{m^2}} \\ &\stackrel{k \leftrightarrow p}{=} I(0, 1, 1) . \end{aligned} \quad (5.130)$$

To calculate explicitly I^ν , we take in eq. (5.115)

$$A = [(p-w)^2 - m^2], \quad B = w^2, \quad n = 1, \quad p = 2 . \quad (5.131)$$

So, in three dimensions,

$$I^\nu = \frac{\Gamma(3)}{\Gamma(1)\Gamma(2)} \int_0^1 dx (1-x) \int d^3 w w^\nu \frac{1}{[Ax + B(1-x)]^3} . \quad (5.132)$$

Let

$$w' = w - px , \quad (5.133)$$

Then,

$$\begin{aligned} D &= Ax + B(1-x) \\ &= w^2 + p^2 x - 2p \cdot wx - xm^2 \\ &= w'^2 + p^2 x - p^2 x^2 - m^2 x \\ &= w'^2 + p^2 x(1-x) - m^2 x , \end{aligned} \quad (5.134)$$

and therefore

$$I^\nu = 2 \int_0^1 dx (1-x) \int (d^3 w) \frac{p^\nu x}{[w^2 + p^2 x(1-x) - m^2 x]^3} . \quad (5.135)$$

Using now (5.18),

$$\begin{aligned}
I^\nu &= 2p^\nu \int_0^1 dx x(1-x)(-1)^3 i\pi^{\frac{3}{2}} \frac{\Gamma(\frac{3}{2})}{\Gamma(3)} [-p^2 x(1-x) + m^2 x]^{\frac{3}{2}} \\
&= -\frac{i\pi^2}{2} p^\nu \int_0^1 dx x^{-\frac{1}{2}} (1-x) [-p^2(1-x) + m^2]^{-\frac{3}{2}} \\
&= -\frac{i\pi^2}{2} p^\nu \left[\int_0^1 dx \left\{ x^{-\frac{1}{2}} - x^{\frac{1}{2}} \right\} [-p^2(1-x) + m^2]^{-\frac{3}{2}} \right] \\
&= -\frac{i\pi^2}{2} [-p^2]^{-\frac{3}{2}} p^\nu \left[-\frac{2m^2}{m^2 - p^2} \sqrt{-\frac{p^2}{m^2}} + 2 \arctan \sqrt{\frac{-p^2}{m^2}} \right]. \quad (5.136)
\end{aligned}$$

So that

$$\begin{aligned}
I(1, 0, 2) &= \frac{2}{p^2 - m^2} \left[p_\nu I^\nu - \frac{1}{2} I(1, 0, 1) \right] \\
&= \frac{2m}{(m^2 - p^2)^2} i\pi^2 \\
&\stackrel{k \leftrightarrow p}{=} I(0, 1, 2). \quad (5.137)
\end{aligned}$$

Finally, substituting into (5.112)

$$I^{(0)} = \frac{1}{\chi} \left\{ q^2(m^2 + k \cdot p) J^{(0)} + i\pi^2 m L \right\}, \quad (5.138)$$

where

$$\begin{aligned}
L &= \frac{q^2(k^2 - m^2) - (k^2 - p^2)(k^2 + m^2)}{(k^2 - m^2)^2} \\
&\quad + \frac{q^2(p^2 - m^2) + (k^2 - p^2)(p^2 + m^2)}{(p^2 - m^2)^2}. \quad (5.139)
\end{aligned}$$

This is our final expression.

The Tensor Integral $I_\mu^{(1)}$

We have

$$I_\mu^{(1)} = \int_M d^3 w \frac{w_\mu}{w^4 [(p-w)^2 - m^2] [(k-w)^2 - m^2]}. \quad (5.140)$$

We write this integral in its most general form as :

$$I_\mu^{(1)} = \frac{i\pi^2}{2} [k_\mu I_A(k, p) + p_\mu I_B(k, p)]. \quad (5.141)$$

Contracting with k^μ and p^μ we obtain the following system of equations :

$$\begin{aligned}
k^\mu I_\mu^{(1)} &= \frac{i\pi^2}{2} [k^2 I_A + (k \cdot p) I_B] \\
p^\mu I_\mu^{(1)} &= \frac{i\pi^2}{2} [(k \cdot p) I_A + p^2 I_B]. \quad (5.142)
\end{aligned}$$

On the other hand

$$\begin{aligned}
p^\mu I_\mu^{(1)} &= \frac{(p^2 - m^2)}{2} \int d^3w \frac{1}{w^4 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\
&- \frac{1}{2} \int d^3w \frac{1}{[(k-w)^2 - m^2]w^4} \\
&+ \frac{1}{2} \int d^3w \frac{1}{w^2 [(p-w)^2 - m^2][(k-w)^2 - m^2]} \\
&= \frac{(p^2 - m^2)}{2} I^{(0)} + \frac{1}{2} J^{(0)} - \frac{1}{2} I(0, 1, 2) ,
\end{aligned} \tag{5.143}$$

where we have used :

$$p \cdot w = \frac{1}{2}(p^2 + w^2 - (p-w)^2 - m^2 + m^2) . \tag{5.144}$$

We also have that :

$$k^\mu I_\mu^{(1)} = \frac{(k^2 - m^2)}{2} I^{(0)} + \frac{1}{2} J^{(0)} - \frac{1}{2} I(1, 0, 2) . \tag{5.145}$$

We solve the system of equations (5.142), to find :

$$\begin{aligned}
I_A(k, p) &= \frac{2}{\Delta^2} \left\{ [k \cdot p(p^2 - m^2) - p^2(k^2 - m^2)] \frac{I_0}{4} + p \cdot q \frac{J_0}{4} \right. \\
&+ \left. \frac{mp^2}{(m^2 - p^2)^2} - \frac{mk \cdot p}{(m^2 - k^2)^2} \right\} , \\
I_B(k, p) &= I_A(p, k) .
\end{aligned} \tag{5.146}$$

Equations (5.141) and (5.146) form the complete solution.

The Tensor Integral $I_{\mu\nu}^{(2)}$

We have,

$$I_{\mu\nu}^{(2)} = \int_M d^3w \frac{w_\mu w_\nu}{w^4 [(p-w)^2 - m^2][(k-w)^2 - m^2]} . \tag{5.147}$$

We express this integral in its most general form as :

$$\begin{aligned}
I_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{g_{\mu\nu}}{3} J_0 + \left(k_\mu k_\nu - g_{\mu\nu} \frac{k^2}{3} \right) I_C + \left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu} \frac{2(k \cdot p)}{3} \right) I_D \right. \\
&+ \left. \left(p_\mu p_\nu - g_{\mu\nu} \frac{p^2}{3} \right) I_E \right\} .
\end{aligned} \tag{5.148}$$

Contracting with p^μ we obtain :

$$\begin{aligned}
p^\mu I_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{p_\nu J_0}{3} + \left((k \cdot p) k_\nu - p_\nu \frac{k^2}{3} \right) I_C \right. \\
&+ \left. \left(p^2 k_\nu + (k \cdot p) p_\nu - p_\nu \frac{2(k \cdot p)}{3} \right) I_D + \left(p^2 p_\nu - p_\nu \frac{p^2}{3} \right) I_E \right\} .
\end{aligned} \tag{5.149}$$

Performing the remaining contractions,

$$p^\nu p^\mu I_{\mu\nu}^{(2)} = \frac{i\pi^3}{2} \left\{ \frac{p^2}{3} J_0 + \left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) I_C + \frac{4}{3} p^2 (k \cdot p) I_D + \frac{2}{3} p^4 I_E \right\} \quad (5.150)$$

and

$$\begin{aligned} k^\nu p^\mu I_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{(k \cdot p)}{3} J_0 + \left(\frac{2}{3} k^2 (k \cdot p) \right) I_C + \left(k^2 p^2 + \frac{(k \cdot p)^2}{3} \right) I_D \right. \\ &\quad \left. + \frac{2}{3} p^2 (k \cdot p) I_E \right\} . \end{aligned} \quad (5.151)$$

In the same way,

$$\begin{aligned} k^\mu I_{\mu\nu}^{(2)} &= \frac{i\pi^3}{2} \left\{ \frac{k_\nu}{3} J_0 + \frac{2}{3} k^2 k_\nu I_C + \left((k \cdot p) k_\nu + k^2 p_\nu - \frac{2}{3} k_\nu (k \cdot p) \right) I_D \right. \\ &\quad \left. + \left((k \cdot p) p_\nu - k_\nu \frac{p^2}{3} \right) I_E \right\} , \end{aligned} \quad (5.152)$$

from where, after a second contraction, we obtain :

$$k^\nu k^\mu I_{\mu\nu}^{(2)} = \frac{i\pi^3}{2} \left\{ \frac{k^2}{3} J_0 + \frac{2}{3} k^4 I_C + \frac{4}{3} k^2 (k \cdot p) I_D + \left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) I_E \right\} \quad (5.153)$$

and

$$\begin{aligned} p^\nu k^\mu I_{\mu\nu}^{(2)} &= \frac{p\pi^3}{2} \left\{ \frac{(k \cdot p)}{3} J_0 + \frac{2}{3} k^2 (k \cdot p) I_C + \left(k^2 p^2 + \frac{1}{3} (k \cdot p)^2 \right) I_D \right. \\ &\quad \left. + \frac{2}{3} p^2 (k \cdot p) I_E \right\} . \end{aligned} \quad (5.154)$$

On the other hand,

$$\begin{aligned} p^\mu I_{\mu\nu}^{(2)} &= \int d^3 w \frac{p \cdot w w_\nu}{w^4 [(p-w)^2 - m^2] [(k-w)^2 - m^2]} \\ &= \frac{(p^2 - m^2)}{2} \int d^3 w \frac{w_\nu}{w^4 [(p-w)^2 - m^2] [(k-w)^2 - m^2]} \\ &\quad + \int d^3 w \frac{w_\nu}{w^2 [(p-w)^2 - m^2] [(k-w)^2 - m^2]} \\ &\quad - \frac{1}{2} \int d^3 w \frac{w_\nu}{w^4 [(k-w)^2 - m^2]} \\ &= \frac{(p^2 - m^2)}{2} I_\nu^{(1)} + \frac{1}{2} J_\nu^{(1)} - \frac{1}{2} E_\nu(k) . \end{aligned} \quad (5.155)$$

Similarly,

$$k^\mu I_{\mu\nu}^{(2)} = \frac{(k^2 - m^2)}{2} I_\nu^{(1)} + \frac{1}{2} J_\nu^{(1)} - \frac{1}{2} E_\nu(p) . \quad (5.156)$$

From the second contraction we obtain :

$$\begin{aligned}
p^\nu p^\mu I_{\mu\nu}^{(2)} &= \frac{(p^2 - m^2)}{2} \frac{i\pi^3}{2} [(k \cdot p) I_A + p^2 I_B] + \frac{1}{2} p^\nu J_\nu^{(1)} - \frac{p^\nu}{2} E_\nu(k) \\
k^\nu p^\mu I_{\mu\nu}^{(2)} &= \frac{(p^2 - m^2)}{2} \frac{i\pi^3}{2} [k^2 I_A + (k \cdot p) I_B] + \frac{1}{2} k^\nu J_\nu^{(1)} - \frac{k^\nu}{2} E_\nu(k) \\
k^\nu k^\mu I_{\mu\nu}^{(2)} &= \frac{(k^2 - m^2)}{2} \frac{i\pi^3}{2} [k^2 I_A + (k \cdot p) I_B] + \frac{1}{2} k^\nu J_\nu^{(1)} - \frac{k^\nu}{2} E_\nu(p) \\
p^\nu k^\mu I_{\mu\nu}^{(2)} &= \frac{(k^2 - m^2)}{2} \frac{i\pi^3}{2} [(k \cdot p) I_A + p^2 I_B] + \frac{1}{2} p^\nu J_\nu^{(1)} - \frac{p^\nu}{2} E_\nu(p) . \quad (5.157)
\end{aligned}$$

In this way, we arrive at the following system of equations :

$$\begin{aligned}
\left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) I_C + \frac{4}{3} p^2 (k \cdot p) I_D + \frac{2}{3} p^4 I_E &= a \\
\frac{2}{3} k^4 I_C + \frac{4}{3} k^2 (k \cdot p) I_D + \left((k \cdot p)^2 - \frac{k^2 p^2}{3} \right) I_E &= b \\
\frac{2}{3} k^2 (k \cdot p) I_C + \left(k^2 p^2 + \frac{(k \cdot p)^2}{3} \right) I_D + \frac{2}{3} p^2 (k \cdot p) I_E &= c , \quad (5.158)
\end{aligned}$$

where

$$\begin{aligned}
a &= \frac{2}{i\pi^3} p^\mu p^\nu I_{\mu\nu}^{(2)} - \frac{p^2}{3} J_0 \\
b &= \frac{2}{i\pi^3} k^\mu k^\nu I_{\mu\nu}^{(2)} - \frac{k^2}{3} J_0 \\
c &= \frac{1}{i\pi^3} \left[k^\nu p^\mu I_{\mu\nu}^{(2)} + p^\nu k^\mu I_{\mu\nu}^{(2)} - \frac{(k \cdot p)}{3} J_0 \right] . \quad (5.159)
\end{aligned}$$

Solution to the system are :

$$\begin{aligned}
I_C(k, p) &= \frac{1}{\Delta^2} \left\{ p^2 J_0 + [p^2 (k \cdot p - 2k^2) - m^2 (k \cdot p - 2p^2)] \frac{I_A}{2} \right. \\
&\quad \left. - p^2 (p^2 - m^2) \frac{I_B}{2} + (k \cdot p - 2p^2) \frac{J_A}{2} - p^2 \frac{J_B}{2} - \frac{k \cdot p}{k^2} I(k) \right. \\
&\quad \left. + \frac{mk \cdot p}{k^2 (m^2 - k^2)} \right\} , \\
I_D(k, p) &= \frac{1}{2\Delta^2} \left\{ -2k \cdot p J_0 + [k^2 (3k \cdot p - p^2) - m^2 (3k \cdot p - k^2)] \frac{I_A}{2} \right. \\
&\quad \left. + [p^2 (3k \cdot p - k^2) - m^2 (3k \cdot p - p^2)] \frac{I_B}{2} + (3k \cdot p - k^2) \frac{J_A}{2} \right. \\
&\quad \left. + (3k \cdot p - p^2) \frac{J_B}{2} + I(k) + I(p) - \frac{m}{m^2 - k^2} - \frac{m}{m^2 - p^2} \right\} , \\
I_E(k, p) &= I_C(p, k) . \quad (5.160)
\end{aligned}$$

This completes the calculation of the vertex to $\mathcal{O}(\alpha)$.

5.2.4 Transverse Vertex to One loop

We can subtract from the full vertex, Eq. (5.11), the longitudinal vertex, Eqs. (5.6) and (5.7), and obtain the transverse vertex to $\mathcal{O}(\alpha)$. Let us recall that the transverse vertex $\Gamma_T^\mu(k, p)$ can be written in terms of 8 basis vectors as follows :

$$\Gamma_T^\mu(k, p) = \sum_{i=1}^8 \tau_i(k^2, p^2, q^2) T_i^\mu(k, p) ,$$

with the basis :

$$\begin{aligned} T_1^\mu &= [p^\mu(k \cdot q) - k^\mu(p \cdot q)] \\ T_2^\mu &= [p^\mu(k \cdot q) - k^\mu(p \cdot q)] (\not{k} + \not{p}) \\ T_3^\mu &= q^2 \gamma^\mu - q^\mu \not{q} \\ T_4^\mu &= q^2 [\gamma^\mu (\not{k} + \not{p}) - k^\mu - p^\mu] - 2(k - p)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} \\ T_5^\mu &= q_\nu \sigma^{\nu\mu} \\ T_6^\mu &= -\gamma^\mu (k^2 - p^2) + (k + p)^\mu \not{q} \\ T_7^\mu &= -\frac{1}{2}(k^2 - p^2) [\gamma^\mu (\not{k} + \not{p}) - k^\mu - p^\mu] + (k + p)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} \\ T_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not{p} - p^\mu \not{k} , \\ \text{with } \sigma_{\mu\nu} &= \frac{1}{2} [\gamma_\mu, \gamma_\nu] . \end{aligned} \tag{5.161}$$

After a lengthy but straightforward algebra, the coefficients τ_i can be identified. We prefer to write these out in the following form :

$$\tau_i(k, p) = \alpha g_i \left[\sum_j^5 a_{ij}(k, p) I(l_j) + \frac{a_{i6}(k, p)}{k^2 p^2} \right] \quad i = 1, \dots, 8 , \tag{5.162}$$

where $l_1^2 = \eta_1^2 \chi / 4$, $l_2^2 = \eta_2^2 \chi / 4$, $l_3^2 = k^2$, $l_4^2 = p^2$ and $l_5^2 = q^2 / 4$. Functions η_1 , η_2 and χ have been defined above. Similarly, the factors g_i are $-g_1 = m \Delta^2 g_2 = 2m \Delta^2 g_3 = 2\Delta^2 g_4 = g_5 = 2m \Delta^2 g_6 = \Delta^2 g_7 = m g_8 = m / 4 \Delta^2$. The coefficients a_{ij} in the one loop perturbative expansion of the τ_i , Eq. (5.162), are tabulated below :

$$\begin{aligned} a_{11}(k, p) &= -(\xi + 2) \eta_1 (m^2 + k \cdot p) \\ a_{12}(k, p) &= a_{11}(p, k) \\ a_{13}(k, p) &= 4(\xi + 2) \frac{(k^2 + k \cdot p)}{(k^2 - p^2)} \\ a_{14}(k, p) &= a_{13}(p, k) \\ a_{15}(k, p) &= -2(\xi + 2) \\ a_{16}(k, p) &= 0 \\ a_{21}(k, p) &= -\eta_1 \left\{ \left[-\frac{q^2}{2} m^4 + \{ (k \cdot p)^2 - (k^2 + p^2)(k \cdot p) + k^2 p^2 \} m^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{q^2}{4} \{ (k \cdot p)^2 + k^2 p^2 \} \Big] \\
& + \frac{(\xi - 1)}{2\chi} \Big[-q^4 m^8 - q^2 \{ (k \cdot p)^2 + 2(k^2 + p^2)k \cdot p - 5k^2 p^2 \} m^6 \\
& + \frac{3}{2} q^2 (k^2 + p^2) \Delta^2 m^4 \\
& + \left\{ 2(k^4 + p^4 + k^2 p^2)(k \cdot p)^3 - 7k^2 p^2 (k^2 + p^2)(k \cdot p)^2 \right. \\
& \left. + 10k^4 p^4 k \cdot p - k^4 p^4 (k^2 + p^2) \right\} m^2 \\
& \left. + \frac{1}{2} k^2 p^2 q^2 \{ (k^2 + p^2)(k \cdot p)^2 - 4k^2 p^2 k \cdot p + k^2 p^2 (k^2 + p^2) \} \right] \Big\} \\
a_{22}(k, p) &= a_{21}(p, k) \\
a_{23}(k, p) &= \frac{1}{(k^2 - p^2)} \Bigg[\xi \{ (k \cdot p)^3 + k^2 (k \cdot p)^2 - 3k^2 p^2 k \cdot p \\
& + 2k^4 k \cdot p + k^4 p^2 - 2k^6 \} m^2 / k^2 \\
& + (k \cdot p)^3 + (2k^2 - p^2)(k \cdot p)^2 + k^2 p^2 k \cdot p - 2k^4 k \cdot p - k^2 p^4 \\
& + (\xi - 1) \{ (k \cdot p)^3 + p^2 (k \cdot p)^2 - 3k^2 p^2 k \cdot p \\
& + 2k^4 k \cdot p + k^2 p^4 - 2k^4 p^2 \} \Big] \\
a_{24}(k, p) &= a_{23}(p, k) \\
a_{25}(k, p) &= q^2 (m^2 + k \cdot p) \\
& + (\xi - 1)(q^2 m^2 + (k \cdot p)^2 - (k^2 + p^2)(k \cdot p) + k^2 p^2) \\
a_{26}(k, p) &= m \Delta^2 \left\{ k \cdot p + \frac{(\xi - 1)}{\chi} \left[q^2 k \cdot p m^4 + 2(k^2 + p^2) \Delta^2 m^2 \right. \right. \\
& \left. \left. - k^2 p^2 \{ 2(k \cdot p)^2 + (k^2 + p^2)k \cdot p - 4k^2 p^2 \} \right] \right\} \\
a_{31}(k, p) &= -\frac{\eta_1}{2} \left\{ \left[\{ -2(k \cdot p)^2 + k^4 + p^4 \} m^4 \right. \right. \\
& + 2 \{ (k^2 + p^2)(k \cdot p)^2 + (k^2 - p^2)^2 k \cdot p - k^2 p^2 (k^2 + p^2) \} m^2 \\
& + \frac{1}{2} \{ -4(k \cdot p)^4 + (k^2 + p^2)^2 (k \cdot p)^2 + k^2 p^2 (k^2 - p^2)^2 \} \Big] \\
& + \frac{(\xi - 1)}{\chi} \left[q^2 \{ -2(k \cdot p)^2 + k^4 + p^4 \} m^8 \right. \\
& \left. + 2 \{ (k^2 + p^2) [-2(k \cdot p)^3 + (k^2 + p^2)(k \cdot p)^2 + (k^4 + p^4)k \cdot p] \right.
\end{aligned}$$

$$\begin{aligned}
& -k^2 p^2 (3k^4 + 3p^4 - 2k^2 p^2) \} m^6 \\
& -\frac{3}{2} q^2 \Delta^2 (k^2 - p^2)^2 m^4 - 2 \left\{ (k^2 + p^2)(k^4 + p^4 - 4k^2 p^2)(k \cdot p)^3 \right. \\
& -k^2 p^2 (k^4 + p^4 - 6k^2 p^2)(k \cdot p)^2 \\
& + 2k^4 p^4 (k^2 + p^2) k \cdot p - k^4 p^4 (k^2 + p^2)^2 \} m^2 \\
& \left. -\frac{1}{2} k^2 p^2 q^2 \left\{ (k^4 + p^4 - 6k^2 p^2)(k \cdot p)^2 + k^2 p^2 (k^2 + p^2)^2 \right\} \right] \Bigg\} \\
a_{32}(k, p) &= a_{31}(p, k) \\
a_{33}(k, p) &= \xi \left\{ (k \cdot p)^3 - k^2 (k \cdot p)^2 - 3k^2 p^2 k \cdot p + 2k^4 k \cdot p - k^4 p^2 + 2k^6 \right\} m^2 / k^2 \\
& + (\xi - 2) \left\{ (k \cdot p)^3 - (2k^2 - p^2)(k \cdot p)^2 \right. \\
& \left. + k^2 p^2 k \cdot p - 2k^4 k \cdot p + k^2 p^4 \right\} \\
a_{34}(k, p) &= a_{33}(k, p) \\
a_{35}(k, p) &= -(k^4 + p^4 - 2(k \cdot p)^2) [\xi m^2 + (\xi - 2) k \cdot p] \\
a_{36}(k, p) &= -m \Delta^2 \left\{ k \cdot p (k^2 + p^2) + 2k^2 p^2 \right. \\
& + \frac{(\xi - 1)}{\chi} \left[q^2 \left\{ (k^2 + p^2) k \cdot p + 2k^2 p^2 \right\} m^4 \right. \\
& \left. \left. + 2(k^2 + p^2)^2 \Delta^2 m^2 - k^2 p^2 (k + p)^2 \left\{ (k^2 + p^2) k \cdot p - 2k^2 p^2 \right\} \right] \right\} \\
a_{41}(k, p) &= -\eta_1 (\xi - 1) \frac{(k^2 - p^2)}{2\chi} \left[-q^4 m^6 + 3q^2 \left\{ -(k^2 + p^2) k \cdot p + 2k^2 p^2 \right\} m^4 \right. \\
& + \left\{ (k \cdot p)^2 [4(k \cdot p)^2 - 3k^4 - 3p^4 - 26k^2 p^2] \right. \\
& + k^2 p^2 [24(k^2 + p^2) k \cdot p - 3k^4 - 3p^4 - 14k^2 p^2] \} \frac{m^2}{2} \\
& + \frac{q^2}{2} \left\{ (k^2 + p^2)(k \cdot p)^3 + 2k^2 p^2 (k \cdot p)^2 \right. \\
& \left. \left. - 3k^2 p^2 (k^2 + p^2) k \cdot p + 2k^4 p^4 \right\} \right] \\
a_{42}(k, p) &= -a_{41}(p, k) \\
a_{43}(k, p) &= \frac{(\xi - 1)}{k^2} [(k^2 + k \cdot p)(k \cdot p)^2 + k^2 (2k^2 - 3p^2) k \cdot p + k^4 (p^2 - 2k^2)] \\
a_{44}(k, p) &= -a_{43}(p, k) \\
a_{45}(k, p) &= (\xi - 1)(k^2 - p^2) q^2 \\
a_{46}(k, p) &= m(\xi - 1)(k^2 - p^2) \frac{\Delta^2}{\chi} [q^2 k \cdot p m^2 + 2(k^2 + p^2)(k \cdot p)^2]
\end{aligned}$$

$$\begin{aligned}
& -2k^2p^2k \cdot p - k^2p^2(k^2 + p^2)] \\
a_{51}(k, p) &= -\eta_1 \left\{ \Delta^2 + \frac{(\xi - 1)}{4\chi} \left[-2q^4m^6 \right. \right. \\
& + 6q^2 \{ 2k^2p^2 - (k^2 + p^2)k \cdot p \} m^4 - 6k^2p^2q^4m^2 \\
& \left. \left. - q^2 \{ (k^2 - p^2)^2(k \cdot p)^2 + 2k^2p^2(k^2 + p^2)k \cdot p - k^2p^2(k^2 + p^2)^2 \} \right] \right\} \\
a_{52}(k, p) &= a_{51}(p, k) \\
a_{53}(k, p) &= \frac{(\xi - 1)}{k^2} [(k \cdot p)^2 + 2k^2k \cdot p - k^2(2k^2 + p^2)] \\
a_{54}(k, p) &= a_{53}(p, k) \\
a_{55}(k, p) &= (\xi - 1)q^2 \\
a_{56}(k, p) &= -m(\xi - 1)\frac{\Delta^2}{\chi} [q^2(k^2 + p^2)m^2 + 2(k^4 + p^4)k \cdot p - 2k^2p^2(k^2 + p^2)] \\
a_{61}(k, p) &= -\eta_1 \frac{(k^2 - p^2)}{2} \left[q^2m^4 - 2 \{ (k \cdot p)^2 - (k^2 + p^2)k \cdot p + k^2p^2 \} m^2 \right. \\
& + \frac{q^2}{2} \{ (k \cdot p)^2 + k^2p^2 \} \\
& + \frac{q^2(\xi - 1)}{\chi} \left[q^2m^8 + 2 \{ (k \cdot p)^2 + (k^2 + p^2)k \cdot p - 3k^2p^2 \} m^6 \right. \\
& - \frac{3}{2}q^2\Delta^2m^4 - 2 \{ (k^2 + p^2)(k \cdot p)^3 - k^2p^2(k \cdot p)^2 - k^4p^4 \} m^2 \\
& \left. \left. - \frac{1}{2}k^2p^2q^2 \{ (k \cdot p)^2 + k^2p^2 \} \right] \right] \\
a_{62}(k, p) &= -a_{61}(p, k) \\
a_{63}(k, p) &= - \left[\xi \{ (k^2 + k \cdot p)(k \cdot p)^2 + k^2(2k^2 - 3p^2)k \cdot p \right. \\
& - k^4(2k^2 - p^2) \} m^2 / k^2 \\
& \left. + (\xi - 2) \{ (2k^2 - p^2 + k \cdot p)(k \cdot p)^2 - k^2(2k^2 - p^2)k \cdot p - k^2p^4 \} \right] \\
a_{64}(k, p) &= -a_{63}(p, k) \\
a_{65}(k, p) &= -q^2(k^2 - p^2) [\xi m^2 - (\xi - 2)k \cdot p] \\
a_{66}(k, p) &= -m(k^2 - p^2)\Delta^2 \left[k \cdot p + \frac{(\xi - 1)}{\chi} (q^2k \cdot pm^4 + 2(k^2 + p^2)\Delta^2m^2 \right. \\
& \left. - k^2p^2q^2k \cdot p) \right]
\end{aligned}$$

$$\begin{aligned}
a_{71}(k, p) &= -\frac{\eta_1(\xi-1)}{4\chi} \left[-2q^6 m^6 - 6q^4 \{ (k^2 + p^2)(k \cdot p) - k^2 p^2 \} m^4 \right. \\
&\quad - 3q^2 \{ ((k \cdot p)^2 + k^2 p^2)(k^4 + p^4 + 6k^2 p^2) - 8k^2 p^2 (k^2 + p^2) k \cdot p \} m^2 \\
&\quad + q^2 \{ (k^2 - p^2)^2 (k \cdot p)^3 + 4k^2 p^2 (k^2 + p^2) (k \cdot p)^2 \\
&\quad \left. - k^2 p^2 (3k^4 + 3p^4 + 10k^2 p^2) k \cdot p + 4k^4 p^4 (k^2 + p^2) \} \right] \\
a_{72}(k, p) &= a_{71}(p, k) \\
a_{73}(k, p) &= (\xi-1) \frac{(k^2 - k \cdot p)}{k^2} ((k \cdot p)^2 + 4k^2 k \cdot p - 2k^4 - 3k^2 p^2) \\
a_{74}(k, p) &= a_{73}(p, k) \\
a_{75}(k, p) &= (\xi-1) q^4 \\
a_{76}(k, p) &= m(\xi-1) \frac{\Delta^2}{\chi} \left[q^2 \left\{ (k^2 + p^2) k \cdot p - 2k^2 p^2 \right\} m^2 \right. \\
&\quad \left. + 2(k^4 + p^4)(k \cdot p)^2 - 4k^2 p^2 (k^2 + p^2) k \cdot p - k^2 p^2 (k^4 + p^4 - 6k^2 p^2) \right] \\
a_{81}(k, p) &= -\eta_1 \frac{(\xi+2)}{2} q^2 (m^2 + k \cdot p) \\
a_{82}(k, p) &= a_{81}(p, k) \\
a_{83}(k, p) &= 2(\xi+2) k \cdot q \\
a_{84}(k, p) &= a_{83}(p, k) \\
a_{85}(k, p) &= -(\xi+2) q^2 \\
a_{86}(k, p) &= 0.
\end{aligned} \tag{5.163}$$

An important point to note is that these coefficients do not contain any trigonometric function, as it has been extracted out for raising the τ_i to a non-perturbative status. The τ_i have the required symmetry under the exchange of vectors k and p . All the τ_i are symmetric except τ_4 and τ_6 which are antisymmetric. Note that the form in which we write the transverse vertex makes it clear that each term in all the τ_i is either proportional to $\alpha I(l)$ or $\alpha/(k^2 p^2)$. We shall see that this form provides us with a natural scheme to arrive at its simple non-perturbative extension.

A few comments in comparison with the work by Davydychev *et. al.* [67], are as follows: (i) None of the τ_i we have calculated has kinematic singularity when $k^2 \rightarrow p^2$. This clearly suggests that the choice of the basis $\{T_i^\mu\}$ suggested by Kızılersü *et. al.* is preferred over the one of Ball and Chiu (in QED3 as well) used by Davydychev *et. al.* [67]. In particular our τ_4 and τ_7 are independent of kinematic singularities. (ii) In three dimensions, their factorization of the common constant factor in Eq. (E.1) is singular. However, as the divergences completely cancel out, we find our expressions more suitable for writing the transverse vertex in three dimensions. (iii) With the way we express J_0 , all the

τ_i are written in terms of basic functions of k and p and a single trigonometric function of the form $I(l)$. This form plays a key role in enabling us to make an easy transition to the possible non-perturbative structure of the vertex, as explained in the next section. Moreover, with the given form of J_0 , a direct comparison can be made with the massless case.

5.3 Non-perturbative Form of the Vertex (Effective Transverse WTI)

5.3.1 On the Gauge Parameter Dependence of the Vertex

Let us first look at the τ_i in the simplified massless case, with the notation $k = \sqrt{-k^2}$, $p = \sqrt{-p^2}$ and $q = \sqrt{-q^2}$ [57, 68],

$$\tau_2 = \frac{\alpha\pi}{4} \frac{1}{kp(k+p)(k+p+q)^2} \left[1 + (\xi - 1) \frac{2k + 2p + q}{q} \right], \quad (5.164)$$

$$\begin{aligned} \tau_3 = & \frac{\alpha\pi}{8} \frac{1}{kpq(k+p+q)^2} [4kp + 3kq + 3pq + 2q^2 \\ & + (\xi - 1)(2k^2 + 2p^2 + kq + pq)] , \end{aligned} \quad (5.165)$$

$$\tau_6 = \frac{\alpha\pi(2 - \xi)}{8} \frac{k - p}{kp(k + p + q)^2}, \quad (5.166)$$

$$\tau_8 = \frac{\alpha\pi(2 + \xi)}{2} \frac{1}{kp(k + p + q)}. \quad (5.167)$$

It is interesting to note that the existence of the factor

$$\frac{k - p}{kp} = - \left(\frac{1}{k} - \frac{1}{p} \right)$$

in eq. (5.166) puts τ_6 on a different footing as compared to the rest of the τ_i . The reason is that in the massless limit, the Fermion Propagator is simply

$$\frac{1}{F(p)} = 1 + \frac{\pi\alpha\xi}{4} \frac{1}{p},$$

implying

$$\frac{1}{F(k)} - \frac{1}{F(p)} \propto \left[\frac{1}{k} - \frac{1}{p} \right].$$

Therefore, the relation of τ_6 with the Fermion Propagator of the type $[1/F(k^2) - 1/F(p^2)]$ seems to arise rather naturally :

$$\tau_6 = -\frac{1}{2\xi} \frac{2 - \xi}{(k + p + q)^2} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right], \quad (5.168)$$

as noticed first by Curtis and Pennington [37] (note however that their coefficient is not the same). In the rest of the τ_i , the factor $1/k - 1/p$ does not arise. However, one could introduce it by hand to arrive at the following expressions :

$$\begin{aligned} \tau_2 &= -\frac{1}{\xi} \frac{1}{(k^2 - p^2)(k + p + q)^2} \left(1 + (\xi - 1) \frac{2k + 2p + q}{q} \right) \\ &\quad \times \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \end{aligned} \quad (5.169)$$

$$\begin{aligned} \tau_3 &= -\frac{1}{2\xi} \frac{[4kp + 3kq + 3pq + 2q^2 + (\xi - 1)(2k^2 + 2p^2 + kq + pq)]}{q(k - p)(k + p + q)^2} \\ &\quad \times \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \end{aligned} \quad (5.170)$$

$$\tau_8 = -\frac{2(2 + \xi)}{\xi} \frac{1}{(k - p)(k + p + q)} \times \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right]. \quad (5.171)$$

Equations. (5.168-5.171) represent a non-perturbative vertex which is in agreement with its complete one-loop expansion. This vertex has been constructed in accordance with the form advocated, e.g., in [7, 37, 42]. There are a couple of important points which need to be discussed here :

- There is an explicit dependence on the gauge parameter, ξ . A widespread belief has been that the gauge dependence of the vertex should solely arise through functions $F(k^2)$ and $F(p^2)$, and there should be *no explicit appearance of the gauge parameter* ξ . Such a belief has been expressed (or is reflected) in various works to date, e.g., [7, 18, 37, 42, 56]. Here we show that at least in massless QED3, such a construction is not possible.
- One of the main reasons that the transverse vertex was believed to be proportional to the factor

$$\left[\frac{1}{F(k)} - \frac{1}{F(p)} \right]$$

was the assumption that the transverse vertex vanishes in the Landau gauge, based upon perturbative results. A complete one-loop calculation reveals that the transverse vertex does not vanish in the Landau gauge. Moreover, an explicit presence of the gauge parameter in the non-perturbative form of the vertex tells us that the presence of the factor $[1/F(k) - 1/F(p)]$ is no longer a guarantee that the transverse vertex vanishes in the Landau gauge.

We now show that the explicit dependence of the vertex on the gauge parameter ξ is unavoidable in massless QED3. We notice that at the one loop level, each of the τ_i can be written in the following form :

$$\tau_i(k, p, q) = \alpha a_i(k, p, q) + \alpha \xi b_i(k, p, q).$$

On the other hand, Eq. (5.5) yields the following form for F :

$$\frac{1}{F(p)} = 1 + \alpha \xi c_i(p) .$$

If we want to write the non-perturbative form of the τ_i in terms of $1/F(p)$ and $1/F(k)$ alone and we do not expect explicit presence of α , the only way to get rid of ξ dependence is to have

$$b_2 T_2^\mu + b_3 T_3^\mu + b_6 T_6^\mu + b_8 T_8^\mu = 0 .$$

It is not possible as T_i^μ form a linearly independent set of basis vectors. Therefore, any construction of the 3-point vertex will surely have an explicit dependence on the gauge parameter. Owing to these reasons, we realize that to demand the transverse vertex to be proportional to $[1/F(k) - 1/F(p)]$ is artificial (apart from τ_6) and is not required. Therefore, we do not pursue this line of action anymore. In the next section, we move on to construct the vertex for the massive case inspired from our perturbative results.

5.3.2 Non-perturbative Vertex (Effective Transverse WTI)

As pointed out in the previous section, each term in all the τ_i is either proportional to the trigonometric function $\alpha I(l)$ or $\alpha/(k^2 p^2)$. On the other hand, the perturbative expressions for $\mathcal{M}(p)$ and $F(p)$, Eqs. (5.5), permit us to write :

$$\frac{1}{F(k)} - \frac{1}{F(p)} = \frac{\alpha}{k^2 p^2} \frac{\xi}{2} [k^2 A(p) - p^2 A(k)] , \quad (5.172)$$

where

$$A(p) = \{m - (m^2 + p^2)I(p)\} \quad (5.173)$$

and

$$\frac{\xi}{2(2+\xi)l^2 I(l)} \left[\frac{\mathcal{M}(l)}{F(l)} - m \right] - \left[1 - \frac{1}{F(l)} \right] = \frac{\xi(m^2 + l^2)}{2l^2} \alpha I(l) . \quad (5.174)$$

In the massless limit, Eq. (5.172) simply reduces to

$$\frac{1}{F(k)} - \frac{1}{F(p)} = \frac{\alpha \pi \xi}{4} \left[\frac{1}{k} - \frac{1}{p} \right]$$

in the Euclidean space, as expected. It was in fact an analogous massless expression in the limit when $k \gg p$ that inspired Curtis and Pennington, [37], to propose their famous vertex in QED4. Here, we are extending the reasoning to all the momentum regimes in the massive QED3. Fortunate simultaneous occurrence of the factor $\alpha/(k^2 p^2)$ in all the 8 Eqs. (5.162) and Eq. (5.172), and the presence of the same trigonometric factor $I(l)$ in the expressions for the vertex as

well as the propagator, one naturally arrives at the following non-perturbative form of τ_i :

$$\begin{aligned} \tau_i = & g_i \left\{ \sum_{j=1}^5 \left(\frac{2a_{ij}(k,p)l_j^2}{\xi(m^2+l_j^2)} \left[\frac{\xi}{2(\xi+2)l_j^2 I(l_j)} \left(\frac{\mathcal{M}(l_j)}{F(l_j)} - m \right) - \left(1 - \frac{1}{F(l_j)} \right) \right] \right) \right. \\ & \left. + \frac{2a_{i6}(k,p)}{\xi[k^2 A(p) - p^2 A(k)]} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \right\}. \end{aligned} \quad (5.175)$$

By construction, in the weak coupling regime, this non-perturbative form of the transverse vertex reduces to its corresponding Feynman expansion at the one loop level in an arbitrary covariant gauge and in all momentum regimes. Note that we have managed to write the transverse vertex solely as a function of the fermion propagator. Therefore, effectively, we have a WTI for this part of the vertex. We would like to emphasize that this is not a unique non-perturbative construction. However, it is probably the most natural and the simplest. A two loop calculation similar to the one presented in this chapter, and the LKF transformation for the vertex should serve as tests of Eq. (5.175) or guides for improvement towards the hunt for the exact non-perturbative vertex. On practical side, the use of our perturbation theory motivated vertex in studies addressing important issues such as Dynamical Mass Generation for fundamental fermions should lead to more reliable results, attempting to preserve key features of gauge field theories, e.g., gauge independence of physical observables. A computational difficulty to use the above vertex in such calculations could arise as the unknown functions F and $\mathcal{M}$ depend on the angle between k and p . This would make it impossible to carry out angular integration analytically in the SDE for the Fermion Propagator. This problem can be circumvented by defining an effective vertex which shifts the angular dependence from the unknown functions F and $\mathcal{M}$ to the known basic functions of k and p . This can be done by re-writing the perturbative results, Eq. (5.162), as follows :

$$\tau_i(k,p) = \alpha g_i \left[b_{i1}(k,p)I(k) + b_{i2}(k,p)I(p) + \frac{a_{i6}(k,p)}{k^2 p^2} \right], \quad (5.176)$$

where

$$b_{i1}(k,p) = a_{i1}(k,p) \frac{I(l_1)}{I(l_3)} + a_{i3}(k,p) + \frac{1}{2} a_{i5}(k,p) \frac{I(l_5)}{I(l_3)}, \quad (5.177)$$

$$b_{i2}(k,p) = a_{i2}(k,p) \frac{I(l_2)}{I(l_4)} + a_{i4}(k,p) + \frac{1}{2} a_{i5}(k,p) \frac{I(l_5^2)}{I(l_4)}. \quad (5.178)$$

This form can now be raised to a non-perturbative level exactly as before, with the only difference that the functions F and $\mathcal{M}$ are independent of the angle between the momenta k and p :

$$\tau_i = g_i \left\{ \sum_{j=1}^2 \left(\frac{2b_{ij}(k,p)\kappa_j^2}{\xi(m^2 + \kappa_j^2)} \left[\frac{\xi}{2(\xi+2)\kappa_j^2 I(\kappa_j)} \left(\frac{\mathcal{M}(\kappa_j)}{F(\kappa_j)} - m \right) \right] \right) \right.$$

$$- \left(1 - \frac{1}{F(\kappa_j)} \right) \Big] \Big) + \frac{2a_{i6}(k, p)}{\xi [k^2 A(p) - p^2 A(k)]} \left[\frac{1}{F(k)} - \frac{1}{F(p)} \right] \Big\}, \quad (5.179)$$

where $\kappa_1 = k$ and $\kappa_2 = p$.

5.4 Comments on the d -dimensional Case

Following Davydychev *et. al.*, [67], two of the Master Integrals needed to calculate are :

$$I(\nu_1, \nu_2, \nu_3) \equiv \int \frac{d^d w}{[(p-w)^2 - m^2]^{\nu_1} [(k-w)^2 - m^2]^{\nu_2} [w^2]^{\nu_3}}, \quad (5.180)$$

$$\mathcal{I}(\nu_1, \nu_2, \nu_3) \equiv \int \frac{d^d w}{[(p-w)^2]^{\nu_1} [(k-w)^2]^{\nu_2} [w^2 - m^2]^{\nu_3}}. \quad (5.181)$$

In terms of these integrals, one can write out the Fermion Propagator and fermion-boson vertex in the context of QED in arbitrary dimensions.

5.4.1 Fermion Propagator

Fermion Propagator in the d -dimensional case at the one-loop order can be written as :

$$\begin{aligned} \frac{1}{F(p)} &= \frac{e^2}{i(2\pi)^d} \frac{(d-2)\xi}{2p^2} [(p^2 + m^2)\mathcal{I}(0, 1, 1) - \mathcal{I}(0, 0, 1)], \\ \frac{\mathcal{M}(p)}{F(p)} &= \frac{e^2}{i(2\pi)^d} m(d-1-\xi)\mathcal{I}(0, 1, 1). \end{aligned} \quad (5.182)$$

The integral $\mathcal{I}(0, 1, 1)$ is the d -dimensional counterpart of our $I(y)$ function in the corresponding equation in three dimensions.

5.4.2 The Transverse Vertex

The corresponding d -dimensional τ 's of the transverse vertex can be expressed as, [67] :

$$\begin{aligned} \tau_i &= \frac{e^2}{i(2\pi)^d} \left\{ t_{i,0} I(1, 1, 1) + t_{i,1} [-(k \cdot q)\mathcal{I}(0, 1, 1) + (p \cdot q)\mathcal{I}(1, 0, 1) \right. \\ &+ q^2 I(1, 1, 0)] + t_{i,2} (\mathcal{I}(0, 1, 1) + \mathcal{I}(1, 0, 1) - 2I(1, 1, 0)) \\ &+ t_{i,3} \left(\mathcal{I}(0, 1, 1) + \mathcal{I}(1, 0, 1) - 2 \frac{\mathcal{I}(0, 0, 1)}{m^2} \right) \\ &\left. + t_{i,4} (\mathcal{I}(0, 1, 1) + \mathcal{I}(1, 0, 1)) + t_{i,5} \frac{\mathcal{I}(0, 1, 1) - \mathcal{I}(1, 0, 1)}{k^2 - p^2} \right\}, \end{aligned} \quad (5.183)$$

where the $t_{i,j}$, $j = 1 \dots 5$ are basic functions of k, p and q , listed in [67]. A straightforward procedure to obtain a non-perturbative vertex in this case is a natural thing to ask for. However, a difficulty arises since the function $I(1, 1, 1)$ in general cannot be rewritten as a linear combination of the integrals which appear in the Fermion Propagator in an obvious way. Note that in three dimensions $I(1, 1, 1)$ has the same structure as the integrals which define the Fermion Propagator, though with more complicated arguments. Therefore, we have not yet been able to write a simple non-perturbative extension of the transverse vertex in arbitrary dimensions.

5.5 Comments on the Two Loop Case

As we have mentioned earlier in this chapter, the three point vertex must reduce to its all order Feynman expansion in the weak coupling regime. What we have achieved in our work is the realization of this statement at the one loop level. To be able to do so beyond this order, we must know the perturbative vertex at the higher loops to have a guide for the construction of its non-perturbative counterpart. A good amount of progress has already been made in the calculation of necessary integrals involved. Most of these attempts surge from the precision analysis in particle physics phenomenology at the next-to-next-to-leading order calculations in perturbation theory. However, the scattering (see for example [85]) or decay processes (see for example [86]) do not in general require the calculation of vertex-type integrals with off-shell fermions and bosons. A complete two-loop calculation of the transverse vertex in arbitrary covariant gauge in arbitrary dimensions, to the best of our knowledge, still does not exist for off-shell particles in the legs.

Using Dimensional regularization with $d = 4 - \epsilon$, the integrals appearing in the calculation of two-loop corrections take the generic form [87] :

$$I(p_1, \dots, p_n) = \int \frac{d^d w_1}{(2\pi)^d} \int \frac{d^d w_2}{(2\pi)^d} \frac{1}{P_1^{m_1} \dots P_t^{m_t}} S_1^{n_1} \dots S_q^{n_q}, \quad (5.184)$$

where P_i are the propagators, depending on w_1, w_2 and the external momenta $p_1 \dots p_n$ while S_i are scalar products of a loop momentum with an external momentum or of the two loop momenta. The integral is specified by the powers m_i , ($m_i \geq 1$) of all propagators ($P_1, \dots, P_t$) and by the section ($S_1, \dots, S_q$) of scalar products and their powers ($n_1 \dots n_q$), ($n_i \geq 0$).

Progress in the calculation of two-loop corrections to four-point amplitudes was based on an efficient procedure to reduce the large number of different scalar integrals to a very limited number of so-called Master Integrals. The scalar integrals are related among each other by various identities. One class of identities are the so-called integration by parts (IBP) identities :

$$\int \frac{d^d w_1}{(2\pi)^d} \int \frac{d^d w_2}{(2\pi)^d} \frac{\partial}{\partial w_2^\mu} V(w, \dots) = 0, \quad (5.185)$$

where V is any combination of propagators, scalar products and loop momentum vectors. V can be a vector or tensor of any rank. IBP identities are a generalization of (5.92) and they follow from the fact that the integral over the total derivative with respect to any loop momentum vanishes.

Another class of identities is obtained from the fact that all integrals under consideration are Lorentz scalars, which are invariant under Lorentz transformations of the external momenta [88]. These Lorentz invariance (LI) identities are obtained from :

$$\left(p_1^\nu \frac{\partial}{\partial p_1^\mu} - p_1^\mu \frac{\partial}{\partial p_1^\nu} + \dots + p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right) I(p_1, \dots, p_n) = 0. \quad (5.186)$$

Master Integrals can be analytically expressed in series of the Dimensional Regulator parameter ϵ using Feynman or Schwinger parametrizations. However, sometimes using these parametric representations does not leave an integral that can be solved easily. Different techniques arose due to this situation, among others :

- The Mellin-Barnes method is based on the representation for a sum to some power, as a contour integral over a complex variable and the integration is then performed on straight contour lines parallel to the imaginary axis. After closing the contour, the result is the sum of all enclosed residues that might be expressed as a hypergeometric series. For the two-loop boxes we have [89].
- The Negative Dimensions technique consists in rewriting the integral over the parameters by introducing new ones through a multinomial expansion. Many conditions have to be satisfied among the parameters, which leads to the restriction that d must be a negative integer. Some results are [90].
- Double-Integral representation, that consist on splitting the loop space into the parallel and perpendicular subspaces to the external momentum. Then, using cylindrical instead of spherical coordinates, one can calculate the real and the imaginary parts of the integrals separately, performing contour integrals. For some results, see [91].
- Strategy of Expansion by regions [92], in which, instead of the integration over the whole space of loop momenta, the integration is performed only over some specific regions. The strategy consists in considering various regions of the loop momenta and expand, in every region, the integrand in Taylor series with respect to the parameters that are considered small in the given region; integrate the integrand expanded in every region and; put to zero any scaleless integral.
- Dispersion relations are used to calculate self-energies for the small external momentum behavior [93].
- Numerical strategies are used, where any loop integral is stripped analytically of its IR singularities so that the finite integrals can be performed numerically. [94].

- The analytic evaluation of MI can also be carried out without explicit integration over loop momenta by deriving differential equations for MI in internal propagator masses or external momenta. The equations can be solved with appropriate boundary conditions [95].

5.5.1 Two-Loops Fermion Propagator

The two-loops contributions to the fermion propagator are depicted in the following Diagram :

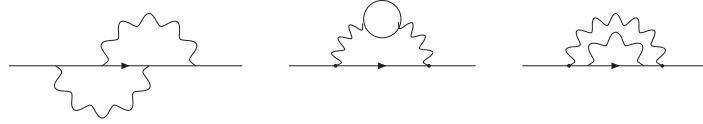


Diagram 9 : Two-Loop Corrections to the Fermion Propagator.

Using the Double-Integral representation method, the Fermion Propagator has been calculated in [91] for arbitrary masses. Scalar self-energies have been calculated using dispersion relations [93]. Using the Pinch Technique [96], off-shell self-energies have been calculated, in terms of Harmonic Polylogarithms. Also, an asymptotic expansion has been used to calculate the on-shell master diagram for the two-loop propagator[97]. This expansion is taken in the limit of the external momenta, and therefore the fermion masses, tend to infinity. On-shell self-energies master integrals with one mass have been calculated in [98]. In the light-cone gauge, the fermion propagator has been calculated in [99]. Following the algorithm developed in [100], these corrections have been calculated in an arbitrary covariant gauge for off-shell massive fermions in arbitrary dimensions[101].

5.5.2 Two-Loops Vertex

The two-loops corrections for the vertex are shown in Diagram (10) :

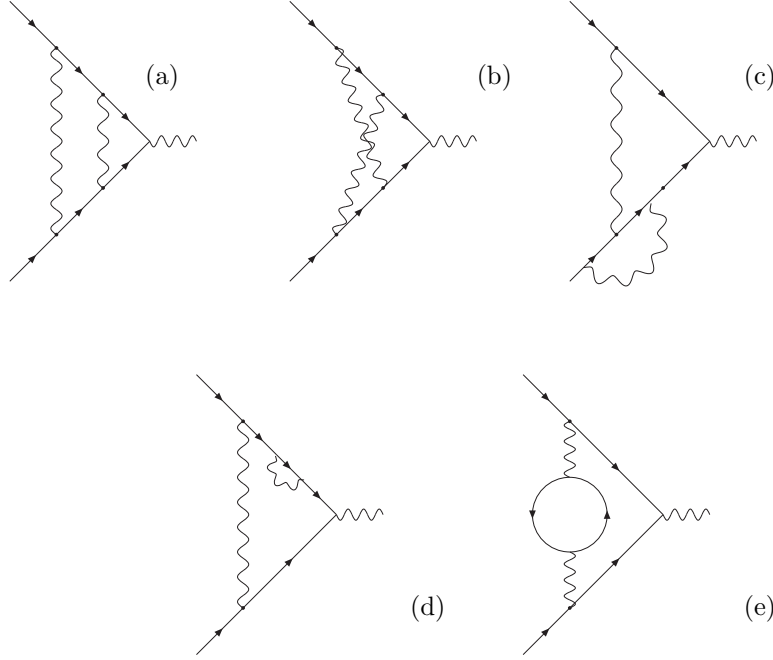


Diagram 10 : Two-Loop Corrections to the Vertex.

Some partial calculations to the full vertex using the double integral representation, planar vertex in four dimensions has been presented in [91]. In Ref. [102] a recent on-shell calculation for the vertex has been done at arbitrary momentum transfer in arbitrary dimensions. The planar vertex ((a) in Figure 5) with essential on shell singularities has been calculated in arbitrary dimensions [103] applying the strategy of expansion by regions. The calculation of a the vertex function at zero momentum transfere and with space-like external momentum [104]. Vertex with two legs on-shell in arbitrary dimensions has been calculated in [105] using the Negative Dimensions technique. Numerically speaking, the massless vertex has been calculated in the Landau gauge [106]. For non-planar vertex diagrams ((b) in the corresponding figure), a numerical method is proposed in [107]. In the Sudakov limit, the Vertex has been calculated using the Mellin-Barnes method in arbitrary dimensions [108]. A full off shell calculation of the two-loops vertex has not been carried out yet. A programme towards the construction of the full two-loop vertex has to take into account the transverse part of it, so that one can check of modify the corres-

ponding basis $\{T_i^\mu\}$. With this basis one can look for a nonperturbative vertex in arbitrary dimensions, or, at least, in our three dimensional case.

Once we are able to construct a three-point vertex which agrees with its perturbative expansion in the weak coupling regime to all orders, we shall rest assured that all the gauge identities are satisfied at every order of perturbation theory. We can then hope that the physical observables associated with non-perturbative physics shall automatically be gauge independent.

Chapter 6

Discussion and Conclusions

In this thesis we have focused our attention in incorporating gauge identities (Ward-Green-Takahashi identities and Landau-Khalatnikov-Fradkin transformations) to the non perturbative study of the Schwinger-Dyson Equations in QED3. We know that Perturbation Theory is the only scheme where in each order of approximation, such identities are satisfied, and also it verified that, indeed, physical observable quantities are independent of the gauge parameter. However, in a non perturbative analysis of the Schwinger-Dyson Equations it has been impossible the conjunction of these two facts.

A trace back for the origin of the gauge dependence points out to the following problems :

1. Violation of the Ward-Green-Takahashi Identity.
2. Incorrect gauge behavior of the Green's functions.
3. Use of wrong technical tools, like gauge dependent regularization schemes.

In order to avoid that the technical details overlap the pure theoretical issues of the study, we choose QED3, because in a theory that lacks of ultraviolet divergences. Therefore, the source for the gauge dependence should be only due to the first two points above mentioned.

In Chapter 3 we considered the simplest scenario, and we calculated the mass and the chiral condensate by solving the Schwinger-Dyson Equation for the Fermion Propagator. We explicitly observed that these quantities depend upon the gauge parameter. Since an incorrect regularization cannot induce such dependence, the only room for improvement is in the three-point vertex ansatz.

An ansatz should be such that :

- It satisfies the Ward-Green-Takahashi Identity,
- The Fermion Propagator obtained from this ansatz and the vertex itself should satisfy their corresponding Landau-Khalatnikov-Fradkin transformation.

The incorporation of the Ward-Green-Takahashi Identity is straightforward. However, as we did in Chapter 4, even in the simplest case, the Landau-Khalatnikov-Fradkin transformation for the Fermion Propagator leads to a gauge behavior extremely complex for this Green's function, and in spite of this, we were able to extract valuable information about its structure beyond the tree level. The search for a non perturbative vertex which leads to this gauge behavior for the Fermion Propagator is not at all a trivial task. It is here that Perturbation Theory plays its role. If we can put forward to a non perturbative ansatz which reduces to its Feynman expansion to all orders in the weak coupling regime, we can hope that it automatically incorporates the Landau-Khalatnikov-Fradkin transformations not only for the Fermion Propagator, but for the Vertex itself.

Therefore, in Chapter 5 we calculated the Fermion-Boson Vertex at one-loop level outside of the quiral phase of QED3, previously studied. We decompose this vertex into its longitudinal and transverse parts. The former guarantees the validity of the Ward-Green-Takahashi Identity, since it is related to the Fermion Propagator calculated at the same order of approximation, while the later ensures that the Landau-Khalatnikov-Fradkin transformations are indeed satisfied. Taking advantage of the lack of additional constraints, we exploited the perturbative form of the Fermion Propagator in such a way that we could write an effective Ward-Green-Takahashi Identity for the transverse vertex. This identity allows us to have the first insight on the non perturbative structure of the Fermion-Boson interaction, since the transverse vertex, related in this way to the Fermion Propagator does not depend explicitly on the electromagnetic coupling. We write, afterwards, the transverse vertex in a convenient form that will allow, to corroborate its non perturbative structure in a future work at the two-loop level.

The systematic construction of the three point vertex guarantees that :

- The Ward-Green-Takahashi identity, which relates the Fermion Propagator to the Fermion-Boson vertex, is satisfied non-perturbatively.
- Both the vertex and the resulting Fermion Propagator satisfy Landau-Khalatnikov-Fradkin transformations to $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$ respectively.
- The vertex does not contain any kinematic singularities when $k^2 \rightarrow p^2$.
- Most importantly, the vertex reduces to its correct perturbative expansion at $\mathcal{O}(\alpha)$ in the weak coupling regime in an arbitrary covariant gauge and for all momentum regimes.
- The vertex has correct symmetry under the parity, charge conjugation and time reversal operations.

This will allow that the technique as well as the reasoning used in our construction could be applied in more complicated theories like QCD in a suitable and realistic way. Finally, it is also possible to incorporate our ideas in some alternative scheme to the Standard Model which pretends to give a solution to

the problem of the origin of the masses. We extend the invitation to whom wishes to do so.

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