Morse & Feshbach Chapter 7 notes PhL 1.13.12

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The intro section is a bit confusing because they make it sound like the normal Green's Function is the same as what Stak calls the Poisson Kernel. True, they have similarities, but they are NOT the same by any means.

In **Section 7.1** M&F finally fess up to this and admit that $PK(x|\xi) = \partial_{\xi n} g(x|\xi)$. For Neumann however it is true that in that case it really is $g(x|\xi)$ that appears in the surface integral, but it is then against $\partial_n u$.

In **Sections 7.2** M&F treat the **Helmholtz** equation Green's Function (they call this "steady waves" since such Helmholtz equation comes from $e^{i\omega t}$ assumed time dependence).

Then in **7.3 and 7.4** they treat the **causal wave and heat** Green's Functions. In each section they state the big Green's Theorem monster and the general of-class solution and they state the propagators for n = 1,2 and 3. This stuff is all summarized in the chapter appendix. Lots and lots of words.

In **Section 7.5** they do their best to **generalize** all this stuff to a general differential operator in "an abstract space" (meaning Hilbert Space, but they never use that term at least in Vol I), not just the ones considered here, and most of the big results follow through. You want your L to be self-adjoint. Then they wander off and try to include integral operators into their generalization. It is all Hilbert Space stuff and this makes you appreciate Stak. And I appreciate Stak's excellent compact notation as well.

It is interesting for me to **compare** this all to Stak. Stak does all the Hilbert Space stuff up front in his Ch 2. He does all the 1D Green's Function stuff in Chapters 1 and 4. Then he does the Laplace Green's Function world in Chapter 6, and the Helmholtz Green's has to wait till Chapter 7. Meanwhile, Chap 7 also does the heat and then the wave causal Green's stuff. I think it was in Chap 5 that Stak does the general Green's Theorem stuff with the surface current and all that, and in this same Chapter he gets all the propagators for all equation types for all n. In contrast, M&F jam all that stuff into this little chapter.

The M&F method seems somewhat disorganized to me. They seem to "wander" into fine details and then back out, and the logic flow is perhaps choppy. But of course I am just skimming. They DO attempt to put in lots of "words" where Stak seems to minimize words.

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An integral rep is better than a series rep as a solution to a problem, say M&F. M&F now define two different kinds of Green's Functions. One is the usual one, the second is the Poisson Kernel type of Green's function. They use the same symbol G for both, but refer to a point on a Dirichlet surface as r_0* whereas a point charge point is called r_0. So this confirms an earlier confusion I had. They refer to these as volume and surface Green's Functions.

But now M&F want to say these two Green's functions are "essentially the same". [But as we see below, they don't really mean they are the same, and it is all just as in Stak.] I think this is how one can interpret that statement. We know from Stak that this is true (for Dirichlet BC)

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi)$$

where $f(\xi)$ is the Dirichlet boundary potential. The Poisson kernel here is $\partial_{\xi n} g(x|\xi)$ and NOT $g(x|\xi)$. But we know we can rewrite the above this way as well

$$u(x) = \int_R d\xi\, E(x|\xi)\, q(\xi) + \int_\sigma dS_\xi\, E(x|\xi)\, \Sigma_x(\xi)$$

where E is the free-space Green's function and $\Sigma_x(\xi)$ is the induced charge. In this second form, yes, the same Green's Function E appears in both terms, but it's the free-space Green's! So let's keep the above two equations in mind as M&F try to sway us. The rest of the intro leaves me confused, discussion of homo and inhomo. But let us proceed. Well perhaps "homo" refers to what I call a "Dirichlet problem": potential is specified on a boundary, there are no "free charges" anywhere. We need a translator here:

homo equation	=	Laplace
inhomo equation	=	Poisson
homo BC's	=	(V specified on boundary)
inhomo BC's	=	(induced σ on boundary)

M&F are a bit "arm-wavy" here I think.

7.1 Source Points and Boundary Points (973)

Longwinded lead-up to 7.1.6 which says for Poisson equation that $F = \int G\rho dV$. M&F like to use $\mathbf{e_n}$ to represent an eigenfunction. There follows a long explanation again of the two "equivalent problems" homo and inhomo equation and BC. But jump to the sample o 797. The G shown in 7.1.8 is really a **Poisson Kernel** in my language and ξ is a point on the boundary of the rectangle where we have some ψ, and we are going to get $\psi = \int_\sigma dS_\xi\, \psi(\xi)\, G_{PK}$ as he shows. Then M&F switch over to the true Green's function. Eventually they develop the true Green's G = bottom page 798 and clearly $G \neq G_{PK}$. On p 799 they admit that these two G's are in fact different. Well, eventually M&F have to fess up and we have

We notice that

$$\left[\frac{\partial}{\partial\eta} G(x,y|\xi,\eta)\right]_{y<\eta} \xrightarrow[y\to b]{} -4\pi G_b(x,y|\xi) \qquad (7.1.13)$$

where G_b is defined in Eq. (7.1.8), so that we can rewrite Eq. (7.1.8) as follows:

$$\psi(x,y) = -\frac{1}{4\pi}\int_{S_0} \psi_0(S_0)\left[\frac{\partial}{\partial n_0} G(x,y|S_0)\right] dS_0 \qquad (7.1.14)$$

Here G_b is the Poisson Kernel G, and G is the real G, and *now we are agreeing with my Stak item* quoted above apart from 4π (because M&F use $G_0 = 1/R$) . Some Stak-like comments about being careful when you let a point approach a surface. Finally they give the "other surface term" for Neumann in 7.1.15. Remember that the full mixed Stak result is this

$$u(\xi) = \int_V dx\, g(\xi|x)q(x) - \int_S dS\, [u(x)\, \partial_{nx}\, g(\xi|x) - g(\xi|x)\partial_n u(x)\,]$$

so both those surface terms are there all the time in effect, but usually we kill one or the other by saying we are doing Dirichlet OR Neumann.

7.2 Green's Functions for Steady Waves

The title of this section I think means they will be dealing with **Helmholtz equation** and maybe Laplace where there is no involvement of a time coordinate t. That must be the meaning of "steady waves". They never seem to use this phrase within the section!

First they arrive at Green's identity with two functions U and V as in 7.2.2. They are a bit vague about the exact meaning of "Green's Theorem". They are back to Helmholtz Green's and we want to find G_k. Various typical Green's integral relations are developed. And we get in 7.2.9 our general $\int_\sigma dS_\xi\, \psi(\xi)\, G_{PK}$ result specifically for Helmholtz. So nothing new here, just an example. General properties of Green's starts p 808. I guess they show **symmetric**, and in 1D they show the **jump** concept. Claim that G_0 has the form e^{ikR}/R for outgoing, etc. Page 812 is **method of images**, source in front of a plane in 2D. Then situation of two parallel planes (**strip** 2D) and all those image charges p 815. H_0 is the 2D E. Use of **Poisson Sum Formula** to improve convergence. I wonder if contour on p 818 is hand-drawn by the authors. A final cos cos expo sum is obtained for the strip problem in 7.2.43, they study convergence.

P 820 then turns to **Green's Function in terms of eigenfunctions** and we get the usual full eigenfunction **bilinear expansion formula** 7.2.39 for Helmholtz where $\lambda = k^2$. Comment on how things look complex but are really real. They go on then to get the various integral **addition theorem formulas** like 7.2.44 following. So this was the "full eigenfunction expansion" of Stak.

On M&F problem is that they wander off with pages of calculations (they way I often do) and you cannot find easily what the main result is, or what the main point is. They end up here with the sum form of the **addition theorems**. This is all the usual business of deciding *which* of the two separated problems you are going to use for partial EF expansion. One has a continuous spectrum K, the other discrete m. Lots of nice special function sum rule formulas are rolled out.

On p 828 in A General Formula, we are moving to **curvilinear coordinates** again, the second appearance in this book. We are in 3D with simple separation with $X_n(\xi_n)$ and ODE_n. So here p 829 is discussion of the **partial EF expansion method**. In 3D we have to **pick two oscillatory and one expo**. The expo one is where you end up with a **1D Green's Function** problem if you are doing Green's. The EF of two of the variables are called $W_p(\xi_2,\xi_3)$ for example as in 7.2.56 orthogonality with some weight. We then have the Stackel separated ODE_n as in 7.2.58 with its M_n and f_n. It is shown now the 1D Green's for coordinate ξ_1 then arises as in 7.2.60. Solve this as in 61 and then full Green's is 63. P 832 gives the M&F

little summary of the 1D SL problem that we just had in ξ_1. They talk about analyticity of G in λ for the 1D Helmholtz with λ, but no development.

Note that they have no formulas of the form $\oint d\lambda\, G$ so far.

7.3 Green's Functions for The Scalar Wave Equation (834)

So this is their wave section as in Stak. They note that the t=0 boundary is going to be "**Cauchy**" so we will need $\psi(0)$ and $\partial_t\psi(0)$ both specified. Unlike Stak, we have a **time's arrow** discussion. In micro realm, you would think either time direction was OK. They do now call the Green's here "causal" as Stak does. The symmetry of G rule requires times to also be reversed as in 7.3.3 which they prove at length. Then on p 837 they are on to the **Green's Theorem statement for the wave situation** as in 7.3.5, Stak like. Then they develop the **retarded potential stuff** as in 7.3.13 p 840 (all in 3D so δ hard shell surfaces). As an example, they consider a **source that is moving at velocity v**. A moving Green's source, not sure I have done that before. The final solution is quite simple as in 7.3.14.

They then back up and do the 2D version of things, but they really use 3D to do this, in the **Hadamard Descent** method idea. In this way they come up with the **2D propagator** $1/\sqrt{\ldots}$ **wake** formula 7.3.15 and they call this thing g. They use the word wake on p 843 top. Then finally they get the **1D solution,** I think u is the Heaviside function in 7.3.16 p 843. yes! But you have to look at p 1909 in Vol II where they list all their symbols. So they have now obtained all three free-space propagators for wave theory.

So various initial value problems are considered in 1D p 844. Something called **Poisson's Solution** is obtained, for $\psi(0,t)$. Then we are on to **Huygen's Principle** p 847. The Principle is first clearly stated and then they prove it.

P 848 and we are back to finite boundary value problems for our causal wave G. They come up with the bilinear expansion formula p 850 which is 7.3.23 with the sin thing. From Stak Ch 7 meta meta I have

$$g(x,t\,;x_0,t_0) = \Sigma_k\, \varphi_k(x)\varphi_k^*(x_0)\,[\,\sin\{\sqrt{\lambda_k}\,(t-t_0)\}\,/\,\sqrt{\lambda_k}\,] \qquad \text{// wave 7.118}$$

$$g(x,t\,;x_0,t_0) = \Sigma_k\, \varphi_k(x)\varphi_k^*(x_0)\,[e^{-\lambda_k(t-t0)}] \qquad \text{// heat 7.57}$$

$$g(x\,;x_0) = \Sigma_k\, \varphi_k(x)\varphi_k^*(x_0)\,[\,1/\lambda_k] \qquad \text{// Laplace 6.108}$$

so they are producing the first one here.

Sample problem: **drumhead membrane** with $\delta(\mathbf{r}-\mathbf{0})$ initial pluck (velocity 0). Have to find the J_n eigenfunctions. The solution is 7.3.31 where sum is the zeros of J_0. They then discuss physically what the solution looks like: there must be an initial with-wave initial ring wave that emits from the origin. The pulse does reform, but it takes twice as long as you would guess.

P 854 addresses the **Klein-Gordon** wave equation. The new feature here is a Helmholtz-like extra term as shown in 7.3.33. If you Fourier out the time, you find that this parameter just shifts ω^2, but with correct relative size you get Yukawa type damping form $e^{-\kappa R}/R$. Claim this arises for a **string embedded** in an elastic medium as p 139 (pretty obvious there in 1D string situation, just a restoring force). If you have a huge restoring force, waves damp out at once on such a string, but with tiny one you can have waves.

7.4 Green's Functions for Diffusion Equation (857)

Discussion of the time direction. **Green's Theorem** for this situation, and the general solution 7.4.9 for the usual problem class of interest. The **free-space** guys I guess are always lower case g, and 7.4.10 then is the free space propagator in n dimensions n = 1,2,3. They have a little plot of how the solution spreads out, but not as good as my 3D plots! Again we obtain the **bilinear expansion** as seen above 7.4.16. [It really is too bad the copier did not use a higher resolution machine! I guess in Russia somewhere). M&F take note of the diffusion issue on p 865 that solutions **seem to imply infinite propagation times**, something I noted as well in Stak. This is an imperfection in the model, and you have to correct your thinking by realizing their is a finite wave velocity in the medium and nothing moves faster than that. I think they may in fact do the details here.

7.5 Green's Functions In Abstract Operator Form (869)

The idea is just to generalize to other PDE's besides wave and heat. The full PDE differential operator is here called $\mathcal{a}$. Notion of **adjoint** comes up and they use $\tilde{\mathcal{a}}$ whereas Stak used L*. The bilinear concomitant I think is the parts stuff you get when you swing over L to get L* as in 7.5.15. They generalize to **n dimensions**, things get a bit messy.

So far they have implied differential operators, but on page 877 they want to talk integral operators, a sudden subject change indeed! In matrix theory we want Hermitian matrices, and for integral equations we want symmetric kernels, and those I guess will also be self-adjoint in some sense. The adjoint is just what you get doing parts. Same idea as symmetry for a Green's function, which is a special case of a kernel. They cannot resist in the end writing things in terms of "abstract operators" and our **dyads** are back once again! They then discuss the Green's function for our abstract operator $\mathcal{a}$ starting p 882 and we get the same bilinear expansion in 7.5.39 that we are used to. Then page 885 talks about how to handle "non-Hermitian operators", I remember this in Stak somewhere. The subject is bi-orthogonal functions. Something called an "idemfactor" top p 886, and this is the "unity dyadic".

Problems for Chapter 7 7.1 through 7.14

A. Table of Green's Functions. Starts with general formulas, then goes to Helmholtz and finally we get the formula I show above,

$$\psi(\mathbf{r}) = \int \rho(\mathbf{r}_0) G_k(\mathbf{r}|\mathbf{r}_0)\, dv_0 + \frac{1}{4\pi} \oint [G_k(\mathbf{r}|\mathbf{r}_0^s) N_0(\mathbf{r}_0^s) - \psi_0(\mathbf{r}_0^s) \frac{\partial}{\partial n_0} G_k(\mathbf{r}|\mathbf{r}_0^s)]\, dA_0$$

where last term is surface integral and they now show both terms.

Helmholtz is like Laplace because the parameter cancels in Green's Theorem. Note the 4π due to definition of G etc. M&F always put a 4π in the Green's ODE and then G = 1/R and you get the 1/4π shown above. The three free-space propagators are given. In 3D the Stackel ODE_n is given and it is assume $\xi_{2,3}$ are oscillatory and we get the general formula for the 3D Green's where ξ_1 has its 1D Green's form p 892 bottom.

Then comes the wave case and again the three propagators are stated for n = 1,2,3.

Finally of course is the diffusion equation.

Bibliography

Batemans PDE book, C&H, Sommerfeld, Kellogg, Webster. No Stak which is 1967 and 1968, whereas M&F are 1953!