Morse & Feshbach Chapter 8 notes PhL 1.5.12

Well, they give lots of examples, but again the organization seems lightweight. Methods have no names, so hard to remember things ("the method they used on page 978"). They had to throw in 100 pages on the IE subject and this is their offering. The term Hilbert-Schmidt does not appear. Hilbert does not appear in this volume! No completely continuous stuff. Somehow this chapter seems like a random collection of integral equation examples. You know they were dying to get WH into the mix. It is a sort of "stream of consciousness" approach. Touch on the major buzzwords and ideas, throw in lots of examples. The student LIKES to see lots of examples, even if they take lots of pages. Again, I am just skimming here. It would not be good to teach IE theory based on this glommy chapter. Again we have the question of whether the book is a teaching tool to be used in a course, or is it a reference work to be consulted when needed. Certainly M&S would fall into the latter category.

Chapter 8: Integral Equations (896-997) 100 pages

Nice little intro page. A PDE is a local thing, an IE is a global thing which builds in the boundary conditions (though I forget how it does this, I recall doing a Richard Price letter example?).

8.1 Integral Equations of Physics, Their Classification (896).

An opening **example** 1 from what I might call "gas physics" produces an integro-differential equation ! The 2nd **example** is another such equation from acoustics, where they have a little membrane which creates sound waves and is affected by same. A 3rd **example** is QM (they call it "wave mechanics") with a velocity dependent potential. This leads to an integral equation $\varphi(\mathbf{p})$ which is $\psi(\mathbf{r})$ in momentum-space.

Their next example (p 900) I think is of the **Stak "integral equation method"** type where you have an integral equation for I which for Laplace would be induced charge density. If you integrate over the entire surface induced layer, you can then set the result equal to ψ on the surface. I am thinking for example of the charged disk where we set V = 0 on the disk with our integral equation. So in this sense, we are "building in" the boundary condition. In fact we are using the BC to obtain I. Then having I, we get solution everywhere. So there you are. Sneddon showed how you could specify a general radial potential on the disk and still solve the problem. These are Dirichlet examples, but I think M&F's example is Neumann.

On p 902 we have the **time-independent SE** which in effect is Helmholtz driven by driving function $V\psi$, and we know we can write $\psi = \int G\, V\, \psi$ in this case, just as we do do Poisson $V = \int G\rho$. But in the SE case we have an integral equation which is the scattering theory starting point often used. They then just consider a 1D example and of an EV problem with some λ and weight r(z) and boom, another integral equation. We know of course that every ODE can be replaced by an integral equation, Stak did that a lot.

M&F p 903 then do some **1D examples**, computing G in each case. The first example is just the 1D string. Here you really see G as "the propagator" since it takes you from $\psi(z_0)$ to $\psi(z)$, they don't use that word of course. Examples involve trig, ln, Legendre, Bessel, Hermite, and finally Laguerre. For each we get the ODE and the integral equation and the Green's Function $G(z|z_0)$. These results are just quoted, not derived. [For each ODE and its special functions, we have an IE and that is what he is discussing here.

We then have the **standard Fredholm classification** and the **Volterra variant**. A Volterra example is done, simple harmonic oscillator in 1D. Discussion of what causes the Volterra form to occur in physics, where things go one way in time.

8.2 General Properties of Integral Equations (907).

With much pain, they write the **integral operator as** $\mathbf{U}^{-1}$ with the idea that it is the inverse of the Green's Differential operator I think. Too hard to just call the operator K as Stak does. Kernel of M&F is K(x,y). So now they want to study the kernel. Is it **positive definite**? Is it **symmetric**? They seem to favor symmetric and "definite" kernels. The **Neumann series iteration** is starting up (but not called that), mention of $K_2 = \int K\,K$. Antisymmetric kernel is a possibility. P 912 section is "properties of the symmetric, definite kernel". The operator is Hermitian they note. They specialize to real and positive definite. The EV integral equation is going to have eigenvalues and eigenfunctions. EV is λ, not μ as in Stak. Symbolic solution Neumann series I think with the resolvent type stuff. They claim that kernels of their type have orthonormal eigenfunctions. They are quiet about completeness. Comment on lower bound for λ.

So above was the Fred EV problem. Now they look at Fred 2 inhomo with λ in there.

P 916 discusses other kinds of kernels. Formulas for eigenvalues in terms of matrix elements in ratio, similar to Stak. P 920 returns to Volterra and certain things are true for that case, iterated kernels.

P 922: what if kernel has **singularities**? **Bounded** kernels?

8.3 Solutions for Fred 1 (925).

Fred 1 means Ku = f in Stak, here it means $K\psi = \varphi$. The g_n are a complete basis, h_n are integrals of the g_n against K. This is "somebody's method", maybe Galerkin. The "**Schmidt** method" means GSO. Now I think they are off doing Fredholm determinants. On and on. Then **Biorthogonal** Series. Some connection to generating functions. Off doing **Gebenbauers** on p 938. The sum of the series is treated as a kernel here, that is the connection For example, kernel = $\Sigma\, z^n P_n(z_0)$.

P 940 and we are talking Green's Functions for Fred 1. On and on.

P 942 and we want to do FT on our Fred 1 equation.

Eventually on p 947 they are doing "the moment problem".

Recap on p 949.

8.4 Solutions for Fred 2 (949).

I see Stak's "**separable kernel**" forms appearing. Three classes of some sort here.

P 959 starts on the inhomo Fred 2. They are just projecting things on basis functions in the usual Stak manner.

8.5 Fourier (and other) Transforms and Integral Equations (960).

On p 961 they get around to the **convolution** kernel form. If you don't have this form, you can FT your IE, but you just get thereby an IE in k-space such as 8.5.3. But convolution form diagonalizes things as in 8.5.5 and you are solved!

P 962 says you might try the **Hankel** transform on (0,∞) in place of the FT.

On p 964 we are shifting **Fourier** integral lines up and down in the complex k plane. Strip of analyticity. Maybe WH here? I could swear they are doing a WH example starting p 967. You take separate contours for different terms.

What if kernel is v(x+y)form instead of v(x-y), p 696. Example on page 971.

P 972 does some **Laplace** transform cases.

P 973 has a convolution integral equation that is Volterra and it seems to diagonalize somehow.

P 976 has the **Mellin** transform appropriate for certain types of integral equations.

P 978 finally does **Wiener-Hopf** followed by a WH example. Recall that their sum and quotient splitting is called "factorization". They treat the Milne problem involving radiation in stars I think.

P 987 has "a general method of factorization" which is that Stak sum-splitting idea. Then they return to the Milne problem.

On p 990 they then turn to doing WH on a Fred 2 inhomo. [Stak did all equation types at one time.]

OK, this WH was their last gasp.

One point of this section is that certain integral equation forms are amenable to certain transforms.

A. Table of Integral Equations and Their Solutions.

This is a long summary of their most important results.

This Chapter has no problems!

Bibliography:

C&H, Lovitt, W&W. Hopf

This is the last chapter in Volume 1 which was pages 1-997. Volume 2 just continues these page numbers.

The **index** is for both volumes and it is 40 pages long.