Morse & Feshbach Chapter 5 notes PhL 1.12.12

This is really the core of the two-volume set I would say. It includes the **Frobenius** theory of series solutions to ODE's and related discussion of **singular points** of an ODE. It contains a discussion of orthogonal **curvilinear coordinates** and **Stackel separation** of solutions in said coordinates. It has a huge section of **integral representations** of special functions and how you derive these things. Much of the chapter is stuff you would find in a Bateman style book, "**properties of special functions**". A large amount of space is spent on **Mathieu** functions. There is a certain "catalog" aspect to this chapter.

Often in a special function book you find results with no derivations, but M&F and Bateman I think have all the derivations, and Bateman has errata. I have no idea where the errors are in M&F, I have found a few. In any discussion they give, you are never sure they are going to provide all of what you really need (they do not for example in the Stackel presentation), so one is inclined perhaps to use some other source for learning and M&F as confirmation and perhaps some enhancement. But some topics just don't appear anywhere else in normal texts, however. M&F are often referenced on current day web pages. Note this "partial teaching" in the first few chapters of this book, where they don't really go into details of the various PDEs.

The ODE is the main act here and I agree with the chapter title. In this approach, there is no attention paid to boundary conditions, we are just trying to find the two orthogonal solutions.

I once wrote (Dec 2009) a separate doc earlier called "an ODE essay on MF Section 5.2" filed in Math/ODE's, This is a pretty high level comment on how one approaches an ODE in a general sense.

There are some items in this Chap 5 which I do not know about, and I will have to put that off to a rainy day (Stokes Phenomena, for example).

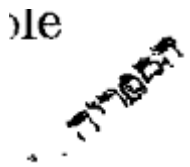
Chapter 5: Ordinary Differential Equations (492-675) 183 pages..... 1
5.1 Separable Coordinates (494)..... 2
5.2 Series Solutions (523)..... 3
5.3 Integral Representations (577)..... 4
Problems for Chapter 5. 5.1 through 5.42..... 5
Table of 3D Separable Coordinates (655)..... 5
Second-order ODE's and their solutions (667)..... 5
Bibliography (674)..... 5

Chapter 5: Ordinary Differential Equations (492-675) 183 pages

Here they will find ODE solutions in general, later they will deal with boundary conditions. All ODE's studied here will be pertinent to the page 271 list of PDE's. All PDE of interest have form $\mathcal{H}\psi=F$ where $\mathcal{H}$ is a differential operator in x and t of at most second order. ψ can be a vector. These equations are all linear. Discussion of the inhomo and homo stuff. **Two general types** of solutions: **separation**, and "**integrated**" by which they mean of the type $V = \int d^3x\ \rho(x)\ (1/4\pi R)$, ie, a Green's type integral. They also include integral solutions over BC surfaces when $\rho = 0$. But integral may be too hard to do! This chapter will deal only with the separation method.

5.1 Separable Coordinates (494)

Discussion of **boundary surfaces** and how you choose a coordinate system to match your BC. Notion of open and closed, the Cauchy BC's that Stak mentions, the "nodal lines", all quick discussion. Then p 497 gets us to the actual separated form (R=1). **Theorem: ALL solutions can be built from the atoms.** Must use separable coordinate system (for your equation of interest). M&F restrict interest in this general section to the **Helmholtz equation**, so that then means Helmholtz-separable. [strange Hebrew word noted on page 497.]



Now on to subject of **2D separable coordinates**. Helmholtz in 2D, they do the usual Cartesian separation and state the solution as an expo, then you can superpose those atoms. α is the "separation constant". On p 499 Helmholtz is written in terms of **complex variables** z and $\bar{z}$, so $\nabla^2 = (1/4)\partial^2/\partial z\partial\bar{z}$, something I don't remember. Now do conformal change from z=x+iy to $w=\xi_1+i\xi_2$. Fine, result is 5.1.5. Specialize to Laplace for a while in this complex notation. Now they are going to go through various **2D conformal cases**, and the first is **parabolics** p 501. Then **polar** and **elliptical** p 503. The notion of **scale factors** h is brought up p 505. Then p 506 on **separation constants**.

Then p 508 starts the **3D discussion**. They note that 2D is simple because only systems in which a certain thing can be written as a sum are viable. (They have not mentioned Stackel yet, but I know what they mean by that sum.) In 3D there are 2 instead of 1 separation constants. R separation is mentioned for 3D, they call R the "**modulation factor**".

Stackel starts on page 509. We get a fast run-through of Stackel theory. The vagueness of this section is why I wrote my own doc on the subject.

The first family of 3D separables are "**confocal quadric surfaces**" which of course means ellipsoidal coordinates. This section is the famous M&F discussion of this subject, word discussion all the geometric details (but no pictures!) The separated equations are just stated on page 512 and the form of an added **potential** is shown (that allows separation to continue). Then M&F do the various "**degenerate cases**": Cartesians, oblates, prolates, cylinder circular and elliptical. Then spherical and conical and paraboloidal. Each system has a Roman numeral number. They claim that **"all 10"** of the other classical systems can be obtained as degenerate forms. I may have said this wrong in my PDF. *********,. I will check on that soon.

On p 515 they look again at the separated ODE_n for ellipsoidals. There are **poles at ±a and ±b and** ∞ (**5 singular points**), and these "**conflow**" as you get to the degenerate cases. M&F compare how ODE singularities affect solutions just as f(z) singularities affect the complex function f(z).

hey then have their long discussion of the various **ways the separation constants can "separate**. The simplest case is this:

$$(A) \qquad \psi = X_1(\xi_1;k_1,k_2,k_3)X_2(\xi_2;k_2)X_3(\xi_3;k_3) \qquad (5.1.41)$$

where recall that k_1^2 is the Helmholtz parameter. For the above, need two Stackel rows each having two zeros, which is true only for Cartesian and cylinder. The worst case is all three appearing in each X_n.

Now back to that factor R. They go through a little **R-separation theory.**

Page 520 has a discussion of "**confocal cyclides**" (as opposed to confocal quadratic surfaces which is the ellipsoidal discussion). These are 4th order surfaces. They write the surface equation, then look at Laplace. People use either "**homogeneous coordinates** λ,μ,ν" or "**pentaspherical coordinates**" x_1,x_2, x_3,x_4,x_5 for cyclides (see top p 520). In these x_i Laplace is as in 5.1.54 which looks like Cartesian but has 5 terms instead of 3, all of the same form. Then they transform things to the ξ_1....ξ_5 **cyclidal coordinates** as on p 521. Then Laplace is the big coupled mess in 5.1.56. They finally end up (taking certain limits) on page 522 with a separable Laplace equation. Then they consider degenerate cases very briefly. I ignored these coordinates in my paper, but M&S discuss them I am sure.

5.2 Series Solutions (523)

They open by writing the general **3D** Stackel separated equation using first p and q, and they relate p to the f_n and q to the Φ_{nm}. They really should be using p_n and q_n (as I did). My p_n differs from theirs:

$$L_n X_n = (1/p_n)\partial_n[p_n(\partial_n X_n)] + q_n X_n = 0 \quad . \qquad (4.1A)$$

$$\mathfrak{L}(\psi) = \frac{d^2\psi}{dz^2} + p(z)\frac{d\psi}{dz} + q(z)\psi = 0 = \frac{1}{f_n}\frac{d}{dz}\left(f_n\frac{d\psi}{dz}\right) + q\psi \quad (5.2.1)$$

$$\text{where} \qquad p = \frac{d}{dz}\ln f_n = \left(\frac{1}{f_n}\right)\left(\frac{df_n}{dz}\right); \quad q = \sum_{m} k_m^2\Phi_{nm}$$

I am more in the Stakgold form and I have then $p_n = f_n$.

But now M&F are going to do **general purpose ODE theory**. Page 524 gets the **Wronskian** mentioned and it as a test for linear independence. **Abel's formula** is stated in some form 5.2.3. Using the Wronskian. M&F show how you can get the **second independent solution** y_2 if you know the first y_1, this is stated in 5.2.4 and later in 5.2.6. M&F then show that there are only 2 independent solutions. Generalization to order n ODE. (I call this second solution their "**trick**" in my other docs).

Then we have one of my favorite subjects: you can look up integrals in a table of same, and we really need a "table of ODE's" to look up ODE solutions. For first order ODE's the simple "**integration factor**" method always works. This sometimes works for second order. Mention of **adjoint** and **bilinear concomitant**.

P 529 starts discussing the "driven" or inhomo equation in the framework of the above integration factor discussion and we end up p 530 with a Green's looking general solution. (I suspect it really is Green's function sitting in there valid for this class of ODE's)

P 530 then starts the series solution discussion. First, about an **ordinary point** where both solutions are power series as in 5.2.20. Then page 532 defines a **regular singular point** (and an irregular one). The indicial equation determines the powers which lead the **Frobenius** solutions (Frobenius is not mentioned!) And if powers are same, one solution involves a log. We have some comments on general theory, then look at special cases.

P 536 discusses "**two regular singular points**", and the next page "one irregular singular point".

Then comes "**three regular singular points**". In each case they write the forms their p and q must have (or perhaps the different options) . We arrive at the famous **Papperitz** equation with its matrix form solution shown 5.2.37. Recursion formulas for the coefficients.

Finally on page 541 we have the idea of moving the three regular singular points to standard locations and then starts the **F(a,b,c,z)** discussion. This is first written in Papperitz notation. Using their general formula from above (what I usually call "the trick formula" for the second solution), they show the second solution to go with F(a,b,c,z) on p 543, it is called y_2,

Special cases of the hypergeometric function: **Tschebyscheff** is the first case (notice their spelling). Then the whole **Legendre** function world (but no mention of P and Q). Then the **connection formulas** which allow analytic continuation of F. So we are spending lots of time doing "special functions" here. Then p 547 looks at the **Gegenbauer** which puts the three singular points at ±1 instead of 0 and 1. Their solutions here they call $T_{\alpha}^{\beta}(z)$ as "**Gegenbauer functions**". . But Legendre is also in this situation. On page 438 we see $P^{\beta}_{\alpha+\beta}(z)$ which they call a "**generalized Legendre function**". Then various Rodriguez type formulas for T. Finally on p 549 they relate T to P for integer β on T. The M&F treatment of Legendre is one of the low points of M&F I am afraid, surely there will be more on this later.

Then "**one regular and one irregular"** singular point. This leads to the **Bessel** world, and to the wave equation in parabolic cylinders, and to the Schrodinger equation with 1/r. I think this is all confluent hypergeometric as on page 551 and we are off on **F(a,c,z).** Special discussion of asymptotic series with regard to **confluents** on p 554. This is the second mention of asymptotic series in the book!

Then **"two regular and one irregular singular point".** This is the **Mathieu** function world. It seems that one of the solutions has the series form 5.2.68 with unknown coefficients a_n. Mention of Floquet's Theorem connected to these functions. Then onto "continued fractions" still in this Mathieu context for ratio of adjacent coefficients a_n/a_{n-1}. This is all "Mathieu theory" which I have never studied. Something called the Hill determinant appears on p 560. Then finally the actual Mathieu functions on p 562. First kind and second kind, on and on for many pages here! I think the literature in 1953 might not have had much on this subject so M&F were filling a hole here.

Next is more general discussion of **Recursion Formulas**, p 568. Then p 570 considers solutions which are not power series, but **series in other functions $f_n(z)$.** The **Legendre polynomials** appear in this context on p 572. And $\mathbf{J}_n$ as well. Then more on **Mathieu** and its recursion formula. Then try instead of a series of functions, an **integral of functions** as on bottom p 575. (An integral representation of the solution) .

5.3 Integral Representations (577)

Little review. Singular points are "geometrical **concentration points**" for coordinate system. Series solutions are fine if you say in your convergence disk and stay away from the next nearest singular point (which slows down series convergence). Mention "**joining" idea** for making a connection between two singular points. The claim is that series joining formulas are not very nice, but integral representations of solutions work better in this joining sense. Integral rep for F(a,b,c,z) is an example. Mention p 585 of the **Euler Transform** and the **Laplace Transform**. Then details of each. Euler is associated with $(1-t)^{\alpha}$ type powers, while Laplace with $e^{\alpha s}$ expo.

Euler transform applied to **hypergeometric** p 587. This leads at once to the standard integral representation of the F function with the three factors with exponents involving a,b,c. A very long section here. For F the integral rep is shown again 5.3.16 p 591 with (0,1) endpoints and this leads to the various shifting formulas for F.

Focus shifts to **Legendre** functions on page 593. Again, integral rep solution forms. This leads to connection between $P_n(z)$ and F, and then this is taken as the definition for complex z and n p 595.

Several kinds of integral reps for $\mathbf{P}_n$ are given, then on p 597 we look at the $\mathbf{Q}_n$ second kind. All kind of fancy contour pictures are given.

Next is **Gegenbauer** and relation of the **T's** to the P's. See page 602 for a very ugly connection for general parameters. I think V is used as a second kind Gegenbauer.

Next is **confluent**.

Suddenly **Laplace transform** appears mysteriously. This was the second of the transforms of the Euler and Laplace type. I think 5.3.47 p 607 is the first example of a Laplace like integral representation since we see in there our exponential, and this is for the confluent. Then back to asymptotic expansions.

New subject is **Stokes Phenomena**. This is the "hide and seek" nature of terms in an asymptotic expansion p 611.

Next is **"third kind"** solutions of ODE's relating to $z=\infty$ behavior. So we have confluents of the third kind. Not sure I remember this from Bateman, but I guess so. using symbol U. Then back to second kind for confluent.

Page 619 takes us to **Bessel Functions** as confluent examples. So we have integral reps for J_n and later for j_n and others. **Hankel** functions p 623. **Neumann** functions are called $N_\nu(z)$ on page 625. It seems we are deviating from our "integral reps" section title here. N is the letter green Jackson uses, while others use Y. M&F then look at **large-order ν asymptotic behavior** of Bessel functions.

Page 633 and we are back to **Mathieu** functions, but not too many integral reps here, just how related to other special functions I think.

P 636 suddenly changes topic to **Laplace transform** because somehow for Helmholtz solutions provide kernels for integral representations. An example is given p 637. I don't follow it but fine, just another way to make integral reps. This method is then applies to **Mathieu** integral reps and others. They arrive at a sum rule for J_o as p 639. Then p 640 is "more on **Mathieu** functions".

P 642 starts into **Spheroidal Wave Functions**.

Problems for Chapter 5. 5.1 through 5.42

A. Table of 3D Separable Coordinates (655)

This gives ∇^2 in general orthogonals, then it gives my Stackel "steps" (R=1 only) leading to the separated ODE_n with its f_n and Φ_{nm}, then the form for an allowed potential V if such a V gets added in. Then the systems are summarized one at a time. For each we get the S matrix, the f_n , the h_n, the coordinate equations and a stereoscopic picture. Again, each system gets a Roman numeral. Also comments on the singular points of the ODE's but those ODE's are not stated. General form for potential V is also stated. In each case he computes the Stackel determinant S.

Then after this list they do some **R separation cases** starting on page 665. Then **bispherical**s and **toroidals** are summarized.

B. Second-order ODE's and their solutions (667)

The table here is classified as in the text by number and type of singular points. Integral reps are included.

Bibliography (674)

Eisenhart on Stackel from 1934, and Robertson in German only. Bateman had a 1932 book "PDE's of math physics",. Forsyth, Ince, W&W, Hobson, Klein, Stratton, Watson Bessel, Jahnke-Emde. Our later Bateman friends Magnus and Oberh. had a special functions book.