Morse & Feshbach Chapter 3 notes PhL 1.11.12

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Chapter 3: Fields and the Variational Principle (275-347)

Little general intro blurb. **My summary**: They review Lagrangian and Hamiltonian dynamics as per Goldstein (variational δI=0 and then EL equations), then generalize it as per G's last chapter to continuous systems, which means the coordinates q_i are replaced by fields $\psi_i(x)$. Scalar field examples are compression waves or wave on string and fluid potential and general diffusion. Vector examples are 3D elastic displacement, EM fields, Dirac "field" (wavefunction). The idea of this entire book I think is that they are doing "math methods" which apply to many unrelated areas of physics, and here they are just trying to paint that broad spectrum of areas a bit before they dive into any methods. The diverse fields of physics are in some sense related or unified by these math methods, such as Lagrangian or Hamiltonian or Variational formulation. Themes of this chapter are: impedance, stress-energy tensor, both not very familiar to me at the moment. Dyadic notation continues in places here.

3.1 The Variational Integral and the Euler Equations (276)

Here we get a quick derivation of the Euler-Lagrange equations from δI = 0 where the integration variable is not dt but $dx_1 dx_2 ....dx_m$ and where L is a function of φ_1. φ_2..... φ_n. M&F call these "the **Euler equations**". They do not discuss the role of "constraints" in such a derivation. The integral should be a scalar ("**invariant**") relative to any involved symmetries. They note that in an area of physics, one usually first finds the PDE's and only later discovers what L is. Constraints are then addresses in a simple example, where they are called "**auxiliary conditions**". Their example has one **Lagrange multiplier λ**.

3.2 Hamilton's Principle and Classical Dynamics (280)

T-V is called "**the kinetic potential**" and it is what gets minimized as integral. The Euler equations for this case are called by M&F "**the Lagrange's equations of motion**". They write this in both G's forms. **Conservative** for them means that mechanical energy is conserved. They don't say what the potential V can be a function of, but on page 281 you can see that they assume V is velocity independent! They then define "momentum" as $\partial T/\partial \dot{q}_i$. The **Hamiltonian** is defined and its relation to Lagrangian L is shown, which M&F call "**the Lagrange function**". They derive then Hamilton's two basic equations as in G. These are "Hamilton's canonical equations". They then define a generalization of "**impedance**" in this framework. I think this relates to small oscillations near some equilibrium. **Dyads** are back! Principle

values, **admittance**, **resonant** frequencies. Fine. Then they are off on **canonical transformations**, and p,q are called conjugate variables. The term "**contact transformation**" is defined better than in G. They define an object S in an example and I think it is the generator of the CT based on page 290. Next come **Poisson Brackets** (u,v). Ah, S is the **action** function they say. Then H(p=∂S/∂q, q) = E which is like G 9-20, so M&F's S is Goldstein's W function. And M&F refer to this last as the **HJE**, and they come up with those **linear time solutions**. Connection is stated but not explored to the **Schrodinger Equation**. 2D oscillator is then taken as an **example** for all the above. Then **charged particle in EM field** is next example. They come up with the same L that G found (and which is in Jackson). Now p = mv + (e/c)A is the new momentum. Then they consider example with constant B field. Then on to a **relativistic particle**. Then on to dissipative systems p 298. Back to impedance and dyads.

3.3 Scalar Fields (301)

Here we are going to generalize the ideas of the previous sections to continuous media, that is why fields comes in, and scalar is the simplest case. Example is a **string** and on p 303 we get L for a string where ψ is the string displacement (the scalar field). Euler equation is then wave equation for string. Comment that min T-V means "difference is kept as small as possible between KE and PE" and you can see what this might mean locally on the string. **Momentum density** is p = $\rho\partial_t\psi$. Then they write H on p 304 and get the same wave equation from the Hamilton method. Suddenly they are in the 4 ness of special relativity. On page 306 we switch subjects to the **Helmholtz Equation**, then p 307 we are doing fluids with the **velocity potential**. Then **compression waves** p 308, and more impedance. We are using the L approach mostly here I think. Acoustic impedance. Plane waves, **stress-energy tensor**. The off onto the diffusion equation in sense of fluid flow. Again L is written for this on p 313. Next comes the SE. Again, the SE comes from a Lagrangian L as in 3.3.20. They note that "second quantization" will not be treated anywhere in M&F. Next is the Klein-Gordon equation. This is the relativistic thing.

3.4 Vector Fields (318)

Appropriate versions of L and Euler-Lagrange are given for this more general case of ψ_i. The term "**vector**" here just means more than one component, it does not mean rotational or Lorentz vector. Comment on general "sort of" **gauge invariance** of L: you can add $\nabla\bullet\mathbf{F}$ to L and things stay the same. On it goes. Then example is "**isotropic elastic medium**" where the vector field is **s**(**x**), the displacement. Again we get EL wave equations, and again we examine plane waves and impedance is back again and the stress-energy tensor. Notice that the "stress energy tensor" did not appear in Goldstein. More dyadic stuff with Hebrew-looking letters on page 325. Next example is the EM field with A and φ as four components they call V_i i = 1,2,3,4. L is written p 327 and then the Euler equations are Maxwell's Equations! More stress-energy tensor. Field momentum E x H stuff p 331. Working in different **gauges**. The Lorentz gauge I think is $\partial_\mu A^\mu = 0$ (they don't use phrase Lorentz gauge), and $\varphi = 0$ I guess might be the Coulomb gauge. More dyadics, more impedance, more stress-energy tensor, more plane waves. Then finally we come to the **Dirac equation**. Again we have an L, and the current J as $\psi^*\alpha\psi$. This Dirac is a very short section, they did not go whole hog on it.

Problems for Chapter 4 (337) Problems 3.1 through 3.14.

Tabulation of the Variational Method for Vector Field.

First they review the major ideas, then for each of the physics situations below they list of L, H, p and so on.

Flexible String or Membrane
Compressible non-viscous fluid
Diffusion Equation
Schrodinger Equation
Klein Gordon Equation
Elastic Wave Equation
EM equations
Dirac Equation

I will quote one section to show what they are doing:

Flexible String or Membrane

Field variable ψ is transverse displacement.
Parameters ξ_α are x and t for string; x, y, and t for membrane.

Lagrange density: $L = \frac{1}{2}\rho\left[\left(\frac{\partial\psi}{\partial t}\right)^2 - c^2\,\mathrm{grad}^2\,\psi\right]$; $\quad c^2 = \frac{T}{\rho}$.

Lagrange-Euler equation:

$$c^2\nabla^2\psi - (\partial^2\psi/\partial t^2) = 0 \text{ (scalar wave equation)}$$

Canonical momentum density: $p = \rho(\partial\psi\ \partial t)$.
Hamiltonian: $H = (1\ 2\rho)p^2 + \frac{1}{2}T\,\mathrm{grad}^2\,\psi$.
Field intensity: $\mathbf{S} = -T(\partial\psi/\partial t)\,\mathrm{grad}\,\psi$.
Field momentum: $\mathbf{P} = \rho(\partial\psi/\partial t)\,\mathrm{grad}\,\psi = -(1\ c^2)\mathbf{S}$.

Bibliography

C&H, Born, Goldstein, Lanczos on variation, Whittaker, L&L, Schiff, Pauli, Heitler, Wentzel