

4 Non-Linear Elasticity

Non-linear elastic materials, in particular elastic materials subjected to large strains, are examined in this chapter. Most of the chapter is concerned with hyperelastic materials, that is, elastic materials for which the stress can be derived from an elastic strain-energy potential function.

Some of the different types of elastic material models are reviewed in §4.1. Hyperelastic materials are studied in the sections which follow. General constitutive relations are derived for both the material and spatial descriptions and for both compressible and incompressible materials. It is shown how the principle of material frame-indifference restricts the form a constitutive equation may take. The more commonly used isotropic material models are discussed in §4.4 and material anisotropy is discussed in §4.5.

A de-coupled formulation of hyperelasticity is presented, which is useful for materials which are nearly incompressible.

Finally, some of the equations for isotropic materials are linearised, showing how they are consistent with the linear elastic model.

4.1 Elastic Solids

In this section is given an overview of the common elasticity models.

4.1.1 The Linear Elastic Solid

The classical **Linear Elastic model**, or **Hookean model**, has the following linear relationship between stress and strain:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}, \quad \sigma_{ij} = C_{ijmn} \varepsilon_{mn} \quad (4.1.1)$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor, §2.7.

Strain Energy

In this purely mechanical theory of elastic materials, there is no dissipation of energy – all the energy of the loads is stored as elastic strain energy in the material as it deforms, and can be recovered.

For the linear elastic material, the rate of deformation is equivalent to the rate of small-strain, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$, so the strain-energy function can be written as

$$dW = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \quad (4.1.2)$$

and the total energy stored per unit volume over the complete history of straining is

$$W = \int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \quad (4.1.3)$$

and the stress can be written as

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \quad (4.1.4)$$

Reduction in the number of Independent Elastic Constants

Since the stress and strain are symmetric, $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{mn} = \varepsilon_{nm}$, the fourth-order elasticity tensor of stiffness coefficients contains the minor symmetries 1.9.65,

$$C_{ijkl} = C_{jikl} = C_{ijlk} \quad (4.1.5)$$

and so the 81 coefficients reduce to 36 independent coefficients. Further, since $\mathbf{C}$ is independent of the strains,

$$W = \int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = \int \mathbf{C} : \boldsymbol{\varepsilon} d\boldsymbol{\varepsilon} = \frac{1}{2} \mathbf{e} : \mathbf{C} : \mathbf{e} \quad (4.1.6)$$

and so

$$\mathbf{C} = \frac{\partial W}{\partial \mathbf{e} \partial \mathbf{e}} \quad (4.1.7)$$

Now

$$\frac{\partial W}{\partial \varepsilon_{ij} \partial \varepsilon_{mn}} = \frac{\partial W}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} \quad (4.1.8)$$

and it follows that $\mathbf{C}$ possesses the *major symmetries*:

$$C_{ijmn} = C_{mnij} \quad (4.1.9)$$

This reduces the number of independent elastic constants from 36 to 21.

Problems involving Elastic Materials

The six constitutive equations 4.1.1, together with the equations of motion and the 6 kinematic relations relating the strains to the displacements, Eqn. 2.7.2,

$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, gives a set of 15 equations in the 15 unknowns: the six stress components, the six strain components and the three displacement components.

To maintain a linear theory, the acceleration term in the equations of motion must be linear; this is achieved by supposing the displacement gradients to be small:

$$v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_k} v_k \approx \frac{\partial u_i}{\partial t}, \quad \frac{d^2 u_i}{dt^2} \approx \frac{\partial^2 u_i}{\partial t^2} \quad (4.1.10)$$

When this acceleration term is included, the problem is **dynamic**. When the equations of equilibrium are used, the problem is **static**.

Compatibility

An alternative method of solution is to remove the displacements from the above system and solve only for the stresses and strains. In this case the strain-displacement relations are replaced by three compatibility equations, and there are then 12 equations in 12 unknowns. Once the system is solved, the displacements can be obtained from the strains by integration.

The Isotropic Linear Elastic Solid

When the material is isotropic, the constitutive equation holds in any coordinate system,

$$\sigma'_{ij} = C_{ijmn} \varepsilon'_{mn} \quad (4.1.11)$$

and so the tensor of elastic constants is isotropic. The most general fourth-order isotropic tensor takes the form 1.10.7,

$$\mathbf{C} = \lambda \mathbf{I} \otimes \mathbf{I} + \mu (\mathbf{I} + \bar{\mathbf{I}}), \quad C_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \quad (4.1.12)$$

with the fourth-order unit tensors given by 1.9.60,

$$\begin{aligned} \mathbf{I} &= \delta_{im} \delta_{jn} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n \\ \bar{\mathbf{I}} &= \delta_{in} \delta_{jm} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n \end{aligned} \quad (4.1.13)$$

which has only *two independent material constants*. Since the strain is symmetric, one has

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{C} : \boldsymbol{\varepsilon} \\ &= (\lambda \mathbf{I} \otimes \mathbf{I} + \mu (\mathbf{I} + \bar{\mathbf{I}})) : \boldsymbol{\varepsilon} \\ &= (\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I}) : \boldsymbol{\varepsilon} \end{aligned} \quad (4.1.14)$$

and the constitutive equation reduces to { **▲ Problem 1** }

$$\boldsymbol{\sigma} = \lambda (\text{tr} \boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}, \quad \sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \quad (4.1.15)$$

and the two elastic constants λ, μ are called **Lamé's constants**.

Problems in Isotropic Elasticity

The 15 equations mentioned earlier can be reduced by eliminating the strains from the constitutive equation and the kinematic equation, and then substituting the resultant expression for stress into the equations of motion, giving **Navier's equations**

$$(\lambda + \mu) \text{grad}(\text{div} \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}}, \quad (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_k^2} + \rho b_i = \rho \frac{\partial^2 u}{\partial t^2} \quad (4.1.16)$$

This set of three partial differential equations is appropriate for problems involving displacement boundary conditions.

The Lamé's constants and the Young's modulus and Poisson's ratio are related through

$$\begin{aligned} E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \\ \lambda &= \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)} \end{aligned} \quad (4.1.17)$$

The linear elastic constitutive equations in terms of the engineering constants reads

$$\begin{aligned}\varepsilon_{ij} &= -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} \\ \sigma_{ij} &= \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right]\end{aligned}\quad (4.1.18)$$

The Bulk Modulus

The tensor of elastic constants can be written in the alternative forms

$$\begin{aligned}\mathbf{C} &= 2\mu\mathbf{I} + \lambda\mathbf{I} \otimes \mathbf{I} \\ &= 2\mu\left[\mathbf{I} + \frac{\nu}{1-2\nu}\mathbf{I} \otimes \mathbf{I}\right] \\ &= \frac{E}{(1+\nu)}\left[\mathbf{I} + \frac{\nu}{1-2\nu}\mathbf{I} \otimes \mathbf{I}\right] \\ &= \kappa\mathbf{I} \otimes \mathbf{I} + 2\mu\left[\mathbf{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right]\end{aligned}\quad (4.1.19)$$

where the new constant introduced is the **bulk modulus** κ . This last expression then leads to the alternative form of the constitutive relations,

$$\boldsymbol{\sigma} = \kappa(\text{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\text{dev}\boldsymbol{\varepsilon} \quad (4.1.20)$$

This expression shows that the stress can be decomposed into a spherical component and a deviatoric component. For a pure volume change, $\text{dev}\boldsymbol{\varepsilon} = 0$, and there are no shear stresses, $\boldsymbol{\sigma} = (\text{tr}\boldsymbol{\sigma})\mathbf{I}$; the bulk modulus is thus a measure of the resistance of the material to volume changes.

4.1.2 Geometrically Non-Linear Elastic Materials

When the strains (displacement gradients) are not small, the behaviour of the material will inevitably be non-linear. This is due to the **geometric non-linearity** of the kinematic strain-displacement relations, for example using the Green-Lagrange strains and the reference configuration,

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\text{Grad}\mathbf{U} + (\text{Grad}\mathbf{U})^T + (\text{Grad}\mathbf{U})^T \text{Grad}\mathbf{U}\right) \\ E_{ij} &= \frac{1}{2}\left\{\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right\}\end{aligned}\quad (4.1.21)$$

The Kirchhoff Material

The Kirchhoff material is an extension of the linear elastic model to the large strain range; the constitutive relation is a linear tensor relation, but non-linearities enter through $\mathbf{E}(\mathbf{u})$:

$$\mathbf{S} = \mathbf{C} : \mathbf{E}, \quad S_{ij} = C_{ijmn} E_{mn} \quad (4.1.22)$$

where $\mathbf{S}$ is the PK2 stress tensor and $\mathbf{E}$ is the Green-Lagrange strain. Since both $\mathbf{S}$ and $\mathbf{E}$ are symmetric, the fourth-order tensor $\mathbf{C}$ has the minor symmetries, $C_{ijmn} = C_{jimn}$ and $C_{ijmn} = C_{ijnm}$, and so has 36 independent coefficients. Following the same arguments as before, one can define a strain energy function (per unit reference volume)

$$dW = \mathbf{S} : d\mathbf{E}, \quad dW = S_{ij} dE_{ij} \quad (4.1.23)$$

and the total energy stored per unit volume over the complete history of straining is

$$W = \int S_{ij} dE_{ij} = \frac{1}{2} C_{ijmn} E_{ij} E_{mn} = \frac{1}{2} \mathbf{E} : \mathbf{C} : \mathbf{E} \quad (4.1.24)$$

and the stress can be written as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}}, \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \quad (4.1.25)$$

Again, the existence of the strain energy function implies that the matrix of elastic coefficients only has 21 independent coefficients.

When the Kirchhoff material is isotropic, the constitutive relation reduces further to

$$\mathbf{S} = \lambda (\text{tr} \mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}, \quad S_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij} \quad (4.1.26)$$

As mentioned in §2.7.2, the linear elastic model can not be used when there are large rigid body rotations, even if the displacement gradients are not large. The Kirchhoff model can be used in these cases.

4.1.3 Materially Non-Linear Elastic Materials

An elastic material might also exhibit **material non-linearity** through a non-linear constitutive equation, for example the Cauchy stress might be some non-linear function of a strain measure, or of the deformation gradient:

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}(t)) \quad (4.1.27)$$

where $\mathbf{f}$ is some tensor function of the deformation gradient $\mathbf{F}$. This constitutive equation is called the **Cauchy Elastic** material model. As can be seen, the stress is dependent on the current state only, and not on the path history, a requirement of elasticity. However, the stress in the case of a Cauchy elastic material cannot in general be written in terms of a strain-energy function. In other words, the work done might be path-dependent.

Objectivity Requirements

The notion of objectivity was introduced in §2.8. When formulating constitutive relations for materials, one must ensure that the **principle of material objectivity**(or the **principle of material frame indifference**) be satisfied. This principle states that

A constitutive law must be independent of the location of the observer (or frame of reference that is taken)

This implies that two observers, even if in relative motion with respect to each other, observe the same stress in a given body. Consider the Cauchy elastic material 4.1.27. The Cauchy stress is an objective tensor. Referring to the example of Eqns. 2.8.47-50 in §2.8.4, objectivity requires that the constitutive relation be of the form

$$\boldsymbol{\sigma} = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^T \quad (4.1.28)$$

The constitutive relation can also be written in terms of other stress measures. For example, using $\mathbf{J}\boldsymbol{\sigma} = \mathbf{P}\mathbf{F}^T = \mathbf{P}(\mathbf{R}\mathbf{U})^T = \mathbf{P}\mathbf{U}^T\mathbf{R}^T$, one has

$$\mathbf{P} = \mathbf{J}\mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{U}^{-1} \quad (4.1.29)$$

For the PK2 stress, one has $\mathbf{S} = \mathbf{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$, so that

$$\mathbf{S} = \det \mathbf{U} \mathbf{U}^{-1}\mathbf{f}(\mathbf{U})\mathbf{U}^{-1} \quad (4.1.30)$$

which does not depend on the rotation. This last relation can also be written in the form

$$\mathbf{S} = \det \mathbf{C}^{1/2} \mathbf{C}^{-1/2} \mathbf{f}_2(\mathbf{C}^{1/2}) \mathbf{C}^{-1/2} \equiv \mathbf{g}(\mathbf{C}) \quad (4.1.31)$$

This is clearly objective, since $\mathbf{S}$ and $\mathbf{C}$ are unaffected by an observer transformation, $\mathbf{S}^* = \mathbf{S}$ and $\mathbf{C}^* = \mathbf{C}$.

4.1.4 Hypoelastic Materials

A **hypoelastic** material is one whose constitutive law relates the rate of stress to the rate of deformation $\mathbf{d}$. This can be written in terms of the Cauchy stress as $\dot{\boldsymbol{\sigma}} = \mathbf{f}(\boldsymbol{\sigma}, \mathbf{d})$.

Consider a simple one-dimensional linear model,

$$\dot{\sigma} = E d \quad (4.1.32)$$

Since, in one-dimension, the stretch is $\lambda = dx/dX$, the rate of deformation is equivalent to the spatial velocity gradient l and the rate of change of a line element dx is $d(dx)/dt = l dx$, dividing through by dX gives $d = \dot{\lambda}/\lambda$ (see Eqn. 2.5.10), so that

$$\dot{\sigma} = E \frac{\dot{\lambda}}{\lambda} \rightarrow \sigma = \int d\sigma = E \int \frac{d\lambda}{\lambda} = E \ln \lambda \quad (4.1.33)$$

This shows that the stress is clearly path-independent, depending only on the current stretch. In fact, the stress can be written as the derivative of a strain energy function according to $\sigma = dW/d\lambda$, where $W = E/\lambda$.

In the three dimensional case, however, the rate of deformation can not in general be written as the rate of change of some simple function, $\mathbf{d} = d(\bullet)/dt$, and so the above calculation cannot be done, implying that the hypoelastic material cannot be written in terms of a potential function, and the work done is path-dependent. The path-dependence is, however, small when the strains are small.

4.1.5 Hyperelastic Materials

A **hyperelastic** material (or **Green elastic** material) is defined to be an elastic material for which a strain-energy function W exists, a scalar function of one of the strain or deformation tensors, whose derivative with respect to a strain component determines the corresponding stress component. From the above, the linear elastic model, the Kirchhoff model and the one-dimensional hypoelastic model are all examples of hyperelastic materials. The hyperelastic material is a subset of the Cauchy-elastic material. Hyperelastic material models for components under large strains will be the subject of the following sections.

4.1.6 Problems

1. Show that

$$\begin{aligned}\mathbf{I} \otimes \mathbf{I} : \boldsymbol{\varepsilon} &= (\text{tr} \boldsymbol{\varepsilon}) \mathbf{I} \\ (\mathbf{I} + \bar{\mathbf{I}}) : \boldsymbol{\varepsilon} &= 2\boldsymbol{\varepsilon}\end{aligned}$$

4.2 Hyperelasticity

The Hyperelastic material is examined in this section.

4.2.1 Constitutive Equations

The rate of change of internal energy W per unit reference volume is given by the stress power, which can be expressed in a number of different ways (see §3.7.6):

$$\dot{W} = J\boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S} : \dot{\mathbf{C}} \quad (4.2.1)$$

The internal energy is regarded as a function of a deformation variable. For example,

The change in energy is due to the deformation which takes place, so take W to be a function of, say, the deformation gradient $\mathbf{F}(t)$, $W(\mathbf{F})$. It is assumed that in the reference configuration the strain energy is zero, $W(\mathbf{I}) = 0$, and that it grows with deformation, $W(\mathbf{F}) \geq 0$ ¹.

The chain rule gives

$$\dot{W} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} \quad (4.2.2)$$

From 4.2.1,

$$\mathbf{P} : \dot{\mathbf{F}} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} \quad (4.2.3)$$

Since $\mathbf{F}$ and $\dot{\mathbf{F}}$ can take on any value independent of the other, one must have

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \quad (4.2.4)$$

This is a constitutive equation relating the kinematic variables to the force variables.

From 4.2.1, alternative forms are:

$$\boxed{\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}}, \quad \mathbf{S} = 2 \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}} \quad (4.2.5)$$

The procedure used in Eqns. 4.2.2-4 cannot be used for the stress power relation $J\boldsymbol{\sigma} : \mathbf{d}$ since there is no function whose derivative is the rate of deformation. Instead, use the relation 3.5.6, $\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T$, 4.2.4, $\mathbf{P} = \partial W / \partial \mathbf{F}$, to get

¹ and that it tends to infinity as the material is either compressed to a point, $J = \det \mathbf{F} \rightarrow 0$ or expanded to an infinite range, $J = \det \mathbf{F} \rightarrow \infty$.

$$\boldsymbol{\sigma} = J^{-1} \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \mathbf{F}^T, \quad \sigma_{ij} = J^{-1} \frac{\partial W}{\partial F_{im}} F_{jm} \quad (4.2.6)$$

An alternative relation can be obtained by first deriving a relationship between the partial derivatives of the strain energy function with respect to the deformation gradient, $\partial W / \partial F_{ij}$, and with respect to the right Cauchy-Green tensor, $\partial W / \partial C_{ij}$. Suppose first that the strain energy is a function of the deformation gradient:

$$\frac{\partial W}{\partial \mathbf{F}} \equiv \frac{\partial W}{\partial F_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j \quad (4.2.7)$$

The chain rule for $W = W(\mathbf{C}(\mathbf{F}))$ gives

$$\frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W}{\partial C_{mn}} \frac{\partial C_{mn}}{\partial F_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j \quad (4.2.8)$$

With $\mathbf{C} = \mathbf{F}^T \mathbf{F}$,

$$\frac{\partial C_{mn}}{\partial F_{ij}} = \frac{\partial F_{km}}{\partial F_{ij}} F_{kn} + F_{km} \frac{\partial F_{kn}}{\partial F_{ij}} = \delta_{ki} \delta_{mj} F_{kn} + \delta_{ki} \delta_{nj} F_{km} = \delta_{mj} F_{in} + \delta_{nj} F_{im} \quad (4.2.9)$$

so that (and using the fact that $\mathbf{C}$ is symmetric),

$$\begin{aligned} \frac{\partial W}{\partial F_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j &= \delta_{mj} F_{in} \frac{\partial W}{\partial C_{mn}} \mathbf{e}_i \otimes \mathbf{e}_j + \delta_{nj} F_{im} \frac{\partial W}{\partial C_{mn}} \mathbf{e}_i \otimes \mathbf{e}_j \\ &= F_{in} \frac{\partial W}{\partial C_{mn}} \mathbf{e}_i \otimes \mathbf{e}_m + F_{im} \frac{\partial W}{\partial C_{mn}} \mathbf{e}_i \otimes \mathbf{e}_n \\ &= F_{in} \frac{\partial W}{\partial C_{nm}} \mathbf{e}_i \otimes \mathbf{e}_m + F_{im} \frac{\partial W}{\partial C_{mn}} \mathbf{e}_i \otimes \mathbf{e}_n \end{aligned} \quad (4.2.10)$$

or

$$\frac{\partial W}{\partial \mathbf{F}} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}}, \quad \frac{\partial W}{\partial F_{ij}} = 2F_{ik} \frac{\partial W}{\partial C_{kj}} \quad (4.2.11)$$

Now 4.2.6 can be re-written as

$$\boxed{\boldsymbol{\sigma} = 2J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^T} \quad (4.2.12)$$

Similarly, Eqn. 4.2.4 can be re-written as

$$\boxed{\mathbf{P} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}}} \quad (4.2.13)$$

The Strain Energy and the Right Stretch Tensor

The right stretch tensor can be expressed as $\mathbf{C} = \mathbf{U}\mathbf{U}$, where $\mathbf{U}$ is the right stretch, or, since $\mathbf{U}$ is also symmetric, $\mathbf{C} = \mathbf{U}^T \mathbf{U}$. One can see the similarity between this relation and the relation $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ so, using the same arguments as given above, one has for $W = W(\mathbf{C}(\mathbf{U}))$, (see 1.11.36)

$$\frac{\partial W}{\partial \mathbf{U}} = 2\mathbf{U} \frac{\partial W}{\partial \mathbf{C}} \quad (4.2.14)$$

Since $\mathbf{U}$ is symmetric,

$$\frac{\partial W}{\partial \mathbf{U}} = \left(\frac{\partial W}{\partial \mathbf{U}} \right)^T = 2 \left(\mathbf{U} \frac{\partial W}{\partial \mathbf{C}} \right)^T = 2 \left(\frac{\partial W}{\partial \mathbf{C}} \right)^T \mathbf{U} = 2 \frac{\partial W}{\partial \mathbf{C}} \mathbf{U} \quad (4.2.15)$$

showing that $\mathbf{U}$ and $\partial W / \partial \mathbf{C}$ are coaxial.

4.2.2 Objectivity of the Constitutive Equations

An observer transformation (see §2.8) results in $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$ and $\mathbf{C} = \mathbf{C}^*$, so $\partial W(\mathbf{C}^*) / \partial \mathbf{C}^* = \partial W(\mathbf{C}) / \partial \mathbf{C}$, and, so, from 4.2.12,

$$\boldsymbol{\sigma}^* = 2J^{-1} \mathbf{Q} \mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^T \mathbf{Q}^T = \mathbf{Q} \boldsymbol{\sigma} \mathbf{Q}^T \quad (4.2.16)$$

which is the objectivity requirement for a spatial tensor and so this constitutive law satisfies the requirement of material frame-indifference, when W is a function of $\mathbf{C}$. This evidently holds true also when W is a function of $\mathbf{E}$. It does not, however, hold true in general when W is a function of $\mathbf{F}$, as in the constitutive law $\mathbf{P} = \partial W(\mathbf{F}) / \partial \mathbf{F}$. However, with W a function of $\mathbf{C}$, the constitutive equation 4.2.13 can be seen to be objective.

The objective constitutive equations in this section are indicated by a box around them.

4.2.3 Elasticity Tensors

The total differential of the PK2 stress can be written as

$$d\mathbf{S} = \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} : d\mathbf{C} = \left(2 \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} \right) : \left(\frac{1}{2} d\mathbf{C} \right) \equiv \mathbf{C} : \frac{1}{2} d\mathbf{C} \quad (4.2.17)$$

where $\mathbf{C}$ is the fourth-order tensor

$$\mathbf{C} = 2 \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \mathbf{S}(\mathbf{E})}{\partial \mathbf{E}} \quad (4.2.18)$$

and is a measure of the change in stress which occurs due to a change in strain. It is called the **elasticity tensor** (in the material description) or the **tangent modulus**.

Since $\mathbf{S}$ and $\mathbf{E}$ are symmetric, $\mathbf{C}$ possesses the minor symmetries, $C_{ijkl} = C_{jikl} = C_{ijlk}$, and so has 36 independent components. However, if hyperelastic conditions hold, so that $\mathbf{S} = \partial W(\mathbf{E}) / \partial \mathbf{E} = 2 \partial W(\mathbf{C}) / \partial \mathbf{C}$, then

$$\mathbf{C} = 4 \frac{\partial^2 W(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}} = \frac{\partial^2 W(\mathbf{E})}{\partial \mathbf{E} \partial \mathbf{E}} \quad (4.2.19)$$

and so $\mathbf{C}$ possesses the major symmetries, $C_{ijkl} = C_{klij}$, and only only 21 independent components.

The rate form of the above equations is

$$\dot{\mathbf{S}} = \frac{d}{dt} \left(2 \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \right) = 2 \frac{\partial^2 W(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}} : \dot{\mathbf{C}} = \frac{\partial^2 W(\mathbf{E})}{\partial \mathbf{E} \partial \mathbf{E}} : \dot{\mathbf{E}} \quad (4.2.20)$$

or

$$\dot{\mathbf{S}} = \mathbf{C} : \frac{1}{2} \dot{\mathbf{C}} = \mathbf{C} : \dot{\mathbf{E}} \quad (4.2.21)$$

4.2.4 A Note on the Strain Energy Function

Some confusion can arise in expressions such as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}}, \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \quad (4.2.22)$$

As mentioned at the end of §1.11.5, one can either

- use the energy function $W^* = W^*(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{33})$, a function of 6 independent variables, in which case

$$S_{11} = \frac{\partial W^*}{\partial E_{11}}, \quad S_{12} = \frac{1}{2} \frac{\partial W^*}{\partial E_{12}}, \quad S_{13} = \frac{1}{2} \frac{\partial W^*}{\partial E_{13}}, \quad \dots \quad (4.2.23)$$

- use the energy function $W = W(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33})$, a function of 9 variables, not all of them independent, in which case

$$S_{11} = \frac{\partial W}{\partial E_{11}}, \quad S_{12} = \frac{\partial W}{\partial E_{12}}, \quad \dots \quad S_{21} = \frac{\partial W}{\partial E_{21}}, \quad \dots \quad (4.2.24)$$

and the symmetry of W with respect to the strains, which must be assumed here, implies that the stress is also symmetric.

4.2.5 Hyperelasticity with Constraints

Consider a Hyperelastic material which is subject to the scalar constraint

$$\phi(\mathbf{F}) = 0 \quad \text{or} \quad \dot{\phi} = \frac{\partial \phi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = 0 \quad (4.2.25)$$

Without the constraint, one has

$$\dot{W} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} = \mathbf{P} : \dot{\mathbf{F}} \quad (4.2.26)$$

The constitutive equation $\mathbf{P} = \partial W / \partial \mathbf{F}$ can be derived from 4.2.26 when $\dot{\mathbf{F}}$ is arbitrary. However, for the material with the constraint, the $\dot{\mathbf{F}}$ cannot be “cancelled” out from each side; one has the conditions

$$\left(\mathbf{P} - \frac{\partial W}{\partial \mathbf{F}} \right) : \dot{\mathbf{F}} = 0, \quad \frac{\partial \phi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = 0 \quad (4.2.27)$$

In general then,

$$\mathbf{P} - \frac{\partial W}{\partial \mathbf{F}} = \alpha \frac{\partial \phi}{\partial \mathbf{F}} \quad (4.2.28)$$

where α is some arbitrary scalar. The stress is therefore given by

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} + \alpha \frac{\partial \phi}{\partial \mathbf{F}} \quad (4.2.29)$$

and the rate of change of internal energy is

$$\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \alpha \frac{\partial \phi}{\partial \mathbf{F}} : \dot{\mathbf{F}} = \dot{W} + \alpha \dot{\phi} \quad (4.2.30)$$

The strain energy is

$$W(\mathbf{F}) + \alpha \phi(\mathbf{F}) \quad (4.2.31)$$

The second term here is evidently zero and so does not contribute to the strain energy. However, the stress is the derivative of the strain energy with respect to a kinematic quantity, and the derivative of this last term, the second term in 4.2.29, is not be zero. The scalar α is, or determines the magnitude of, a **workless** reaction stress. It ensures that the constraint is satisfied; it is not set a value in the constitutive equation, rather, it is

determined by considering particular problems, through equilibrium and boundary conditions.

For N constraints $\phi_i(\mathbf{F}) = 0, i = 1 \dots N$, 4.2.29 generalises to

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} + \sum_{i=1}^N \alpha_i \frac{\partial \phi_i}{\partial \mathbf{F}} \quad (4.2.32)$$

Incompressible Materials

An incompressible material is one whose volume remains constant throughout a motion, and so has the following constraint:

$$J = \det \mathbf{F} = 1 \quad \text{or} \quad \dot{J} = 0 \quad (4.2.33)$$

From 1.11.34, $\partial(\det \mathbf{A}) / \partial \mathbf{A} = (\det \mathbf{A}) \mathbf{A}^{-T}$, one has $dJ / d\mathbf{F} = J\mathbf{F}^{-T}$, which equals $\mathbf{F}^{-T}$ at $J = 1$. Thus the PK1 stresses can be written as

$$\mathbf{P} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}} - p\mathbf{F}^{-T} \quad (4.2.34)$$

and the strain energy function is of the form

$$W(\mathbf{F}) = p(J(\mathbf{F}) - 1) \quad (4.2.35)$$

with $p = -\alpha$.

Using 3.5.9, $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$, one also has

$$\mathbf{S} = 2 \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} - p\mathbf{C}^{-1} \quad (4.2.36)$$

Using 2.5.20 and 2.5.18b,

$$\dot{J} = J \text{tr} \mathbf{d} = J \text{tr}(\mathbf{F}^{-T} \dot{\mathbf{E}} \mathbf{F}^{-1}) = J \text{tr}(\mathbf{C}^{-1} \dot{\mathbf{E}}) = J \mathbf{C}^{-1} : \dot{\mathbf{E}} = \frac{1}{2} J \mathbf{C}^{-1} : \dot{\mathbf{C}} \quad (4.2.37)$$

and the stress power is

$$\begin{cases} \mathbf{S} : \frac{1}{2} \dot{\mathbf{C}} = \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} : \dot{\mathbf{C}} - \frac{1}{2} p \mathbf{C}^{-1} : \dot{\mathbf{C}} \\ \mathbf{P} : \dot{\mathbf{F}} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} - p \mathbf{F}^{-T} : \dot{\mathbf{F}} \end{cases} = \dot{W} - p \dot{J} \quad (4.2.38)$$

consistent with 4.2.30.

The strain energy of 4.2.35 is expressed in terms of $\mathbf{F}$. This can be re-expressed in terms of $\mathbf{C}$. Since $J = \det \mathbf{F} = \sqrt{\det \mathbf{C}} = \sqrt{\text{III}_C}$, the incompressibility constraint can be expressed as

$$\phi(\mathbf{C}) = \text{III}_C - 1 = 0 \quad (4.2.39)$$

and the strain energy can be written as

$$W(\mathbf{C}) - \frac{1}{2} p(\text{III}_C(\mathbf{C}) - 1) \quad (4.2.40)$$

The factor of 1/2 has been included so that the p in 4.2.35 is the same as the p in 4.2.40, since, from 1.11.33,

$$\frac{\partial \text{III}_C}{\partial \mathbf{C}} = \mathbf{C}^{-1} \quad (4.2.41)$$

leading to the same expression for the PK2 stress as before, Eqn.4.2.36.

Similarly, in terms of the Cauchy stress, one has { **▲ Problem 1** }

$$\boxed{\boldsymbol{\sigma} = 2\mathbf{F} \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \mathbf{F}^T - p\mathbf{I}} \quad (4.2.42)$$

Because of its role in this equation, the scalar p is called the **hydrostatic pressure**.

Inextensible Constraint

Consider a material which is inextensible in a direction defined by a unit vector $\hat{\mathbf{A}}$ in the reference configuration. The constraint for this material is given by Eqn. 2.2.60,

$$\phi(\mathbf{F}) = \hat{\mathbf{A}}\mathbf{C}(\mathbf{F})\hat{\mathbf{A}} - 1 = 0 \quad (4.2.43)$$

Then, from 4.2.11,

$$\begin{aligned} \frac{\partial \phi}{\partial \mathbf{F}} &= 2\mathbf{F} \frac{\partial \phi}{\partial \mathbf{C}} \\ &= 2\mathbf{F}\hat{\mathbf{A}} \frac{\partial \mathbf{C}}{\partial \mathbf{C}} \hat{\mathbf{A}} \\ &= 2\mathbf{F}\hat{\mathbf{A}} \otimes \hat{\mathbf{A}} \end{aligned} \quad (4.2.44)$$

The stress is then

$$\boxed{\mathbf{P} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}} + 2\alpha \mathbf{F}\hat{\mathbf{A}} \otimes \hat{\mathbf{A}}} \quad (4.2.45)$$

Post-contracting with $\mathbf{F}^T$ (with $J = 1$) then gives

$$\boxed{\boldsymbol{\sigma} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^T + 2\alpha \mathbf{F} \hat{\mathbf{A}} \otimes \mathbf{F} \hat{\mathbf{A}}} \quad (4.2.46)$$

For inextensibility in two directions,

$$\phi_1(\mathbf{F}) = \hat{\mathbf{A}}_1 \mathbf{C}(\mathbf{F}) \hat{\mathbf{A}}_1 - 1 = 0, \quad \phi_2(\mathbf{F}) = \hat{\mathbf{A}}_2 \mathbf{C}(\mathbf{F}) \hat{\mathbf{A}}_2 - 1 = 0 \quad (4.2.47)$$

one has the stress

$$\boxed{\boldsymbol{\sigma} = 2\mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^T + \alpha_1 \mathbf{F} \hat{\mathbf{A}}_1 \otimes \mathbf{F} \hat{\mathbf{A}}_1 + \alpha_2 \mathbf{F} \hat{\mathbf{A}}_2 \otimes \mathbf{F} \hat{\mathbf{A}}_2} \quad (4.2.48)$$

4.2.6 Problems

1. Use the equation

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} - p \mathbf{F}^{-T}$$

to show that the constitutive equation for an incompressible hyperelastic material can be written in terms of the Cauchy stress as

$$\boldsymbol{\sigma} = 2\mathbf{F} \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \mathbf{F}^T - p \mathbf{I}$$

4.3 Isotropic Elastic Materials

4.3.1 Constitutive Equations for Isotropic Elastic Materials

Cauchy Stress and Isotropy

A material is said to be **isotropic** if a rotation of a particle (in the undeformed state) has no influence on the stress tensor. From §2.8.6, the condition of isotropy is then

$$\boldsymbol{\sigma}(\mathbf{F}^\diamond) = \boldsymbol{\sigma}(\mathbf{F}) \rightarrow \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q}^\mathsf{T}) = \boldsymbol{\sigma}(\mathbf{F}) \quad (4.3.1)$$

where the superscript $\diamond$ refers to deformations relative to the rotated reference configuration, Fig. 2.8.6. (This is often expressed in the equivalent form $\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q})$.)

A Cauchy elastic material automatically satisfies the isotropy condition when the Cauchy stress is an arbitrary tensor-valued function of the left Cauchy-Green tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{b}) \quad (4.3.2)$$

To see this, note that $\boldsymbol{\sigma}(\mathbf{b}(\mathbf{F})) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{F}^\mathsf{T})$, so

$$\boldsymbol{\sigma}(\mathbf{b}(\mathbf{F}\mathbf{Q}^\mathsf{T})) = \boldsymbol{\sigma}((\mathbf{F}\mathbf{Q}^\mathsf{T})(\mathbf{F}\mathbf{Q}^\mathsf{T})^\mathsf{T}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q}^\mathsf{T}\mathbf{Q}\mathbf{F}^\mathsf{T}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{F}^\mathsf{T}) = \boldsymbol{\sigma}(\mathbf{b}(\mathbf{F})) \quad (4.3.3)$$

Note that the condition of isotropy 4.3.1 must be satisfied for the particular rotation $\mathbf{Q} = \mathbf{R}$, so a condition to be satisfied by isotropic materials is

$$\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{R}^\mathsf{T}) = \boldsymbol{\sigma}(\mathbf{v}) = \boldsymbol{\sigma}(\mathbf{b}^{1/2}) = \boldsymbol{\sigma}(\mathbf{b}) \quad (4.3.4)$$

where $\mathbf{v}$ is the left stretch tensor, and again one has 4.3.2.

In addition to satisfying the condition of isotropy, the stress must satisfy the condition of objectivity: $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}(\mathbf{b}^*)$, where the superscript $*$ refers to objectivity transformations, §2.8.3. This requirement is automatically satisfied since both the Cauchy stress and the left Cauchy-Green tensors are objective spatial tensors:

$$\boxed{\mathbf{Q}\boldsymbol{\sigma}(\mathbf{b})\mathbf{Q}^\mathsf{T} = \boldsymbol{\sigma}(\mathbf{Q}\mathbf{b}\mathbf{Q}^\mathsf{T})} \quad \text{Isotropy Condition for the} \\ \text{(objective) Cauchy Stress} \quad (4.3.5)$$

PK2 Stress and Isotropy

From 3.5.7 and 2.8.58, a rigid body rotation of the reference configuration alters the PK2 stress according to

$$\mathbf{S}^\diamond = J^\diamond \mathbf{F}^{\diamond-1} \boldsymbol{\sigma}^\diamond \mathbf{F}^{\diamond-\mathsf{T}} = J(\mathbf{F}\mathbf{Q}^\mathsf{T})^{-1} \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q}^\mathsf{T})^{-\mathsf{T}} = \mathbf{Q}\mathbf{S}\mathbf{Q}^\mathsf{T} \quad (4.3.6)$$

On the other hand, $\mathbf{S}$ is objective when written as a function of the material tensor $\mathbf{C}$. Then, since $\mathbf{C}^\diamond = \mathbf{Q}\mathbf{C}\mathbf{Q}^T$ (see Eqns. 2.8.58),

$$\boxed{\mathbf{Q}\mathbf{S}(\mathbf{C})\mathbf{Q}^T = \mathbf{S}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T)} \quad \text{Isotropy Condition for the} \\ \text{(objective) PK2 Stress} \quad (4.3.7)$$

Isotropic Tensor Functions

The constitutive relations 4.3.5 for the Cauchy stress and 4.3.7 for the PK2 stress are very similar. In general, the second-order tensor-valued function $\mathbf{T}$ of the second-order tensor variable $\mathbf{B}$ is an **isotropic tensor functions** if

$$\boxed{\mathbf{T}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) = \mathbf{Q}\mathbf{T}(\mathbf{B})\mathbf{Q}^T} \quad \text{Isotropic Tensor Function} \quad (4.3.8)$$

for all orthogonal tensors $\mathbf{Q}$. Thus, $\boldsymbol{\sigma}$ is an isotropic tensor function of the tensor variable $\mathbf{b}$ and $\mathbf{S}$ is an isotropic tensor function of the tensor variable $\mathbf{C}$. Isotropic functions are discussed in the Appendix to this Chapter, §4.A, and there it is shown that, for symmetric $\mathbf{T}$ and symmetric $\mathbf{B}$, $\mathbf{T}$ must take the form

$$\boxed{\mathbf{T}(\mathbf{B}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2} \quad \text{Form for a symmetric isotropic tensor} \\ \text{function of a symmetric tensor} \quad (4.3.9)$$

where $\alpha_0, \alpha_1, \alpha_2$ are scalar functions of the principal scalar invariants of $\mathbf{B}$, 1.9.38, 1.9.46:

$$\alpha_i = \alpha_i \{I_{\mathbf{B}}, II_{\mathbf{B}}, III_{\mathbf{B}}\} \quad (4.3.10)$$

Since the set of principal scalar invariants, the set $\{\text{tr}\mathbf{S}, \text{tr}\mathbf{S}^2, \text{tr}\mathbf{S}^3\}$ and the set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ uniquely determine one another, the coefficients α_i can be taken to be functions of any one of these three sets.

Equation 4.3.9 can be rewritten in various alternative forms using the Cayley-Hamilton theorem, 1.9.45:

$$\mathbf{B}^3 - I_{\mathbf{B}}\mathbf{B}^2 + II_{\mathbf{B}}\mathbf{B} - III_{\mathbf{B}}\mathbf{I} = \mathbf{0}, \quad (4.3.11)$$

for example

$$\mathbf{T}(\mathbf{B}) = \beta_0 \mathbf{I} + \beta_1 \mathbf{B} + \beta_{-1} \mathbf{B}^{-1} \quad (4.3.12)$$

where

$$\beta_0 = \alpha_0 - II_{\mathbf{B}}\alpha_2, \quad \beta_1 = \alpha_1 + I_{\mathbf{B}}\alpha_2, \quad \beta_{-1} = III_{\mathbf{B}}\alpha_2. \quad (4.3.13)$$

The Cauchy-Elastic Solid

Since for an isotropic Cauchy-elastic solid, the Cauchy stress is an isotropic tensor function of the left Cauchy-Green strain, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{b})$, Eqn. 4.3.5 (a consequence of isotropy and objectivity) and since $\boldsymbol{\sigma}$ and $\mathbf{b}$ are symmetric, it follows from 4.3.9 that the Cauchy stress takes the form

$$\boldsymbol{\sigma} = \alpha_0(I_b, II_b, III_b)\mathbf{I} + \alpha_1(I_b, II_b, III_b)\mathbf{b} + \alpha_2(I_b, II_b, III_b)\mathbf{b}^2 \quad (4.3.14)$$

or, alternatively, from 4.3.12, the form

$$\boldsymbol{\sigma} = \beta_0(I_b, II_b, III_b)\mathbf{I} + \beta_1(I_b, II_b, III_b)\mathbf{b} + \beta_{-1}(I_b, II_b, III_b)\mathbf{b}^{-1} \quad (4.3.15)$$

and these scalar functions are related through 4.3.13.

Similar forms hold for the PK2 stress as a function of the right Cauchy-Green strain.

4.3.2 Strain Energy and Isotropy

Consider the strain energy $W = W(\mathbf{F})$. From §2.8.4, objectivity requires that (see Eqn. 2.8.46),

$$W(\mathbf{F}) = W(\mathbf{U}) \quad (4.3.16)$$

Also, the right stretch tensor is the square-root of the right Cauchy-Green tensor, so one can write $W = W(\mathbf{C})$, which is clearly objective, since $W(\mathbf{C}) = W(\mathbf{C}^*)$. In addition to the objectivity requirement, isotropy requires that $W(\mathbf{C}) = W(\mathbf{C}^\diamond)$, or, 2.8.58b

$$\boxed{W(\mathbf{C}) = W(\mathbf{Q}\mathbf{C}\mathbf{Q}^T)} \quad \text{Isotropy Condition for the} \\ \text{(objective) Strain Energy} \quad (4.3.17)$$

Isotropic Scalar Functions

The strain energy of Eqn. 4.3.17 is also an isotropic function; in general, the scalar function ϕ of the second-order tensor variable $\mathbf{B}$ is an **isotropic scalar functions** if

$$\boxed{\phi(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) = \phi(\mathbf{B})} \quad \text{Isotropic Scalar Function} \quad (4.3.18)$$

for all orthogonal tensors $\mathbf{Q}$. Thus, W is an isotropic scalar function of the tensor variable $\mathbf{C}$. It is shown in the Appendix to this Chapter, §4.A, that, for symmetric $\mathbf{B}$, ϕ must take the form

$$\boxed{\phi(\mathbf{B}) = \phi(\{I_B, II_B, III_B\})} \quad \text{Form for an isotropic scalar} \\ \text{function of a symmetric tensor} \quad (4.3.19)$$

Thus the strain energy for an isotropic elastic material must be a function only of the three principal scalar invariants of the right Cauchy-Green strain (or of the three principal values λ_i). Since the invariants for the right- and left-Cauchy-Green strain tensors are the same (see Eqn. 2.2.15), the strain energy can also be expressed as a function of the invariants of $\mathbf{b}$.

4.4 Isotropic Hyperelasticity

Attention is now restricted to the case of isotropic hyperelastic materials.

4.4.1 Constitutive Equations in Material Form

Consider the general hyperelastic constitutive law 4.2.5:

$$\mathbf{S} = 2 \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \quad (4.4.1)$$

From §4.3.2, the strain energy for an isotropic hyperelastic material must be a function of the principal invariants of $\mathbf{C}$:

$$W = W(I_c(\mathbf{C}), II_c(\mathbf{C}), III_c(\mathbf{C})) \quad (4.4.2)$$

Then, using the relations 1.11.33 for the derivatives of the invariants (taking into account that $\mathbf{C}$ is symmetric),

$$\begin{aligned} \frac{\partial W}{\partial \mathbf{C}} &= \frac{\partial W}{\partial I_c} \frac{\partial I_c}{\partial \mathbf{C}} + \frac{\partial W}{\partial II_c} \frac{\partial II_c}{\partial \mathbf{C}} + \frac{\partial W}{\partial III_c} \frac{\partial III_c}{\partial \mathbf{C}} \\ &= \left(\frac{\partial W}{\partial I_c} + \frac{\partial W}{\partial II_c} I_c \right) \mathbf{I} - \frac{\partial W}{\partial II_c} \mathbf{C} + \frac{\partial W}{\partial III_c} III_c \mathbf{C}^{-1} \end{aligned} \quad (4.4.3)$$

Further, the $\mathbf{C}^{-1}$ term can be replaced by a $\mathbf{C}^2$ term using the Cayley-Hamilton theorem. In summary then, expressing $\mathbf{S}$ in the form 4.3.9 (and 4.3.12),

$$\mathbf{S} = 2 \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} = \begin{cases} 2\alpha_0 \mathbf{I} + 2\alpha_1 \mathbf{C} + 2\alpha_2 \mathbf{C}^2 \\ 2\beta_0 \mathbf{I} + 2\beta_1 \mathbf{C} + 2\beta_{-1} \mathbf{C}^{-1} \end{cases}$$

$$\begin{aligned} \alpha_0 &= \frac{\partial W}{\partial I_c} + \frac{\partial W}{\partial II_c} I_c + \frac{\partial W}{\partial III_c} II_c & \beta_0 &= \frac{\partial W}{\partial I_c} + \frac{\partial W}{\partial II_c} I_c \\ \alpha_1 &= -\frac{\partial W}{\partial II_c} - \frac{\partial W}{\partial III_c} I_c & \beta_1 &= -\frac{\partial W}{\partial II_c} \\ \alpha_2 &= \frac{\partial W}{\partial III_c} & \beta_{-1} &= \frac{\partial W}{\partial III_c} III_c \end{aligned}$$

Constitutive Equation in material description (4.4.4)

4.4.2 Constitutive Equations in Spatial Form

As mentioned at the end of §4.3.2, the invariants of $\mathbf{C}$ and $\mathbf{b}$ are the same and the strain energy can be expressed in terms of the invariants of $\mathbf{b}$, $W = W(I_b(\mathbf{b}), II_b(\mathbf{b}), III_b(\mathbf{b}))$.

Then

$$\frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} = \begin{cases} \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2 \\ \beta_0 \mathbf{I} + \beta_1 \mathbf{b} + \beta_{-1} \mathbf{b}^{-1} \end{cases} \quad (4.4.5)$$

and the coefficients α_i, β_i are the same as in 4.4.4

Taking the expression for $\partial W / \partial \mathbf{C}$ in 4.4.4, pre-contracting with $\mathbf{F}$, post-contracting with $\mathbf{F}^T$, using the definitions $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, $\mathbf{b} = \mathbf{F} \mathbf{F}^T$, and (pre- or post-) contracting the above similar expression for $\partial W / \partial \mathbf{b}$ in terms of the invariants of $\mathbf{b}$ with $\mathbf{b}$, one finds that

$$\frac{\partial W}{\partial \mathbf{b}} \mathbf{b} = \mathbf{b} \frac{\partial W}{\partial \mathbf{b}} = \mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^T \quad (4.4.6)$$

Therefore, using $\mathbf{S} = J \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}^{-T}$, the constitutive equation in the material formulation, $\mathbf{S} = 2 \partial W(\mathbf{C}) / \partial \mathbf{C}$, can be transformed into a spatial constitutive equation:

$$\boldsymbol{\sigma} = 2J^{-1} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} \mathbf{b} = 2J^{-1} \mathbf{b} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} \quad (4.4.7)$$

It should be emphasised that this expression, unlike the relation $\mathbf{S} = 2 \partial W(\mathbf{C}) / \partial \mathbf{C}$, holds only for isotropic materials.

Differentiating the strain energy with respect to $\mathbf{b}$ (exactly as in Eqn. 4.4.3) and pre- or post-contracting the result with $2J^{-1} \mathbf{b}$ then leads to

$$\boxed{\boldsymbol{\sigma} = 2J^{-1} \mathbf{b} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} = \begin{cases} \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2 \\ \beta_0 \mathbf{I} + \beta_1 \mathbf{b} + \beta_{-1} \mathbf{b}^{-1} \end{cases}}$$

$$\begin{aligned} \alpha_0 &= 2J^{-1} \left[\frac{\partial W}{\partial III_b} III_b \right] & \beta_0 &= 2J^{-1} \left[\frac{\partial W}{\partial II_b} II_b + \frac{\partial W}{\partial III_b} III_b \right] \\ \alpha_1 &= 2J^{-1} \left[\frac{\partial W}{\partial I_b} + \frac{\partial W}{\partial II_b} I_b \right] & \beta_1 &= 2J^{-1} \left[\frac{\partial W}{\partial I_b} \right] \\ \alpha_2 &= 2J^{-1} \left[-\frac{\partial W}{\partial II_b} \right] & \beta_{-1} &= 2J^{-1} \left[-\frac{\partial W}{\partial II_b} III_b \right] \end{aligned}$$

Constitutive Equation in spatial description (4.4.8)

Principal Directions of the Cauchy Stress and Left Cauchy-Green Tensor

Recall that the eigenvalues of the right stretch tensor $\mathbf{U}$ are the principal stretches λ_i , and that the eigenvalues of the right Cauchy-Green tensor $\mathbf{C}$ and the left Cauchy-Green tensor $\mathbf{b}$ are the squares of the principal stretches, λ_i^2 (the eigenvectors of $\mathbf{b}$ are those of $\mathbf{C}$ rotated with the rotation tensor $\mathbf{R}$). Taking $\hat{\mathbf{n}}$ to be an eigenvector of $\mathbf{b}$ and λ^2 the corresponding eigenvalue, and using the constitutive equation $\boldsymbol{\sigma} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2$, one has

$$\boldsymbol{\sigma} \hat{\mathbf{n}} = \alpha_0 \mathbf{I} \hat{\mathbf{n}} + \alpha_1 \mathbf{b} \hat{\mathbf{n}} + \alpha_2 \mathbf{b}^2 \hat{\mathbf{n}} = (\alpha_0 + \alpha_1 \lambda^2 + \alpha_2 \lambda^4) \hat{\mathbf{n}} \quad (4.4.9)$$

Thus $\boldsymbol{\sigma} \hat{\mathbf{n}} = \sigma \hat{\mathbf{n}}$ where σ is an eigenvalue of $\boldsymbol{\sigma}$, the principal stress, given by term inside the brackets, and so the eigenvectors, or principal directions, of the Cauchy stress and $\mathbf{b}$ coincide.

If one takes a deformation for which $b_{13} = b_{23} = 0$, it follows from 4.4.8 that, for arbitrary coefficient β_{-1} , one also has $\sigma_{13} = \sigma_{23} = 0$. The tensors $\mathbf{b}$ and $\boldsymbol{\sigma}$ are coaxial, that is, $\mathbf{b}\boldsymbol{\sigma} = \boldsymbol{\sigma}\mathbf{b}$. From this relation, one finds that one must have

$$\frac{\sigma_{11} - \sigma_{22}}{\sigma_{12}} = \frac{b_{11} - b_{22}}{b_{12}} \quad (4.4.10)$$

This relation holds for all compressible isotropic hyperelastic materials under the stress and deformation conditions stated, regardless of the particular constitutive relation.

4.4.3 Constitutive Equations in terms of the Stretches

The derivatives of the stretch with respect to $\mathbf{b}$ are given in §2.3.3. Introducing the principal stresses, and noting that $\mathbf{b}\hat{\mathbf{n}}_i = \lambda_i^2 \hat{\mathbf{n}}_i$ (no sum over the i) it follows then that (no sum over the i in what follows)

$$\begin{aligned} \sigma_i \hat{\mathbf{n}}_i &= \boldsymbol{\sigma} \hat{\mathbf{n}}_i = 2J^{-1} \mathbf{b} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} \hat{\mathbf{n}}_i = 2J^{-1} \mathbf{b} \frac{\partial W(\lambda)}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial \mathbf{b}} \hat{\mathbf{n}}_i \\ &= 2J^{-1} \mathbf{b} \frac{\partial W(\lambda)}{\partial \lambda_k} \frac{1}{2\lambda_k} (\hat{\mathbf{n}}_k \otimes \hat{\mathbf{n}}_k) \hat{\mathbf{n}}_i = 2J^{-1} \frac{\partial W(\lambda)}{\partial \lambda_i} \frac{1}{2\lambda_i} \mathbf{b} \hat{\mathbf{n}}_i \\ &= J^{-1} \lambda_i \frac{\partial W(\lambda)}{\partial \lambda_i} \hat{\mathbf{n}}_i \end{aligned} \quad (4.4.11)$$

or

$$\sigma_1 = J^{-1} \lambda_1 \frac{\partial W(\lambda)}{\partial \lambda_1}, \quad \sigma_2 = J^{-1} \lambda_2 \frac{\partial W(\lambda)}{\partial \lambda_2}, \quad \sigma_3 = J^{-1} \lambda_3 \frac{\partial W(\lambda)}{\partial \lambda_3} \quad (4.4.12)$$

Similarly, for the Piola-Kirchhoff stresses, one has

$$P_i = \frac{\partial W}{\partial \lambda_i}, \quad S_i = \frac{1}{\lambda_i} \frac{\partial W}{\partial \lambda_i} \quad (4.4.13)$$

Example (Uniaxial Stretch)

Consider a specimen of isotropic hyperelastic material under a uniaxial tension σ_{11} , with

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (4.4.14)$$

Since $\boldsymbol{\sigma}$ and $\mathbf{b}$ are coaxial (see 4.4.9), the principal stresses are, from 4.4.8,

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta_1 \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} + \beta_{-1} \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix} \quad (4.4.15)$$

Subtracting the second of these equations from the first leads to

$$\sigma_{11} = (\lambda_1^2 - \lambda_2^2) \left(\beta_1 - \beta_{-1} \frac{1}{\lambda_1^2 \lambda_2^2} \right) \quad (4.4.16)$$

Subtracting the third from the second leads to

$$0 = (\lambda_2^2 - \lambda_3^2) \left(\beta_1 - \beta_{-1} \frac{1}{\lambda_2^2 \lambda_3^2} \right) \quad (4.4.17)$$

which implies that $\lambda_2 = \lambda_3$ (the terms inside the second bracket can also be zero, but not if $\beta_1 > 0$, $\beta_{-1} \leq 0$, as is usually the case).

The material parameters β_0 , β_1 , β_{-1} can be obtained from such a test by measuring σ_{11} , λ_1 and λ_2 . On the other hand, if the constitutive relation, that is, the parameters, are known, then one can obtain the stretches by specifying the uniaxial stress.

■

4.4.4 Incompressible Materials

When the material is incompressible, then

$$J = \det \mathbf{F} = \text{III}_{\mathbf{C}} = \text{III}_{\mathbf{b}} = 1 \quad (4.4.18)$$

and the strain energy function is only dependent on two invariants:

$$W = W(I_C, II_C) = W(I_b, II_b), \quad (4.4.19)$$

and the constitutive equations are the same as those given earlier for the compressible material, with $III_b = 1$, $J^{-1} = 1$. However, the derivative $\partial W / \partial III_C$ is an unknown and in particular it is unknown at $III_C = 1$.

Returning to the more general constitutive equations for an incompressible material, 4.2.36, 4.2.39, one has

$$\mathbf{S} = 2 \frac{\partial W(I_C, II_C)}{\partial \mathbf{C}} - p \mathbf{C}^{-1}, \quad \boldsymbol{\sigma} = 2 \mathbf{F} \frac{\partial W(I_C, II_C)}{\partial \mathbf{C}} \mathbf{F}^T - p \mathbf{I} \quad (4.4.20)$$

Carrying out the differentiation as before, one now has, directly from these equations,

$$\begin{aligned} \mathbf{S} &= \beta_0 \mathbf{I} + \beta_1 \mathbf{C} - p \mathbf{C}^{-1} \\ \beta_0 &= 2 \left[\frac{\partial W}{\partial I_C} + \frac{\partial W}{\partial II_C} I_C \right] \quad \beta_1 = -2 \frac{\partial W}{\partial II_C} \end{aligned}$$

Constitutive Equation for an Isotropic Incompressible Hyperelastic Material (Material Form) (4.4.21)

and

$$\begin{aligned} \boldsymbol{\sigma} &= \alpha_1 \mathbf{b} + \alpha_{-1} \mathbf{b}^{-1} - p \mathbf{I} \\ \alpha_1 &= 2 \frac{\partial W}{\partial I_b} \quad \alpha_{-1} = -2 \frac{\partial W}{\partial II_b} \end{aligned}$$

Constitutive Equation for an Isotropic Incompressible Hyperelastic Material (Spatial Form) (4.4.22)

Comparing these equations with the more general compressible equations: in the $\mathbf{S}$ equation, 4.4.4, the hydrostatic pressure is equivalent to $-2 \partial W / \partial III_C$ (with $III_C = 1$); in the $\boldsymbol{\sigma}$ equation, 4.4.8, it is $p = -2 [\partial W / \partial III_b + II_b \partial W / \partial II_b]$.

Similarly, the equations can be written in terms of the principal stretches and principal stresses (the incompressibility constraint is now $\lambda_1 \lambda_2 \lambda_3 = 1$):

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad P_i = \frac{\partial W}{\partial \lambda_i} - \frac{1}{\lambda_i} p, \quad S_i = \frac{1}{\lambda_i} \frac{\partial W}{\partial \lambda_i} - \frac{1}{\lambda_i^2} p \quad (4.4.23)$$

Example (Strip Biaxial Tension)

Consider an incompressible isotropic hyperelastic material which is stretched in the $\mathbf{e}_1$ direction but its dimensions in the $\mathbf{e}_2$ direction is held constant, with

$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} \quad (4.4.24)$$

With $\sigma_3 = 0$, one finds that

$$p = 2 \frac{\partial W}{\partial I_b} \frac{1}{\lambda^2} - 2 \frac{\partial W}{\partial \Pi_b} \lambda^2 \quad (4.4.25)$$

and the stresses are

$$\sigma_i = 2 \left[\frac{\partial W}{\partial I_b} \left(\lambda_i^2 - \frac{1}{\lambda^2} \right) + \frac{\partial W}{\partial \Pi_b} \left(\lambda^2 - \frac{1}{\lambda_i^2} \right) \right], \quad i = 1, 2 \quad (4.4.26)$$

■

Example (Biaxial Stretch)

Consider the biaxial stretch of a thin sheet of incompressible isotropic hyperelastic material, with

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1/\lambda_1 \lambda_2 \end{bmatrix} \quad (4.4.27)$$

From 4.4.18:

$$\sigma_i = \alpha_1 \lambda_i^2 + \alpha_{-1} \lambda_i^{-2} - p, \quad i = 1, 2, 3, \quad \left(\alpha_1 = 2 \frac{\partial W}{\partial I_b}, \alpha_{-1} = -2 \frac{\partial W}{\partial \Pi_b} \right) \quad (4.4.28)$$

Taking the stretches to be in-plane, with the larger faces stress-free, one has $\sigma_3 = 0$ so that, with $\lambda_3 = 1/\lambda_1 \lambda_2$, one can solve for p ,

$$p = \alpha_1 \frac{1}{\lambda_1^2 \lambda_2^2} + \alpha_{-1} \lambda_1^2 \lambda_2^2$$

and the stresses in the sheet are

$$\sigma_1 = (\alpha_1 - \alpha_{-1}\lambda_2^2) \left(\lambda_1^2 - \frac{1}{\lambda_1^2 \lambda_2^2} \right), \quad \sigma_2 = (\alpha_1 - \alpha_{-1}\lambda_1^2) \left(\lambda_2^2 - \frac{1}{\lambda_1^2 \lambda_2^2} \right) \quad (4.4.29)$$

If the strain energy is given in terms of the stretches, one would use 4.4.23 rather than 4.4.22.

The PK1 stresses, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$, are, directly from these equations,

$$P_1 = \frac{\sigma_1}{\lambda_1} = (\alpha_1 - \alpha_{-1}\lambda_2^2) \left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2} \right), \quad P_2 = \frac{\sigma_2}{\lambda_2} = (\alpha_1 - \alpha_{-1}\lambda_1^2) \left(\lambda_2 - \frac{1}{\lambda_1^2 \lambda_2^3} \right) \quad (4.4.30)$$

The total forces acting on the lateral surfaces are then, from 3.5.4,

$$F_1 = P_1 A_0 = A_0 (\alpha_1 - \alpha_{-1}\lambda_2^2) \left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2} \right), \quad F_2 = P_2 A_0 = A_0 (\alpha_1 - \alpha_{-1}\lambda_1^2) \left(\lambda_2 - \frac{1}{\lambda_1^2 \lambda_2^3} \right) \quad (4.4.31)$$

where A_0 is the area in the reference configuration.

The material parameters α_1, α_{-1} can be determined from such a test by measuring the applied forces. This is helped by controlling the stretches λ_1 and λ_2 in such a way that the invariant I_b is held constant, or the invariant II_b is held constant.

When the two applied forces are equal, $F_1 = F_2$, one finds that

$$(\lambda_1 - \lambda_2) \left(\left(1 + \lambda_1^3 \lambda_2^2 \right) \left(1 + \frac{\alpha_{-1}}{\alpha_1} \lambda_1 \lambda_2 \right) - \frac{\alpha_{-1}}{\alpha_1} (\lambda_1 + \lambda_2)^2 \right) = 0 \quad (4.4.32)$$

The term inside the first brackets gives the expected symmetric solution $\lambda_1 = \lambda_2$. However, the term inside the second brackets can also be zero for $\lambda_1 \neq \lambda_2$, but this will only occur for fairly large stretches for typical values of the material parameters. The symmetric equal-stretch solution holds up until this happens. For example, for the typical value $-\alpha_{-1}/\alpha_1 = 0.1$, the second term becomes zero when $\lambda_1 = \lambda_2 \approx 3.17$. The symmetric solution is unstable, in the sense that the forces must be precisely equal for it to hold. If there is even the slightest imbalance between F_1 and F_2 , then $\lambda_1 \approx \lambda_2$ up to about a value of 3.17 but then the near-square will begin to deform into a rectangle, with the response now keeping close to the asymmetric solution. The long side of the rectangle will be along the direction of the slightly larger force and will keep lengthening until the stretch in the other direction returns back to $\lambda = 1$.

This large strain problem differs from the problems of linear elasticity, which have unique solutions.

■

Example (Simple Shear)

The case of simple shear was discussed in detail in §2.2.6. In the equations following 2.2.40, let

$$\cos 2\beta = \sin \theta, \quad \sin 2\beta = \cos \theta, \quad \tan 2\beta = \frac{1}{\tan \theta} = \frac{2}{k} \quad (4.4.33)$$

Then the principal material directions can be expressed as

$$\hat{\mathbf{N}}_1 = \sin \beta \mathbf{E}_1 + \cos \beta \mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = -\cos \beta \mathbf{E}_1 + \sin \beta \mathbf{E}_2 \quad (4.4.34)$$

With $\mathbf{n} = \mathbf{R}\mathbf{N}$, the principal spatial directions are (see figure 4.4.1 below)

$$\hat{\mathbf{n}}_1 = \cos \beta \mathbf{E}_1 + \sin \beta \mathbf{E}_2, \quad \hat{\mathbf{n}}_2 = -\sin \beta \mathbf{E}_1 + \cos \beta \mathbf{E}_2 \quad (4.4.35)$$

The principal stretches are

$$\lambda_1 = \frac{1 + \cos 2\beta}{\sin 2\beta}, \quad \lambda_2 = \frac{1 - \cos 2\beta}{\sin 2\beta} \quad \text{and} \quad 1 \quad (4.4.36)$$

and the principal stresses are, from 4.4.18,

$$\begin{aligned} \sigma_1 &= \alpha_1 \lambda_1^2 + \alpha_{-1} \frac{1}{\lambda_1^2} - p \\ \sigma_2 &= 2\alpha_1 \lambda_2^2 + \alpha_{-1} \frac{1}{\lambda_2^2} - p \\ \sigma_3 &= \alpha_1 + \alpha_{-1} - p \end{aligned} \quad \left(\alpha_1 = 2 \frac{\partial W}{\partial \text{I}_{\mathbf{b}}}, \alpha_{-1} = -2 \frac{\partial W}{\partial \text{II}_{\mathbf{b}}} \right) \quad (4.4.37)$$

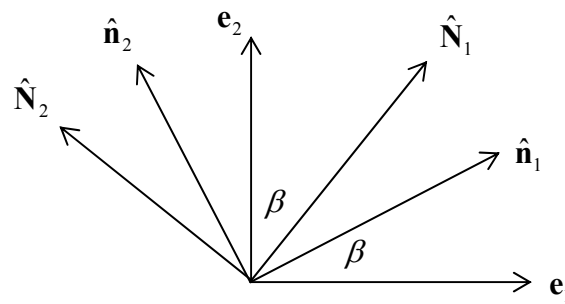


Figure 4.4.1: principal directions for pure shear

The strain energy here is a function of the invariants of $\mathbf{b}$, which are

$$\text{I}_{\mathbf{b}} = 3 + k^2, \quad \text{II}_{\mathbf{b}} = 3 + k^2 \quad (\text{III}_{\mathbf{b}} = 1) \quad (4.4.38)$$

In the original Cartesian coordinates, the stresses are

$$\begin{aligned}
\sigma_{11} &= \sigma_1 \cos^2 \beta + \sigma_2 \sin^2 \beta \\
\sigma_{22} &= \sigma_1 \sin^2 \beta + \sigma_2 \cos^2 \beta \\
\sigma_{12} &= (\sigma_1 - \sigma_2) \sin \beta \cos \beta
\end{aligned} \tag{4.4.39}$$

It follows that (this is a universal relation, independent of the constitutive relation)

$$\sigma_{11} - \sigma_{22} = (2 / \tan 2\beta) \sigma_{12} = k \sigma_{12} \tag{4.4.40}$$

The fact that $\sigma_{11} \neq \sigma_{22}$ is called the **Poynting effect**. The stress are now

$$\begin{aligned}
\sigma_{11} &= \alpha_1 (1 + k^2) + \alpha_{-1} - p \\
\sigma_{22} &= \alpha_1 + \alpha_{-1} (1 + k^2) - p \\
\sigma_{12} &= (\alpha_1 - \alpha_{-1}) k \\
\sigma_{33} &= \alpha_1 + \alpha_{-1} - p
\end{aligned} \tag{4.4.41}$$

The proportionality factor between the shear stress and the shear strain measure k is called the shear modulus:

$$\mu = 2 \left(\frac{\partial W}{\partial I_b} + \frac{\partial W}{\partial II_b} \right) \tag{4.4.42}$$

In the case of plane stress, the hydrostatic pressure p can be evaluated, and one finds that

$$\sigma_{11} = \alpha_1 k^2, \quad \sigma_{22} = \alpha_{-1} k^2, \quad \sigma_{12} = \mu k \tag{4.4.43}$$

The stresses acting on the deformed material are shown in Fig. 4.4.2. The unit normal $\mathbf{e}$ to the sloping surface (on the right hand side) is

$$\mathbf{e} = \frac{1}{\sqrt{1+k^2}} (\mathbf{e}_1 - k \mathbf{e}_2) \tag{4.4.44}$$

and, from Cauchy's law, the traction acting on that surface is

$$\mathbf{t} = \frac{1}{\sqrt{1+k^2}} \begin{bmatrix} \sigma_{11} - k \sigma_{12} \\ \sigma_{12} - k \sigma_{22} \\ 0 \end{bmatrix} \tag{4.4.45}$$

The normal and shear traction are then (using the Poynting effect)

$$\begin{aligned}
t_N &= \frac{1}{1+k^2} [(\sigma_{11} - k \sigma_{12}) - k(\sigma_{12} - k \sigma_{22})] = \sigma_{11} - \frac{k(2+k^2)}{1+k^2} \sigma_{12} = \sigma_{22} - \frac{k}{1+k^2} \sigma_{12} \\
t_S &= \frac{1}{1+k^2} [k(\sigma_{11} - k \sigma_{12}) + (\sigma_{12} - k \sigma_{22})] = \frac{1}{1+k^2} \sigma_{12}
\end{aligned}$$

(4.4.46)

and tend to σ_{11} and σ_{12} as $k \rightarrow 0$.

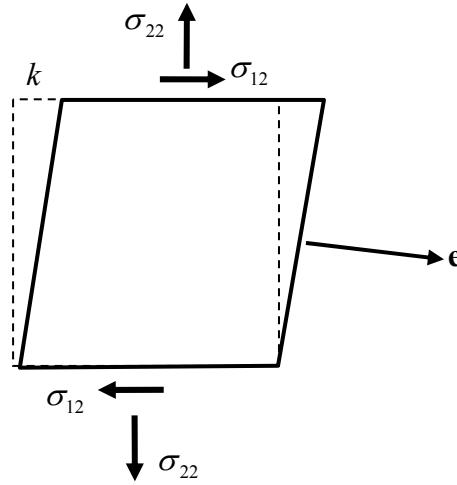


Figure 4.4.2: stresses acting on the material in simple shear

It is clear then that the pure shear of material discussed here can only occur when normal stresses are applied to the faces. There is of course no volume change for an incompressible material; for a compressible material, however, there will in general be a volume change unless normal stresses are applied, a phenomenon known as the **Kelvin effect**.

■

See also the shear problem using convected coordinates discussed in the Appendix to this Chapter, §4.A.3.

4.4.5 Series Expansion of the Strain Energy Function

It is sometimes helpful when seeking specific constitutive laws for isotropic hyperelastic materials to expand the strain energy function into an infinite power series of the form

$$W = W(I_C, II_C, III_C) = \sum_{p,q,r=0}^{\infty} \alpha_{pqr} (I_C - 3)^p (II_C - 3)^q (III_C - 1)^r \quad (4.4.47)$$

In the undeformed configuration, $I_C = II_C = 3$, $III_C = 1$, and the strain energy is zero. For incompressible materials, this reads as

$$W = \sum_{p,q=0}^{\infty} \alpha_{pq} (I_C - 3)^p (II_C - 3)^q \quad (4.4.48)$$

The series can be written in terms of the principal stretches by noting that

$$\begin{aligned}
I_C &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
II_C &= \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \\
III_C &= \lambda_1^2 \lambda_2^2 \lambda_3^2
\end{aligned} \tag{4.4.49}$$

Next are considered some particular forms of the strain energy function.

4.4.6 Incompressible Material Models

First, models of isotropic hyperelastic incompressible materials are examined.

The Ogden Model (Incompressible)

The strain energy for the **Ogden model** is defined to be

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^n \frac{\mu_i}{\alpha_i} \{ \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \} \tag{4.4.50}$$

where the μ_i, α_i are experimentally determined material constants. Note that this model does not include terms involving the products of different stretches, which appear in the most general power series 4.4.48. However, the Ogden model allows for fractional powers of α_i . Only three terms in the series are usually needed to correlate the model with experimental data of incompressible rubber. Typical values are

$$\begin{aligned}
\alpha_1 &= 1.3 & \mu_1 &= 6300 \text{ kPa} \\
\alpha_2 &= 5.0 & \mu_2 &= 1.2 \text{ kPa} \\
\alpha_3 &= -2.0 & \mu_3 &= -10 \text{ kPa}
\end{aligned}$$

The Mooney-Rivlin Model (Incompressible)

The **Mooney-Rivlin** model, applicable to incompressible materials, is

$$W = c_1(I_C - 3) + c_2(II_C - 3) \tag{4.4.51}$$

And, from 4.4.22, the stresses are then

$$\boldsymbol{\sigma} = 2c_1 \mathbf{b} - 2c_2 \mathbf{b}^{-1} - p \mathbf{I} \tag{4.4.52}$$

From the simple shear example discussed in §4.4.4, the shear modulus for the Mooney-Rivlin material is $2(c_1 + c_2)$, implying that the shear stress to shear strain measure k is linear.

The Mooney-Rivlin model is often used to model rubber-like materials and the numerical value of c_2 is usually much smaller than c_1 .

Note that the Mooney-Rivlin model can be viewed as a special case of the Ogden model, since setting $n = 2$, $\alpha_1 = 2$, $\alpha_2 = -2$ in that 4.4.50, and using the incompressibility constraint $\lambda_1^2 \lambda_2^2 \lambda_3^2 = 1$ gives

$$\begin{aligned} W &= \frac{\mu_1}{2} \{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\} - \frac{\mu_2}{2} \{\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\} \\ &= \frac{\mu_1}{2} \{I_C - 3\} - \frac{\mu_2}{2} \{\Pi_C - 3\} \end{aligned} \quad (4.4.53)$$

The Neo-Hookean Model (Incompressible)

The **Neo-Hookean** is a special case of the Mooney-Rivlin model, wherein $c_2 = 0$, giving

$$W = c_1 (I_C - 3) \quad (4.4.54)$$

It is useful as a model of non-linear elasticity within the small-strain range.

The Varga Model (Incompressible)

The **Varga** model is another special case of the Ogden model:

$$W = c_1 (\lambda_1 + \lambda_2 + \lambda_3 - 3) \quad (4.4.55)$$

The Yeoh Model (Incompressible)

The **Yeoh** model was motivated by the problem of predicting the behaviour of rubbers containing reinforcing fillers such as carbon black, for which the Mooney-Rivlin model does not succeed too well. The strain energy here takes the form

$$W = c_1 (I_C - 3) + c_2 (I_C - 3)^2 + c_3 (I_C - 3)^3 \quad (4.4.56)$$

As usual, $W(\mathbf{I}) = 0$ in the reference configuration and the strain energy is a convex function, increasing monotonically with I_C (and attains a minimum in the reference configuration, so that $\partial W / \partial I_C = 0$ must contain no real roots, which places restrictions on the values that the constants c_1, c_2, c_3 can take).

The Arruda-Boyce Model (Incompressible)

The Arruda-Boyce model is based on a proposed molecular network structure to a material, and the strain energy is

$$W = \mu \left[\frac{1}{2} (I_C - 1) + \frac{1}{20n} (I_C^2 - 9) + \frac{11}{1050n^2} (I_C^3 - 27) + \dots \right] \quad (4.4.57)$$

The two parameters are μ , the shear modulus, and n , the number of segments in a polymer chain.

4.4.7 Compressible Material Models

The Mooney-Rivlin Model (Compressible)

The compressible Mooney-Rivlin model reads as

$$W = c(J-1)^2 - 2(c_1 + c_2) \ln J + c_1(I_b - 3) + c_2(II_b - 3) \quad (4.4.58)$$

The stresses are then, from 4.4.8, with $J = \sqrt{III_b}$,

$$\boldsymbol{\sigma} = \frac{2}{\sqrt{III_b}} \left\{ \left[c(III_b - \sqrt{III_b}) - 4(c_1 + c_2) \right] \mathbf{I} + [c_1 + c_2(\text{tr} \mathbf{b})] \mathbf{b} - c_2 \mathbf{b}^2 \right\} \quad (4.4.59)$$

The Neo-Hookean Model (Compressible)

Neo-Hookean models are characterised by their dependence on the first (as in the incompressible case) and third invariants, but not on the second invariant. There are quite a number of Neo-Hookean models available, for example (the last term here is the incompressible Neo-Hookean term)

$$W = \frac{\lambda}{2} (\ln J)^2 - \mu \ln J + \frac{1}{2} \mu (I_b - 3) \quad (4.4.60)$$

With corresponding stresses

$$\begin{aligned} \boldsymbol{\sigma} &= 2J^{-1} \frac{\partial W}{\partial III_b} III_b \mathbf{I} + 2J^{-1} \frac{\partial W}{\partial I_b} \mathbf{b} \\ &= J^{-1} [\lambda \ln J + \mu(\mathbf{b} - \mathbf{I})] \end{aligned} \quad (4.4.61)$$

The λ, μ of this compressible Neo-Hookean model are the classical Lamé constants, which can be seen by particularising the constitutive equation to the small strain range.

Another Neo-Hookean model is

$$W = \frac{1}{2} \kappa (\ln J)^2 + \frac{1}{2} \mu (J^{-2/3} I_b - 3) \quad (4.4.62)$$

with corresponding stresses

$$\begin{aligned}\boldsymbol{\sigma} &= J^{-1} \left(\kappa \ln J + \mu J^{-2/3} \left(\mathbf{b} - \frac{1}{3} (\text{tr} \mathbf{b}) \mathbf{I} \right) \right) \\ &= J^{-1} \left(\kappa \ln J + \mu J^{-2/3} \text{dev} \mathbf{b} \right)\end{aligned}\quad (4.4.63)$$

The Blatz and Ko Model (Compressible)

Blatz and Ko proposed a compressible model based on a combination of theoretical arguments and experimental work on foamed materials:

$$W = f \frac{\mu}{2} \left[(I_b - 3) + \frac{1}{\beta} (III_b^{-\beta} - 1) \right] + (1-f) \frac{\mu}{2} \left[\left(\frac{II_b}{III_b} - 3 \right) + \frac{1}{\beta} (III_b^{+\beta} - 1) \right] \quad (4.4.64)$$

where

$$\beta = \frac{\nu}{1-2\nu} \quad (4.4.65)$$

and μ, ν denote the shear modulus and Poisson's ratio, and $f \in [0,1]$ is an interpolation parameter. When the material is incompressible, $III_b = 1$, and the model reduces to the Mooney-Rivlin model. Also, by letting $f = 1$, and replacing III_b with J^2 , the Blatz and Ko model reduces to the Neo-Hookean model

$$W = \frac{\mu}{2} (I_b - 3) + \frac{\mu}{2} \frac{1}{\beta} (J^{-2\beta} - 1) \quad (4.4.66)$$

4.4.8 Problems

1. Evaluate the PK2 stress for the Mooney-Rivlin material (incompressible)

4.5 Material Anisotropy

4.5.1 Material Symmetry

The isotropic material was defined as one whose material response was unaffected by rigid body rotations of the reference configuration. Other material symmetries are possible; to generalise the notion, instead of considering orthogonal transformations, consider an arbitrary deformation $\mathbf{F}_0$ of the reference configuration S_0 bringing it to a new configuration $S^\diamond$, Fig. 4.5.1 (compare with the isotropic case, Fig. 2.8.6).

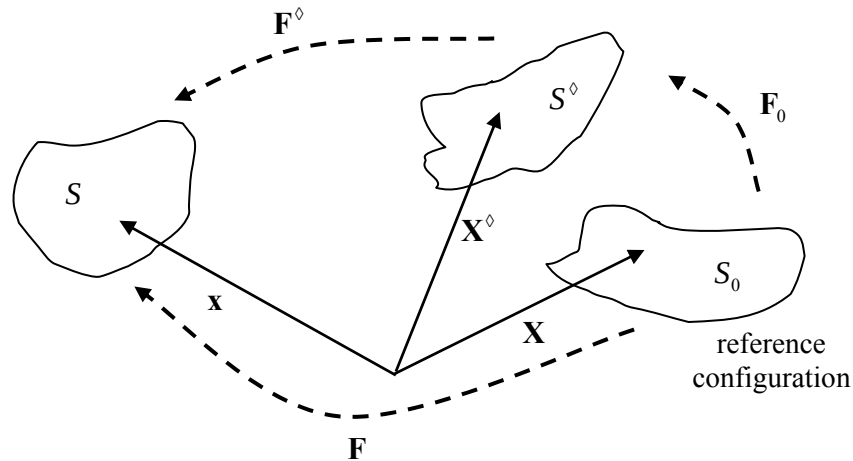


Figure 4.5.1: a deformation of the reference configuration

Considering the Cauchy-elastic material, if the deformation $\mathbf{F}_0$ has no effect on the response of the material, then

$$\boldsymbol{\sigma}(\mathbf{F}^\diamond) = \boldsymbol{\sigma}(\mathbf{F}) \rightarrow \boldsymbol{\sigma}(\mathbf{F}\mathbf{F}_0^{-1}) = \boldsymbol{\sigma}(\mathbf{F}) \quad (4.5.1)$$

When $\mathbf{F}_0 = \mathbf{Q}$, one has the isotropic material. Setting $\mathbf{G} = \mathbf{F}_0^{-1}$, 4.5.1 can be cast in the most usual form:

$$\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{G}) \quad (4.5.2)$$

Note that the restriction $\det \mathbf{G} = \pm 1$ is assumed, since otherwise arbitrary dilatations could occur with no change in material response, which seems physically unreasonable.

Note that the set of all tensors $\mathbf{G}$ which satisfy 4.5.2 forms a **group** (see the Appendix to this Chapter, §4.A.2) and hence is called the **symmetry group** of the material (with respect to the configuration S_0).

Apart from isotropy, the two most important practical cases of material symmetry are **transverse isotropy** and **orthotropy**.

4.5.2 Transverse Isotropy

Consider first the transversely isotropic material. Such a material has a single preferred direction, defined by a unit vector $\mathbf{a}_0$ in the reference configuration. Such a vector is illustrated in Fig. 4.5.2, showing also the unit vectors $\hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3$ completing an orthonormal set. The symmetry group of the transversely isotropic material is the set of orthogonal tensors $\mathbf{Q}$ which transform the set $\{\mathbf{a}_0, \mathbf{n}_2, \mathbf{n}_3\}$ into the new orthonormal set $\{\pm \mathbf{a}_0, \mathbf{n}'_2, \mathbf{n}'_3\}$. In particular,

$$\mathbf{Q}\mathbf{a}_0 = \pm \mathbf{a}_0 \quad (4.5.3)$$

In order to ensure that the sense of $\mathbf{Q}\mathbf{a}_0$ is immaterial, it is best to introduce the **structural tensor** $\mathbf{a}_0 \otimes \mathbf{a}_0$, which transforms as the axes change according to

$$\mathbf{Q}\mathbf{a}_0 \otimes \mathbf{Q}\mathbf{a}_0 = \pm \mathbf{a}_0 \otimes \pm \mathbf{a}_0 \quad (4.5.4)$$

or

$$\mathbf{Q}(\mathbf{a}_0 \otimes \mathbf{a}_0)\mathbf{Q}^T = \mathbf{a}_0 \otimes \mathbf{a}_0 \quad (4.5.5)$$

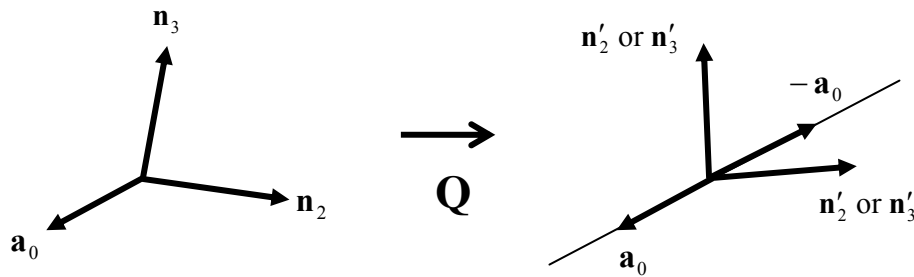


Figure 4.5.2: an orthonormal set of vectors

The strain energy can now be taken to be a function of $\mathbf{C}$, as in the isotropic case, and $\mathbf{a}_0 \otimes \mathbf{a}_0$, which characterises the structure of the material:

$$W = W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) \quad (4.5.6)$$

Allowing for transformations of the undeformed configuration,

$$W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = S(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{a}_0 \otimes \mathbf{Q}\mathbf{a}_0) \quad (4.5.7)$$

with $\mathbf{Q}$ here restricted to the symmetry group defined by 4.5.3. Then W is an isotropic scalar function of two symmetric tensors and so, from Table 4.A.1, takes the form

$$W = W\left(\text{tr}\mathbf{C}, \text{tr}\mathbf{C}^2, \text{tr}\mathbf{C}^3, \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0), \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2, \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^3, \text{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0), \text{tr}\mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0), \text{tr}\mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2, \text{tr}\mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0)^2\right) \quad (4.5.8)$$

Since { **▲**Problem 1 }

$$\begin{aligned} \text{tr}(\mathbf{a} \otimes \mathbf{a}) &= 1, & \text{tr}(\mathbf{C}(\mathbf{a} \otimes \mathbf{a})) &= \mathbf{a}\mathbf{C}\mathbf{a}, & \text{tr}(\mathbf{C}^2(\mathbf{a} \otimes \mathbf{a})) &= \mathbf{a}\mathbf{C}^2\mathbf{a} \\ \mathbf{a} \otimes \mathbf{a} &= (\mathbf{a} \otimes \mathbf{a})^2 = (\mathbf{a} \otimes \mathbf{a})^3 \end{aligned} \quad (4.5.9)$$

one arrives at the representation

$$W = W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)) \quad (4.5.10)$$

where the fourth and fifth scalar (pseudo-) invariants I_4, I_5 are defined by

$$I_4 = \mathbf{a}\mathbf{C}\mathbf{a}, \quad I_5 = \mathbf{a}\mathbf{C}^2\mathbf{a} \quad (4.5.11)$$

Note also that, from the definition of the stretch, Eqn. 2.2.17,

$$I_4 = \mathbf{a}_0 \mathbf{C} \mathbf{a}_0 = \lambda_a^2 \quad (4.5.12)$$

where λ_a is the stretch of the unit line element $\mathbf{a}_0$.

If the preferred direction is $\mathbf{e}_3$, then the fourth and fifth invariants in terms of components are

$$\begin{aligned} I_4 &= \mathbf{a}_0 \mathbf{C} \mathbf{a}_0 = C_{33} \\ I_5 &= \mathbf{a}_0 \mathbf{C}^2 \mathbf{a}_0 = C_{13}^2 + C_{23}^2 + C_{33}^2 \end{aligned} \quad (4.5.13)$$

in which case the five invariants can be taken as $\{I_1, I_2, I_3, C_{33}, C_{13}^2 + C_{23}^2\}$.

Using the relations { **▲**Problem 2 }

$$\frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{a}_0, \quad \frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{C} \mathbf{a}_0 + \mathbf{a}_0 \mathbf{C} \otimes \mathbf{a}_0 \quad (4.5.14)$$

the PK2 stresses for a hyperelastic material are then

$$\begin{aligned} \mathbf{S} &= 2 \sum_{i=1}^5 \frac{\partial W(\mathbf{C}, \mathbf{a}_0)}{\partial I_i} \frac{\partial I_i}{\partial \mathbf{C}} \\ &= 2 \left[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \mathbf{I} - \frac{\partial W}{\partial I_2} \mathbf{C} + I_3 \frac{\partial W}{\partial I_3} \mathbf{C}^{-1} \right. \\ &\quad \left. + \frac{\partial W}{\partial I_4} \mathbf{a}_0 \otimes \mathbf{a}_0 + \frac{\partial W}{\partial I_5} (\mathbf{a}_0 \otimes \mathbf{C} \mathbf{a}_0 + \mathbf{a}_0 \mathbf{C} \otimes \mathbf{a}_0) \right] \end{aligned} \quad (4.5.15)$$

Let $\mathbf{a}$ be a unit vector in the current configuration, in the direction of $\mathbf{F}\mathbf{a}_0$, that is,

$$\lambda_a \mathbf{a} = \mathbf{F}\mathbf{a}_0 \quad (4.5.16)$$

Then, using Eqn. 3.5.7, $\boldsymbol{\sigma} = J^{-1} \mathbf{F} \mathbf{S} \mathbf{F}^T$, with $\mathbf{F} \mathbf{I} \mathbf{F}^T = \mathbf{b}$, $\mathbf{F} \mathbf{C} \mathbf{F}^T = \mathbf{b}^2$ (see Eqn. 2.2.14), $\mathbf{F} \mathbf{C}^{-1} \mathbf{F}^T = \mathbf{I}$ and noting that $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants, 4.5.13 becomes

$$\begin{aligned} \boldsymbol{\sigma} = 2J^{-1} \left[I_3 \frac{\partial W}{\partial I_3} \mathbf{I} + \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \mathbf{b} - \frac{\partial W}{\partial I_2} \mathbf{b}^2 \right. \\ \left. + I_4 \frac{\partial W}{\partial I_4} \mathbf{a} \otimes \mathbf{a} + I_4 \frac{\partial W}{\partial I_5} (\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} \otimes \mathbf{a}) \right] \end{aligned} \quad (4.5.17)$$

Using the Cayley-Hamilton theorem allows one to re-write the Cauchy stress as

$$\begin{aligned} \boldsymbol{\sigma} = 2J^{-1} \left[\left(I_2 \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \right) \mathbf{I} + \frac{\partial W}{\partial I_1} \mathbf{b} - I_3 \frac{\partial W}{\partial I_2} \mathbf{b}^{-1} \right. \\ \left. + I_4 \frac{\partial W}{\partial I_4} \mathbf{a} \otimes \mathbf{a} + I_4 \frac{\partial W}{\partial I_5} (\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} \otimes \mathbf{a}) \right] \end{aligned} \quad (4.5.18)$$

Expressing the scalar invariants in terms of $\mathbf{b}$ rather than $\mathbf{C}$, the coefficients of the tensors in 4.5.17-18 are functions of the set

$$\{\text{tr} \mathbf{b}, \text{tr} \mathbf{b}^2, \text{tr} \mathbf{b}^3, \lambda_a^2 \mathbf{a} \otimes \mathbf{a}, \lambda_a^2 \mathbf{a} \mathbf{b} \mathbf{a}\} \quad (4.5.19)$$

Transversely Isotropic Materials with Constraints

For an incompressible material, $I_3 = 1$, and, analogous to 4.2.40, the strain energy takes the form

$$W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)) - \frac{1}{2} p(I_3 - 1) \quad (4.5.20)$$

For a material which is inextensible in the direction of $\mathbf{a}_0$, from 4.5.13, $I_4 = 0$, and the strain energy takes the form

$$W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_5(\mathbf{C}, \mathbf{a}_0)) - \frac{1}{2} q(I_4 - 1) \quad (4.5.21)$$

4.5.3 Orthotropy

Consider now a material which is dependent on two characteristic directions, $\mathbf{a}_0$ and $\mathbf{b}_0$; again the sense of these directions is immaterial. The strain energy is now of the form

$$W = W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0, \mathbf{b}_0 \otimes \mathbf{b}_0) \quad (4.5.22)$$

As isotropic scalar function of three symmetric tensors depends on the following traces (see Table 4.A.1)

$$\begin{aligned} & \text{tr} \mathbf{C}, \text{tr} \mathbf{C}^2, \text{tr} \mathbf{C}^3 \\ & \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0), \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2, \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^3, \text{tr}(\mathbf{b}_0 \otimes \mathbf{b}_0), \text{tr}(\mathbf{b}_0 \otimes \mathbf{b}_0)^2, \text{tr}(\mathbf{b}_0 \otimes \mathbf{b}_0)^3 \\ & \text{tr} \mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0), \text{tr} \mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0), \text{tr} \mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2, \text{tr} \mathbf{C}^2(\mathbf{a}_0 \otimes \mathbf{a}_0)^2 \\ & \text{tr} \mathbf{C}(\mathbf{b}_0 \otimes \mathbf{b}_0), \text{tr} \mathbf{C}^2(\mathbf{b}_0 \otimes \mathbf{b}_0), \text{tr} \mathbf{C}(\mathbf{b}_0 \otimes \mathbf{b}_0)^2, \text{tr} \mathbf{C}^2(\mathbf{b}_0 \otimes \mathbf{b}_0)^2 \\ & \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)(\mathbf{b}_0 \otimes \mathbf{b}_0), \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2(\mathbf{b}_0 \otimes \mathbf{b}_0), \\ & \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)(\mathbf{b}_0 \otimes \mathbf{b}_0)^2, \text{tr}(\mathbf{a}_0 \otimes \mathbf{a}_0)^2(\mathbf{b}_0 \otimes \mathbf{b}_0)^2 \\ & \text{tr} \mathbf{C}(\mathbf{a}_0 \otimes \mathbf{a}_0)(\mathbf{b}_0 \otimes \mathbf{b}_0) \end{aligned} \quad (4.5.23)$$

Using 4.5.9 (see Eqn. 1.9.10e) this reduces to the set of nine invariants

$$\begin{aligned} & \text{tr} \mathbf{C}, \text{tr} \mathbf{C}^2, \text{tr} \mathbf{C}^3 \\ & \mathbf{a}_0 \mathbf{C} \mathbf{a}_0, \mathbf{a}_0 \mathbf{C}^2 \mathbf{a}_0, \mathbf{b}_0 \mathbf{C} \mathbf{b}_0, \mathbf{b}_0 \mathbf{C}^2 \mathbf{b}_0, (\mathbf{a}_0 \cdot \mathbf{b}_0)^2, (\mathbf{a}_0 \cdot \mathbf{b}_0) \mathbf{a}_0 \mathbf{C} \mathbf{b}_0 \end{aligned} \quad (4.5.24)$$

with $\mathbf{a}_0 \mathbf{C} \mathbf{b}_0 = \mathbf{b}_0 \mathbf{C} \mathbf{a}_0$. The term $\mathbf{a}_0 \cdot \mathbf{b}_0$ is the cosine of the angle between the two characteristic directions; this does not change during the deformation and so this term can be omitted, leaving

$$\text{tr} \mathbf{C}, \text{tr} \mathbf{C}^2, \text{tr} \mathbf{C}^3, \mathbf{a}_0 \mathbf{C} \mathbf{a}_0, \mathbf{a}_0 \mathbf{C}^2 \mathbf{a}_0, \mathbf{b}_0 \mathbf{C} \mathbf{b}_0, \mathbf{b}_0 \mathbf{C}^2 \mathbf{b}_0, (\mathbf{a}_0 \cdot \mathbf{b}_0) \mathbf{a}_0 \mathbf{C} \mathbf{b}_0 \quad (4.5.25)$$

An orthotropic material is one for which $\mathbf{a}_0$ and $\mathbf{b}_0$ are perpendicular, $\mathbf{a}_0 \cdot \mathbf{b}_0 = 0$, making the last term here zero. This also then defines a third preferred direction, $\mathbf{c}_0$, orthogonal to both $\mathbf{a}_0$ and $\mathbf{b}_0$, which introduces extra terms $\mathbf{c}_0 \mathbf{C} \mathbf{c}_0, \mathbf{c}_0 \mathbf{C}^2 \mathbf{c}_0$. But

$$\begin{aligned} \text{tr} \mathbf{C} &= \mathbf{a}_0 \mathbf{C} \mathbf{a}_0 + \mathbf{b}_0 \mathbf{C} \mathbf{b}_0 + \mathbf{c}_0 \mathbf{C} \mathbf{c}_0 \\ \text{tr} \mathbf{C}^2 &= \mathbf{a}_0 \mathbf{C}^2 \mathbf{a}_0 + \mathbf{b}_0 \mathbf{C}^2 \mathbf{b}_0 + \mathbf{c}_0 \mathbf{C}^2 \mathbf{c}_0 \end{aligned} \quad (4.5.26)$$

so that $\mathbf{c}_0 \mathbf{C} \mathbf{c}_0, \mathbf{c}_0 \mathbf{C}^2 \mathbf{c}_0$ are redundant. Finally, the strain energy is of the form

$$W = W(\text{tr} \mathbf{C}, \text{tr} \mathbf{C}^2, \text{tr} \mathbf{C}^3, \mathbf{a}_0 \mathbf{C} \mathbf{a}_0, \mathbf{a}_0 \mathbf{C}^2 \mathbf{a}_0, \mathbf{b}_0 \mathbf{C} \mathbf{b}_0, \mathbf{b}_0 \mathbf{C}^2 \mathbf{b}_0) \quad (4.5.27)$$

As before, the stresses in a hyperelastic material can now be obtained by differentiation.

4.5.4 Problems

1. Show that, for unit vector $\mathbf{a}$,

$$\begin{aligned} \text{(i)} \quad & (\mathbf{a} \otimes \mathbf{a})^2 = \mathbf{a} \otimes \mathbf{a}, & \text{(ii)} \quad & \text{tr}(\mathbf{a} \otimes \mathbf{a}) = 1 \\ \text{(iii)} \quad & \text{tr}(\mathbf{C}(\mathbf{a} \otimes \mathbf{a})) = \mathbf{a} \mathbf{C} \mathbf{a} & \text{(iv)} \quad & \text{tr}(\mathbf{C}^2(\mathbf{a} \otimes \mathbf{a})) = \mathbf{a} \mathbf{C}^2 \mathbf{a} \end{aligned}$$

2. Show that

$$\frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{a}_0, \quad \frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{C} \mathbf{a}_0 + \mathbf{a}_0 \mathbf{C} \otimes \mathbf{a}_0$$

3. Show that

$$\begin{aligned} \text{(i)} \quad & \mathbf{F}(\mathbf{a}_0 \otimes \mathbf{a}_0) \mathbf{F}^T = \lambda_a^2 \mathbf{a} \otimes \mathbf{a} \\ \text{(ii)} \quad & \mathbf{F}(\mathbf{a}_0 \otimes \mathbf{C} \mathbf{a}_0) \mathbf{F}^T = \lambda_a^2 \mathbf{a} \otimes \mathbf{b} \mathbf{a} \end{aligned}$$

For (ii), it might help to note the following relations (for vector $\mathbf{b}$ and second-order tensors $\mathbf{A}$, $\mathbf{B}$):

$$\begin{aligned} (\mathbf{A} \mathbf{b}) \mathbf{B} &\neq \mathbf{A} (\mathbf{b} \mathbf{B}) \\ (\mathbf{A} \mathbf{B}) \mathbf{b} &= \mathbf{A} (\mathbf{B} \mathbf{b}) \\ (\mathbf{A}^T \mathbf{b}) \mathbf{A}^T &= (\mathbf{A} \mathbf{A}^T) \mathbf{b} \end{aligned}$$

4.A Appendix to Chapter 4

4.A.1 Isotropic Functions

The scalar-, vector- and tensor-valued functions ϕ , $\mathbf{a}$ and $\mathbf{T}$ of the scalar variable ϕ , vector variable $\mathbf{v}$ and second-order tensor variable $\mathbf{B}$ are **isotropic functions** if

	$\phi(\mathbf{Q}\mathbf{v}) = \phi(\mathbf{v})$	$\phi(\mathbf{QBQ}^T) = \phi(\mathbf{B})$
$\mathbf{a}(\phi) = \mathbf{Qa}(\phi)$	$\mathbf{a}(\mathbf{Qv}) = \mathbf{Qa}(\mathbf{v})$	$\mathbf{a}(\mathbf{QBQ}^T) = \mathbf{Qa}(\mathbf{B})$
$\mathbf{T}(\phi) = \mathbf{QT}(\phi)\mathbf{Q}^T$	$\mathbf{T}(\mathbf{Qv}) = \mathbf{QT}(\mathbf{v})\mathbf{Q}^T$	$\mathbf{T}(\mathbf{QBQ}^T) = \mathbf{QT}(\mathbf{B})\mathbf{Q}^T$

Isotropic Functions (4.A.1)

for all orthogonal tensors $\mathbf{Q}$.

Isotropic functions are also called **isotropic invariants**. Here follow some examples.

Examples (of Isotropic Functions)

1. The scalar-valued function of a second order tensor $\phi(\mathbf{T}) = \det \mathbf{T}$ is an isotropic function since

$$\phi(\mathbf{QTQ}^T) = \det(\mathbf{QTQ}^T) = \det \mathbf{T}$$

2. The scalar-valued function of two second order tensors $\phi(\mathbf{A}, \mathbf{B}) = \text{tr}(\mathbf{AB})$ is an isotropic function of its two tensor variables since

$$\phi(\mathbf{QAQ}^T, \mathbf{QBQ}^T) = \text{tr}(\mathbf{QAQ}^T \mathbf{QBQ}^T) = \text{tr}(\mathbf{QABQ}^T) = \text{tr}(\mathbf{AB})$$

More generally, the function $\text{tr}(\mathbf{A}^m \mathbf{B}^m)$, m an integer, is isotropic

3. The vector-valued function $\mathbf{a}$ of a vector $\mathbf{v}$ and a second order tensor $\mathbf{T}$, $\mathbf{a}(\mathbf{v}, \mathbf{T}) = \mathbf{Av}$ is an isotropic function since

$$\mathbf{a}(\mathbf{Qv}, \mathbf{QT}^m \mathbf{Q}^T) = \mathbf{QT}^m \mathbf{Q}^T \mathbf{Qv} = \mathbf{QT}^m \mathbf{v} = \mathbf{Qa}(\mathbf{v}, \mathbf{T})$$

Indeed the function $\mathbf{a}(\mathbf{v}, \mathbf{T}) = \mathbf{T}^m \mathbf{v}$, m an integer is an isotropic function.

4. The tensor-valued function of a second order tensor: $\mathbf{A}(\mathbf{T}) = \mathbf{T}^2$ is an isotropic function since

$$\mathbf{A}(\mathbf{QTQ}^T) = (\mathbf{QTQ}^T)^2 = (\mathbf{QTQ}^T)(\mathbf{QTQ}^T) = \mathbf{QT}^2 \mathbf{Q}^T = \mathbf{QA}(\mathbf{T})\mathbf{Q}^T$$

Indeed the function $\mathbf{A}(\mathbf{T}) = \mathbf{T}^m$, m an integer is an isotropic function.

■

Restrictions on the form that isotropic functions can take is next examined.

Isotropic Scalar-valued Functions

Consider first an isotropic scalar-valued function of a vector $\mathbf{u}$, $\phi(\mathbf{u})$, so that $\phi(\mathbf{u}) = \phi(\mathbf{Qu})$. Since only the magnitude of $\mathbf{u}$ is invariant under an orthogonal tensor transformation, it follows that ϕ depends on $\mathbf{u}$ only through $|\mathbf{u}| = \mathbf{u} \cdot \mathbf{u}$, so $\phi \equiv \phi(\mathbf{u} \cdot \mathbf{u})$. Here, $\mathbf{u} \cdot \mathbf{u}$ is called the **integrity basis** of ϕ .

Similarly, an isotropic scalar-valued function of two arguments is defined through

$$\phi(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{Qu}, \mathbf{Qv}) \quad (4.A.2)$$

for every orthogonal $\mathbf{Q}$, and its integrity basis consists of the three scalar invariants

$$\mathbf{u} \cdot \mathbf{u}, \quad \mathbf{u} \cdot \mathbf{v}, \quad \mathbf{v} \cdot \mathbf{v} \quad (4.A.3)$$

since only the lengths of the two vectors and the angle between them are preserved under a rotation.

Consider next a scalar-valued isotropic function ϕ of a *symmetric* second-order tensor $\mathbf{S}$. Since $\mathbf{S}$ is symmetric, it has the spectral decomposition representation $\mathbf{S} = \sum \lambda_i \mathbf{s}_i \otimes \mathbf{s}_i$, where $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues and $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\}$ are the eigenvectors of $\mathbf{S}$. Since $\mathbf{S}$ is isotropic,

$$\phi(\mathbf{S}) = \phi(\mathbf{QSQ}^T) = \phi(\mathbf{Q}(\sum \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i)\mathbf{Q}^T) = \phi(\sum \lambda_i \mathbf{Q}\hat{\mathbf{n}}_i \otimes \mathbf{Q}\hat{\mathbf{n}}_i) \quad (4.A.4)$$

Thus ϕ is independent of the orientation of the principal directions of $\mathbf{S}$ and so must depend only on the three principal values,

$$\phi(\mathbf{S}) = f(\lambda_1, \lambda_2, \lambda_3) \quad (4.A.5)$$

Note also that f must be a symmetric function of the eigenvalues. For example, take $\mathbf{Q}$ to be a positive rotation about $\hat{\mathbf{n}}_3$. Then $\mathbf{Q}\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$, $\mathbf{Q}\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$ and $\mathbf{Q}\hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_3$, so

$$\phi(\mathbf{S}) = \phi(\mathbf{QSQ}^T) = \phi(\lambda_2 \hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \lambda_1 \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2 + \lambda_3 \hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3) = f(\lambda_2, \lambda_1, \lambda_3) \quad (4.A.6)$$

and, similarly, the subscripts on any pair of eigenvalues in 4.A.5 can be interchanged.

Since the set $\{\text{tr}\mathbf{S}, \text{tr}\mathbf{S}^2, \text{tr}\mathbf{S}^3\}$, the set of three principal scalar invariants $\{I_s, II_s, III_s\}$ and the set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ uniquely determine one another, any of these sets can be regarded as the integrity basis of $\phi(\mathbf{S})$.

Some important isotropic scalar-valued functions and their integrity bases are listed in Table 4.A.1 below. The integrity basis consists of that entry together with appropriate entries from higher up in the Table, for example the integrity basis for a tensor $\mathbf{A}$ and two vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{u}, \quad \mathbf{v} \cdot \mathbf{v}, \quad \mathbf{u} \cdot \mathbf{v}, \quad \text{tr} \mathbf{A}, \quad \text{tr} \mathbf{A}^2, \quad \text{tr} \mathbf{A}^3 \\ \mathbf{u} \mathbf{A} \mathbf{u}, \quad \mathbf{u} \mathbf{A}^2 \mathbf{u}, \quad \mathbf{v} \mathbf{A} \mathbf{v}, \quad \mathbf{v} \mathbf{A}^2 \mathbf{v}, \quad \mathbf{u} \mathbf{A} \mathbf{v}, \quad \mathbf{u} \mathbf{A}^2 \mathbf{v}$$

	Isotropic Function		Integrity Basis
Scalar-valued functions	$\phi(\mathbf{u})$	$= \phi(\mathbf{Q}\mathbf{u})$	$\mathbf{u} \cdot \mathbf{u}$
	$\phi(\mathbf{u}, \mathbf{v})$	$= \phi(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{v})$	$\mathbf{u} \cdot \mathbf{v}$
$\mathbf{A}, \mathbf{B}, \mathbf{C}$ are symmetric tensors	$\phi(\mathbf{A})$	$= \phi(\mathbf{Q}\mathbf{A}\mathbf{Q}^T)$	$\text{tr} \mathbf{A}, \text{tr} \mathbf{A}^2, \text{tr} \mathbf{A}^3$
	$\phi(\mathbf{u}, \mathbf{A})$	$= \phi(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{A}\mathbf{Q}^T)$	$\mathbf{u} \mathbf{A} \mathbf{u}, \mathbf{u} \mathbf{A}^2 \mathbf{u}$
	$\phi(\mathbf{A}, \mathbf{B})$	$= \phi(\mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{Q}\mathbf{B}\mathbf{Q}^T)$	$\text{tr} \mathbf{A} \mathbf{B}, \text{tr} \mathbf{A}^2 \mathbf{B}, \text{tr} \mathbf{A} \mathbf{B}^2, \text{tr} \mathbf{A}^2 \mathbf{B}^2$
	$\phi(\mathbf{u}, \mathbf{A}, \mathbf{v})$	$= \phi(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{Q}\mathbf{v})$	$\mathbf{u} \mathbf{A} \mathbf{v}, \mathbf{u} \mathbf{A}^2 \mathbf{v}$
	$\phi(\mathbf{A}, \mathbf{B}, \mathbf{C})$		$\text{tr} \mathbf{A} \mathbf{B} \mathbf{C}$
	Four or more tensors		redundant

Table 4.A.1: Isotropic Scalar Functions and Integrity Bases

Isotropic Vector-valued Functions

Next, consider a vector-valued isotropic function $\mathbf{a}$ of a vector $\mathbf{v}$, so $\mathbf{Q}\mathbf{a}(\mathbf{v}) = \mathbf{a}(\mathbf{Q}\mathbf{v})$. To find the dependence of $\mathbf{a}$ on $\mathbf{v}$, consider the scalar-valued function ϕ given by (note that ϕ here is linear in its first argument, $\mathbf{u}$):

$$\phi(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{a}(\mathbf{v}) \quad (4.A.7)$$

It follows that

$$\phi(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{u} \cdot \mathbf{a}(\mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{a}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{a}(\mathbf{v}) = \phi(\mathbf{u}, \mathbf{v}) \quad (4.A.8)$$

and so ϕ is an isotropic function of its two vector arguments and must depend only on the three invariants 4.A.3, and so takes the general form

$$\phi(\mathbf{u} \cdot \mathbf{u}, \mathbf{u} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{v}) = \mathbf{u} \cdot \alpha(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} \quad (4.A.9)$$

Finally, $\mathbf{a}$ must take the form

$$\mathbf{a}(\mathbf{v}) = \alpha \mathbf{v} \quad (4.A.10)$$

where the coefficient α is a function of the scalar invariant of $\mathbf{v}$, i.e. $\mathbf{v} \cdot \mathbf{v}$.

Note that the only isotropic vector function $\mathbf{a}$ of a tensor $\mathbf{B}$ is the null vector $\mathbf{a} = \mathbf{0}$.

Another important isotropic vector-valued functions is that of a vector and symmetric tensor. This and their integrity bases are listed in Table 4.A.2 below.

	Isotropic Function		Integrity Basis
Vector-valued functions	$\mathbf{a}(\mathbf{v})$	$\mathbf{a}(\mathbf{Qv}) = \mathbf{Qa}(\mathbf{v})$	$\mathbf{v}$
	$\mathbf{a}(\mathbf{T})$	$\mathbf{a}(\mathbf{QTQ}^T) = \mathbf{Qa}(\mathbf{T})$	$\mathbf{0}$
$\mathbf{S}$ is a symmetric tensor	$\mathbf{a}(\mathbf{v}, \mathbf{S})$	$\mathbf{a}(\mathbf{Qv}, \mathbf{QSQ}^T) = \mathbf{Qa}(\mathbf{v}, \mathbf{S})$	$\mathbf{Sv}, \mathbf{S}^2\mathbf{v}$

Table 4.A.2: Isotropic Vector Functions and Integrity Bases

Isotropic Tensor-valued Functions

Consider next a second-order tensor-valued function $\mathbf{T}$ of a tensor $\mathbf{B}$. To find how $\mathbf{T}$ depends on $\mathbf{B}$, this time consider the scalar-valued function ϕ given by (again, note that by definition ϕ is linear in its first argument, $\mathbf{A}$)

$$\phi(\mathbf{A}, \mathbf{B}) = \text{tr}[\mathbf{AT}(\mathbf{B})] \quad (4.A.11)$$

It follows that

$$\begin{aligned} \phi(\mathbf{QAQ}^T, \mathbf{QBQ}^T) &= \text{tr}[\mathbf{QAQ}^T \mathbf{T}(\mathbf{QBQ}^T)] \\ &= \text{tr}[\mathbf{QAQ}^T \mathbf{QT}(\mathbf{B})\mathbf{Q}^T] \\ &= \text{tr}[\mathbf{QAT}(\mathbf{B})\mathbf{Q}^T] \\ &= \text{tr}[\mathbf{AT}(\mathbf{B})] \\ &= \phi(\mathbf{A}, \mathbf{B}) \end{aligned} \quad (4.A.12)$$

Thus ϕ is an isotropic function of its two tensor arguments and so, if $\mathbf{A}$ and $\mathbf{B}$ are symmetric, is a function of the ten invariants listed in Table 4.A.1. Since ϕ is linear in $\mathbf{A}$, it can only depend on six of these ten invariants, namely $\text{tr}\mathbf{A}$, $\text{tr}\mathbf{B}$, $\text{tr}\mathbf{B}^2$, $\text{tr}\mathbf{B}^3$, $\text{tr}\mathbf{AB}$, $\text{tr}\mathbf{AB}^2$, and so takes the form

$$\phi = \text{tr}[\mathbf{AT}(\mathbf{B})] = \text{tr}[\mathbf{A}(\alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2)] \quad (4.A.13)$$

and so $\mathbf{T}$ takes the form

$$\boxed{\mathbf{T}(\mathbf{B}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2} \quad \text{Form for a symmetric isotropic tensor function of a symmetric tensor (4.A.14)}$$

where $\alpha_0, \alpha_1, \alpha_2$ are scalar functions of the invariants of $\mathbf{B}$. Equation 4.A.13 can be rewritten in various alternative forms using the Cayley-Hamilton theorem, 1.9.45.

Some important symmetric isotropic tensor-valued functions are listed in Table 4.A.3 below.

	Isotropic Function		Integrity Basis
Tensor-valued functions T, A, B are symmetric tensors	$\mathbf{T}(\mathbf{v})$	$\mathbf{T}(\mathbf{Qv}) = \mathbf{QT}(\mathbf{v})\mathbf{Q}^T$	$\mathbf{I}, \mathbf{v} \otimes \mathbf{v}$
	$\mathbf{T}(\mathbf{A})$	$\mathbf{T}(\mathbf{QAQ}^T) = \mathbf{QT}(\mathbf{A})\mathbf{Q}^T$	$\mathbf{I}, \mathbf{A}, \mathbf{A}^2$
	$\mathbf{T}(\mathbf{u}, \mathbf{v})$	$\mathbf{T}(\mathbf{Qu}, \mathbf{Qv}) = \mathbf{QT}(\mathbf{u}, \mathbf{v})\mathbf{Q}^T$	$\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}$
	$\mathbf{T}(\mathbf{u}, \mathbf{A})$	$\mathbf{T}(\mathbf{Qu}, \mathbf{QAQ}^T) = \mathbf{QT}(\mathbf{u}, \mathbf{A})\mathbf{Q}^T$	$\mathbf{u} \otimes \mathbf{Au} + \mathbf{Au} \otimes \mathbf{u}, \mathbf{Au} \otimes \mathbf{Au}$
	$\mathbf{T}(\mathbf{u}, \mathbf{S}, \mathbf{v})$	$\mathbf{T}(\mathbf{Qu}, \mathbf{QAQ}^T, \mathbf{Qv}) = \mathbf{QT}(\mathbf{u}, \mathbf{A}, \mathbf{v})\mathbf{Q}^T$	$\mathbf{u} \otimes \mathbf{Av} + \mathbf{Av} \otimes \mathbf{u}$ $\mathbf{v} \otimes \mathbf{Au} + \mathbf{Au} \otimes \mathbf{v}$
	$\mathbf{T}(\mathbf{A}, \mathbf{B})$		$\mathbf{AB} + \mathbf{BA}, \mathbf{ABA}, \mathbf{BAB}$

Table 4.A.3: Isotropic (Symmetric) Tensor Functions and Integrity Bases

Some Results for Isotropic Functions

Here follow some other important results regarding isotropic functions.

1. The principal values of an isotropic tensor function $\mathbf{T}$ of a tensor $\mathbf{B}$ are scalar invariants of $\mathbf{B}$.

To show this, let $t_i(\mathbf{B})$ be the principal values of $\mathbf{T}(\mathbf{B})$ and let $t_i(\mathbf{QBQ}^T)$ be the principal values of $\mathbf{T}(\mathbf{QBQ}^T)$. Then

$$\det(\mathbf{T}(\mathbf{B}) - t_i(\mathbf{B})\mathbf{I}) = 0, \quad \det(\mathbf{T}(\mathbf{QBQ}^T) - t_i(\mathbf{QBQ}^T)\mathbf{I}) = 0$$

Because of the isotropy, and using the relation 1.9.13a, $\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$, the second of these can be written as

$$\begin{aligned} \det(\mathbf{QT}(\mathbf{B})\mathbf{Q}^T - t_i(\mathbf{QBQ}^T)\mathbf{I}) &= \det(\mathbf{QT}(\mathbf{B})\mathbf{Q}^T - t_i(\mathbf{QBQ}^T)\mathbf{QIQ}^T) \\ &= \det(\mathbf{T}(\mathbf{B}) - t_i(\mathbf{QBQ}^T)\mathbf{I}) \end{aligned}$$

This holds for all orthogonal $\mathbf{Q}$ and hence

$$t_i(\mathbf{B}) = t_i(\mathbf{QBQ}^T) \quad (4.A.15)$$

which is the definition of an isotropic scalar invariant of $\mathbf{B}$.

2. An isotropic tensor function $\mathbf{T}$ of a tensor $\mathbf{B}$ is coaxial with $\mathbf{B}$.

This follows directly from 4.A.14, since $\mathbf{B}^n$ has the same principal directions as $\mathbf{B}$.

3. Let $\mathbf{T}$ be a symmetric isotropic tensor function of the symmetric tensor $\mathbf{B}$; if in addition the function $\mathbf{T}$ is a *linear* function of $\mathbf{B}$, then it has the representation

$$\mathbf{T}(\mathbf{B}) = \alpha(\text{tr } \mathbf{B})\mathbf{I} + \beta\mathbf{B} \quad (4.A.16)$$

where α, β are arbitrary constants (independent of $\mathbf{B}$).

This follows directly from 4.A.14, noting that only the first invariant, $\text{tr } \mathbf{B}$, is linear in $\mathbf{B}$. It will be noted that this is the form of the (isotropic) linear elastic material model, 4.1.15.

4. Let $\mathbf{C}$ be a fourth-order isotropic function, that is

$$C_{ijkl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} C_{mnpq} \quad (4.A.17)$$

with the minor symmetries 1.9.65, $C_{ijkl} = C_{jikl} = C_{ijlk}$. Then it has the representation

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (4.A.18)$$

In terms of the identity tensors of §1.9.16 (compare with Eqn. 1.10.7),

$$\mathbf{C} = \lambda \mathbf{I} \otimes \mathbf{I} + \mu (\mathbf{I} + \bar{\mathbf{I}}) \quad (4.A.19)$$

To show this, consider a symmetric second-order tensor $\mathbf{S}$ and define $\mathbf{A} = \mathbf{C} : \mathbf{S}$. Then the index notation for $\mathbf{A}(\mathbf{S})$ is $C_{ijkl} S_{kl}$, and $\mathbf{A}$ is clearly symmetric. Then

$$\begin{aligned} \mathbf{A}(\mathbf{Q}\mathbf{S}\mathbf{Q}^T) : C_{ijkl} (Q_{km} S_{mn} Q_{ln}) &= Q_{ip} Q_{jq} Q_{kr} Q_{ls} C_{pqrs} Q_{km} S_{mn} Q_{ln} \\ &= Q_{ip} Q_{jq} \delta_{rm} \delta_{sn} C_{pqrs} S_{mn} \\ &= Q_{ip} Q_{jq} C_{pqmn} S_{mn} \\ \mathbf{Q}\mathbf{A}(\mathbf{S})\mathbf{Q}^T : Q_{im} C_{mnkl} S_{kl} Q_{jn} \end{aligned} \quad (4.A.20)$$

from which it can be seen that $\mathbf{A}$ is a symmetric isotropic tensor function of the tensor variable $\mathbf{S}$. Further, $\mathbf{A}$ is linear in $\mathbf{S}$, and for $\mathbf{S}$ symmetric, it follows that $\mathbf{A}$ takes the representation 4.A.16,

$$\mathbf{A}(\mathbf{S}) = \lambda(\text{tr } \mathbf{S})\mathbf{I} + 2\mu\mathbf{S} \quad (4.A.21)$$

In component form, this is

$$\begin{aligned} A_{ij} &= \lambda \delta_{ij} S_{kk} + 2\mu S_{ij} \\ &= \lambda \delta_{ij} S_{kk} + \mu (S_{ij} + S_{ji}) \\ &= \lambda \delta_{ij} \delta_{kl} S_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) S_{kl} \end{aligned} \quad (4.A.22)$$

from which 4.A.18 follows.

4.A.2 The Symmetry Group

The nonempty set G with a binary operation, that is, to each pair of elements $a, b \in G$ there is assigned an element $ab \in G$, is called a **group** if the following axioms hold:

1. **associative law**: $(ab)c = a(bc)$ for any $a, b, c \in G$
2. **identity element**: there exists an element $e \in G$, called the identity element, such that $ae = ea = a$
3. **inverse**: for each $a \in G$, there exists an element $a^{-1} \in G$, called the inverse of a , such that $aa^{-1} = a^{-1}a = e$

Consider the set of tensors $\mathbf{G}$ of 4.3.2. Since for two tensors $\mathbf{G}_1$ and $\mathbf{G}_2$ in G ,

$$\sigma(\mathbf{F}\mathbf{G}_1\mathbf{G}_2) = \sigma(\mathbf{F}\mathbf{G}_1) = \sigma(\mathbf{F}) \quad (4.A.23)$$

$\mathbf{G}_1\mathbf{G}_2 \in G$. The associative law clearly holds, the identity element is $\mathbf{I}$ and the inverse of $\mathbf{G}$ is $\mathbf{G}^{-1}$. Thus the set of tensors $\mathbf{G}$ forms a group.

4.A.3 Shear of an Isotropic Square Block

Consider a combined stretch and simple shear of an isotropic hyperelastic material, Fig. 4.A.1. Relative to the Cartesian coordinate system

$$x_1 = \lambda_1 X_1 + k\lambda_2 X_2, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \quad (4.A.24)$$

Then

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & k\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (4.A.25)$$

and so can be considered to be a homogeneous stretch followed by a simple shear. The left Cauchy-Green strain and inverse are

$$\mathbf{b} = \begin{bmatrix} \lambda_1^2 + k^2\lambda_2^2 & k\lambda_2^2 & 0 \\ k\lambda_2^2 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad \mathbf{b}^{-1} = \begin{bmatrix} 1/\lambda_1^2 & -k/\lambda_1^2 & 0 \\ -k/\lambda_1^2 & 1/\lambda_2^2 + k^2/\lambda_1^2 & 0 \\ 0 & 0 & 1/\lambda_3^2 \end{bmatrix} \quad (4.A.26)$$

The compressible and incompressible isotropic relations are (4.4.8 and 4.4.22 respectively)

$$\begin{aligned}\boldsymbol{\sigma}^{(c)} &= \beta_0 \mathbf{I} + \beta_1 \mathbf{b} + \beta_{-1} \mathbf{b}^{-1} \\ \boldsymbol{\sigma}^{(i)} &= -p \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_{-1} \mathbf{b}^{-1}\end{aligned}\tag{4.A.27}$$

Substituting in the Cauchy-Green strains, one finds that $\sigma_{13} = \sigma_{23} = 0$ and

$$\sigma_{12}^{(c)} = k \left(\beta_1 \lambda_2^2 - \beta_{-1} \frac{1}{\lambda_1^2} \right), \quad \sigma_{12}^{(i)} = k \left(\alpha_1 \lambda_2^2 - \alpha_{-1} \frac{1}{\lambda_1^2} \right) \quad (4.A.28)$$

Using this relation, it can then be seen that

$$\sigma_{11} - \sigma_{22} = \frac{\lambda_1^2 - \lambda_2^2 + k^2 \lambda_2^2}{k \lambda_2^2} \sigma_{12} \quad (4.A.29)$$

which holds for both compressible and incompressible materials, and is the universal relation analogous to 4.4.40. Here, however, the stretches can be chosen so as to make the normal stress-difference zero.

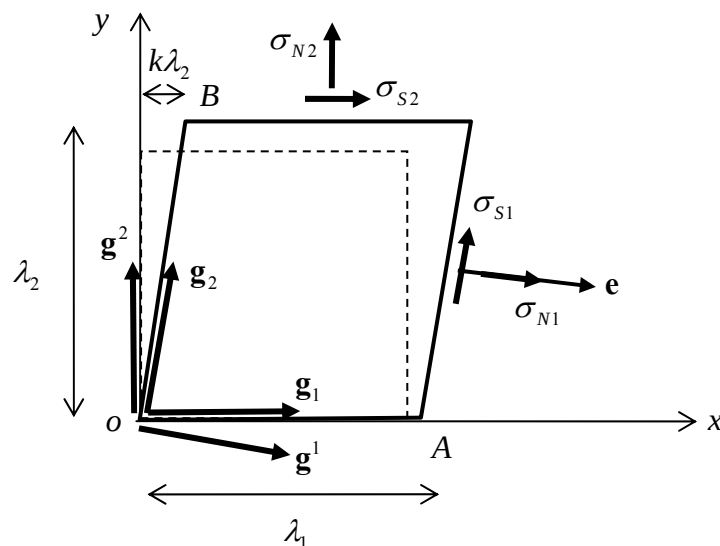


Figure 4.A.1: block under stretch and simple shear

Introduce now base vectors $\mathbf{g}_1, \mathbf{g}_2$ along the edges of the deformed block, with corresponding contravariant base vectors $\mathbf{g}^1$ and $\mathbf{g}^2$, Fig. 4.A.1, so that

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{e}_1, & \mathbf{g}_2 &= k\mathbf{e}_1 + \mathbf{e}_2, & \mathbf{g}_3 &= \mathbf{e}_3 \\ \mathbf{g}^1 &= \mathbf{e}_1 - k\mathbf{e}_2, & \mathbf{g}^2 &= \mathbf{e}_2, & \mathbf{g}^3 &= \mathbf{e}_3 \end{aligned} \quad (4.A.30)$$

The metric coefficients are

$$g_{ij} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g = 1 \quad (4.A.31)$$

From 3.9.2, the unit normals to the block surfaces are (see 4.4.44)

$$\bar{\mathbf{n}}^1 = \hat{\mathbf{g}}^1 = \frac{\mathbf{g}^1}{\sqrt{g^{11}}} = \frac{\mathbf{e}_1 - k\mathbf{e}_2}{\sqrt{1+k^2}}, \quad \bar{\mathbf{n}}^2 = \hat{\mathbf{g}}^2 = \frac{\mathbf{g}^2}{\sqrt{g^{22}}} = \mathbf{e}_2 \quad (4.A.32)$$

The stress components with respect to the curvilinear system can be obtained from the transformation rule in §1.13.1:

$$[\bar{\sigma}^{ij}] = [A_m^i]^T [\sigma^{mn}] [A_n^j], \quad A_i^j = \mathbf{e}_i \cdot \mathbf{g}^j = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.A.33)$$

leading to

$$[\bar{\sigma}^{ij}] = \begin{bmatrix} \sigma_{11} - 2k\sigma_{12} + k^2\sigma_{22} & \sigma_{12} - k\sigma_{22} & 0 \\ \sigma_{21} - k\sigma_{22} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \quad (4.A.34)$$

The normal and shear stresses acting on the surfaces of the block are (see Fig. 4.A.1) are

$$\sigma_{N2} = \sigma_{22}, \quad \sigma_{S2} = \sigma_{12}, \quad \sigma_{N1} = \bar{\sigma}^{11}, \quad \sigma_{S1} = \bar{\sigma}^{12} \quad (4.A.35)$$

In order that the normal stresses acting on the block are zero, then, one requires

$$\sigma_{22} = 0, \quad \sigma_{11} - 2k\sigma_{12} = 0 \quad (4.A.36)$$

From 4.A.29, this means that

$$\lambda_1^2 = (1+k^2)\lambda_2^2 \quad (4.A.37)$$

A physical interpretation of this results is that the lengths of the sides of the deformed block are equal, $|oA| = |oB|$ in Fig. 4.A.1. In this case, 4.A.34 reduces to, using 4.A.28,

$$\begin{aligned} \left\{ \begin{bmatrix} \bar{\sigma}^{ij} \end{bmatrix}^{(c)} \right\} &= \left\{ \begin{bmatrix} \sigma_{12}^{(c)} \\ \sigma_{12}^{(i)} \end{bmatrix} \right\} (\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1) \\ &= k \left[\left\{ \begin{bmatrix} \beta_1 \\ \alpha_1 \end{bmatrix} \right\} \frac{\lambda_1^2}{1+k^2} - \left\{ \begin{bmatrix} \beta_{-1} \\ \alpha_{-1} \end{bmatrix} \right\} \frac{1}{\lambda_1^2} \right] (\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1) \end{aligned} \quad (4.A.38)$$

Thus a state of pure shear is achieved, with only shear stresses acting on the faces, and a square block deforms into a rhombic block.

Consider now the (incompressible) Neo-Hookean model, Eqn. 4.4.54, for which

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2c_1\mathbf{b} \quad (4.A.39)$$

The stress components are then

$$\bar{\sigma}^{ij} = -pg^{ij} + 2c_1b^{ij} \quad (4.A.39)$$

The metric components g^{ij} are given by 4.A.31. The contravariant components of the left Cauchy-Green strain can be obtained from coordinate transformation equations similar to 4.A.33 (with $b_{ij} = b^{ij}$ in the Cartesian system), leading to

$$[b^{ij}] = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 + k^2\lambda_2^2 & k\lambda_2^2 & 0 \\ k\lambda_2^2 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad (4.A.40)$$

with $\lambda_1\lambda_2\lambda_3 = 1$. Then, with the stress taking the representation 4.A.38, with $\alpha_1 = 2c_1$, $\alpha_{-1} = 0$,

$$\begin{bmatrix} 0 & 2kc_1\lambda_1^2/(1+k^2) & 0 \\ 2kc_1\lambda_1^2/(1+k^2) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -p \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2c_1 \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad (4.A.41)$$

Solving leads to

$$\begin{aligned} p &= 2c_1(1+k^2)^{-1/3} \\ \lambda_1 &= (1+k^2)^{1/3}, \quad \lambda_2 = \lambda_3 = (1+k^2)^{-1/6} \end{aligned} \quad (4.A.42)$$

The solution shows that $\lambda_1 > 1$ and $\lambda_2 < 1$ and so the block deforms as in Fig. 4.A.2.

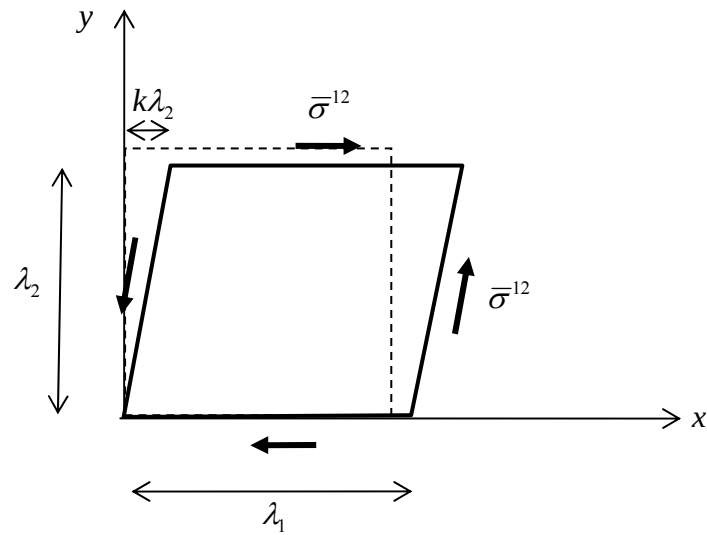


Figure 4.A.2: simple shear of a Neo-Hookean block

Note that, in contrast to the decomposition

5.2 Plasticity I: Hypoelastic-Plastic Models

The two main types of classical plasticity model for large strains are the **hypoelastic-plastic** model and the **hyperelastic-plastic** model. The first of these is discussed in this section.

5.2.1 Hypoelasticity

In the hypoelastic-plastic models of plasticity, the elastic response is assumed to be hypoelastic. As discussed in §4.1.4, hypoelastic materials are characterised by constitutive relations of the form

$$\dot{\boldsymbol{\sigma}} = f(\boldsymbol{\sigma}, \mathbf{d}) \quad (5.2.1)$$

and, in mutiaxial problems, this implies that the response cannot be expressed in terms of an elastic strain energy function. Thus the response is path-dependent and dissipation may occur even though the material is supposed to be elastic. However, the idea with the hypoelastic-plastic models is that the elastic strains are assumed to be relatively small so that any error in the conservation of energy is very small, and so can be neglected.

The stress-rate $\dot{\boldsymbol{\sigma}}$ in the hypoelastic equation 5.2.1 must be objective. Thus the material derivative of the Cauchy stress can not be used, but any of the many objective rates, for example the Jaumann, Green-Naghdi, Truesdell, etc., can be used.

A large class of hypoelastic materials is encompassed in the *linear* relation between objective stress-rate and the rate of deformation:

$$\boldsymbol{\sigma}^\nabla = \mathbf{C}^\nabla : \mathbf{d} \quad (5.2.2)$$

where $\boldsymbol{\sigma}^\nabla$ is an objective stress rate and $\mathbf{C}^\nabla$ is the corresponding fourth order tensor of elastic moduli, which may itself depend on the stress, in which case it must be an objective function of the stress. For a given finitely deformed state, the (small) increments in stress and strain are linearly related and are recovered upon unloading. However, for finite deformations, the work done in a closed path may not be zero.

The elastic modulus tensor $\mathbf{C}^\nabla$ is also called the **tangent modulus**. It possesses the minor symmetries due to the symmetry of $\mathbf{d}$ and $\boldsymbol{\sigma}^\nabla$. It is usually assumed to possess also the major symmetries.

Example

Consider the following hypoelastic constitutive equation:

$$\boldsymbol{\sigma}^{\nabla J} = \mathbf{C}^{\sigma J} : \mathbf{d} \quad (5.2.3)$$

where $\boldsymbol{\sigma}^{\nabla J}$ is the Jaumann stress-rate, Eqn. 3.5.20, $\dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}$. The Truesdell stress-rate $\boldsymbol{\sigma}^{\nabla T}$, defined by Eqn. 3.5.22, $\dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^T + \text{tr}(\mathbf{d})\boldsymbol{\sigma}$, is then

$$\begin{aligned}
\boldsymbol{\sigma}^{\nabla T} &= \boldsymbol{\sigma}^{\nabla J} + \mathbf{w}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{w} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^T + \text{tr}(\mathbf{d})\boldsymbol{\sigma} \\
&= \boldsymbol{\sigma}^{\nabla J} - \mathbf{d}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{d}^T + \text{tr}(\mathbf{d})\boldsymbol{\sigma} \\
&= (\mathbf{C}^{\sigma J} - \hat{\mathbf{C}} + \boldsymbol{\sigma} \otimes \mathbf{I}) : \mathbf{d}
\end{aligned} \tag{5.2.4}$$

where $\hat{\mathbf{C}} : \mathbf{d} = \mathbf{d}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{d}$. Thus 5.2.3 can be expressed as $\boldsymbol{\sigma}^{\nabla T} = \mathbf{C}^{\sigma T} : \mathbf{d}$ with the Truesdell tangent modulus given by $\mathbf{C}^{\sigma J} - \hat{\mathbf{C}} + \boldsymbol{\sigma} \otimes \mathbf{I}$.

It is interesting to note that, if $\mathbf{C}^{\sigma J}$ is constant, then $\boldsymbol{\sigma}^{\nabla T}$ is not. Further, if $\mathbf{C}^{\sigma J}$ has the major symmetries, then $\boldsymbol{\sigma}^{\nabla T}$ does not (since $\boldsymbol{\sigma} \otimes \mathbf{I}$ does not). For this reason one often uses the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ rather than the Cauchy stress. For example, the Jaumann rate of the Kirchhoff stress is $\boldsymbol{\tau}^{\nabla J} = \dot{\boldsymbol{\tau}} - \mathbf{w}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{w}$. Then, with $\dot{J} = J\text{tr}(\mathbf{d})$,

$$\begin{aligned}
\boldsymbol{\sigma}^{\nabla T} &= J^{-1}(\boldsymbol{\tau}^{\nabla J} - J\mathbf{d}\boldsymbol{\sigma} - J\boldsymbol{\sigma}\mathbf{d}^T) \\
&= (J^{-1}\mathbf{C}^{\tau J} - \hat{\mathbf{C}}) : \mathbf{d} \\
&\equiv \mathbf{C}^{\sigma T} : \mathbf{d}
\end{aligned} \tag{5.2.5}$$

Now if $\mathbf{C}^{\sigma T}$ has the major symmetries, then so does $\mathbf{C}^{\tau J}$.

5.2.2 Hypoelastic – Plastic Model

Additive Decomposition of the Rate of deformation

The rate of deformation is now decomposed additively into elastic and plastic parts according to

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p \tag{5.2.6}$$

The elastic response is then

$$\boldsymbol{\sigma}^\nabla = \mathbf{C}_e^\sigma : (\mathbf{d} - \mathbf{d}^p) \tag{5.2.7}$$

The yield condition is

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = 0 \tag{5.2.8}$$

where $\boldsymbol{\alpha}$ represents any other variable(s) besides $\boldsymbol{\sigma}$ upon which f depends.

The flow rule is

$$\mathbf{d}^p = \dot{\lambda} \mathbf{G}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) \tag{5.2.9}$$

The tensor $\mathbf{G}$ is usually expressed in the form $\mathbf{G} = \partial g(\boldsymbol{\sigma}, \boldsymbol{\alpha}) / \partial \boldsymbol{\sigma}$, where g is the plastic potential; λ is the plastic multiplier. For associative flow-rules, $f = g$. The variables $\boldsymbol{\alpha}$ are assumed to follow an evolution law of the form

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\alpha}) \quad (5.2.10)$$

The loading and unloading conditions may be expressed as

$$\dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{\lambda} f = 0 \quad (5.2.11)$$

These are known as the **Kuhn-Tucker conditions**. The first of these states that the plastic multiplier rate is non-negative; for plastic loading, $\dot{\lambda} > 0$, otherwise $\dot{\lambda} = 0$. The second states that the stress must lie on or inside the yield surface. The last condition assures that the stress state remains on the yield surface during plastic flow. This last condition can also be expressed in rate form, $\dot{f} = 0$. This is known as the **consistency condition**:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} = 0 \quad (5.2.12)$$

Isotropic Materials

From the objectivity requirement, the yield function $f(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ in Eqn. 5.2.8 must be an objective scalar function of $\boldsymbol{\sigma}$. This implies that f must be a function only of the invariants of $\boldsymbol{\sigma}$. Thus the form $f(\boldsymbol{\sigma})$ necessarily represents the yield function of an isotropic material. For anisotropic materials, one must use a different stress measure, for example the yield function could be expressed in term of the PK2 stress, $\bar{f}(\mathbf{S}) = 0$.

Consider then an isotropic material, with $f = f(I_1, I_2, I_3)$, where I_i are the invariants of $\boldsymbol{\sigma}$. Then it can be shown that { **▲ Problem 1** }

$$\boldsymbol{\sigma} \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \boldsymbol{\sigma} \quad (5.2.13)$$

that is, the tensors $\boldsymbol{\sigma}$ and $\partial f / \partial \boldsymbol{\sigma}$ are coaxial.

Consider now a formulation in terms of the Jaumann stress-rate. From 5.2.13, it follows that { **▲ Problem 2** }

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^{\nabla J} \quad (5.2.14)$$

The consistency condition 5.2.12 can now be expressed in terms of the Jaumann stress rate:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^{\nabla J} + \frac{\partial f}{\partial \mathbf{a}} : \dot{\mathbf{a}} = 0 \quad (5.2.15)$$

The plastic multiplier can now be solved for, using the hypoelastic relation 5.2.7, the flow rule 5.2.9, the evolution equation 5.2.10 and 5.2.15 {▲ Problem 3}:

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} : \mathbf{d}}{-\frac{\partial f}{\partial \mathbf{a}} : \mathbf{A} + \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} : \mathbf{G}} \quad (5.2.16)$$

Substituting this expression into the flow rule 5.2.9 and using the hypoelastic relation 5.2.7 then leads to {▲ Problem 4}

$$\boldsymbol{\sigma}^{\nabla J} = \mathbf{C}^{\sigma J} : \mathbf{d}, \quad \mathbf{C}^{\sigma J} = \mathbf{C}_e^{\sigma J} - \frac{(\mathbf{C}_e^{\sigma J} : \mathbf{G}) \otimes \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} \right)}{-\frac{\partial f}{\partial \mathbf{a}} : \mathbf{A} + \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C}_e^{\sigma J} : \mathbf{G}} \quad (5.2.17)$$

The tensor $\mathbf{C}^{\sigma J}$ is the **elasto-plastic tangent modulus**. It consists of an elastic component $\mathbf{C}_e^{\sigma J}$ and a component which results from plastic flow.

When the flow is associative, so that $\mathbf{G} = \partial f / \partial \boldsymbol{\sigma}$, the elasto-plastic modulus possesses the major symmetries. The plasticity equations can also be based upon other stress-rates; for example, the Truesdell stress-rate. For the same reasons as discussed earlier, although $\mathbf{C}^{\sigma J}$ has the major symmetries for associative flow rules, the Truesdell modulus will not. If, however, the equations 5.2.6-17 are formulated in terms of the Kirchhoff stress, then the Truesdell modulus will possess the major symmetries (in fact, all the relevant relations above are valid for the Kirchhoff stress, with $\boldsymbol{\sigma}$ simply being replaced by $\boldsymbol{\tau}$). Note that, if the elastic strains are small and plastic deformations are isochoric (volume preserving), then $J \approx 1$ and $\boldsymbol{\tau} \approx \boldsymbol{\sigma}$.

Small Strains

In the case of small strains, in the above relations one replaces $\mathbf{d}$ with $\dot{\boldsymbol{\varepsilon}}$ and decomposes the small strain rate according to $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$. The stress rate is the time derivative of the Cauchy stress, since objective stress-rates are not now a consideration. The hypoelastic relation is $\dot{\boldsymbol{\sigma}} = \mathbf{C} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p)$ and the remaining relations follow, for example the elasto-plastic tangent modulus is given by 5.2.17b with $\mathbf{C}_e^{\sigma}$ replaced with the small-strain elastic modulus tensor $\mathbf{C}$. The small-strain formulation is valid for anisotropic elastic moduli.

5.2.3 J_2 Flow Theory

In the J_2 Flow Theory, one assumes the material is isotropic, plastic flow is independent of the hydrostatic pressure, plastic flow is incompressible and the yield surface used is the

Von Mises yield surface. An effective stress is used to generalise the uniaxial behaviour to multiaxial stress states.

First to be examined is the isotropic hardening model.

Isotropic Hardening

It is useful to express the equations 5.2.6-17 now in terms of the Kirchhoff stress and the Jaumann stress-rate. In that case, one again has Eqn. 5.2.6, $\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p$, with the hypoelastic Eqn. 5.2.7 now reading

$$\boldsymbol{\tau}^{\nabla J} = \mathbf{C}_e^{\nabla J} : (\mathbf{d} - \mathbf{d}^p) \quad (5.2.18)$$

The Von Mises yield function can be expressed as

$$f(\boldsymbol{\tau}, \alpha) = \sqrt{3J_2} - Y = \sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}} - Y = 0 \quad (5.2.19)$$

where J_2 is the second invariant of the deviatoric Kirchhoff stress and $\mathbf{s} = \text{dev}\boldsymbol{\tau}$. The term $\sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}}$ acts as an effective stress $\hat{\sigma}$.

The (associative) flow rule 5.2.9 can be expressed as (see the Appendix to this section for details of the differentiation involved here)

$$\mathbf{d}^p = \dot{\lambda} \mathbf{G}(\boldsymbol{\tau}), \quad \mathbf{G}(\boldsymbol{\tau}) = \frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}}} \quad (5.2.20)$$

The only variable α necessary in the model is the scalar accumulated effective plastic strain:

$$\alpha \equiv \hat{\varepsilon}^p, \quad \hat{\varepsilon}^p = \int d\hat{\varepsilon}^p = \int \dot{\hat{\varepsilon}}^p dt, \quad d\hat{\varepsilon}^p = \sqrt{\frac{2}{3} d\boldsymbol{\varepsilon}^p : d\boldsymbol{\varepsilon}^p} \quad (5.2.21)$$

The proposed evolution law for α is simply (Eqn. 5.2.10 with the function $A = 1$)

$$\dot{\alpha} = \dot{\lambda} \quad (= \dot{\hat{\varepsilon}}^p) \quad (5.2.22)$$

With the definition of the plastic modulus being

$$H(\hat{\varepsilon}^p) = \frac{dY(\hat{\varepsilon}^p)}{d\hat{\varepsilon}^p} \quad (5.2.23)$$

the consistency condition 5.2.12 is, from 5.2.19,

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} : \dot{\boldsymbol{\tau}} - H(\hat{\varepsilon}^p) \dot{\hat{\varepsilon}}^p = 0 \quad (5.2.24)$$

The plastic multiplier rate is now given by Eqn. 5.2.16,

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \mathbf{d}}{H(\hat{\varepsilon}^p) + \frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}}} \quad (5.2.25)$$

Finally, the elasto-plastic tangent modulus, Eqn. 5.2.17, is

$$\boldsymbol{\tau}^{\nabla J} = \mathbf{C}^{\tau J} : \mathbf{d}, \quad \mathbf{C}^{\tau J} = \mathbf{C}_e^{\tau J} - \frac{\left(\mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}} \right) \otimes \left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} \right)}{H(\hat{\varepsilon}^p) + \frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}}} \quad (5.2.26)$$

It is useful to decompose the response into volumetric and deviatoric components. First express the elastic moduli as in Eqns. 4.1.19,

$$\mathbf{C}_e^{\tau J} = \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu \left[\mathbf{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right] \quad (5.2.27)$$

where $\mathbf{I} = \delta_{im} \delta_{jn} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n$. Note then that {▲Problem 5}

$$\mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}} = 2\mu \frac{\partial f}{\partial \boldsymbol{\tau}}, \quad \frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{\tau J} : \frac{\partial f}{\partial \boldsymbol{\tau}} = 3\mu \quad (5.2.28)$$

so that the tangent modulus 5.2.26b can be expressed as

$$\mathbf{C}^{\tau J} = \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu \left[\mathbf{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right] - \frac{(4/3)\mu}{1 + H/3\mu} \frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}} \quad (5.2.29)$$

Kinematic Hardening

To accommodate kinematic hardening, one introduces as another hardening parameter the back stress $\bar{\boldsymbol{\alpha}}$. The yield condition 5.2.8 is now (in terms of the Kirchhoff stress)

$$f(\boldsymbol{\tau}, \bar{\boldsymbol{\alpha}}, \hat{\varepsilon}^p) = \sqrt{\frac{3}{2}(\mathbf{s} - \bar{\boldsymbol{\alpha}}) : (\mathbf{s} - \bar{\boldsymbol{\alpha}})} - Y(\hat{\varepsilon}^p) = 0 \quad (5.2.30)$$

The flow rule is

$$\mathbf{d}^p = \dot{\lambda} \mathbf{G}(\boldsymbol{\tau}, \bar{\boldsymbol{\alpha}}), \quad \mathbf{G}(\boldsymbol{\tau}, \bar{\boldsymbol{\alpha}}) = \frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{3(\mathbf{s} - \bar{\boldsymbol{\alpha}})}{2\sqrt{\frac{3}{2}(\mathbf{s} - \bar{\boldsymbol{\alpha}}) : (\mathbf{s} - \bar{\boldsymbol{\alpha}})}} \quad (5.2.31)$$

The evolution equation for the back-stress is, for example using a linear hardening rule and the Jaumann stress-rate,

$$\bar{\mathbf{a}}^{\nabla J} = c \mathbf{d}^p (= c \dot{\lambda} \mathbf{G}) \quad (5.2.32)$$

Note that the use of the Jaumann rate in the back-stress evolution law can lead to physically unreasonable stress oscillations in simple shear for large deformations (Nagtegaal and DeJong, 1981). However, this behaviour is not significant provided the elastic strains are not too large. A formulation based upon the Green-Naghdi strain can eliminate such stress oscillations (Johnson and Bammann, 1984).

5.2.4 Drucker – Prager Model

The yield criterion for a Drucker-Prager material is a modification of the Von Mises function so as to incorporate a plastic response due to a hydrostatic pressure:

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \sqrt{3J_2} + \alpha I_1 - Y = 0 \quad (5.2.33)$$

where $I_1 = \text{tr} \boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbf{I}$. The associated flow rule is then

$$\mathbf{d}^p = \dot{\lambda} \mathbf{G}(\boldsymbol{\sigma}), \quad \mathbf{G}(\boldsymbol{\sigma}) = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}}\mathbf{s} : \mathbf{s}} + \alpha \mathbf{I} \quad (5.2.34)$$

where $\mathbf{s} = \text{dev} \boldsymbol{\sigma}$. A suitable non-associative flow-rule might be

$$\mathbf{d}^p = \dot{\lambda} \mathbf{G}(\boldsymbol{\sigma}), \quad \mathbf{G}(\boldsymbol{\sigma}) = \frac{\partial g}{\partial \boldsymbol{\sigma}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}}\mathbf{s} : \mathbf{s}}, \quad g = \sqrt{3J_2} \quad (5.2.35)$$

5.2.5 Elastic Moduli and Objectivity

It was mentioned above that objectivity requires that a yield function of the form $f(\boldsymbol{\sigma})$ necessarily represents an isotropic material. Objectivity also places restrictions on the elastic moduli. First, consider a *constant* modulus tensor $\mathbf{C}_e^{\tau J}$. Objectivity of a constitutive equation requires that

$$(\boldsymbol{\tau}^{\nabla J})^* = \mathbf{C}_e^{\tau J} : (\mathbf{d}^e)^* \quad (5.2.36)$$

or

$$\mathbf{Q} \boldsymbol{\tau}^{\nabla J} \mathbf{Q}^T = \mathbf{C}_e^{\tau J} : (\mathbf{Q} \mathbf{d}^e \mathbf{Q}^T) \quad (5.2.37)$$

so, using the index notation,

$$(\boldsymbol{\tau}^{\nabla J})_{ij} = [Q_{pi} Q_{qj} Q_{rk} Q_{sl} (\mathbf{C}_e^{\tau J})_{pqrs}] ((\mathbf{d}^e)_{kl}) \quad (5.2.38)$$

In order that objectivity be satisfied, one must have

$$(\mathbf{C}_e^{\tau})_{ijkl} = Q_{pi} Q_{qj} Q_{rk} Q_{sl} (\mathbf{C}_e^{\tau})_{pqrs} \quad (5.2.39)$$

which shows that, if the modulus is constant, *it must be isotropic*.

If one requires an anisotropic elastic modulus, one must use a formulation based on a configuration other than the spatial configuration, as discussed in the next sub-section.

5.2.6 Corotational Stress Formulation

The models discussed thus far have two drawbacks:

1. the yield function must be an isotropic function of the stress
2. if the elastic moduli are constant, then the moduli must be isotropic

To overcome these restrictions to isotropic materials, one can formulate a plasticity model in terms of, for example, the corotational stress.

The corotational stress is defined by Eqn. 3.5.12,

$$\boldsymbol{\sigma}_U = \mathbf{R}^T \boldsymbol{\sigma} \mathbf{R} \quad (5.2.40)$$

Using a formulation based on the Kirchhoff stress, the corotational Kirchhoff stress is

$$\boldsymbol{\tau}_U = \mathbf{R}^T \boldsymbol{\tau} \mathbf{R} \quad (5.2.41)$$

Recall from §3.5.3 that the PK2 stress $\mathbf{S}$ is the pull-back of the Kirchhoff stress $\boldsymbol{\tau}$, $\mathbf{S} = \chi_*^{-1}(\boldsymbol{\tau})^\# = \mathbf{F}^{-1} \boldsymbol{\tau} \mathbf{F}^{-T}$; the corotational stress $\boldsymbol{\tau}_U$ is the pull-back of $\boldsymbol{\tau}$ but with respect to $\mathbf{R}$ and not $\mathbf{F}$: $\boldsymbol{\tau}_U = \chi_*^{-1}(\boldsymbol{\tau})_{\mathbf{F}=\mathbf{R}}^\# = \mathbf{R}^{-1} \boldsymbol{\tau} \mathbf{R}^{-T}$.

Define now the corotational rate of deformation:

$$\mathbf{d}_U = \mathbf{R}^T \mathbf{d} \mathbf{R} \quad (5.2.42)$$

with the decomposition

$$\mathbf{d}_U = \mathbf{d}_U^e + \mathbf{d}_U^p \quad (5.2.43)$$

Note that the corotational stress and rate of deformation are insensitive to rigid body rotations $\mathbf{Q}$ to the current configuration:

$$\begin{aligned} (\boldsymbol{\tau}_U)^* &= (\mathbf{R}^T \boldsymbol{\tau} \mathbf{R})^* = (\mathbf{Q} \mathbf{R})^T (\mathbf{Q} \boldsymbol{\tau} \mathbf{Q}^T) (\mathbf{Q} \mathbf{R}) = \boldsymbol{\tau}_U \\ (\mathbf{d}_U)^* &= (\mathbf{R}^T \mathbf{d} \mathbf{R})^* = (\mathbf{Q} \mathbf{R})^T (\mathbf{Q} \mathbf{d} \mathbf{Q}^T) (\mathbf{Q} \mathbf{R}) = \mathbf{d}_U \end{aligned} \quad (5.2.44)$$

The corotational stress-rate is

$$\dot{\boldsymbol{\tau}}_U = \overline{\mathbf{R}^T \boldsymbol{\tau} \mathbf{R}} = \dot{\mathbf{R}}^T \boldsymbol{\tau} \mathbf{R} + \mathbf{R}^T \dot{\boldsymbol{\tau}} \mathbf{R} + \mathbf{R}^T \boldsymbol{\tau} \dot{\mathbf{R}} \quad (5.2.45)$$

A rigid body rotation $\mathbf{Q}$ to the current configuration also results in { **▲ Problem 6** }

$$(\dot{\boldsymbol{\tau}}_U)^* = \dot{\boldsymbol{\tau}}_U \quad (5.2.46)$$

With the corotational stress-rate objective, the elastic response given by

$$\dot{\boldsymbol{\tau}}_U = \mathbf{C}_{Ue}^\tau : \mathbf{d}_U^e \quad (5.2.47)$$

Note that rigid body rotations to the current configuration leave $\dot{\boldsymbol{\tau}}_U$ and $\mathbf{d}_U^e$ unchanged and so $\mathbf{C}_{Ue}^\tau$ remains unchanged also. Thus, unlike the isotropic restriction 5.2.39, there is no objectivity restriction here on the form of $\mathbf{C}_{Ue}^\tau$ and so $\mathbf{C}_{Ue}^\tau$ can represent in general an anisotropic response.

Another way of looking at this is as follows: for an anisotropic material and using a *fixed* coordinate system, the components of an elastic modulus tensor $\mathbf{C}$ will in general change as the material rotates. With the corotational formulation, however, the axes rotate with the material, and so material rotation has no effect on $\mathbf{C}_{Ue}^\tau$.

The remainder of the formulation follows as before, for example the yield condition would be $f(\boldsymbol{\tau}_U, \mathbf{a}_U) = 0$ and the flow rule $\mathbf{d}_U^e = \dot{\lambda} \mathbf{G}(\boldsymbol{\tau}_U, \mathbf{a}_U)$. The evolution law for the $\mathbf{a}_U$ variable(s) is $\dot{\mathbf{a}}_U = \dot{\lambda} \mathbf{A}(\boldsymbol{\tau}_U, \mathbf{a}_U)$. Again, the scalar yield function may now represent in general anisotropic material behaviour. The loading and unloading conditions are again given by 5.2.11.

The plastic multiplier rate is now (see 5.2.16)

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\tau}_U} : \mathbf{C}_{Ue}^\tau : \mathbf{d}_U}{-\frac{\partial f}{\partial \mathbf{a}_U} : \mathbf{A} + \frac{\partial f}{\partial \boldsymbol{\tau}_U} : \mathbf{C}_{Ue}^\tau : \mathbf{G}} \quad (5.2.48)$$

and

$$\boldsymbol{\tau}^{\nabla J} = \mathbf{C}_U^\tau : \mathbf{d}_U, \quad \mathbf{C}_U^\tau = \mathbf{C}_{Ue}^\tau - \frac{(\mathbf{C}_{Ue}^\tau : \mathbf{G}) \otimes \left(\frac{\partial f}{\partial \boldsymbol{\tau}_U} : \mathbf{C}_{Ue}^\tau \right)}{-\frac{\partial f}{\partial \mathbf{a}_U} : \mathbf{A} + \frac{\partial f}{\partial \boldsymbol{\tau}_U} : \mathbf{C}_{Ue}^\tau : \mathbf{G}} \quad (5.2.49)$$

5.2.7 Problems

1. The derivatives of the invariants of the stress tensor are (see Eqn. 1.11.33)

$$\frac{\partial I_1}{\partial \boldsymbol{\sigma}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \boldsymbol{\sigma}} = I_1 \mathbf{I} - \boldsymbol{\sigma}, \quad \frac{\partial I_3}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma}^2 - I_1 \boldsymbol{\sigma} + I_2 \mathbf{I} = I_3 \boldsymbol{\sigma}^{-1}$$

Use these relations to show that, for the yield function $f = f(I_1, I_2, I_3)$, the tensors $\boldsymbol{\sigma}$ and $\partial f / \partial \boldsymbol{\sigma}$ are coaxial.

2. Use the fact that $\boldsymbol{\sigma}$ and $\partial f / \partial \boldsymbol{\sigma}$ are coaxial, and Eqn. 1.9.3h,

$$\mathbf{A} : (\mathbf{BC}) = (\mathbf{B}^T \mathbf{A}) : \mathbf{C} = (\mathbf{AC}^T) : \mathbf{B}, \text{ to show that}$$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}^{\nabla J}$$

where $\boldsymbol{\sigma}^{\nabla J} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}$.

3. Derive Eqn. 5.2.16
4. Use the index notation to verify that

$$\mathbf{C} : (\mathbf{A} : \mathbf{C} : \mathbf{d}) : \mathbf{G} = (\mathbf{C} : \mathbf{G}) \otimes (\mathbf{A} : \mathbf{C}) : \mathbf{d}$$

where $\mathbf{A}, \mathbf{d}, \mathbf{G}$ are second order tensors and $\mathbf{C}$ is a fourth-order tensor. Hence derive the elasto-plastic modulus 5.2.17b.

5. Use the relations 1.9.62 and 1.9.64 to show that, for an arbitrary tensor $\mathbf{A}$,

$$\mathbf{C}_e^{TJ} : \mathbf{A} = \kappa(\text{tr} \mathbf{A}) \mathbf{I} + 2\mu \text{dev} \mathbf{A}$$

$$\mathbf{A} : \mathbf{C}_e^{TJ} : \mathbf{A} = \kappa(\text{tr} \mathbf{A})^2 + 2\mu \text{dev} \mathbf{A} : \text{dev} \mathbf{A}$$

with $\mathbf{C}_e^{TJ}$ given by 5.2.27. Hence, using 1.9.31, show that, for a deviatoric tensor $\mathbf{A}$,

$$\mathbf{C}_e^{TJ} : \mathbf{A} = 2\mu \mathbf{A}$$

$$\mathbf{A} : \mathbf{C}_e^{TJ} : \mathbf{A} = 2\mu \mathbf{A} : \mathbf{A}$$

Finally, use 5.2.20,

$$\frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{3\mathbf{s}}{2\sqrt{\frac{3}{2}} \mathbf{s} : \mathbf{s}}$$

to derive Eqns. 5.2.28,

$$\frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{C}_e^{TJ} : \frac{\partial f}{\partial \boldsymbol{\tau}} = 3\mu$$

6. The application of a rigid body rotation $\mathbf{Q}$ to the current configuration results in a change to the corotational stress rate 5.2.45,

$$(\dot{\boldsymbol{\tau}}_U)^* = \dot{\mathbf{R}}^* \boldsymbol{\tau}^* \mathbf{R}^* + \mathbf{R}^{*T} \dot{\boldsymbol{\tau}}^* \mathbf{R}^* + \mathbf{R}^{*T} \boldsymbol{\tau}^* \dot{\mathbf{R}}^*$$

$$= \left(\overline{\dot{\mathbf{Q}\mathbf{R}}} \right)^T (\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T)(\mathbf{Q}\mathbf{R}) + (\mathbf{Q}\mathbf{R})^T \left(\overline{\dot{\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T}} \right) (\mathbf{Q}\mathbf{R}) + (\mathbf{Q}\mathbf{R})^T (\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T) \left(\overline{\dot{\mathbf{Q}\mathbf{R}}} \right)$$

Use the relation $\overline{\dot{\mathbf{Q}^T \mathbf{Q}}} = \dot{\mathbf{Q}^T} \mathbf{Q} + \mathbf{Q}^T \dot{\mathbf{Q}} = \mathbf{0}$ to show that $(\dot{\boldsymbol{\tau}}_U)^* = \overline{\mathbf{R}^T \boldsymbol{\tau}^* \dot{\mathbf{R}}^*} = \dot{\boldsymbol{\tau}}_U$.

5.2.8 Appendix to §5.2

Differentiation of the Von Mises Yield Function

The Von Mises yield criterion is $f = \sqrt{J_2} - k = 0$ where $J_2 = \frac{1}{2} s_{ij} s_{ij}$. Using the product rule of differentiation, in index notation:

$$\begin{aligned}
\frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} &= \frac{\partial \sqrt{\frac{1}{2} s_{mn} s_{mn}}}{\partial \sigma_{ij}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{2} s_{pq} s_{pq}}} \frac{1}{2} \frac{\partial (s_{mn} s_{mn})}{\partial \sigma_{ij}} \\
&= \frac{1}{2\sqrt{\frac{1}{2} s_{pq} s_{pq}}} s_{mn} \frac{\partial s_{mn}}{\partial \sigma_{ij}} \\
&= \frac{s_{mn}}{\sqrt{2 s_{pq} s_{pq}}} \frac{\partial (\sigma_{mn} - \frac{1}{3} \sigma_{kk} \delta_{mn})}{\partial \sigma_{ij}} \\
&= \frac{s_{mn}}{\sqrt{2 s_{pq} s_{pq}}} (\delta_{mi} \delta_{nj} - \frac{1}{3} \delta_{ki} \delta_{kj} \delta_{mn}) \\
&= \frac{s_{mn}}{\sqrt{2 s_{pq} s_{pq}}} (\delta_{mi} \delta_{nj} - \frac{1}{3} \delta_{ij} \delta_{mn}) \\
&= \frac{s_{ij} - \frac{1}{3} \delta_{ij} s_{mn}}{\sqrt{2 s_{pq} s_{pq}}} \\
&= \frac{s_{ij}}{\sqrt{2 s_{pq} s_{pq}}} = \frac{s_{ij}}{2\sqrt{J_2}}
\end{aligned} \tag{5.2.50}$$

In tensor notation:

$$\begin{aligned}
\frac{\partial \sqrt{J_2}}{\partial \boldsymbol{\sigma}} &= \frac{\partial \sqrt{\frac{1}{2} \mathbf{s} : \mathbf{s}}}{\partial \boldsymbol{\sigma}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{2} \mathbf{s} : \mathbf{s}}} \frac{1}{2} \frac{\partial (\mathbf{s} : \mathbf{s})}{\partial \boldsymbol{\sigma}} \\
&= \frac{1}{2\sqrt{\frac{1}{2} \mathbf{s} : \mathbf{s}}} \mathbf{s} : \frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} \\
&= \frac{\mathbf{s}}{\sqrt{2 \mathbf{s} : \mathbf{s}}} : \frac{\partial (\boldsymbol{\sigma} - \frac{1}{3} (\text{tr} \boldsymbol{\sigma}) \mathbf{I})}{\partial \boldsymbol{\sigma}} \\
&= \frac{\mathbf{s}}{\sqrt{2 \mathbf{s} : \mathbf{s}}} : (\mathbf{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I}) \\
&= \frac{\mathbf{s} - \frac{1}{3} (\text{tr} \mathbf{s}) \mathbf{I}}{\sqrt{2 \mathbf{s} : \mathbf{s}}} \\
&= \frac{\mathbf{s}}{\sqrt{2 \mathbf{s} : \mathbf{s}}} = \frac{\mathbf{s}}{2\sqrt{J_2}}
\end{aligned} \tag{5.2.51}$$

5.3 Plasticity II: The Multiplicative Decomposition of $\mathbf{F}$

Whereas the hypoelastic-plastic formulation involved an *additive* decomposition (of the deformation rate), the hyperelastic-plastic formulation involves a *multiplicative* decomposition (of the deformation gradient).

5.3.1 The Multiplicative Decomposition of $\mathbf{F}$

The key feature of the hyperelastic-plastic formulation is the multiplicative decomposition of the deformation gradient according to

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \quad (5.3.1)$$

where $\mathbf{F}^e$ and $\mathbf{F}^p$ are, respectively, the **elastic part of the deformation gradient** and the **plastic part of the deformation gradient**. $\mathbf{F}^p$ brings line elements $d\mathbf{X}$ from the undeformed configuration to an **intermediate configuration** $d\bar{\mathbf{x}} = \mathbf{F}^p d\mathbf{X}$ and then $\mathbf{F}^e$ brings them to their current location, Fig. 5.3.1.

$$d\mathbf{x} = \mathbf{F}^e d\bar{\mathbf{x}} = \mathbf{F}^e \mathbf{F}^p d\mathbf{X}. \quad (5.3.2)$$

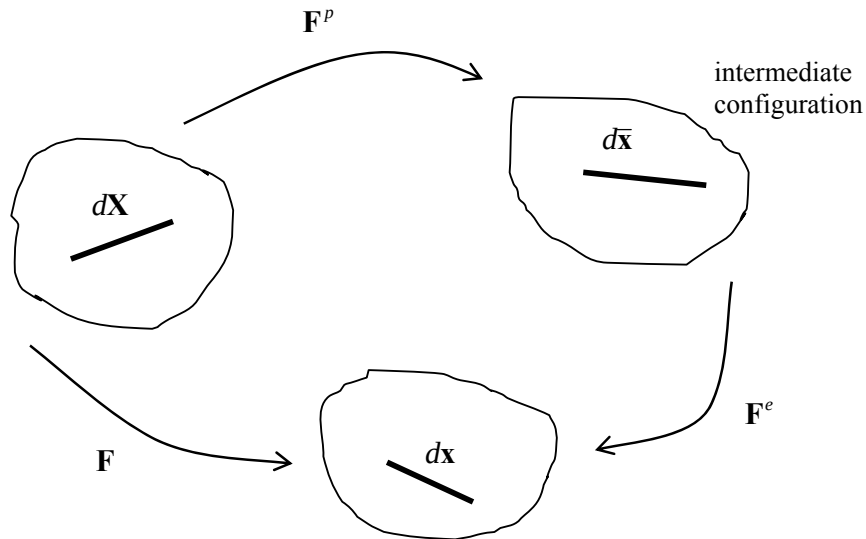


Figure 5.3.1: Decomposition of the deformation gradient

The idea now is to relate the current state of stress to the elastic part of the deformation:

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}^e) \quad (5.3.3)$$

where

$$\mathbf{f}(\mathbf{I}) = 0 \quad (5.3.4)$$

In other words, the intermediate configuration is stress-free. At any stage of the deformation, one can imagine unloading the body instantaneously, arriving at the stress-free intermediate configuration. Thus it appears that the intermediate configuration has a clear physical interpretation – a configuration which encompasses any permanent deformation after elastic unloading. Unfortunately, there are difficulties associated with the intermediate configuration which makes it a more abstract concept than this. For example, consider a stressed element in the current configuration. Unload the element so that it is stress free. According to 5.3.3, this is accomplished through $(\mathbf{F}^e)^{-1}$ back to the intermediate configuration. Since each element in the current configuration can undergo this process independently, there is no reason that the individual elements will be compatible in the intermediate configuration, and so the intermediate configuration is in general an incompatible configuration. Thus, although mappings of line elements between the various configurations are given by Eqn. 5.3.2, the $\mathbf{F}^e$ and $\mathbf{F}^p$ in 5.3.2 cannot necessarily be expressed as gradients of position vectors; they can be expressed as gradients only in the special case where the deformation does lead to a compatible intermediate configuration.

When the material is unloaded from the continuous current configuration (as in a thought experiment), the body moves from a Euclidean space to the non-Euclidean space of the intermediate configuration. The elastic part of the deformation gradient restores compatibility, bringing the body back into the Euclidean space of the current configuration.

It is thus best to consider the decomposition 5.3.1 as merely a mathematical means by which the constitutive response of a material can be conveniently described. It is not to say that a material actually does undergo a “plastic” deformation to the intermediate configuration and then subsequently undergoes an “elastic” deformation to reach the current configuration. Rather, the intermediate configuration is a useful abstract configuration.

Issues:

1. Conceptual de-straining
2. no unique intermediate configuration
3. if the yield surface has shifted, unloading to zero stress might necessitate plastic flow

The decomposition 5.3.1 was introduced into finite-strain elastoplasticity by Lee and Liu in (1967). Since then has followed theory development, explanation and criticism. See, for example, Lee (1969), Hahn (1974), Nemat-Nasser (1979), Lee and McMeeking (1980), Lee (1981), Menat-Nasser (1982), Simon and Ortiz (1985),

5.3.2 Strain Tensors

In terms of the elastic and plastic parts of the deformation gradient, the Right Cauchy-Green strain $\mathbf{C}$ and Left Cauchy-Green strain $\mathbf{b}$ are

$$\begin{aligned}\mathbf{C} &= \mathbf{F}^T \mathbf{F} = (\mathbf{F}^p)^T (\mathbf{F}^e)^T \mathbf{F}^e \mathbf{F}^p \\ \mathbf{b} &= \mathbf{F} \mathbf{F}^T = \mathbf{F}^e \mathbf{F}^p (\mathbf{F}^p)^T (\mathbf{F}^e)^T\end{aligned}\quad (5.3.5)$$

The Euler-Lagrange and Euler-Almansi strains are

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}((\mathbf{F}^p)^T (\mathbf{F}^e)^T \mathbf{F}^e \mathbf{F}^p - \mathbf{I}) \\ \mathbf{e} &= \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1}) = \frac{1}{2}(\mathbf{I} - (\mathbf{F}^e)^{-T} (\mathbf{F}^p)^{-T} (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1})\end{aligned}\quad (5.3.6)$$

Define now elastic and plastic Right Cauchy-Green strains and elastic and plastic Left Cauchy-Green strains:

$$\begin{aligned}\overline{\mathbf{C}}^e &= (\mathbf{F}^e)^T \mathbf{F}^e, & \mathbf{b}^e &= \mathbf{F}^e (\mathbf{F}^e)^T \\ \mathbf{C}^p &= (\mathbf{F}^p)^T \mathbf{F}^p, & \overline{\mathbf{b}}^p &= \mathbf{F}^p (\mathbf{F}^p)^T\end{aligned}\quad (5.3.7)$$

and elastic and plastic Green-Lagrange strains and elastic and plastic Euler-Almansi strains:

$$\begin{aligned}\overline{\mathbf{E}}^e &= \frac{1}{2}(\overline{\mathbf{C}}^e - \mathbf{I}), & \mathbf{e}^e &= \frac{1}{2}(\mathbf{I} - (\mathbf{b}^e)^{-1}) \\ \mathbf{E}^p &= \frac{1}{2}(\mathbf{C}^p - \mathbf{I}), & \overline{\mathbf{e}}^p &= \frac{1}{2}(\mathbf{I} - (\overline{\mathbf{b}}^p)^{-1})\end{aligned}\quad (5.3.8)$$

The overbars in Eqns. 5.3.7-8 are used to signify that the tensor is associated with the intermediate configuration. The action of these twelve tensors on line elements in the reference, intermediate and current configurations can be summarised as {▲ Problem 1}

$$\begin{aligned}d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= d\mathbf{X}^{(1)} \mathbf{C} d\mathbf{X}^{(2)} & d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= d\mathbf{x}^{(1)} \mathbf{b}^{-1} d\mathbf{x}^{(2)} \\ d\overline{\mathbf{x}}^{(1)} \cdot d\overline{\mathbf{x}}^{(2)} &= d\mathbf{X}^{(1)} \mathbf{C}^p d\mathbf{X}^{(2)}, & d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= d\overline{\mathbf{x}}^{(1)} (\overline{\mathbf{b}}^p)^{-1} d\overline{\mathbf{x}}^{(2)} \\ d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= d\overline{\mathbf{x}}^{(1)} \overline{\mathbf{C}}^e d\overline{\mathbf{x}}^{(2)} & d\overline{\mathbf{x}}^{(1)} \cdot d\overline{\mathbf{x}}^{(2)} &= d\mathbf{x}^{(1)} (\mathbf{b}^e)^{-1} d\mathbf{x}^{(2)}\end{aligned}\quad (5.3.9)$$

and {▲ Problem 2}

$$\begin{aligned}\frac{1}{2}(|d\mathbf{x}|^2 - |d\mathbf{X}|^2) &= d\mathbf{X} \mathbf{E} d\mathbf{X} & \frac{1}{2}(|d\mathbf{x}|^2 - |d\mathbf{X}|^2) &= d\mathbf{x} \mathbf{e} d\mathbf{x} \\ \frac{1}{2}(|d\overline{\mathbf{x}}|^2 - |d\mathbf{X}|^2) &= d\mathbf{X} \mathbf{E}^p d\mathbf{X}, & \frac{1}{2}(|d\overline{\mathbf{x}}|^2 - |d\mathbf{X}|^2) &= d\overline{\mathbf{x}} \overline{\mathbf{e}}^p d\overline{\mathbf{x}} \\ \frac{1}{2}(|d\mathbf{x}|^2 - |d\overline{\mathbf{x}}|^2) &= d\overline{\mathbf{x}} \overline{\mathbf{E}}^e d\overline{\mathbf{x}} & \frac{1}{2}(|d\mathbf{x}|^2 - |d\overline{\mathbf{x}}|^2) &= d\mathbf{x} \mathbf{e}^e d\mathbf{x}\end{aligned}\quad (5.3.10)$$

They are illustrated in Fig. 5.3.2. Note that $\mathbf{C} \neq \overline{\mathbf{C}}^e \mathbf{C}^p$, $\mathbf{E} \neq \overline{\mathbf{E}}^e \mathbf{E}^p$.

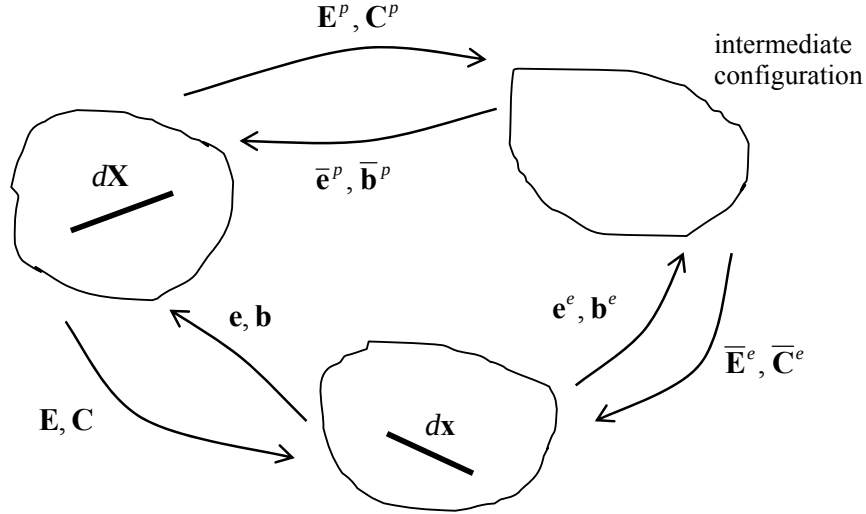


Figure 5.3.2: Strain Tensors and associated configurations

Define now a new strain measure $\bar{\mathbf{E}}$, which can be called the **intermediate strain**, associated with the intermediate configuration, through

$$\bar{\mathbf{E}} = (\mathbf{F}^p)^{-T} \mathbf{E} (\mathbf{F}^p)^{-1} \quad (5.3.11)$$

This is the push-forward for covariant components of the Green-Lagrange strain from the reference configuration to the intermediate configuration (see Eqn. 2.10.18a):

$$\bar{\mathbf{E}} = \chi_{*r \rightarrow i}(\mathbf{E})^b \quad (5.3.12)$$

Using 5.3.7-8, this strain can be decomposed into an elastic component and a plastic component according to {▲ Problem 3}

$$\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \bar{\mathbf{e}}^p \quad (5.3.13)$$

This compares with the push-forward of the Green-Lagrange strain from the reference configuration to the *current* configuration, which equals the Euler-Almansi strain, $\mathbf{e} = \chi_{*r \rightarrow c}(\mathbf{E})^b = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1}$.

The push-forward of the strain $\bar{\mathbf{E}}$ to the current configuration is the Euler-Almansi strain (compare with 5.3.11):

$$\mathbf{e} = \chi_{*i \rightarrow c}(\bar{\mathbf{E}}) = (\mathbf{F}^e)^{-T} \bar{\mathbf{E}} (\mathbf{F}^e)^{-1} \quad (5.3.14)$$

which can be verified using Eqns. 5.3.11, 5.3.1 and the relation $\mathbf{e} = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1}$.

Finally, introduce the tensors

$$\begin{aligned}\mathbf{E}^e &= \mathbf{E} - \mathbf{E}^p = \frac{1}{2}(\mathbf{C} - \mathbf{C}^p) \\ \mathbf{e}^p &= \mathbf{e} - \mathbf{e}^e = \frac{1}{2}((\mathbf{b}^e)^{-1} - \mathbf{b}^{-1})\end{aligned}\quad (5.3.15)$$

In each configuration, there is now a set of elastic and plastic strain tensors, with their sum equaling a total strain. These sets are related by push-forward/pull-back operations {▲ Problem 4}:

$$\begin{aligned}\bar{\mathbf{E}} &= \chi_{*r \rightarrow i}(\mathbf{E})^b = (\mathbf{F}^p)^{-T} \mathbf{E} (\mathbf{F}^p)^{-1} & \mathbf{E} &= \chi_{*r \leftarrow i}^{-1}(\bar{\mathbf{E}})^b = (\mathbf{F}^p)^T \bar{\mathbf{E}} \mathbf{F}^p \\ \bar{\mathbf{E}}^e &= \chi_{*r \rightarrow i}(\mathbf{E}^e)^b = (\mathbf{F}^p)^{-T} \mathbf{E}^e (\mathbf{F}^p)^{-1} & \mathbf{E}^e &= \chi_{*r \leftarrow i}^{-1}(\bar{\mathbf{E}}^e)^b = (\mathbf{F}^p)^T \bar{\mathbf{E}}^e \mathbf{F}^p \\ \bar{\mathbf{e}}^p &= \chi_{*r \rightarrow i}(\mathbf{E}^p)^b = (\mathbf{F}^p)^{-T} \mathbf{E}^p (\mathbf{F}^p)^{-1} & \mathbf{E}^p &= \chi_{*r \leftarrow i}^{-1}(\bar{\mathbf{e}}^p)^b = (\mathbf{F}^p)^T \bar{\mathbf{e}}^p \mathbf{F}^p\end{aligned}\quad (5.3.16a)$$

$$\begin{aligned}\mathbf{e} &= \chi_{*r \rightarrow c}(\mathbf{E})^b = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1} & \mathbf{E} &= \chi_{*r \leftarrow c}^{-1}(\mathbf{e})^b = \mathbf{F}^T \mathbf{e} \mathbf{F} \\ \mathbf{e}^e &= \chi_{*r \rightarrow c}(\mathbf{E}^e)^b = \mathbf{F}^{-T} \mathbf{E}^e \mathbf{F}^{-1} & \mathbf{E}^e &= \chi_{*r \leftarrow c}^{-1}(\mathbf{e}^e)^b = \mathbf{F}^T \mathbf{e}^e \mathbf{F} \\ \mathbf{e}^p &= \chi_{*r \rightarrow c}(\mathbf{E}^p)^b = \mathbf{F}^{-T} \mathbf{E}^p \mathbf{F}^{-1} & \mathbf{E}^p &= \chi_{*r \leftarrow c}^{-1}(\mathbf{e}^p)^b = \mathbf{F}^T \mathbf{e}^p \mathbf{F}\end{aligned}\quad (5.3.16b)$$

$$\begin{aligned}\mathbf{e} &= \chi_{*i \rightarrow c}(\bar{\mathbf{E}})^b = (\mathbf{F}^e)^{-T} \bar{\mathbf{E}} (\mathbf{F}^e)^{-1} & \bar{\mathbf{E}} &= \chi_{*i \leftarrow c}^{-1}(\mathbf{e})^b = (\mathbf{F}^e)^T \mathbf{e} \mathbf{F}^e \\ \mathbf{e}^e &= \chi_{*i \rightarrow c}(\bar{\mathbf{E}}^e)^b = (\mathbf{F}^e)^{-T} \bar{\mathbf{E}}^e (\mathbf{F}^e)^{-1} & \bar{\mathbf{E}}^e &= \chi_{*i \leftarrow c}^{-1}(\mathbf{e}^e)^b = (\mathbf{F}^e)^T \mathbf{e}^e \mathbf{F}^e \\ \mathbf{e}^p &= \chi_{*i \rightarrow c}(\bar{\mathbf{e}}^p)^b = (\mathbf{F}^e)^{-T} \bar{\mathbf{e}}^p (\mathbf{F}^e)^{-1} & \bar{\mathbf{e}}^p &= \chi_{*i \leftarrow c}^{-1}(\mathbf{e}^p)^b = (\mathbf{F}^e)^T \mathbf{e}^p \mathbf{F}^e\end{aligned}\quad (5.3.16c)$$

These nine strain tensors and their relationships through push-forward (pull-back) operations are illustrated in Fig. 5.3.3.

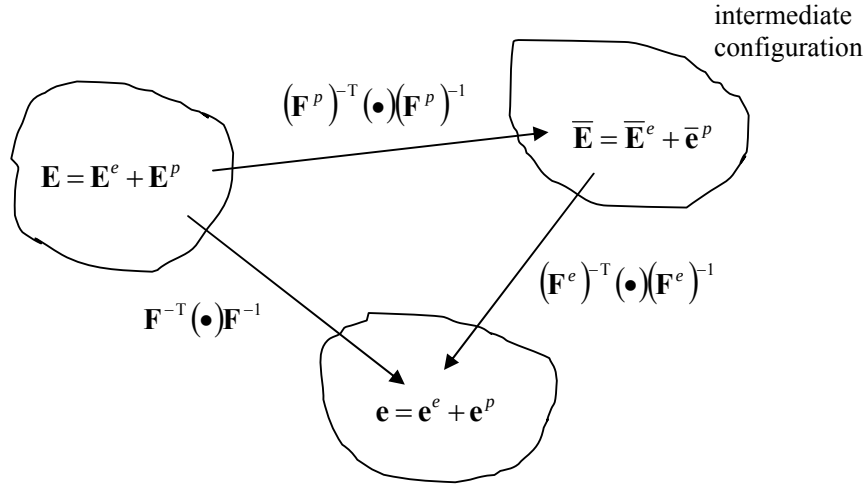


Figure 5.3.3: Push-Forward operations on strain tensors

The action of the three new tensors introduced above, $\bar{\mathbf{E}}$, $\mathbf{E}^e$, $\mathbf{e}^p$, analogous to 5.3.10, is given by

$$\begin{aligned}
\frac{1}{2}(|d\mathbf{x}|^2 - |d\bar{\mathbf{x}}|^2) &= d\mathbf{X}\mathbf{E}^e d\mathbf{X} \\
\frac{1}{2}(|d\bar{\mathbf{x}}|^2 - |d\mathbf{X}|^2) &= d\mathbf{x}\mathbf{e}^p d\mathbf{x} \\
\frac{1}{2}(|d\mathbf{x}|^2 - |d\mathbf{X}|^2) &= d\bar{\mathbf{x}}\bar{\mathbf{E}} d\bar{\mathbf{x}}
\end{aligned} \tag{5.3.17}$$

and

$$\begin{aligned}
d\mathbf{X}\mathbf{E}d\mathbf{X} &= d\bar{\mathbf{x}}\bar{\mathbf{E}}d\bar{\mathbf{x}} = d\mathbf{x}e d\mathbf{x} \\
d\mathbf{X}\mathbf{E}^p d\mathbf{X} &= d\bar{\mathbf{x}}\bar{\mathbf{e}}^p d\bar{\mathbf{x}} = d\mathbf{x}\mathbf{e}^p d\mathbf{x} \\
d\mathbf{X}\mathbf{E}^e d\mathbf{X} &= d\bar{\mathbf{x}}\bar{\mathbf{E}}^e d\bar{\mathbf{x}} = d\mathbf{x}\mathbf{e}^e d\mathbf{x}
\end{aligned} \tag{5.3.18}$$

The various tensors are summarised in Table 5.3.1.

Configuration	Green-Lagrange	Euler-Almansi	Other
Reference	$\mathbf{E}, \mathbf{E}^p$		$\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p$
Intermediate	$\bar{\mathbf{E}}^e$	$\bar{\mathbf{e}}^p$	$\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \bar{\mathbf{e}}^p$
Current		$\mathbf{e}, \mathbf{e}^e$	$\mathbf{e}^p = \mathbf{e} - \mathbf{e}^e$

Table 5.3.1: Strain Tensors for the Multiplicative Decomposition

5.3.3 Strain-Rate Tensors

The velocity gradient $\mathbf{l}$ is

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{5.3.19}$$

In terms of the elastic and plastic deformation gradients { **▲** Problem 5 },

$$\mathbf{l} = \dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1} + \mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1} \tag{5.3.20}$$

The rate of deformation and spin are

$$\mathbf{d} = \text{sym} \mathbf{l} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T), \quad \mathbf{w} = \text{skew} \mathbf{l} = \frac{1}{2}(\mathbf{l} - \mathbf{l}^T) \tag{5.3.21}$$

One can now define elastic and plastic velocity gradients, and decompose the total velocity gradient according to

$$\mathbf{l} = \mathbf{l}^e + \mathbf{l}^p \tag{5.3.22}$$

There are, however, a number of different possible definitions of $\mathbf{l}^e$ and $\mathbf{l}^p$. The two most prevalent definitions are discussed in what follows.

Elastic and Plastic Velocity Gradients 1

The first term in Eqn. 5.3.20 is analogous to Eqn. 5.3.19, $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. It only involves elastic deformations and hence can be called the elastic part of the velocity gradient, $\mathbf{l}^e$. The second term is denoted by $\mathbf{l}^p$ and is called the plastic part of the velocity gradient (although it involves both elastic and plastic deformations). Thus one has

$$\begin{aligned}\mathbf{l}^e &= \dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1} \\ \mathbf{l}^p &= \mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1}\end{aligned}\quad (5.3.23)$$

The rate of deformation and spin are then

$$\begin{aligned}\mathbf{d} &= \text{sym}[\dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1}] + \text{sym}[\mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1}] \\ \mathbf{w} &= \text{skew}[\dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1}] + \text{skew}[\mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1}]\end{aligned}\quad (5.3.24)$$

With the corresponding elastic and plastic parts:

$$\begin{aligned}\mathbf{d}^e &= \frac{1}{2}(\mathbf{l}^e + (\mathbf{l}^e)^T), & \mathbf{w}^e &= \frac{1}{2}(\mathbf{l}^e - (\mathbf{l}^e)^T) \\ \mathbf{d}^p &= \frac{1}{2}(\mathbf{l}^p + (\mathbf{l}^p)^T), & \mathbf{w}^p &= \frac{1}{2}(\mathbf{l}^p - (\mathbf{l}^p)^T)\end{aligned}\quad (5.3.25)$$

Elastic and Plastic Velocity Gradients 2

The rate of deformation is the Lie derivative of the Euler-Almansi strain,

$$\mathbf{L}_v^b(\mathbf{e}) = \chi_{*r \rightarrow c} \left(\frac{d}{dt} [\chi_{*r \leftarrow c}^{-1}(\mathbf{e})^b] \right)^b = \chi_{*r \rightarrow c} (\dot{\mathbf{e}})^b = \mathbf{F}^{-T} \dot{\mathbf{E}} \mathbf{F}^{-1} = \mathbf{d} \quad (5.3.26)$$

In the same way, define the elastic and plastic rates of deformation to be the Lie derivatives of the tensors $\mathbf{e}^e$ and $\mathbf{e}^p$ respectively:

$$\begin{aligned}\mathbf{d}^e &\equiv \mathbf{L}_v^b(\mathbf{e}^e) = \chi_{*r \rightarrow c} \left(\frac{d}{dt} [\chi_{*r \leftarrow c}^{-1}(\mathbf{e}^e)^b] \right)^b = \chi_{*r \rightarrow c} (\dot{\mathbf{e}}^e)^b = \mathbf{F}^{-T} \dot{\mathbf{E}}^e \mathbf{F}^{-1} \\ \mathbf{d}^p &\equiv \mathbf{L}_v^b(\mathbf{e}^p) = \chi_{*r \rightarrow c} \left(\frac{d}{dt} [\chi_{*r \leftarrow c}^{-1}(\mathbf{e}^p)^b] \right)^b = \chi_{*r \rightarrow c} (\dot{\mathbf{e}}^p)^b = \mathbf{F}^{-T} \dot{\mathbf{E}}^p \mathbf{F}^{-1}\end{aligned}\quad (5.3.27)$$

One can also define rates of deformation in the intermediate configuration by taking Lie derivatives of the strain tensors in the intermediate configuration:

$$\bar{\mathbf{d}} \equiv \mathbf{L}_v^b(\bar{\mathbf{E}}) = \chi_{*r \rightarrow i} \left(\frac{d}{dt} [\chi_{*r \leftarrow i}^{-1}(\bar{\mathbf{E}})^b] \right)^b = \chi_{*r \rightarrow i} (\dot{\bar{\mathbf{E}}})^b = (\mathbf{F}^p)^{-T} \dot{\bar{\mathbf{E}}} (\mathbf{F}^p)^{-1} \quad (5.3.28)$$

and

$$\begin{aligned}
\bar{\mathbf{d}}^e &\equiv L_v^b(\bar{\mathbf{E}}^e) = \chi_{*r \rightarrow i} \left(\frac{d}{dt} \left[\chi_{*r \leftarrow i}^{-1} (\bar{\mathbf{E}}^e)^b \right] \right)^b = \chi_{*r \rightarrow c} (\dot{\mathbf{E}}^e)^b = (\mathbf{F}^p)^{-T} \dot{\mathbf{E}}^e (\mathbf{F}^p)^{-1} \\
\bar{\mathbf{d}}^p &\equiv L_v^b(\bar{\mathbf{e}}^p) = \chi_{*r \rightarrow i} \left(\frac{d}{dt} \left[\chi_{*r \leftarrow i}^{-1} (\bar{\mathbf{e}}^p)^b \right] \right)^b = \chi_{*r \rightarrow c} (\dot{\mathbf{e}}^p)^b = (\mathbf{F}^p)^{-T} \dot{\mathbf{e}}^p (\mathbf{F}^p)^{-1}
\end{aligned} \tag{5.3.29}$$

Note that

$$\bar{\mathbf{d}} = \bar{\mathbf{d}}^e + \bar{\mathbf{d}}^p \tag{5.3.30}$$

These rates of deformations as Lie derivatives are illustrated in Fig. 5.3.4.

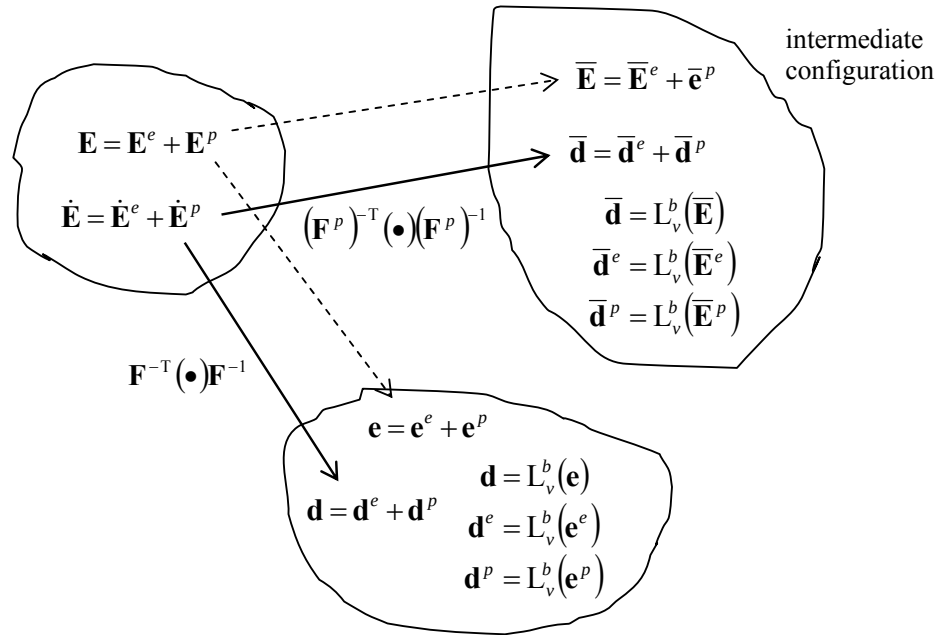


Figure 5.3.4: Rate of Deformation and Lie Derivatives

Note also that the rates of deformations in the intermediate and current configurations are related through the push-forward/pull-back operations { **▲ Problem 6** }

$$\begin{aligned}
\mathbf{d} &= \chi_{*i \rightarrow c} (\bar{\mathbf{d}})^b = (\mathbf{F}^e)^{-T} \bar{\mathbf{d}} (\mathbf{F}^e)^{-1} \\
\mathbf{d}^e &= \chi_{*i \rightarrow c} (\bar{\mathbf{d}}^e)^b = (\mathbf{F}^e)^{-T} \bar{\mathbf{d}}^e (\mathbf{F}^e)^{-1} \\
\mathbf{d}^p &= \chi_{*i \rightarrow c} (\bar{\mathbf{d}}^p)^b = (\mathbf{F}^e)^{-T} \bar{\mathbf{d}}^p (\mathbf{F}^e)^{-1}
\end{aligned} \tag{5.3.31}$$

Finally, note that direct relationships between $\mathbf{d}$ and $\dot{\mathbf{e}}$, and between $\bar{\mathbf{d}}$ and $\dot{\bar{\mathbf{E}}}$. From 2.5.18,

$$\dot{\mathbf{e}} = \mathbf{d} - \mathbf{l}^T \mathbf{e} - \mathbf{e} \mathbf{l} \tag{5.3.32}$$

Similarly, analogous to 5.3.19, introduce a plastic velocity gradient,

$$\bar{\mathbf{l}}^p = \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} \quad (5.3.33)$$

so that { **▲**Problem 7}

$$\dot{\bar{\mathbf{E}}} = \bar{\mathbf{d}} - (\bar{\mathbf{l}}^p)^T \bar{\mathbf{E}} - \bar{\mathbf{E}} \bar{\mathbf{l}}^p \quad (5.3.34)$$

Similarly { **▲**Problem 8},

$$\begin{aligned} \dot{\mathbf{e}}^e &= \mathbf{d}^e - \mathbf{l}^T \mathbf{e}^e - \mathbf{e}^e \mathbf{l} & \dot{\bar{\mathbf{E}}}^e &= \bar{\mathbf{d}}^e - (\bar{\mathbf{l}}^p)^T \bar{\mathbf{E}}^e - \bar{\mathbf{E}}^e \bar{\mathbf{l}}^p \\ \dot{\mathbf{e}}^p &= \mathbf{d}^p - \mathbf{l}^T \mathbf{e}^p - \mathbf{e}^p \mathbf{l} & \dot{\bar{\mathbf{e}}}^p &= \bar{\mathbf{d}}^p - (\bar{\mathbf{l}}^p)^T \bar{\mathbf{e}}^p - \bar{\mathbf{e}}^p \bar{\mathbf{l}}^p \end{aligned} \quad (5.3.35)$$

These can all be expressed as Cotter-Rivlin rates of the strains:

$$\begin{aligned} \mathbf{d} &= \dot{\mathbf{e}} + \mathbf{l}^T \mathbf{e} + \mathbf{e} \mathbf{l} & \bar{\mathbf{d}} &= \dot{\bar{\mathbf{E}}} + (\bar{\mathbf{l}}^p)^T \bar{\mathbf{E}} + \bar{\mathbf{E}} \bar{\mathbf{l}}^p \\ \mathbf{d}^e &= \dot{\mathbf{e}}^e + \mathbf{l}^T \mathbf{e}^e + \mathbf{e}^e \mathbf{l} & \bar{\mathbf{d}}^e &= \dot{\bar{\mathbf{E}}}^e + (\bar{\mathbf{l}}^p)^T \bar{\mathbf{E}}^e + \bar{\mathbf{E}}^e \bar{\mathbf{l}}^p \\ \mathbf{d}^p &= \dot{\mathbf{e}}^p + \mathbf{l}^T \mathbf{e}^p + \mathbf{e}^p \mathbf{l} & \bar{\mathbf{d}}^p &= \dot{\bar{\mathbf{e}}}^p + (\bar{\mathbf{l}}^p)^T \bar{\mathbf{e}}^p + \bar{\mathbf{e}}^p \bar{\mathbf{l}}^p \end{aligned} \quad (5.3.36)$$

Push forward now the plastic velocity gradient to the current configuration:

$$\chi_{*i \rightarrow c} (\bar{\mathbf{l}}^p)^b = (\mathbf{F}^e)^{-T} \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1} = (\mathbf{F}^e)^{-T} \dot{\mathbf{F}}^p (\mathbf{F}^e \mathbf{F}^p)^{-1} \quad (5.3.33)$$

Alternatively, introduce a velocity gradient $\bar{\mathbf{l}}$ which is the pull-back for covariant components of the spatial velocity gradient. For covariant components,

$$\bar{\mathbf{l}} = \chi_{*i \leftarrow c}^{-1} (\mathbf{l})^b = (\mathbf{F}^e)^T \mathbf{l} \mathbf{F}^e \quad (5.3.26)$$

so that

$$d\bar{\mathbf{x}} \bar{\mathbf{l}} d\bar{\mathbf{x}} = d\bar{\mathbf{x}} (\mathbf{F}^e)^T \mathbf{l} \mathbf{F}^e d\bar{\mathbf{x}} = \mathbf{F}^e d\mathbf{x} \mathbf{l} \mathbf{F}^e d\bar{\mathbf{x}} = d\mathbf{x} \mathbf{l} d\mathbf{x} \quad (5.3.27)$$

Here, so the rate of change of a line element $d\bar{\mathbf{x}}$ in the intermediate configuration is

$$(\dot{d\bar{\mathbf{x}}}) = \bar{\mathbf{l}}^p d\bar{\mathbf{x}}$$

The velocity gradients $\mathbf{l}, \mathbf{l}^e, \mathbf{l}^p$ are defined on the current configuration. These can be pulled back to the intermediate configuration. The pull-backs to be used are for mixed components $l_{.j}^i, (l^e)_{.j}^i, (l^p)_{.j}^i$ (see the Appendix to this section, §5.3.3, which gives the component forms of the tensors used in this section). These pull-backs are (see Eqn. 2.10.20c):

$$\begin{aligned}\bar{\mathbf{l}} &= \chi_{*i \leftarrow c}^{-1}(\mathbf{l})^\vee = (\mathbf{F}^e)^{-1} \mathbf{l} \mathbf{F}^e \\ \bar{\mathbf{l}}^e &= \chi_{*i \leftarrow c}^{-1}(\mathbf{l}^e)^\vee = (\mathbf{F}^e)^{-1} \mathbf{l}^e \mathbf{F}^e \\ \bar{\mathbf{l}}^p &= \chi_{*i \leftarrow c}^{-1}(\mathbf{l}^p)^\vee = (\mathbf{F}^e)^{-1} \mathbf{l}^p \mathbf{F}^e\end{aligned}\tag{5.3.20}$$

so, using 5.3.19,

$$\bar{\mathbf{l}} = \bar{\mathbf{l}}^e + \bar{\mathbf{l}}^p\tag{5.3.21}$$

where

$$\begin{aligned}\bar{\mathbf{l}}^e &= (\mathbf{F}^e)^{-1} \dot{\mathbf{F}}^e \\ \bar{\mathbf{l}}^p &= \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1}\end{aligned}\tag{5.3.22}$$

the second of these being 5.3.14.

The push-forwards for the velocity gradients defined on the intermediate configuration, the inverse of the operations 5.3.20, are

$$\begin{aligned}\mathbf{l} &= \chi_{*i \rightarrow c}(\bar{\mathbf{l}})^\vee = \mathbf{F}^e \bar{\mathbf{l}} (\mathbf{F}^e)^{-1} \\ \mathbf{l}^e &= \chi_{*i \rightarrow c}(\bar{\mathbf{l}}^e)^\vee = \mathbf{F}^e \bar{\mathbf{l}}^e (\mathbf{F}^e)^{-1} \\ \mathbf{l}^p &= \chi_{*i \rightarrow c}(\bar{\mathbf{l}}^p)^\vee = \mathbf{F}^e \bar{\mathbf{l}}^p (\mathbf{F}^e)^{-1}\end{aligned}\tag{5.3.23}$$

and 5.3.18-19 are recovered.

These velocity gradients are summarised in Fig. 5.3.5.

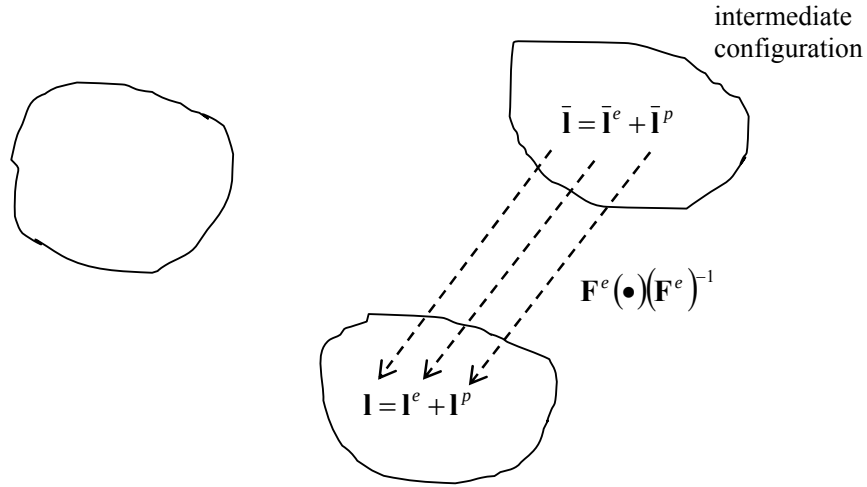


Figure 5.3.5: Velocity Gradients

Rate of Deformation and Spin

The rate of deformation and spin are given by the symmetric and skew-symmetric parts of the velocity gradient. From 5.3.17,

$$\begin{aligned} \mathbf{d} &= \left[\dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1} \right]_s + \left[\mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1} \right]_s \\ \mathbf{w} &= \left[\dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1} \right]_a + \left[\mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1} \right]_a \end{aligned} \quad (5.3.?)$$

The velocity gradients can be decomposed into their symmetric and skew-symmetric parts, that is, the rate of deformation and spin:

$$\begin{aligned} \mathbf{d}^e &= \frac{1}{2} (\mathbf{l}^e + (\mathbf{l}^e)^T), & \mathbf{w}^e &= \frac{1}{2} (\mathbf{l}^e - (\mathbf{l}^e)^T) \\ \mathbf{d}^p &= \frac{1}{2} (\mathbf{l}^p + (\mathbf{l}^p)^T), & \mathbf{w}^p &= \frac{1}{2} (\mathbf{l}^p - (\mathbf{l}^p)^T) \end{aligned} \quad (5.3.24)$$

If the velocity gradients have the mixed components $l^i_{,j}, (l^e)^i_{,j}, (l^p)^i_{,j}$ then, in order that the rate of deformation and spin tensors have covariant components, one should include the metric tensor $\mathbf{g} = \mathbf{I}$ in 5.3.24 as follows (see the Appendix to this section, §5.3.3):

$$\begin{aligned} \mathbf{d}^e &= \frac{1}{2} (\mathbf{g} \mathbf{l}^e + (\mathbf{l}^e)^T \mathbf{g}), & \mathbf{w}^e &= \frac{1}{2} (\mathbf{g} \mathbf{l}^e - (\mathbf{l}^e)^T \mathbf{g}) \\ \mathbf{d}^p &= \frac{1}{2} (\mathbf{g} \mathbf{l}^p + (\mathbf{l}^p)^T \mathbf{g}), & \mathbf{w}^p &= \frac{1}{2} (\mathbf{g} \mathbf{l}^p - (\mathbf{l}^p)^T \mathbf{g}) \end{aligned} \quad (5.3.25)$$

For two tensors $\mathbf{a}, \mathbf{b}$, the push forward and pull-back (for covariant components) of their contraction can be expressed as

$$\begin{aligned}
\chi_*(\mathbf{ab})^b &= \mathbf{F}^{-T} \mathbf{a} \mathbf{b} \mathbf{F}^{-1} = (\mathbf{F}^{-T} \mathbf{a} \mathbf{F}^T) (\mathbf{F}^{-T} \mathbf{b} \mathbf{F}^{-1}) = \chi_*(\mathbf{a})^{\vee} \chi_*(\mathbf{b})^b \\
&= (\mathbf{F}^{-T} \mathbf{a} \mathbf{F}^{-1}) (\mathbf{F} \mathbf{b} \mathbf{F}^{-1}) = \chi_*(\mathbf{a})^b \chi_*(\mathbf{b})^{\vee} \\
\chi_*^{-1}(\mathbf{ab})^b &= \mathbf{F}^T \mathbf{a} \mathbf{b} \mathbf{F} = (\mathbf{F}^T \mathbf{a} \mathbf{F}^{-T}) (\mathbf{F}^T \mathbf{b} \mathbf{F}) = \chi_*^{-1}(\mathbf{a})^{\vee} \chi_*^{-1}(\mathbf{b})^b \\
&= (\mathbf{F}^T \mathbf{a} \mathbf{F}) (\mathbf{F}^{-1} \mathbf{b} \mathbf{F}) = \chi_*^{-1}(\mathbf{a})^b \chi_*^{-1}(\mathbf{b})^{\vee}
\end{aligned} \tag{5.3.26}$$

Using the relations

$$\begin{aligned}
\chi_*^{-1}(\mathbf{g})^b &= \mathbf{F}^T \mathbf{g} \mathbf{F} = \mathbf{C} \\
\chi_*^{-1}(\mathbf{g})^{\vee} &= \mathbf{F}^{-1} \mathbf{g} \mathbf{F} = \mathbf{I} \\
\chi_*^{-1}(\mathbf{g})^{\vee} &= \mathbf{F}^T \mathbf{g} \mathbf{F}^{-T} = \mathbf{I}
\end{aligned} \tag{5.3.27}$$

and Eqns. 5.3.20, the pull-back of the rate of deformation and spin tensors of 5.3.25 are

$$\begin{aligned}
\bar{\mathbf{d}} &= \chi_{*i \leftarrow c}^{-1}(\mathbf{d})^b = \frac{1}{2} \chi_{*i \leftarrow c}^{-1}(\mathbf{gl} + \mathbf{l}^T \mathbf{g})^b \\
&= \frac{1}{2} \left(\chi_{*i \leftarrow c}^{-1}(\mathbf{g})^b \chi_{*i \leftarrow c}^{-1}(\mathbf{l})^{\vee} + \chi_*^{-1}(\mathbf{l}^T)^{\vee} \chi_{*i \leftarrow c}^{-1}(\mathbf{g})^b \right) \\
&= \frac{1}{2} \left(\mathbf{C}^e \bar{\mathbf{l}} + (\mathbf{F}^e)^T \mathbf{l}^T (\mathbf{F}^e)^{-T} \mathbf{C}^e \right) \\
&= \frac{1}{2} \left(\mathbf{C}^e \bar{\mathbf{l}} + \left((\mathbf{F}^e)^{-1} \mathbf{l}^T (\mathbf{F}^e) \right)^T \mathbf{C}^e \right) \\
&= \frac{1}{2} \left(\mathbf{C}^e \bar{\mathbf{l}} + \bar{\mathbf{l}}^T \mathbf{C}^e \right)
\end{aligned} \tag{5.3.28}$$

Similar calculations lead to, in summary,

$$\begin{aligned}
\bar{\mathbf{d}} &= \frac{1}{2} (\mathbf{C}^e \bar{\mathbf{l}} + \bar{\mathbf{l}}^T \mathbf{C}^e), & \bar{\mathbf{w}} &= \frac{1}{2} (\mathbf{C}^e \bar{\mathbf{l}} - \bar{\mathbf{l}}^T \mathbf{C}^e) \\
\bar{\mathbf{d}}^e &= \frac{1}{2} (\mathbf{C}^e \bar{\mathbf{l}}^e + (\bar{\mathbf{l}}^e)^T \mathbf{C}^e), & \bar{\mathbf{w}}^e &= \frac{1}{2} (\mathbf{C}^e \bar{\mathbf{l}}^e - (\bar{\mathbf{l}}^e)^T \mathbf{C}^e) \\
\bar{\mathbf{d}}^p &= \frac{1}{2} (\mathbf{C}^e \bar{\mathbf{l}}^p + (\bar{\mathbf{l}}^p)^T \mathbf{C}^e), & \bar{\mathbf{w}}^p &= \frac{1}{2} (\mathbf{C}^e \bar{\mathbf{l}}^p - (\bar{\mathbf{l}}^p)^T \mathbf{C}^e)
\end{aligned} \tag{5.3.29}$$

The tensor $\mathbf{C}^e$ plays the role of the metric tensor on the intermediate configuration and appears here so as to ensure that the mixed components of the velocity gradients become covariant components, consistent with the covariant components of $\mathbf{d}$ and $\mathbf{w}$.

Note that

$$\bar{\mathbf{d}} = \bar{\mathbf{d}}^e + \bar{\mathbf{d}}^p, \quad \bar{\mathbf{w}} = \bar{\mathbf{w}}^e + \bar{\mathbf{w}}^p \tag{5.3.30}$$

5.3.4 Stress Tensors

The pull-back of the Kirchhoff stress $\boldsymbol{\tau}$ is the PK2 stress $\mathbf{S}$:

$$\mathbf{S} = \chi_{*r \leftarrow c}^{-1}(\boldsymbol{\tau})^{\#} = \mathbf{F}^{-1} \boldsymbol{\tau} \mathbf{F}^{-T} \quad (5.3.)$$

Introduce a new stress tensor $\bar{\mathbf{S}}$ to be the analogous pull-back of $\boldsymbol{\tau}$ to the intermediate configuration (see Eqn. 2.10.20b),

$$\bar{\mathbf{S}} = \chi_{*i \leftarrow c}^{-1}(\boldsymbol{\tau})^{\#} = (\mathbf{F}^e)^{-1} \boldsymbol{\tau} (\mathbf{F}^e)^{-T} \quad (5.3.)$$

or, equivalently, the push-forward of $\mathbf{S}$ to the intermediate configuration (see Eqn. 2.10.18b),

$$\bar{\mathbf{S}} = \chi_{*r \rightarrow i}(\mathbf{S})^{\#} = (\mathbf{F}^p) \mathbf{S} (\mathbf{F}^p)^T \quad (5.3.)$$

Stress Rates

Define $\dot{\bar{\mathbf{S}}}$ to be the material time derivative of $\bar{\mathbf{S}}$:

$$\dot{\bar{\mathbf{S}}} = \frac{d}{dt} \bar{\mathbf{S}} \quad (5.3.)$$

This is equivalent to the pull-back of the Kirchhoff stress to the intermediate configuration:

$$\begin{aligned} L_{v,i \leftarrow c}^{\#} \boldsymbol{\tau} &= \dot{\boldsymbol{\tau}} - \mathbf{l}^e \boldsymbol{\tau} - \boldsymbol{\tau} (\mathbf{l}^e)^T \\ &= \chi_{*i \rightarrow c} \left(\frac{d}{dt} [\chi_{*i \leftarrow c}^{-1}(\boldsymbol{\tau})^{\#}] \right)^{\#} \\ &= \chi_{*i \rightarrow c} (\dot{\bar{\mathbf{S}}})^{\#} \\ &= \mathbf{F}^e \dot{\bar{\mathbf{S}}} (\mathbf{F}^e)^T \end{aligned} \quad (5.3.)$$

Another useful Lie derivative is that of $\bar{\mathbf{S}}$, where the pull back is to the reference configuration:

$$\begin{aligned} L_{v,r \leftarrow i}^{\#} \bar{\mathbf{S}} &= \dot{\bar{\mathbf{S}}} - \bar{\mathbf{l}}^p \bar{\mathbf{S}} - \bar{\mathbf{S}} (\bar{\mathbf{l}}^p)^T \\ &= \chi_{*r \rightarrow i} \left(\frac{d}{dt} [\chi_{*r \leftarrow i}^{-1}(\bar{\mathbf{S}})^{\#}] \right)^{\#} \\ &= \chi_{*r \rightarrow i} (\dot{\mathbf{S}})^{\#} \\ &= \mathbf{F}^p \dot{\mathbf{S}} (\mathbf{F}^p)^T \end{aligned} \quad (5.3.)$$

These are related through

$$\chi_{*i \rightarrow c} \left(L_{v,r \leftarrow i}^{\#} \bar{\mathbf{S}} \right)^{\#} = L_{v,r \rightarrow c}^{\#} \boldsymbol{\tau} = \mathbf{F} \dot{\mathbf{S}} \mathbf{F}^T \quad (5.3.)$$

5.3.5 Problems

1. Use Eqns. 5.3.2 to show that the action of the Right Cauchy-Green strain tensors $\mathbf{C}$, $\mathbf{C}^p$, $\bar{\mathbf{C}}^e$ and the Left Cauchy-Green strain tensors $\mathbf{b}$, $\bar{\mathbf{b}}^p$, $\mathbf{b}^e$ on line elements in the reference, intermediate and current configurations are given by Eqns. 5.3.9.
2. Show that the action of the Green-Lagrange strain tensors $\mathbf{E}$, $\mathbf{E}^p$, $\bar{\mathbf{E}}^e$ and the Euler-Almansi strain tensors $\mathbf{e}$, $\bar{\mathbf{e}}^p$, $\mathbf{e}^e$ on line elements in the reference, intermediate and current configurations are given by Eqns. 5.3.10.
3. Derive the decomposition 5.3.13, $\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \bar{\mathbf{e}}^p$.
4. Verify all nine push-forward/pull-back operations, 5.3.16a-c.
5. Use Eqn. 2.5.4, $\mathbf{L} = \dot{\mathbf{F}} \mathbf{F}^{-1}$, and 5.3.1, to show that $\mathbf{l} = \dot{\mathbf{F}}^e (\mathbf{F}^e)^{-1} + \mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} (\mathbf{F}^e)^{-1}$.
6. Use relations 5.3.26-29 to verify relations 5.3.31.
7. Use the relation 5.3.11, $\bar{\mathbf{E}} = (\mathbf{F}^p)^{-T} \mathbf{E} (\mathbf{F}^p)^{-1}$, and Eqns. 2.5.5, i.e. $\overline{(\dot{\mathbf{F}}^p)^{-1}} = -(\mathbf{F}^p)^{-1} (\dot{\mathbf{I}}^p)$, $\overline{(\dot{\mathbf{F}}^p)^{-T}} = -(\dot{\mathbf{I}}^p)^T (\mathbf{F}^p)^{-T}$, to show that $\dot{\bar{\mathbf{E}}} = \bar{\mathbf{d}} - (\dot{\mathbf{I}}^p)^T \bar{\mathbf{E}} - \bar{\mathbf{E}} \dot{\mathbf{I}}^p$.

5.3.6 Appendix to §5.3: Convected Coordinates

Deformation Gradients

Using a convected coordinate formulation, the line elements can be expressed as

$$d\mathbf{X} = d\Theta^i \mathbf{G}_i, \quad d\bar{\mathbf{x}} = d\Theta^i \bar{\mathbf{g}}_i, \quad d\mathbf{x} = d\Theta^i \mathbf{g}_i \quad (5.3.5)$$

The deformation gradients are

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i, \quad \mathbf{F}^p = \bar{\mathbf{g}}_i \otimes \mathbf{G}^i, \quad \mathbf{F}^e = \mathbf{g}_i \otimes \bar{\mathbf{g}}^i \quad (5.3.6)$$

and the inverse deformation gradients are

$$\mathbf{F}^{-1} = \mathbf{G}_i \otimes \mathbf{g}^i, \quad (\mathbf{F}^p)^{-1} = \mathbf{G}_i \otimes \bar{\mathbf{g}}^i, \quad (\mathbf{F}^e)^{-1} = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i \quad (5.3.6)$$

Strain Tensors

From these, the Cauchy-Green tensors are

$$\begin{aligned} \mathbf{C} &= g_{ij} \mathbf{G}^i \otimes \mathbf{G}^j, & \mathbf{C}^p &= \bar{g}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j, & \bar{\mathbf{C}}^e &= g_{ij} \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j \\ \mathbf{b} &= G^{ij} \mathbf{g}_i \otimes \mathbf{g}_j, & \bar{\mathbf{b}}^p &= G^{ij} \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j, & \mathbf{b}^e &= \bar{g}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \end{aligned} \quad (5.3.6)$$

and their inverses are

$$\begin{aligned} \mathbf{C}^{-1} &= g^{ij} \mathbf{G}_i \otimes \mathbf{G}_j, & (\mathbf{C}^p)^{-1} &= \bar{g}^{ij} \mathbf{G}_i \otimes \mathbf{G}_j, & (\bar{\mathbf{C}}^e)^{-1} &= g^{ij} \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}_j \\ \mathbf{b}^{-1} &= G_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, & (\bar{\mathbf{b}}^p)^{-1} &= G_{ij} \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j, & (\mathbf{b}^e)^{-1} &= \bar{g}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \end{aligned} \quad (5.3.6)$$

with the metric coefficients given by

$$\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j, & \bar{g}_{ij} &= \bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j, & G_{ij} &= \mathbf{G}_i \cdot \mathbf{G}_j \\ g^{ij} &= \mathbf{g}^i \cdot \mathbf{g}^j, & \bar{g}^{ij} &= \bar{\mathbf{g}}^i \cdot \bar{\mathbf{g}}^j, & G^{ij} &= \mathbf{G}^i \cdot \mathbf{G}^j \end{aligned} \quad (5.3.6)$$

The Green-Lagrange and Euler-Almansi strains are

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}(g_{ij} - G_{ij}) \mathbf{G}^i \otimes \mathbf{G}^j, & \mathbf{E}^p &= \frac{1}{2}(\bar{g}_{ij} - G_{ij}) \mathbf{G}^i \otimes \mathbf{G}^j, & \bar{\mathbf{E}}^e &= \frac{1}{2}(g_{ij} - \bar{g}_{ij}) \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j \\ \mathbf{e} &= \frac{1}{2}(g_{ij} - G_{ij}) \mathbf{g}^i \otimes \mathbf{g}^j, & \bar{\mathbf{e}}^p &= \frac{1}{2}(\bar{g}_{ij} - G_{ij}) \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j, & \mathbf{e}^e &= \frac{1}{2}(g_{ij} - \bar{g}_{ij}) \mathbf{g}^i \otimes \mathbf{g}^j \end{aligned} \quad (5.3.6)$$

Also,

$$\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \bar{\mathbf{e}}^p = \frac{1}{2}(g_{ij} - G_{ij}) \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j \quad (5.3.6)$$

Strain-Rate Tensors

The time rates of change of the Green-Lagrange strain and the plastic Green-Lagrange strain are

$$\dot{\mathbf{E}} = \frac{1}{2} \dot{g}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j, \quad \dot{\mathbf{E}}^p = \frac{1}{2} \dot{\bar{g}}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j \quad (5.3.6)$$

The spatial velocity gradient is

$$\mathbf{l} = l_j^i \mathbf{g}_i \otimes \mathbf{g}^j = \frac{\partial v^i}{\partial x^j} \mathbf{g}_i \otimes \mathbf{g}^j \quad (5.3.6)$$

The rate of deformation is the symmetric part of $\mathbf{l}$:

$$\mathbf{d} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T), \quad d_{\cdot j}^i = \frac{1}{2}(l_{\cdot j}^i + l_j^i) \quad (5.3.6)$$

For covariant components,

$$d_{ij} = g_{im} d_{\cdot j}^m = \frac{1}{2}(g_{im} l_{\cdot j}^m + g_{im} l_j^{\cdot m}) = \frac{1}{2}(l_{ij} + l_{ji}) \quad (5.3.6)$$

When $\mathbf{l}$ is expressed in terms of mixed components, as in ???, one ends up with mixed components for the rate of deformation, ????. One can obtain the covariant components for $\mathbf{d}$ directly, by expressing $\mathbf{d}$ in the form

$$\begin{aligned} \mathbf{d} &= \frac{1}{2}(\mathbf{gl} + \mathbf{l}^T \mathbf{g}) = \frac{1}{2}((\mathbf{g}^i \otimes \mathbf{g}_i)(l_{\cdot j}^m \mathbf{g}_m \otimes \mathbf{g}^j) + (l_j^{\cdot m} \mathbf{g}^j \otimes \mathbf{g}_m)(\mathbf{g}_i \otimes \mathbf{g}^i)) \\ &= \frac{1}{2}(l_{ij} + l_{ji}) \mathbf{g}^i \otimes \mathbf{g}^j \end{aligned} \quad (5.3.6)$$

where $\mathbf{g}$ is the metric tensor.

Note that the rate of deformation and the plastic rate of deformation are the push-forwards of the corresponding Green-Lagrange strains, so

$$\mathbf{d} = \chi_* (\dot{\mathbf{E}})^b = \frac{1}{2} \dot{g}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \quad \bar{\mathbf{d}} = \chi_* (\dot{\mathbf{E}}^p)^b = \frac{1}{2} \dot{\bar{g}}_{ij} \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j \quad (5.3.6)$$

or $\mathbf{d} = \mathbf{F}^{-T} \dot{\mathbf{E}} \mathbf{F}^{-1}$ and $\bar{\mathbf{d}} = (\mathbf{F}^p)^{-T} \dot{\mathbf{E}}^p (\mathbf{F}^p)^{-1}$.

2.2 Thermoelasticity

In this section, thermoelasticity is considered. By definition, the constitutive relations for such a material depend only on the set of field variables $\{\mathbf{F}, \theta, \text{Grad}\theta\}$. This general case will be considered toward the end of this section. Simpler cases, isothermal $\{\mathbf{F}\}$, small-strain $\{\boldsymbol{\varepsilon}\}$, one-dimensional $\{\varepsilon\}$, will be considered first.

In what follows, upper case letters will be used to denote the energy potentials *per unit volume*. As before, lower case denotes *per unit mass* so, for example, $U = \rho u$ for internal energy.

2.2.1 Isothermal Small Strain Elasticity

For isothermal deformations, the basic thermodynamic statement can be written as (see Eqn. 1.6.24)

$$\begin{aligned}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} &= \rho \dot{\psi} + \rho \dot{\phi} \\ &= \dot{\Psi} + \Phi\end{aligned}\tag{2.2.1}$$

where

$$\begin{aligned}\dot{\Psi} &= \rho \dot{\psi} && \text{(rate of change of the free energy per unit volume)} \\ \Phi &= \rho \dot{\phi} \geq 0 && \text{(dissipation per unit volume)}\end{aligned}$$

In isothermal elasticity¹, $\Phi = 0$, and so one need only specify a free energy function.

One-Dimensional Analysis

Consider a simple one-dimensional extension of material. The free energy is a function of kinematic state variables. The only state variable is the single strain ε and the only stress is σ and hence

$$\sigma \dot{\varepsilon} = \dot{\Psi}(\varepsilon)\tag{2.2.2}$$

i.e.

$$\sigma \dot{\varepsilon} = \frac{d\Psi(\varepsilon)}{d\varepsilon} \dot{\varepsilon}\tag{2.2.3}$$

so that, since $\dot{\varepsilon}$ is arbitrary,

$$\sigma = \frac{d\Psi(\varepsilon)}{d\varepsilon}\tag{2.2.4}$$

¹ this is discussed more rigorously in §2.2.2 below

which agrees with the general relation 1.6.33d. This is the stress-strain (constitutive) equation. In the context of elasticity theory, the Helmholtz free energy function is the elastic strain energy function (Eqn. 2.2.4 is the same as Part I, Eqn. 5.4.9).

The classical linear elastic material corresponds to choosing a simple quadratic free energy function:

$$\Psi(\varepsilon) = \frac{1}{2} E \varepsilon^2 \quad (2.2.5)$$

so

$$\sigma = E \varepsilon, \quad \dot{\sigma} = E \dot{\varepsilon} \quad (2.2.6)$$

A nonlinear example is

$$\begin{aligned} \Psi(\varepsilon) &= E_0 \varepsilon + E_1 \frac{\varepsilon^2}{2} + E_2 \frac{\varepsilon^3}{3} + E_3 \frac{\varepsilon^4}{4} \\ \sigma(\varepsilon) &= E_0 + E_1 \varepsilon + E_2 \varepsilon^2 + E_3 \varepsilon^3 \\ \dot{\sigma} &= (\partial \sigma / \partial \varepsilon) \dot{\varepsilon} \\ &= (E_1 + 2E_2 \varepsilon + 3E_3 \varepsilon^2) \dot{\varepsilon} \end{aligned} \quad (2.2.7)$$

One should ensure that (at *least* (1))

- (1) $\sigma(0) = 0$ (zero strain at zero stress) which implies that $E_0 = 0$
- (2) $(d\sigma/d\varepsilon)_{\varepsilon=0} \neq 0$ (so one has a non-zero initial modulus)
- (3) $(d^2\sigma/d\varepsilon^2)_{\varepsilon=0} = 0$ (so that, with (2), the origin is a point of inflection, ensuring similar tensile and compressive strain behaviour)

Note that Ψ has the dimensions of stress, so that the E_i also have the dimensions of stress.

In the **rate formulation**: $\dot{\sigma} = D \dot{\varepsilon}$, D is the **instantaneous elastic modulus** (or **stiffness**) $d\sigma/d\varepsilon$ or $d^2\Psi/d\varepsilon^2$. D is a constant in the linear model, but is, in general, a function of strain, as in the above example, where $D = E_1 + 2E_2 \varepsilon + 3E_3 \varepsilon^2$.

Three-dimensional Analysis

Generalising the above to three dimensions, the free energy function is

$$\Psi(\varepsilon_{ij}) \quad \Psi(\boldsymbol{\varepsilon}) \quad (2.2.8)$$

so that the basic thermomechanical identity becomes

$$\sigma_{ij} \dot{\varepsilon}_{ij} = \dot{\Psi}(\varepsilon_{ij}) = \frac{\partial \Psi}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} \quad \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \dot{\Psi}(\boldsymbol{\varepsilon}) = \frac{d\Psi}{d\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \quad (2.2.9)$$

Since this again holds for all strain rates one can deduce that

$$\sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}} \quad \boldsymbol{\sigma} = \frac{d\Psi}{d\boldsymbol{\varepsilon}} \quad (2.2.10)$$

Again, for the linear elastic material, the free energy function is assumed to be a quadratic function of the strain:

$$\Psi = \frac{1}{2} D_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \quad \Psi = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{D} : \boldsymbol{\varepsilon} \quad (2.2.11)$$

so that

$$\begin{aligned} \sigma_{mn} &= \frac{\partial \Psi}{\partial \varepsilon_{mn}} \\ &= \frac{1}{2} D_{ijkl} \frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{mn}} \varepsilon_{kl} + \frac{1}{2} D_{ijkl} \varepsilon_{ij} \frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}} \\ &= \frac{1}{2} D_{mnkl} \varepsilon_{kl} + \frac{1}{2} D_{ijmn} \varepsilon_{ij} \\ &= D_{mnkl} \varepsilon_{kl} \end{aligned} \quad (2.2.12)$$

provided the fourth order stiffness tensor $\mathbf{D}$ possesses the major symmetry:

$$D_{ijkl} = D_{klij} \quad (2.2.13)$$

($\mathbf{D}$ satisfies the minor symmetries $D_{ijkl} = D_{jikl}$ and $D_{ijkl} = D_{ijlk}$ by virtue of the symmetry of the stress and strain.)

Eqn. 2.1.12 is the stress-strain equation for an anisotropic linear elastic material. The rate form is formally

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} \quad (2.2.14)$$

For a nonlinear material Ψ is no longer quadratic, and the instantaneous modulus tensor is a function of the strains.

If one starts by proposing the rate form of the elastic law, with a given modulus function $\mathbf{D}(\boldsymbol{\varepsilon})$, then in general one cannot integrate Eqn. 2.2.14 to find a free energy function (although one can achieve this in one-dimension). Such a model is termed **hypoelastic**. A model in which Ψ does exist is called **hyperelastic**. Hyperelastic models automatically satisfy the laws of thermomechanics, hypoelastic ones do not.

Note that the strain energy must be positive, $\Psi > 0$. From 2.2.11, this implies that

$$D_{ijkl} \varepsilon_{ij} \varepsilon_{kl} > 0 \quad \boldsymbol{\varepsilon} : \mathbf{D} : \boldsymbol{\varepsilon} > 0 \quad (2.2.15)$$

for all strain tensors, that is, the stiffness tensor must be positive definite.

Isotropic Elasticity

The theory of isotropic elasticity with small strains is a special case of that for large strains discussed in Part III, §4.4. The theory discussed there applies here, only now with the left Cauchy Green strain replaced by the small strain tensor. Thus, the free energy function must be a function of a set of three invariants of the strain tensor $\boldsymbol{\varepsilon}$, $\Psi(E_1, E_2, E_3)$, which here are taken to be

$$\begin{aligned} E_1 &= \varepsilon_{ii} = \text{tr} \boldsymbol{\varepsilon} \\ E_2 &= \varepsilon_{ij} \varepsilon_{ij} = \text{tr} \boldsymbol{\varepsilon}^2 \\ E_3 &= \varepsilon_{ij} \varepsilon_{jk} \varepsilon_{ki} = \text{tr} \boldsymbol{\varepsilon}^3 \end{aligned} \quad (2.2.16)$$

The stress-strain relation is hence given by:

$$\begin{aligned} \sigma_{ij} &= \frac{\partial \Psi}{\partial \varepsilon_{ij}} \\ &= \frac{\partial \Psi}{\partial E_1} \frac{\partial E_1}{\partial \varepsilon_{ij}} + \frac{\partial \Psi}{\partial E_2} \frac{\partial E_2}{\partial \varepsilon_{ij}} + \frac{\partial \Psi}{\partial E_3} \frac{\partial E_3}{\partial \varepsilon_{ij}} \end{aligned} \quad (2.2.17)$$

It is readily verified that {▲ Problem 1}

$$\begin{aligned} \partial E_1 / \partial \varepsilon_{ij} &= \delta_{ij} & \partial E_1 / \partial \boldsymbol{\varepsilon} &= \mathbf{I} \\ \partial E_2 / \partial \varepsilon_{ij} &= 2\varepsilon_{ij} & \partial E_2 / \partial \boldsymbol{\varepsilon} &= 2\boldsymbol{\varepsilon} \\ \partial E_3 / \partial \varepsilon_{ij} &= 3\varepsilon_{ik} \varepsilon_{kj} & \partial E_3 / \partial \boldsymbol{\varepsilon} &= 3\boldsymbol{\varepsilon}^2 \end{aligned} \quad (2.2.18)$$

so that

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial E_1} \mathbf{I} + 2 \frac{\partial \Psi}{\partial E_2} \boldsymbol{\varepsilon} + 3 \frac{\partial \Psi}{\partial E_3} \boldsymbol{\varepsilon}^2 \quad (2.2.19)$$

which is a quadratic polynomial in the strain tensor. If the free energy function does not depend on the third strain invariant E_3 , the tensor relation is actually linear, though it should be appreciated that the coefficients $\partial \Psi / \partial E_k$, $k = 1, 2, 3$, will in general depend in a nonlinear manner on the strain invariants.

The classical linear model is recovered by choosing

$$\Psi = \frac{1}{2} \lambda E_1^2 + \mu E_2 \quad \Psi = \frac{1}{2} \lambda (\text{tr} \boldsymbol{\varepsilon})^2 + \mu \text{tr} \boldsymbol{\varepsilon}^2 \quad (2.2.20)$$

for then

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad \boldsymbol{\sigma} = \lambda (\text{tr} \boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon} \quad (2.2.21)$$

The Gibbs Function and the Complementary Energy

From section 1.3, the Helmholtz free energy and the Gibbs energy functions are related by a Legendre transformation. Formally, consider again the one-dimensional stress-strain relation 2.2.4:

$$\sigma = \frac{d\Psi(\varepsilon)}{d\varepsilon} \quad (2.2.22)$$

To invert this expression and express the strain as a function of stress, first partially integrate the equation with respect to strain to obtain

$$\sigma\varepsilon = \Psi(\varepsilon) + C(\sigma) \quad (2.2.23)$$

where $C(\sigma)$ is an arbitrary function of integration, dependent only on stress. Differentiating with respect to σ leads to

$$\varepsilon = \frac{dC(\sigma)}{d\sigma} \quad (2.2.24)$$

which is the inverse of the original equation. From Table 1.3.1, it is clear that the function C is the negative of the Gibbs energy function (per unit volume). In this context, C is the complementary energy discussed in Part I, §5.4. One says that the free energy is the **dual** of the complementary energy, and *vice versa*.

For example, if

$$\Psi = K\varepsilon^3, \quad \sigma = 3K\varepsilon^2 \quad (2.2.25)$$

then {▲ Problem 2}

$$C(\sigma) = 2K(\sigma/3K)^{3/2}. \quad (2.2.26)$$

This theory readily extends to the general three dimensional situation. The original stress strain equation is given by 2.2.10 so that

$$\sigma_{ij}\varepsilon_{ij} = \Psi(\varepsilon_{ij}) + C(\sigma_{ij}) \quad \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \Psi(\boldsymbol{\varepsilon}) + C(\boldsymbol{\sigma}) \quad (2.2.27)$$

and the strains are now

$$\varepsilon_{ij} = \frac{dC(\sigma_{ij})}{d\sigma_{ij}}, \quad \boldsymbol{\varepsilon} = \frac{dC(\boldsymbol{\sigma})}{d\boldsymbol{\sigma}} \quad (2.2.28)$$

For the linear material, $\Psi = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{D} : \boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} = \mathbf{D}^{-1} : \boldsymbol{\sigma}$ and the dual Complementary function is $C = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{D}^{-1} : \boldsymbol{\sigma}$. The fact that $\mathbf{D}$ in the linear case is positive definite ensures that it is invertible. The inverse $\mathbf{D}^{-1}$ is called the **elastic flexibility tensor C** (see 2.2.11):

$$C = \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} \quad C = \frac{1}{2} \boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma} \quad (2.2.29)$$

2.2.2 Large Strain Thermoelasticity

In the general large strain thermoelastic case, the constitutive relations are of the form

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}(\mathbf{F}, \theta, \nabla \theta) \\ \mathbf{q} &= \mathbf{q}(\mathbf{F}, \theta, \nabla \theta) \\ u &= u(\mathbf{F}, \theta, \nabla \theta) \\ \psi &= \psi(\mathbf{F}, \theta, \nabla \theta) \\ s &= s(\mathbf{F}, \theta, \nabla \theta) \end{aligned} \quad (2.2.30)$$

Here, for convenience, the material gradient of 2.1.13 has been replaced with the spatial gradient $\nabla \theta$ (through $\nabla \theta = \mathbf{F}^{-1} \text{Grad} \theta$).

The dissipation inequality 1.6.20 is

$$\phi = -s\dot{\theta} - \dot{\psi} + \frac{1}{\rho} \boldsymbol{\sigma} : \mathbf{d} - \frac{1}{\rho\theta} \mathbf{q} \cdot \nabla \theta \geq 0 \quad (2.2.31)$$

Using the relation Part III, 3.7.25, $\boldsymbol{\sigma} : \mathbf{d} = \boldsymbol{\sigma} \mathbf{F}^{-T} : \dot{\mathbf{F}}$, this can be expressed as

$$\rho s \dot{\theta} + \rho \dot{\psi} - \boldsymbol{\sigma} \mathbf{F}^{-T} : \dot{\mathbf{F}} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \leq 0 \quad (2.2.32)$$

With

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \frac{\partial \psi}{\partial (\nabla \theta)} \cdot \dot{\overline{\nabla \theta}} \quad (2.2.33)$$

one has

$$\rho \left(s + \frac{\partial \psi}{\partial \theta} \right) \dot{\theta} - \left(\boldsymbol{\sigma} \mathbf{F}^{-T} - \rho \frac{\partial \psi}{\partial \mathbf{F}} \right) : \dot{\mathbf{F}} + \rho \frac{\partial \psi}{\partial (\nabla \theta)} \cdot \dot{\overline{\nabla \theta}} + \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \leq 0 \quad (2.2.34)$$

For any given $\{\mathbf{F}, \theta, \nabla \theta\}$, this inequality must hold for any arbitrary values of $\dot{\mathbf{F}}, \dot{\theta}$ and $\dot{\overline{\nabla \theta}}$. Thus the coefficients must be zero:

$$s = -\frac{\partial \psi}{\partial \theta}, \quad \boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^T, \quad \frac{\partial \psi}{\partial (\nabla \theta)} = 0 \quad (2.2.35)$$

The third of these implies that the free energy is independent of the temperature gradient:

$$\psi = \psi(\mathbf{F}, \theta) \quad (2.2.36)$$

and so the entropy and stress are also; further, the internal energy is

$$u = \psi - \theta \frac{\partial \psi}{\partial \theta} \quad (2.2.37)$$

Stress-Strain Relation

The stress-deformation relation 2.2.35b can be expressed in a number of different ways, for example, in terms of the PK1 stress $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\text{T}}$ and the PK2 stress $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\text{T}}$ (the first of these follows from the symmetry of the Cauchy stress)

$$\begin{aligned} \boldsymbol{\sigma} &= \rho \frac{\partial \psi}{\partial \mathbf{F}} \mathbf{F}^{\text{T}} = \rho \mathbf{F} \left(\frac{\partial \psi}{\partial \mathbf{F}} \right)^{\text{T}} \\ \mathbf{P} &= \rho_0 \frac{\partial \psi}{\partial \mathbf{F}} \\ \mathbf{S} &= \rho_0 \mathbf{F}^{-1} \frac{\partial \psi}{\partial \mathbf{F}} \end{aligned} \quad (2.2.38)$$

Introducing the free energy per unit *reference* volume, Ψ , these read as

$$\boldsymbol{\sigma} = J^{-1} \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^{\text{T}}, \quad \mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}, \quad \mathbf{S} = \mathbf{F}^{-1} \frac{\partial \Psi}{\partial \mathbf{F}} \quad (2.2.39)$$

In the case of material symmetries, they take reduced forms as discussed in Part III, Chapter 4. For example, for isotropic materials the Cauchy stress is a function of the left Cauchy-Green strain only:

$$\boldsymbol{\sigma} = 2J^{-1} \frac{\partial \Psi}{\partial \mathbf{b}} \mathbf{b} \quad (2.2.40)$$

Gibbs Relations

One also has the useful **Gibbs relations** {▲ Problem 3}

$$\begin{aligned}
\dot{\psi} &= \frac{1}{\rho_0} \mathbf{P} : \dot{\mathbf{F}} - s \dot{\theta} \\
\dot{u} &= \frac{1}{\rho_0} \mathbf{P} : \dot{\mathbf{F}} + \theta \dot{s} \\
\dot{h} &= -\frac{1}{\rho_0} \mathbf{F} : \dot{\mathbf{P}} + \theta \dot{s} \\
\dot{g} &= -\frac{1}{\rho_0} \mathbf{F} : \dot{\mathbf{P}} - s \dot{\theta}
\end{aligned} \tag{2.2.41}$$

Isothermal Elasticity

For isothermal materials, the free energy becomes the strain energy W per unit reference volume, e.g. in terms of the PK1 stress, the stress and stress power can be expressed as

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}, \quad \dot{W} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}} = \mathbf{P} : \dot{\mathbf{F}} \tag{2.2.42}$$

Since there are no temperature gradients, the entropy inequality 2.2.34 becomes an equality and any process is completely reversible. For a reversible process, the dissipation is $\phi = \theta \dot{s} = 0$ and so, as mentioned in §1.6.4, such a process is also isentropic.

Adiabatic Elasticity

For adiabatic conditions, the heat flux is zero, so again the entropy inequality 2.2.34 becomes an equality and any process is reversible and isentropic. Working with the internal energy, from 2.2.41b, with its canonical variables, $u = u(\mathbf{F}, s)$, one has

$$\dot{u} = \frac{1}{\rho} \boldsymbol{\sigma} \mathbf{F}^{-T} : \dot{\mathbf{F}} = \frac{1}{\rho_0} \mathbf{P} : \dot{\mathbf{F}} \tag{2.2.43}$$

The internal energy thus becomes the strain energy and one again has 2.2.42.

Energy Potentials and Canonical Variables

Following on from §1.3, the thermodynamic potentials (per unit reference volume) in terms of canonical variables take the form

$$\Psi = \Psi(\mathbf{F}, \theta), \quad U = U(\mathbf{F}, s), \quad G = G(\mathbf{P}, \theta), \quad H = H(\mathbf{P}, s) \tag{2.2.44}$$

The various relationships for these potentials are given in Table 2.2.1, which is a reproduction of Table 1.3.1 for finite strain reversible processes.

The fourth column contains the Gibbs relations 2.2.41.

Thermo-dynamic	Symbol and	Definition	Rate-form relationship	Maxwell relation
----------------	------------	------------	------------------------	------------------

potential	appropriate variables			
Internal energy	$U(S, \mathbf{F})$		$\dot{U} = \mathbf{P} : \dot{\mathbf{F}} + \theta \dot{S}$	$\theta = \left(\frac{\partial U}{\partial S} \right)_{\mathbf{F}}, \mathbf{P} = \left(\frac{\partial U}{\partial \mathbf{F}} \right)_S$ $\left(\frac{\partial \theta}{\partial \mathbf{F}} \right)_S = \left(\frac{\partial \mathbf{P}}{\partial S} \right)_{\mathbf{F}}$
Enthalpy	$H(S, \mathbf{P})$	$H = U - \mathbf{P} : \mathbf{F}$	$\dot{H} = -\mathbf{F} : \dot{\mathbf{P}} + \theta \dot{S}$	$\theta = \left(\frac{\partial H}{\partial S} \right)_{\mathbf{P}}, \mathbf{F} = -\left(\frac{\partial H}{\partial \mathbf{P}} \right)_S$ $\left(\frac{\partial \theta}{\partial \mathbf{P}} \right)_S = -\left(\frac{\partial \mathbf{F}}{\partial S} \right)_{\mathbf{P}}$
Helmholtz free energy	$\Psi(\theta, \mathbf{F})$	$\Psi = U - \theta S$	$\dot{\Psi} = \mathbf{P} : \dot{\mathbf{F}} - S \dot{\theta}$	$s = -\left(\frac{\partial \Psi}{\partial \theta} \right)_{\mathbf{F}}, \mathbf{P} = \left(\frac{\partial \Psi}{\partial \mathbf{F}} \right)_{\theta}$ $\left(\frac{\partial S}{\partial \mathbf{F}} \right)_{\theta} = -\left(\frac{\partial \mathbf{P}}{\partial \theta} \right)_{\mathbf{F}}$
Gibbs free energy	$G(\theta, \mathbf{P})$	$G = U - \theta S - \mathbf{P} : \mathbf{F}$	$\dot{G} = -\mathbf{F} : \dot{\mathbf{P}} - S \dot{\theta}$	$s = -\left(\frac{\partial G}{\partial \theta} \right)_{\mathbf{P}}, \mathbf{F} = -\left(\frac{\partial G}{\partial \mathbf{P}} \right)_{\theta}$ $\left(\frac{\partial S}{\partial \mathbf{P}} \right)_{\theta} = \left(\frac{\partial \mathbf{F}}{\partial \theta} \right)_{\mathbf{P}}$

Table 2.2.1: Thermodynamic Potentials (per unit reference volume)

The Gibbs Relation and Classical Thermodynamics

In increment form, the second of the Gibbs relations 2.2.41 can be expressed as

$$ds = \frac{1}{\theta} \left(du - \frac{1}{\rho} \boldsymbol{\sigma} \mathbf{F}^{-\text{T}} : d\mathbf{F} \right) \quad (2.2.45)$$

Consider an elastic (inviscid) fluid, for which by definition the stress is a function of $\mathbf{F}$ only through the density ρ :

$$\boldsymbol{\sigma} = -p(\rho, \theta) \mathbf{I} \quad (2.2.46)$$

where p is the hydrostatic pressure. Using {▲ Problem 4}

$$\frac{\dot{\rho}}{\rho} = -\mathbf{F}^{-\text{T}} : \mathbf{F}, \quad (2.2.47)$$

relation 2.2.45 reduces to the classical thermodynamics Gibbs relation

$$ds = \frac{1}{\theta} \left(du - \frac{p}{\rho^2} d\rho \right) \quad (2.2.48)$$

Heat Flux

The first law is

$$\begin{aligned}\dot{u} &= \frac{1}{\rho_0} \mathbf{P} : \dot{\mathbf{F}} - \frac{1}{\rho_0} \text{Div} \mathbf{Q} \\ &= \frac{1}{\rho} \boldsymbol{\sigma} : \mathbf{d} - \frac{1}{\rho} \text{div} \mathbf{q}\end{aligned}\quad (2.2.49)$$

Recall that, in this expression, the divergence of the (Cauchy) heat flux represents the heat entering a material element, Eqn. 1.6.4,

$$\frac{\delta Q}{dt} = - \int_v \text{div} \mathbf{q} dv \quad (2.2.50)$$

For a homogeneous material, $(\delta Q / dt)(1/v) = -\text{div} \mathbf{q}$.

Using 2.2.41b, one has the useful relation between the rate of entropy and the heat flux vector for a thermoelastic material:

$$\theta \dot{s} = - \frac{1}{\rho_0} \text{Div} \mathbf{Q} = - \frac{1}{\rho} \text{div} \mathbf{q} \quad (2.2.51)$$

Heat Conduction

From 2.2.34, the dissipation inequality reads

$$\mathbf{q}(\mathbf{F}, \theta, \nabla \theta) \cdot \nabla \theta \leq 0 \quad (2.2.52)$$

This is **Fourier's inequality** and merely states that heat flows from a hot region to a cold region. One usually assumes **Fourier's law of heat conduction**:

$$\mathbf{q} = -\mathbf{K}(\mathbf{F}, \theta) \nabla \theta \quad (2.2.53)$$

where $\mathbf{K}$ is the **thermal conductivity** tensor. This constitutive law is sometimes referred to as **Duhamel's law of heat conduction**, to distinguish it from the case of a thermally isotropic material, for which $\mathbf{K} = k\mathbf{I}$, which is then referred to as Fourier's law.

If one considers the left hand side of Eqn. 2.2.52 to be a function of $\nabla \theta$, this function attains a maximum (of zero) when $\nabla \theta = 0$. Thus

$$\left. \frac{\partial}{\partial \nabla \theta} (\mathbf{q} \cdot \nabla \theta) \right|_{\nabla \theta = 0} = \left(\frac{\partial \mathbf{q}}{\partial \nabla \theta} \right)^T \nabla \theta + \mathbf{q} \Big|_{\nabla \theta = 0} = \mathbf{q}(\mathbf{F}, \theta, \mathbf{0}) = \mathbf{0} \quad (2.2.54)$$

This implies that, without a temperature gradient, there is no heat flow. Since it is a maximum,

$$\left. \frac{\partial^2}{\partial(\nabla\theta)^2} (\mathbf{q} \cdot \nabla\theta) \right|_{\nabla\theta=0} = \left(\frac{\partial\mathbf{q}}{\partial\nabla\theta} \right)^T + \frac{\partial\mathbf{q}}{\partial\nabla\theta} \quad (2.2.55)$$

must be negative semi-definite. This implies that the symmetric part of the thermal conductivity tensor $\mathbf{K}$ in 2.2.53 must be positive semi-definite.

Consider next the quantity $\mathbf{K} : \partial^2\theta / \partial\mathbf{x}^2$. Since $\partial(\partial\theta / \partial x_i) / \partial x_j = \partial(\partial\theta / \partial x_j) / \partial x_i$, $\partial^2\theta / \partial\mathbf{x}^2$ is symmetric. Further, the double contraction of a symmetric tensor and a skew tensor is zero. Thus $\mathbf{K} : \partial^2\theta / \partial\mathbf{x}^2 = \text{sym}\mathbf{K} : \partial^2\theta / \partial\mathbf{x}^2$. The heat flux appears in the first law only as $\text{div}\mathbf{q}$. Evaluating this gives, with Fourier's law,

$$\text{div}\mathbf{q} = \frac{\partial}{\partial x_i} \left(-K_{ij} \frac{\partial\theta}{\partial x_j} \right) = -\frac{\partial K_{ij}}{\partial x_i} \frac{\partial\theta}{\partial x_j} - K_{ij} \frac{\partial^2\theta}{\partial x_i \partial x_j} \quad (2.2.56)$$

For a homogeneous material, $\partial K_{ij} / \partial x_i = 0$, and so

$$\text{div}\mathbf{q} = -\text{sym}\mathbf{K} : \frac{\partial^2\theta}{\partial\mathbf{x}^2} \quad (2.2.57)$$

Thus, for homogeneous materials, it is only the symmetric part of the thermal conductivity tensor which enters the energy balance. In fact, it is almost universally accepted that $\mathbf{K}$ is symmetric, whether the material is homogeneous or not.

Heat Capacity and Latent Heat

The specific heat capacity is the heat required to raise the temperature of a body by unit temperature (per unit mass). From the first law, 2.2.49, when there is no deformation ($\dot{\mathbf{F}} = \mathbf{0}$), all the heat absorbed is transferred to internal energy. The **specific heat capacity at constant deformation** is then by definition (see Eqn. 1.1.15)

$$c_F = \frac{\partial u(\mathbf{F}, \theta)}{\partial \theta} \quad (2.2.58)$$

The heat capacity can also be expressed in terms of the free energy { **▲ Problem 5** }:

$$c_F = \theta \frac{\partial s(\mathbf{F}, \theta)}{\partial \theta} = -\theta \frac{\partial^2 \psi(\mathbf{F}, \theta)}{\partial \theta^2} \quad (2.2.59)$$

From 2.2.51, one then has the **equation of heat conduction** { **▲ Problem 6** }

$$-\theta \frac{\partial^2 \psi}{\partial \theta \partial \theta} \dot{\theta} - \theta \frac{\partial^2 \psi}{\partial \theta \partial \mathbf{F}} : \dot{\mathbf{F}} = -\frac{1}{\rho_0} \text{Div}\mathbf{Q} \quad (2.2.60)$$

Some of the heat energy results in a temperature change, the first term, which is the heat capacity term. The second term represents a thermoelastic coupling effect whereby a

deformation induces a temperature change; this term is often neglected as it is usually small in applications. This second term can be interpreted as a latent heat term, whereby heat can be absorbed without a change in temperature.

In terms of the enthalpy, the first law 2.2.49 can be expressed as

$$\dot{h} = \frac{1}{\rho_0} \mathbf{F} : \dot{\mathbf{P}} - \frac{1}{\rho_0} \text{Div} \mathbf{Q} \quad (2.2.61)$$

At constant stress, the heat energy now results in an increase in enthalpy. The **specific heat capacity at constant stress** is then by definition

$$c_p = \frac{\partial h(\mathbf{P}, \theta)}{\partial \theta} \quad (2.2.62)$$

The heat capacity can also be expressed in terms of the Gibbs energy (using the relations in Table 2.2.1):

$$c_p = \theta \frac{\partial s(\mathbf{P}, \theta)}{\partial \theta} = -\theta \frac{\partial^2 g(\mathbf{P}, \theta)}{\partial \theta^2} \quad (2.2.63)$$

For isotropic engineering materials, e.g. metals, concrete, which undergo small strains, $c_F = c_\varepsilon \mathbf{I}$, with $c_\varepsilon \approx 10^3 \text{ Nm/kgK}$ and $c_p = c_\sigma \mathbf{I}$, with c_σ being slightly greater than c_ε (by a few percent).

2.2.3 Small Strain Thermoelasticity

The Free Energy and Thermal Strains

In the linear theory, the free energy is a function of the small-strain and the temperature, $\psi(\boldsymbol{\varepsilon}, \theta)$, and, as with the strains, temperature changes are assumed to be small. Assume that in the reference state the temperature is θ_0 and the material is strain-free. Then, expanding the free energy in a Taylor series and retaining only terms up to second order,

$$\begin{aligned} \psi(\boldsymbol{\varepsilon}, \theta) = & \psi|_0 + \left. \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \right|_0 : \boldsymbol{\varepsilon} + \left. \frac{\partial \psi}{\partial \theta} \right|_0 (\theta - \theta_0) + \frac{1}{2} \boldsymbol{\varepsilon} : \left. \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^2} \right|_0 : \boldsymbol{\varepsilon} \\ & + (\theta - \theta_0) \left. \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \theta} \right|_0 : \boldsymbol{\varepsilon} + \frac{1}{2} \left. \frac{\partial^2 \psi}{\partial \theta^2} \right|_0 (\theta - \theta_0)^2 \end{aligned} \quad (2.2.64)$$

where the notation $|_0$ means that the quantity is evaluated at the reference state. Note that the specific heat at constant deformation (strain) occurs in the last term here (see Eqn. 2.2.59).

The stress is now

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \bigg|_0 + \rho \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^2} \bigg|_0 : \boldsymbol{\varepsilon} + \rho (\theta - \theta_0) \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \theta} \bigg|_0 \quad (2.2.65)$$

Assuming the material is stress-free in the reference state, $\boldsymbol{\varepsilon} = \mathbf{0}$, $\theta = \theta_0$, then $(\partial \psi / \partial \boldsymbol{\varepsilon})|_0 = \mathbf{0}$, and 2.2.65 can be expressed as

$$\sigma_{ij} = D_{ijkl} \varepsilon_{kl} - \beta_{ij} (\theta - \theta_0) \quad \boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon} - \boldsymbol{\beta} (\theta - \theta_0) \quad (2.2.66)$$

where the stiffness tensor is

$$D_{ijkl} = \rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \quad \mathbf{D} = \rho \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^2} \quad (2.2.67)$$

and

$$\beta_{ij} = -\rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \theta} \quad \boldsymbol{\beta} = -\rho \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \theta} \quad (2.2.68)$$

It was seen earlier that $\mathbf{D}$ is positive definite and so is always invertible. Define then the tensor $\boldsymbol{\alpha}$ through $\mathbf{D} : \boldsymbol{\alpha} = \boldsymbol{\beta}$ (so $\boldsymbol{\alpha} = \mathbf{C} : \boldsymbol{\beta}$, where $\mathbf{C}$ is the flexibility tensor) so that

$$\sigma_{ij} = D_{ijkl} [\varepsilon_{kl} - \alpha_{kl} (\theta - \theta_0)] \quad \boldsymbol{\sigma} = \mathbf{D} : [\boldsymbol{\varepsilon} - \boldsymbol{\alpha} (\theta - \theta_0)] \quad (2.2.69)$$

The quantity $\boldsymbol{\alpha}(\theta - \theta_0)$ is called the **thermal strain** and $\boldsymbol{\alpha}$ is the tensor of **thermal expansion coefficients**. For isotropic engineering materials, $\boldsymbol{\alpha} = \alpha \mathbf{I}$, with $\alpha \approx 10^{-5} \text{ K}^{-1}$.

Note that, from 2.2.67-8, with the derivatives evaluated at the reference configuration, $\mathbf{D}$ and $\boldsymbol{\alpha}$ are constant tensors. One can generalize to a theory with temperature-dependent properties, by expressing 2.2.64 in the form

$$\begin{aligned} \psi(\boldsymbol{\varepsilon}, \theta) = & A + B(\theta)(\theta - \theta_0) + \frac{1}{2} \boldsymbol{\varepsilon} : \frac{1}{\rho} \mathbf{D}(\theta) : \boldsymbol{\varepsilon} \\ & - (\theta - \theta_0) \frac{1}{\rho} \boldsymbol{\beta}(\theta) : \boldsymbol{\varepsilon} + \frac{1}{2} E(\theta) (\theta - \theta_0)^2 \end{aligned} \quad (2.2.70)$$

The Gibbs Energy

The temperature dependence of a thermoelastic material can be conveniently expressed using the Gibbs free energy, $g = g(\boldsymbol{\sigma}, \theta)$:

$$\varepsilon_{ij} = -\rho \frac{\partial g}{\partial \sigma_{ij}} \quad \boldsymbol{\varepsilon} = -\rho \frac{\partial g}{\partial \boldsymbol{\sigma}} \quad (2.2.71)$$

Then, immediately,

$$\dot{\epsilon}_{ij} = C_{ijkl} \dot{\sigma}_{kl} + \alpha_{ij} \dot{\theta} \quad \dot{\boldsymbol{\epsilon}} = \mathbf{C} : \dot{\boldsymbol{\sigma}} + \boldsymbol{\alpha} \dot{\theta} \quad (2.2.72)$$

where now

$$\begin{aligned} C_{ijkl} &= -\rho \frac{\partial^2 g}{\partial \sigma_{ij} \partial \sigma_{kl}} & \mathbf{C} &= -\rho \frac{\partial^2 g}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} \\ \alpha_{ij} &= -\rho \frac{\partial^2 g}{\partial \sigma_{ij} \partial \theta} & \boldsymbol{\alpha} &= -\rho \frac{\partial^2 g}{\partial \boldsymbol{\sigma} \partial \theta} \end{aligned} \quad (2.2.73)$$

To complete the set of second derivatives of the Gibbs function, one also has the heat capacity at constant stress (see Eqn. 2.2.63):

$$c_{\sigma} = -\theta \frac{\partial^2 g}{\partial \theta \partial \theta} \quad (2.2.74)$$

Heat Capacity

Just as the stress, Eqn. 2.2.65, was expressed in terms of the free energy 2.2.64, the entropy can be expressed as the linear function

$$s = -\frac{\partial \psi}{\partial \theta} = -\frac{\partial \psi}{\partial \theta} \Big|_0 - \frac{\partial^2 \psi}{\partial \boldsymbol{\epsilon} \partial \theta} \Big|_0 : \boldsymbol{\epsilon} - (\theta - \theta_0) \frac{\partial^2 \psi}{\partial \theta^2} \Big|_0 \quad (2.2.75)$$

The first term on the right represents the reference entropy, at the reference state, and can be omitted. The last term involves the specific heat at constant strain. The second term involves the tensor $\boldsymbol{\beta}$ (see Eqn. 2.2.68). Thus

$$s = \frac{1}{\rho} \boldsymbol{\alpha} : \mathbf{D} : \boldsymbol{\epsilon} + \frac{\theta - \theta_0}{\theta} c_{\epsilon} \quad (2.2.76)$$

For an isotropic material, the stiffness tensor is given by Part III, Eqn. 4.1.12,

$$\mathbf{D} = \lambda \mathbf{I} \otimes \mathbf{I} + \mu (\mathbf{I} + \bar{\mathbf{I}}) \quad D_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \quad (2.2.77)$$

in which case

$$\alpha_{ij} D_{ijkl} \epsilon_{kl} = \alpha \delta_{ij} (\lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}) = \alpha (3\lambda + 2\mu) \epsilon_{kk} = 3K \alpha \epsilon_{kk} \quad (2.2.78)$$

where K is the bulk modulus. With $(\theta - \theta_0)/\theta \approx (\theta - \theta_0)/\theta_0$, the entropy is now

$$s(\boldsymbol{\epsilon}, \theta) = \frac{3K\alpha}{\rho} \text{tr} \boldsymbol{\epsilon} + \frac{\theta - \theta_0}{\theta_0} c_{\epsilon} \quad (2.2.79)$$

With Fourier's law, $\mathbf{q} = -k\text{grad}\theta$, the equation of heat conduction, 2.2.60, can be expressed as (with $\theta \approx \theta_0$)

$$c_\varepsilon \dot{\theta} + \theta_0 \frac{3K\alpha}{\rho} \text{tr}\dot{\boldsymbol{\varepsilon}} = \frac{k}{\rho} \text{div}(\text{grad}\theta) \quad (2.2.80)$$

In order to express these relations in terms of stress, from 2.2.69, the isotropic stiffness tensor 2.2.77 and Part III, 4.1.20,

$$\boldsymbol{\sigma} = 2\mu \text{dev}\boldsymbol{\varepsilon} + K(\text{tr}\boldsymbol{\varepsilon})\mathbf{I} - 3\alpha K(\theta - \theta_0)\mathbf{I} \quad (2.2.81)$$

Then, with the trace of a deviator zero,

$$\text{tr}\boldsymbol{\sigma} = 3K(\text{tr}\boldsymbol{\varepsilon}) - 9\alpha K(\theta - \theta_0) \quad (2.2.82)$$

Thus 2.2.79 can be expressed as

$$s(\boldsymbol{\sigma}, \theta) = \frac{\alpha}{\rho} \text{tr}\boldsymbol{\sigma} + \frac{9\alpha^2 K}{\rho} (\theta - \theta_0) + \frac{\theta - \theta_0}{\theta_0} c_\varepsilon \quad (2.2.83)$$

The specific heat capacity at constant stress is then, from 2.2.63,

$$c_\sigma = \theta \frac{\partial s(\boldsymbol{\sigma}, \theta)}{\partial \theta} \approx \theta_0 \frac{\partial s(\boldsymbol{\sigma}, \theta)}{\partial \theta} \quad (2.2.84)$$

leading to the relation

$$c_\sigma - c_\varepsilon = \theta_0 \frac{9\alpha^2 K}{\rho} \quad (2.2.85)$$

In terms of stress, the equation of heat conduction 2.2.80 reads

$$c_\sigma \dot{\theta} + \theta_0 \frac{\alpha}{\rho} \text{tr}\dot{\boldsymbol{\sigma}} = \frac{k}{\rho} \text{div}(\text{grad}\theta) \quad (2.2.86)$$

2.2.4 Problems

1. Differentiate the invariants 2.2.16 to derive relations 2.2.18.
2. Derive the complementary function 2.2.26.
3. Derive the Gibbs relations 2.2.41 for an elastic material.

4. Use the conservation of mass relation $\rho_0 = J\rho$ and the relation Part III, 2.5.19, $\dot{J} = J\mathbf{F}^{-T} : \dot{\mathbf{F}}$, to derive the relation 2.2.47, $\dot{\rho}/\rho = -\mathbf{F}^{-T} : \dot{\mathbf{F}}$.
5. Show that the heat capacity is proportional to the second derivative of the free energy, Eqn. 2.2.59.
6. By taking the entropy s to be a function of the variables $(\mathbf{F}, \theta)$, use the relation 2.2.51, $\rho_0 \theta \dot{s} = -\text{Div} \mathbf{Q}$, and the relation between entropy and free energy, Eqn. 2.2.35a, to derive the equation of heat conduction 2.2.60,

$$-\theta \frac{\partial^2 \psi}{\partial \theta \partial \theta} \dot{\theta} - \theta \frac{\partial^2 \psi}{\partial \theta \partial \mathbf{F}} : \dot{\mathbf{F}} = -\frac{1}{\rho_0} \text{Div} \mathbf{Q}$$

7. Derive the equation of heat conduction 2.2.86.