Morse & Feshbach Chapter 6 notes PhL 1.12.12

Well, it is because of boundary conditions that you get Eigenfunctions, so I guess it is fine to mix these two subjects into a single chapter. But EF's are almost always 1D affairs, whereas the BV subject is more general. This Chapter is a nice "second source" for the Cauchy data problem, characteristics and all that stuff (hyperbolic, etc). The eigenfunction part is old hat stuff for me. The "digital" middle section is an interesting twist!

In the book plan, the previous chapter discusses how to find solutions to ODE's without BC's, and here we have a little bit on "adding in" the BC's, but really just in 1D. The whole complicated subject of those surface integrals in Stakgold is not brought up in this chapter. Nothing is done in n dimensions here. This is a 2D and 3D book, period.

Chapter 6: Boundary Conditions and Eigenfunctions (676-790) 114 pages 1
6.1 Types of Equations and of Boundary Conditions (676) 1
6.2 Difference Equations and Boundary Conditions (692) 1
6.3 Eigenfunctions and their Uses (706) 2
Problems for Chapter 6 6.1 through 6.12 4
A. Table of some useful eigenfunctions and their properties 4
B. Eigenfunctions by the Factorization Method 4
Bibliography. 4

Chapter 6: Boundary Conditions and Eigenfunctions (676-790) 114 pages

6.1 Types of Equations and of Boundary Conditions (676)

This is a very long discussion of the material Stak addressed in his book. The notion of "**the Cauchy data**" for the boundary conditions. The idea that a PDE has certain characteristic curves, and whether things are under or overdetermined for an open piece of boundary condition depend on how the boundary curve or surface is positioned with respect to those characteristic curves or surfaces. The distinction of **elliptical, parabolic and hyperbolic** is made. So I can regard this as a pretty solid "second source" for the Stakgold discussion. I don't want to relearn this subject today, I just want to know that it is here in M&F. M&F to talk about various simple examples.

6.2 Difference Equations and Boundary Conditions (692)

This is an *unexpected* topic. Symbols Δ and Δ^2 for first and second differences. ODE series recursor equations are difference equations. M&F produce a connection between such a recursor difference equation for coefficients and the original ODE. They start in 1D and on p 695 go to 2D, ie, a PDE. They discuss how an equation has a "reaction" to a BC. This all bears on lattice gauge theory I presume (yes, there is a nice wiki page on same).

This M&F demo section will use the 2D Laplace equation as its **elliptic** example, and so on for the other two. Value at a lattice point is average of neighbors for Laplace. By a simple demo, they show that the value of ψ at an interior point is mainly due to the values at the boundaries, because the effect of other interior point keeps getting cut down by factors of 1/4 arising from the averaging program. Fascinating! So then take lattice$\rightarrow$0 limit and you have proven "Dirichlet". Then they do similarly for Neumann and have some further comments. A little digital eigenfunction section has a conclusion that I miss. They then

do a digital Green's function example. I guess things are 0 on the boundary. The max and min rules for Laplace fall out as well.

Hyperbolic is next on p 703. They do this talking about a different averaging rule in terms of diagonal lattice points.

Parabolic is next.

Everything is summarized in a nice picture I have never seen before:

Conditions	Boundary	Hyperbolic equation	Elliptic equation	Parabolic equation
Dirichlet or Neumann (Value or slope specified)	Open	Insufficient	Insufficient	**Unique, stable solution in positive direction, unstable in negative direction**
	Closed	Solution not unique	**Unique, stable solution** (see page 698 for Neumann conditions)	Solution over-specified
Cauchy (Value and slope specified)	Open	**Unique, stable solution**	Solution unstable	Solution over-specified
	Closed	Solution over-specified	Solution over-specified	Solution over-specified

On the left are the three kinds of boundary "data", and along the top are the three kinds of equations, and for each the boundary can be open or closed. Notice that the heat equation is parabolic and goes in the upper right box, so stable going forwards in time, unstable going backwards! The Wave equation on the other hand needs full Cauchy boundary data (Stak's two initial conditions!), it is hyperbolic. This Cauchy data is of course on an open boundary (t = 0 plane). Then finally Laplace is fine with closed boundary. In fact green **Jackson** stole the above table and put it on page 17, and credits M&F.

Now I suspect the following: if I were to read all of the above in full detail, it would be insufficient to **fully teach** the subject. I think I would need another source to get the full picture, BUT their discussion is relatively simple and physical and describes the general pay of the land (same for Stak's discussion). Fascinating that this ancient 1953 book makes use of digital versions of the equations to discuss features. They do not use the word "computer" anywhere (at least in vol 1). They would enjoy the idea of writing a little program to do what they presented. I keep thinking I might get to this some day.

6.3 Eigenfunctions and their Uses (706)

Comment on how sometimes there is no cookbook and you have to do a little guessing. They open with 2D Laplace. The BC of interest is ψ being **0 on three edges of a rectangle** and specified on fourth. Separation gives the atomic solution 6.3.1 with unknown A_n coefficients, a **Smythian form**. This is an early example of osc in one direction and expo in the other, no M&F comment on that yet.

Now how does a Green's Function relate to this example? This is a little strange to me. What he is really writing is a result like Stak 6.11 Vol II p 92. This is not really a Green's Function kernel, it is a **Poisson kernel** where your "point source" is really a point of boundary value on the boundary. But M&F go ahead and call this kernel a Green's Function. It is NOT something that vanishes on the entire boundary. Apart from nomenclature, all is well.

Eigenfunctions beings p 711. General discussion given, EF's and EV's. In our example above, the x direction had EF's sine. On page 714 the above circle problem is mentioned with series solution. It is just the **Fourier Series** theorem. Next is a Legendre example, but all in terms of those T functions, yech.

P 716 starts on **abstract vector space** where EF's are spanning vectors. Dot product, **orthogonality**, all the usual stuff. Basis vectors called e_n but these really mean EF's.

The Sturm Liouville Problem is addressed by name on p 719, hurray! The weight function is called r(z), λ is the EV, we have the usual p and q functions. This ODE form is called "**the Liouville Equation**". Saxon-like argument here I think regarding what happens if λ is a little off one way or the other (you miss at the other end). Various **theorems** are proven, such as ∞ is the only λ_n accumulation point. Transcendental example. (very few pictures!) The issue of degenerate solutions. Then p 726 and we are making a series of EF's as a candidate solution. Question of whether EF's are a **complete set**. EF's of different λ_n are orthogonal, the usual derivation. Use the word factorization with this SL discussion. Some sort of operators G are flying around too. Oh yes, this is the **raising/lowering operator method** as in Saxon for harmonic oscillator. (I am not sure they have required the 0 BV for "EF".)

Page 736 comments on EF's and the **variational principle**. EF's minimize a certain integral they call Ω in 6.3.20. Perhaps this is in Stak in his approximation method stuff to find eigenvalues. They use this Ω thing to prove that EF's form a complete set. No "completeness relation" yet. Something now about asymptotic nature of EF's relative to "order". Then this is applied to Bessel on p 741 and that **Stokes** asymptotic stuff is back for a reprise. They get the usual large order form for Bessel functions p 742. General EF's are compared to the Fourier Series.

The **Gibb's Phenomenon** is next p 745. This relates to ringing of partial sums near discontinuities of the function you are fitting with the EF's, such as a square wave, see wiki. Interesting history.

Legendre polys and generating functions are next p 748 and we are back to "special function" material. That is, P_n is a particular case of EF's and a S-L problem. The PQ series for a/(z-ξ) appears. A generating function for Q_n is produced p 753.

P 753 starts into **2D EF's** as you might have for a membrane. Square boundary is sin sin. **Triangular** boundary has interesting results as shown on p 755.

P 757 seems to address the question of what happens when you have a 2D SL problem that does not factorize. An example is Helmholtz in elliptical coordinates μ and φ bottom page 757. You try to separate, but the BC's tangle both coordinates. This is long section again bringing in **Mathieu** functions So and Se. I am not really sure of the point of this section, maybe my conjecture is wrong.

P 579 is on the **density of EV's** in EV space. I am familiar with this but don't know what point they are making.

Then we switch to the case of a **continuous spectrum** on p 762. The **Fourier-Bessel transform** of order m appears on page 765 and they take the limit to get the **Hankel transform**, but they don't call it that.

P 766 starts into the **Schrodinger equation** as an EV equation. They use an example with some very strange potential shape which gives some F(a,b,c,z) eigenfunctions. I guess they just wanted something other than the usual traditional examples.

Then on to differential and integral operators and vector to vector space. The **dyads are back** for another act! I think we are just doing Hermitian operators in **matrix mechanics**. On and on. Principle axes means you have diagonalized the matrix.

Problems for Chapter 6 6.1 through 6.12

A. Table of some useful eigenfunctions and their properties.

I. Strangely they omit the ODE here, and Case I is the **Gegenbauer** functions T_n^{β} : generating function, special cases, recurrences, low order specific functions like T_2^0, normalization, range is (-1.1), addition theorem.

II. (0,∞) the **Laguerre** polys, as in hydrogen atom.

III. (-∞,∞) **Hermites**.

B. Eigenfunctions by the Factorization Method

This is the G+ and G- raising/lowering operator stuff again, with a potential present, the SE.

Bibliography.

Bateman book on PDEs in math physics. C&H, Hadamard, Somerfeld, Szego