

DESIGN OF SPACE ELEVATOR

by

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ABSTRACT

A space elevator is a tall structure that starts from the surface of the Earth and reaches out beyond geostationary orbit (GEO) in space. The space elevator could be used to deliver payloads to any Earth's orbit or send them to any other planet in the solar system. Currently, rocket propulsion is the only system that can deliver payloads to various destinations in space. The space elevator can provide easier, safer, faster and cheaper access to space compared to rocket propulsion system.

To date, most of the designs have been based on the simple linear decoupled dynamic analysis of the space elevator. These simple linear models neglect the Coriolis acceleration, elongation, and the coupling between the longitudinal and transverse motions. The simple models are useful for calculating the natural frequencies of the elevator, but they cannot give the accurate dynamic response. A three dimensional nonlinearly coupled model that includes the Coriolis acceleration, elongation, and the coupling between the longitudinal and transverse motions is used to accurately model the dynamic response of the elevator.

It is found that the linear models significantly underestimate the response of the system. Therefore, a non-linearly coupled model must be used to accurately predict the dynamic response of the system. However, the frequency contents obtained from the linear models are similar to those identified from the nonlinearly coupled models. Therefore, the linear models can be used to predict the natural frequency of the system for preliminary designs.

Two design cases have been proposed in this thesis. These designs satisfy various design criteria such as acceptable taper ratio, point mass, maximum tensile strength and maximum displacements. The design also takes care that the natural periods are sufficiently away from the tidal periods due to the Moon and the Sun. The first design case of the space elevator having a length of 66,000 *km* will be useful for reaching

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destinations within the Earth's orbit and the second design case having a length of 91,000 *km* can also reach the destinations in solar orbit.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Currently, rocket propulsion is the only system to deliver payloads to their destinations in space. A Space elevator provides an alternative means to rocket propulsion. Space elevator is a physical connection from a point on earth to a point beyond geostationary orbit (GEO) in space. The GEO is one of the geosynchronous orbits that is directly above the Earth's equator with a period equal to the Earth's rotational period and an orbital eccentricity of approximately zero. The GEO is 42,164 *km* away from center of the earth. At the GEO, the Earth's gravitational force and the centripetal acceleration are balanced so that an object can remain stationary with respect to the Earth.

For an object on an orbit lower than the GEO, the object will have to rotate faster than the Earth to prevent itself from falling, and for an object on an orbit higher than the GEO, the object will have to rotate slower than the Earth. Therefore, the portion of the space elevator below GEO experiences a net force toward the Earth, and the portion of the space elevator above GEO experiences a net force away from the Earth. The space elevator is a tension structure that is in unstable equilibrium about the GEO. Stability can be provided by putting more mass on the portion above the GEO and anchoring it to the ground. Mathematical details of the forces on the GEO are given in Chapter 2.

A space elevator is preferred to rocket propulsion system because it can provide easier, safer, faster and cheaper access to space. Using the rocket propulsion system, it costs about \$400 million to launch a satellite into geo synchronous orbit [1]. Space elevator could reduce this launch costs to as low as \$10 per *kg* of material transported into space [2]. Not only does it provide a cheaper passage to space, space elevator is a much safer alternative to rockets. Until now, NASA has launched 120 space shuttle

missions, and two missions, Challenger in 1986 and Columbia in 2003, resulted in catastrophic failures. This failure rate is consistent with the failure rate of 2% in unmanned rockets (such as missiles). With the space elevator, the dangerous accelerations at lift-off and reentry associated with rocket flights can be eliminated. With the introduction of a space elevator, launching a satellite into the Earth's GEO becomes merely ascending along the elevator. The space elevator could be used to deliver payloads to any of Earth's orbits or send them to any other planet in the solar system.

One of the major hurdles which stands in the way of the feasibility of a space elevator is the material required for its construction. No building material was found to be suitable for the construction of space elevator until 1991, when the first carbon nanotubes were made [3]. The high tensile strength and low density of the carbon nanotubes (CNT) make them perfectly suitable for space elevator construction. Since then a lot of research have been performed in the area of carbon nanotubes. Recent developments in this area include the first measured values for the tensile strength of CNT by Yu [4, 5]. Yu found measured tensile strengths ranging from 11 to 63 *GPa* for individual CNT.

1.2 Literature Review

The space elevator concept was first proposed, about fifty years ago, by Russian Scientist Artsutanov [6]. His idea was to use a geo stationary satellite as a base for constructing a space elevator. In 1966, oceanographers at the Scripps Institute of Oceanography and Woods Hole Oceanographic Institution led by John Isaacs, provided the first engineering analysis of a space elevator [7]. They named the space elevator Sky Hook. At that time no material was available which could satisfy the strength requirements of their proposed sky hook.

In 1975, Jerome Pearson published the most cited and extensive analysis of the Space Elevator concept [8]. In this paper he discussed the vibrational modes of the space elevator due to pay loads, gravity and tidal forces. Arthur C. Clarke introduced this concept to a popular audience with his novel, "Fountains of Paradise" [9].

Bradley Edwards conducted a feasibility analysis on the space elevator in his report submitted to NASA under the NASA Institute for Advanced Concepts (NIAC) program [1]. This report presented a detailed study of the problems that would be encountered in designing, constructing, deploying and utilizing the space elevator. Edwards proposed a deployment strategy, consisting of sending a Spacecraft to GEO carrying the spooled cable [10]. Initially the cable will be released in both upwards and downward directions with the help of two spindles. The cable that is released in downward direction is pulled by gravity and it is anchored to the Earth after it is fully extended. The other cable that is released in the upward direction is pulled by outward centrifugal acceleration and after it reaches the desired location, the Spacecraft at GEO moves on that cable to act as a countermass. Finally climbers with the cable will ascend and descend along this initial cable until a sufficiently thick string is created.

1.3 Motivation

1. So far, only Simple decoupled linear dynamic analysis of a space elevator is available. The simple model doesn't take either the Coriolis force or the coupling between the longitudinal and transverse motions into account. The simple model only provides the natural frequencies of the elevator, but cannot give an acceptable dynamic response.
2. To model the dynamic response accurately, a three dimensional nonlinearly coupled model that includes the Coriolis acceleration is needed.
3. The purpose of this thesis is to
 - a. accurately model the dynamic response of the space elevator using a three-dimensional coupled model.
 - b. propose feasible designs for the space elevator

1.4 Thesis Outline

In Chapter 2, Pearson's model is explained and two design modifications proposed by previous studies are also implemented. The first is to make the cross-sectional area of elevator tapered to maintain the constant stress level along the elevator. The second is to place a counter mass at the tip of the elevator to decrease its length. Simple linear equations of motion in the longitudinal and transverse directions have been derived for the space elevator. Linear models are decoupled from each other, and the Coriolis accelerations, elongation, and the geometric coupling between the two motions are neglected. Natural periods of the space elevator in the longitudinal and transverse directions are calculated.

In Chapter 3, the coupled and non-linear partial differential equations of motion for the three dimensional motion of the elevator are derived using the Hamilton's principle. Using the static solution of these equations, significance of the elongation factor is explained.

In Chapter 4, the tidal forces from the Moon and the Sun are obtained. Equations for these tidal accelerations are derived by defining the co-ordinate systems for the Earth-Moon and the Earth-Sun systems. The forcing periods of the tidal forces are found from their power spectral density plots.

In Chapter 5, response of the elevator in the longitudinal and transverse directions under the influence of tidal forces is calculated by solving the linear equations numerically using the finite difference method first. The maximum possible displacements for the elevator are plotted for various lengths of elevator using the modal analysis. Using different design criterion such as the maximum point mass that can be carried up, possible resonance and accessible destinations for various lengths of elevator, two design cases have been proposed. These two design cases of the elevator will be serving two different purposes.

CHAPTER 2

LINEAR MATHEMATICAL MODEL OF SPACE ELEVATOR

2.1 Introduction

In this chapter, we will start with the basic model of the space elevator proposed by Pearson [8]. Two design ideas have been proposed to Pearson's model to maintain the lower constant stress level along the cable and to reduce the length of the elevator. Two-dimensional equations of motion for the space elevator are derived by decoupling the longitudinal and transverse motion. These equations have been numerically solved to find the natural periods of the first three modes of vibration of elevator in the transverse and longitudinal directions for various lengths of the elevator.

2.2 Pearson's Model

Pearson's version of space elevator is a long uniform structure having its base on the surface of the Earth and extending up to GEO [8]. Consider a small incremental element of the elevator of arbitrary length dr , which is located at a distance r from the Earth's center as shown in Figure 2.1. The forces acting on the element are gravitational force acting towards the Earth and the centripetal force acting away from the Earth. The elevator is located at the equator so that the centripetal acceleration acts along the cable. Let us consider an incremental element of mass dm as shown in Figure 2.1. The forces acting on this element are given by

$$\uparrow \Sigma F_{radial} = dT - F_g = dm a_r,$$

$$dT = \frac{GM_e dm}{r^2} - \frac{dm v^2}{r} \quad (2.1)$$

where dT = Internal tension which varies along the elevator,

$$F_g = \text{Gravitational pull given by } F_g = G M_e dm/r^2,$$

a_r = Radial acceleration given by $a_r = -v^2/r$.

M_e = Mass of the Earth,

dm = Mass of element = $\rho A dr$,

ρ = Density of the material used for elevator,

A = Cross sectional area of the elevator,

G = Newton's gravitational constant = $6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$,

r = distance from center of the Earth to the element of elevator, and

v = velocity of the elevator element due to the Earth's rotation.

The GEO is defined as the orbit where the gravitational force is balanced by the centripetal acceleration, or $dF = 0$. The radial distance to the GEO is given by

$$v_s^2 = \frac{GM_e}{r_s}, \quad (2.2)$$

where r_s = radius $GEO = 42164 \times 10^3 m$, and

v_s = velocity of elevator at GEO.

Eqn 2.1 can be written as,

$$dT = \left(\frac{GM_e \rho A}{r^2} - \frac{\rho A v^2}{r} \right) dr. \quad (2.3)$$

Assuming that the elevator is stationary with respect to the Earth, the relationship between the velocity of elevator and its distance from the Earth is linear with the proportionality constant of Ω . Therefore, we can write

$$\Omega = \frac{v(r)}{r} = \frac{v_s}{r_s}, \quad (2.4)$$

where $v(r)$ is the velocity of elevator at a distance r from the center of the Earth and

Ω is the angular velocity of the Earth's rotation and is given by

$$\Omega = \frac{2\pi}{24 \times 60 \times 60} = 7.27 \times 10^{-5} \text{ rad/s}$$

The gravitational acceleration, g , is defined by

$$g = \frac{GM_e}{r_0^2}, \quad (2.5)$$

where g = acceleration due to gravity on surface of the Earth = 9.81 m/s^2 and

r_0 = radius of the Earth = $6378 \times 10^3 \text{ m}$.

Substituting the Eqns 2.2, 2.4 and 2.5 into Eqn 2.3, the incremental force acting on the elevator becomes

$$dT = \rho A g r_0^2 \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) dr. \quad (2.6)$$

The tension at a distance r can be obtained by integrating dT from r_0 to r or

$$T(r) = T(r_0) - \rho A g r_0^2 \left(\frac{1}{r} + \frac{r^2}{2r_s^3} \right) + \rho A g r_0^2 \left(\frac{1}{r_0} + \frac{r_0^2}{2r_s^3} \right), \quad (2.7)$$

where $T(r_0)$ is the force at the ground to anchor the elevator to ground.

Another important observation from Eqn 2.7 is that the incremental force dF on the incremental element, dr , is positive for $r < r_s$ and negative for $r > r_s$. That is, for the elevator to rotate at the Earth's angular velocity about its own axis, an incremental element below GEO requires an incremental upward force and an incremental element

above GEO requires an incremental downward force. Therefore, for the elevator to be self supporting, it must extend beyond GEO.

The length of the self supporting tower can be obtained by finding r_f such that $F(r_f) = 0$. Assuming that the force at the ground (to anchor the elevator to the ground) is zero, we find that

$$\rho A g r_0^2 \left(\frac{1}{r_0} + \frac{r_0^2}{2r_s^3} \right) - \rho A g r_0^2 \left(\frac{1}{r_f} + \frac{r_f^2}{2r_s^3} \right) = 0, \quad (2.8)$$

the length of the elevator to be 144,000 *km*.

2.3 Design Ideas

From previous studies, there are two design ideas that we will utilize [8, 10]. The first is that the cross-sectional area should be tapered to maintain low constant stress level along the cable. The second is that the length of the elevator could be shortened if a counterweight is attached to the end of the elevator. In the following, we will discuss these two ideas.

2.3.1 Tapered Cross-section

Let us consider a uniform tower (constant cross sectional area) such that the stress varies along the length of the tower and is given by

$$\sigma(r) = \frac{T(r)}{A} = \frac{T(r_0)}{A} - \rho g r_0^2 \left(\frac{1}{r} + \frac{r^2}{2r_s^3} \right) + \rho g r_0^2 \left(\frac{1}{r_0} + \frac{r_0^2}{2r_s^3} \right). \quad (2.9)$$

If we assume that the force applied at the ground is zero, the location on the elevator which experiences the maximum stress could be found by equating $d\sigma/dr$ to zero. The value of r found to be r_s and the corresponding value of σ is obtained by substituting $r = r_s$ in Eqn 2.9 or

$$\sigma_{max} = \rho g r_0^2 \left(\frac{1}{r_0} + \frac{r_0^2}{2 r_s^3} - \frac{3}{2 r_s} \right) = \rho \times 4.8424 \times 10^7. \quad (2.10)$$

For steel, the maximum stress (using $\rho=7850 \text{ kg/m}^3$) is about 358 *GPa*. The tensile strength of steel is typically about 250 *MPa* [11]. Therefore, it is impossible to build a uniform elevator using steel. For carbon nanotubes with density 1300 kg/m^3 , the maximum tensile strength is about 62.9 *GPa*, which is slightly under the experimentally measured strength of 63 *GPa* [4, 5].

The advantage of using a tapered cross-section is that it is theoretically possible to build an elevator using any material if we allow the cross-sectional area to vary, by making the elevator thickest at GEO. However, as the allowable stress decreases, the taper ratio (defined by the ratio of the cross-sectional area at GEO to that at the Earth's surface) increases. For steel, we will show that the taper ratio is too large to be practical.

The cross-sectional area as a function of the radial distance is obtained as follows. Starting with Equation 2.6, let us replace A with $A(r)$. The goal is to obtain $A(r)$ that will make the stress at any location constant, or $dF = \sigma dA$. Then, we can write

$$\begin{aligned} \sigma dA &= \rho g r_0^2 \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) A(r) dr \\ \frac{dA}{A(r)} &= \frac{r_0^2}{h} \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) dr, \end{aligned} \quad (2.11)$$

where $h = \sigma/\rho g$ is the specific strength of the elevator and it has the units of length.

By integrating the Eqn 2.11 from r_0 to r , cross-sectional area can be written as

$$A(r) = A_0 e^{\left(\left(\frac{r_0}{h} \right) \left(1 - \frac{r_0}{r} \right) + \left(\frac{r_0^2}{2hr_s^3} \right) (r_0^2 - r^2) \right)}, \quad (2.12)$$

where A_0 is the cross-sectional area of the tower at the surface of the Earth.

The above area equation in Eqn 2.12 is plotted (Assuming $A_0 = 1$) in Figure 2.2 with r varying from $r_0 = 6378 \text{ km}$ to $150,000 \text{ km}$. From figure 2.2, we can conclude that the cross-sectional area of the elevator increases exponentially up to GEO and decreases exponentially from GEO towards the end. Therefore the elevator will have maximum cross-sectional area at GEO.

If we substitute $r = r_s$ in the Eqn 2.12, the taper ratio can be expressed as

$$\frac{A_s}{A_0} = e^{\left(\frac{r_0}{h}\right)\left(1 + \frac{r_0^3}{2r_s^3} - \frac{3r_0}{2r_s}\right)}. \quad (2.13)$$

If we substitute the corresponding values of r_0 and r_s in Eqn 2.13, the taper ratio becomes

$$\frac{A_s}{A_0} = e^{\left(\frac{0.776 r_0}{h}\right)}. \quad (2.14)$$

The taper ratio decreases with increasing h . Theoretically, we can reduce the taper ratio by using material with high strength to weight ratio, which effectively increases h , but there is a practical limit on the value of h because of the upper limit on the stress that a certain material can endure. Another way to think about this is that we can reduce the stress by increasing the taper ratio, so that any material can be used to make the space elevator if it is tapered. However, taper ratio for a material with low specific strength, h , may be impractical. Taper ratio values for different materials are given in Table 2.1.

Taper ratio for steel is so high that it is practically impossible to build an elevator. Theoretically, carbon nanotube can have the tensile strength of 300 GPa . But the maximum measured value of tensile strength is 64.3 GPa . For this value of σ the corresponding taper ratio is 2.72, which is quite possible.

2.3.2 Counter mass with Tapered Cross-section

Counter mass is a lump of mass placed at the tip of the elevator. The basic purpose of using counter mass is to shorten the length of the elevator. By extending the elevator

beyond GEO, elevator experiences the required centripetal force to balance the net weight of the tower. Instead of increasing the height of the elevator, we can put a lump of mass at a certain height above GEO such that it creates the centripetal force required to keep the elevator in tension about GEO. The length of the elevator depends on the weight of the counter mass, the relation between them is shown in Figure 2.3. Height of the elevator without counter mass is 144,000 km [8]. The counter mass is assumed as a point mass. The net force on the tower with length, $L = r_f - r_0$, is given by

$$W = \left(\int_{r_0}^{r_f} \left(\frac{GM_e}{r^2} - \Omega^2 r \right) dm \right) + \left(\frac{GM_e M_p}{r_f^2} \right) - (\Omega^2 r_f M_p). \quad (2.15)$$

In Eqn 2.15, the second term $\frac{GM_e M_p}{r_f^2}$ represents the gravitational force of attraction between counter mass and the Earth and the third term $\Omega^2 r_f M_p$ represents centripetal force acting on counter mass due to the Earth's rotation.

The counter mass that will make the net force, W , zero is given by

$$\begin{aligned} M_p(r_f) &= \frac{\left(\int_{r_0}^{r_f} \left(\frac{GM_e}{r^2} - \Omega^2 r \right) dm \right)}{-\left(\frac{GM}{r_f^2} \right) + (\Omega^2 r_f)} \\ &= \frac{\left(\int_{r_0}^{r_f} \rho A(r) g r_0^2 \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) dr \right)}{-g r_0^2 \left(\frac{1}{r_f^2} - \frac{r_f}{r_s^3} \right)}. \end{aligned} \quad (2.16)$$

The ratio of the point mass to the cable mass is given by

$$\frac{M_p(r_f)}{\text{cable mass}} = \frac{\left(\int_{r_0}^{r_f} \rho A(r) g r_0^2 \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) dr \right)}{-g r_0^2 \left(\frac{1}{r_f^2} - \frac{r_f}{r_s^3} \right) \int_{r_0}^{r_f} \rho A(r) dr} \quad (2.17)$$

$$= \frac{\left(\int_{r_0}^{r_f} A(r) g r_0^2 \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) dr \right)}{-g r_0^2 \left(\frac{1}{r_f^2} - \frac{r_f}{r_s^3} \right) \int_{r_0}^{r_f} A(r) dr},$$

where $A(r)$ is given by Equation 2.12.

In Eqn. 2.17, point mass/cable mass will be a function of specific strength h and length of the elevator L . The relationship between the mass ratio and the length of the cable is shown in Figure 2.3. From Figure 2.3, we can observe that the Weight of the counter mass needed increases with the decrease in length of the elevator i.e. heavier mass is needed to shorten the length of the elevator.

2.4 Equations of Motion of Space Elevator- Linear Decoupled Model

As a first approximation, let us assume that the transverse motion and the longitudinal motion are decoupled. In addition, let us assume that the motions in either direction are small so that the linear models are adequate. The transverse motion is modeled using the string model, and the longitudinal motion is modeled using longitudinal beam model. The string model is valid for such a long and thin member where moment and shear are negligible [12].

The equation of motion is given by

$$m(x) \frac{\partial^2 y}{\partial t^2} = H \frac{\partial}{\partial x} \left(A(x) \frac{\partial y}{\partial x} \right) + f(x, t), 0 \leq x \leq L, \quad (2.18)$$

with boundary conditions given by

$$y(0, t) = 0, \\ M_p \frac{\partial^2 y}{\partial t^2} \Big|_{L, t} + H A(L) \frac{\partial y}{\partial x} \Big|_{L, t} = 0.$$

In Eqn 2.18, $m(x)$ is mass per unit length, $y(x, t)$ is deflection of string and is function of both space and time. When the transverse motion is modeled, $y(x, t)$ and H can be replaced by the transverse deflection, $v(x, t)$, and tensile strength, σ . When the

longitudinal motion is modeled, $y(x, t)$ and H can be replaced by the longitudinal deflection, $u(x, t)$, and Young's modulus, E .

2.5 Natural Periods of the Space Elevator

To find the natural frequencies of the elevator, homogeneous part of the Eqn. 2.18 is considered and it can be written as,

$$\frac{\rho}{H} \ddot{y} - \frac{A'(x)}{A(x)} y'(x) - y''(x) = 0. \quad (2.19)$$

In Eqn. 2.19, $(\)'$ represents the derivative with respect to x $\frac{\partial}{\partial x}(\)$

$(\)^\cdot$ represents the derivative with respect to time t $\frac{\partial}{\partial t}(\)$.

Using the method of separation of variables, $y(x, t)$ can be expressed as

$$y(x, t) = Y(x) f(t). \quad (2.20)$$

Substituting Eqn. 2.20 in Eqn. 2.19 gives

$$\frac{\rho}{H} Y(x) \ddot{f}(t) - \frac{A'(x)}{A(x)} Y'(x) f(t) - Y''(x) f(t) = 0. \quad (2.21)$$

Dividing Eqn. 2.21 by $Y(x) f(t)$ gives

$$\frac{\rho}{H} \frac{\ddot{f}(t)}{f(t)} - \frac{A'(x)}{A(x)} \frac{Y'(x)}{Y(x)} - \frac{Y''(x)}{Y(x)} = 0. \quad (2.22)$$

We can write, $\rho \ddot{f}(t)/(H f(t)) = -\lambda$, so the time equation for Eqn. 2.22 becomes

$$\ddot{f}(t) + \lambda \frac{H}{\rho} f(t) = 0. \quad (2.23)$$

where $\lambda H/\rho = \omega^2$.

The spatial equation for Eqn. 2.22 can be written as

$$Y''(x) + \frac{A'(x)}{A(x)} Y'(x) + \frac{\rho}{H} \omega^2 Y(x) = 0. \quad (2.24)$$

The wave number β is defined as

$$\beta^2 = \omega^2 \frac{\rho}{H}. \quad (2.25)$$

Substituting Eqn. 2.25 in Eqn. 2.24 gives

$$Y''(x) + \frac{A'(x)}{A(x)} Y'(x) + \beta^2 Y(x) = 0. \quad (2.26)$$

Substituting Eqn. 2.20 in the boundary conditions in Eqn. 2.18, boundary condition at the surface of the Earth will be $Y(0) = 0$ and the boundary condition at the end of the elevator will be

$$M_p Y(L) \ddot{f}(t) + H A(L) Y'(L) \dot{f}(t) = 0. \quad (2.27)$$

Dividing Eqn. 2.27 by $\dot{f}(t)$ gives

$$-M_p \omega^2 Y(L) + H A(L) Y'(L) = 0. \quad (2.28)$$

Above Eqn. can be written in terms of wave number as,

$$Y'(L) - \left(\frac{M_p}{\rho A(L)} \right) (\beta^2) Y(L) = 0 \quad (2.29)$$

Cross-sectional area of the elevator is given by

$$A(r) = A_L e^{\left(\left(\frac{r_0^2}{h} \right) \left(\frac{1}{L+r_0} - \frac{1}{r} \right) + \left(\frac{r_0^2}{2hr_s^3} \right) ((L+r_0)^2 - r^2) \right)}. \quad (2.30)$$

where A_L is the cross-sectional area at the tip of the elevator.

Substituting Eqn. 2.30 in the point mass equation derived in Eqn. 2.16

$$M_p = \rho A_L \frac{\left(\int_{r_0}^{r_f} e^{\left(\left(\frac{r_0^2}{h} \right) \left(\frac{1}{L+r_0} - \frac{1}{r} \right) + \left(\frac{r_0^2}{2hr_s^3} \right) ((L+r_0)^2 - r^2) \right)} g r_0^2 \left(\frac{1}{r^2} - \frac{r}{r_s^3} \right) dr \right)}{-g r_0^2 \left(\frac{1}{r_f^2} - \frac{r_f}{r_s^3} \right)}. \quad (2.31)$$

From Eqn. 2.31, we can observe that $M_p/\rho A(L)$ is a function of length of the elevator L and specific strength h . Since $A'(x)/A(x)$ is function of h , the spatial equation of the elevator in Eqn. 2.26 is also a function of L and h . Therefore, the wave number for equations of motion of the elevator is also a function of L and h only.

It is difficult to solve the equations of motion in Eqn. 2.26 analytically due to varying cross-sectional area. Therefore, a numerical solution via finite difference method is used. The details of this method are given in Appendix B. We let the forcing terms in Eqn. B.9 be zero for natural frequency calculations. The spatial equation in Eqn. 2.26 is

turned into an eigen value problem $([K] - \lambda[M]) \{Y\} = \{0\}$. The natural frequencies of the system are square root of eigen values, λ .

Figure 2.4 shows the variation of wave number with the length of the elevator for different values of h .

To find the natural frequencies for longitudinal motion of the elevator, the H term in $[K]$ and $[M]$ matrices defined in Appendix B is replaced by Young's modulus E . In Figure 2.5, the first three natural periods of longitudinal vibration as functions of the elevator length for different values of h are plotted. From this figure we can make the following observations.

- The first mode longitudinal natural periods for most of the elevator lengths will be around 6 *hr*. Natural periods for second and third modes of vibration are close to each other and under 2 *hr*.
- The natural periods are increasing with the increase in h , but the rate of increase is very small.

Similarly, the natural periods for transverse vibrations are also calculated using the $[K]$ and $[M]$ matrices from Eqn.B.9 in Appendix B. In this case, the H term is replaced by the tensile strength σ . The natural periods of the transverse model are simply $\sqrt{E/\sigma}$ times that of the longitudinal model. In Figure 2.6, the first three natural periods of transverse vibration are plotted as functions of elevator length for different values of h . Following observations can be made from Figure 2.6.

- The natural periods of first mode of vibration for most of the elevator lengths are around 20 *hr*. Natural periods for second and third modes of vibration are around 5 *hr*.
- In contrast to the longitudinal case, natural periods are decreasing with the increase in h for the transverse vibration.

Table 2.1: Taper ratio for different materials

Material	Tensile strength $\sigma(\text{GPa})$	Density $\rho(\text{kg}/\text{m}^3)$	Specific strength $h(\text{m})$	Taper ratio
Carbon steel	0.25	7850	3250	e^{1523}
Tool steel	2	7850	26×10^3	e^{190}
Carbon nanotube [4, 5]	63	1300	5.05×10^6	2.72
Carbon nanotube [4, 5]	130	1300	10.2×10^6	1.62
Carbon nanotube [13]	300	1300	23.5×10^6	1.23

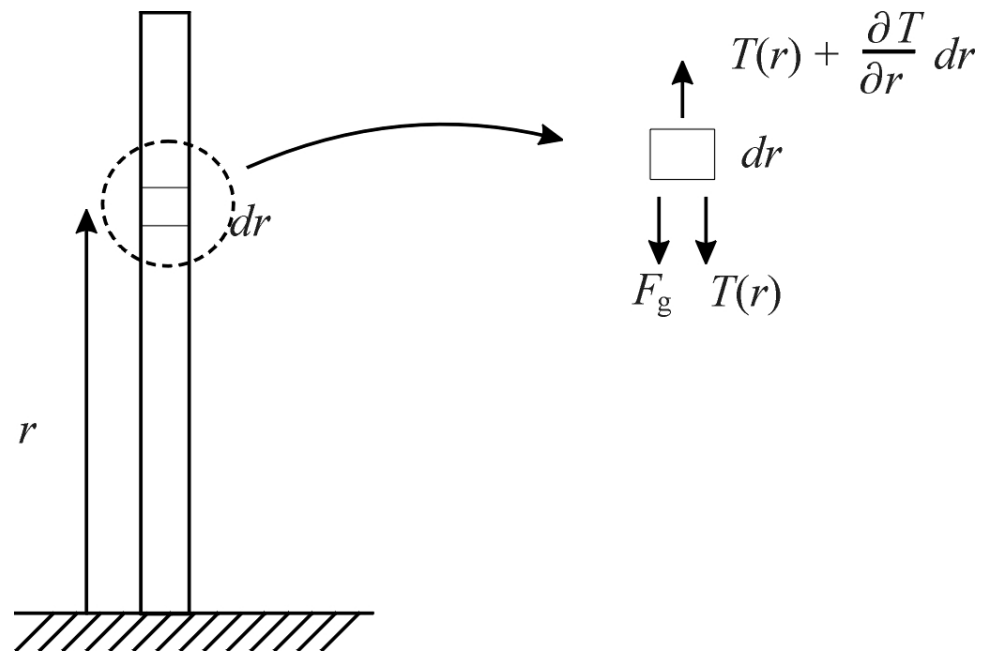


Figure 2.1: Forces acting on an incremental element of length dr

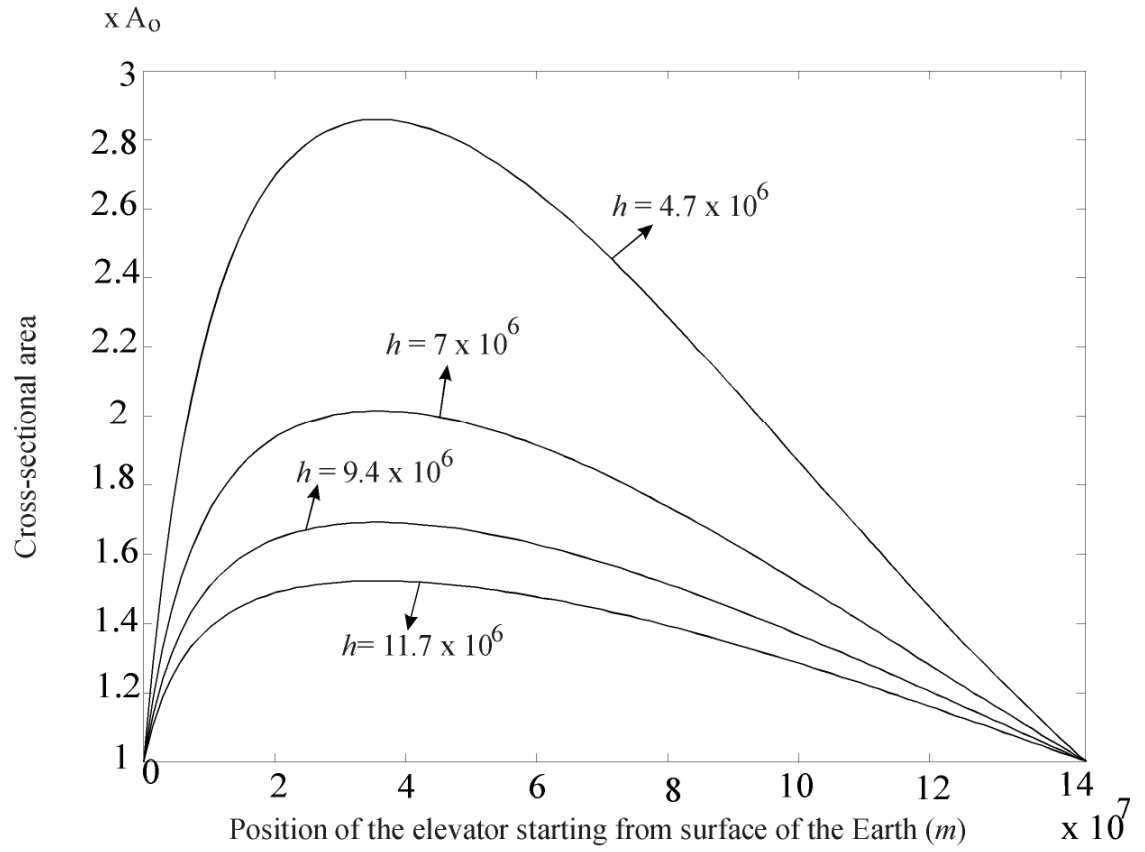


Figure 2.2: Cross-sectional area at various positions of the elevator for different values of h

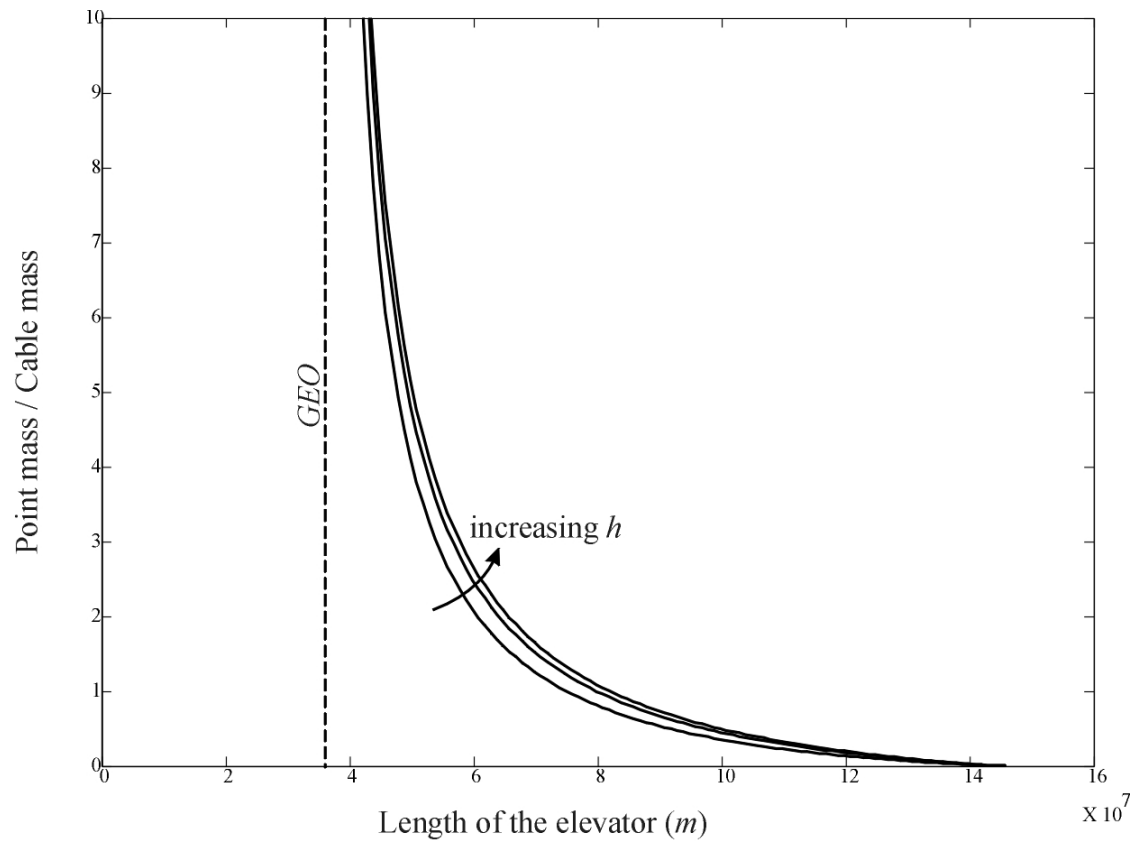


Figure 2.3: Relation between point mass and elevator length for different values of h

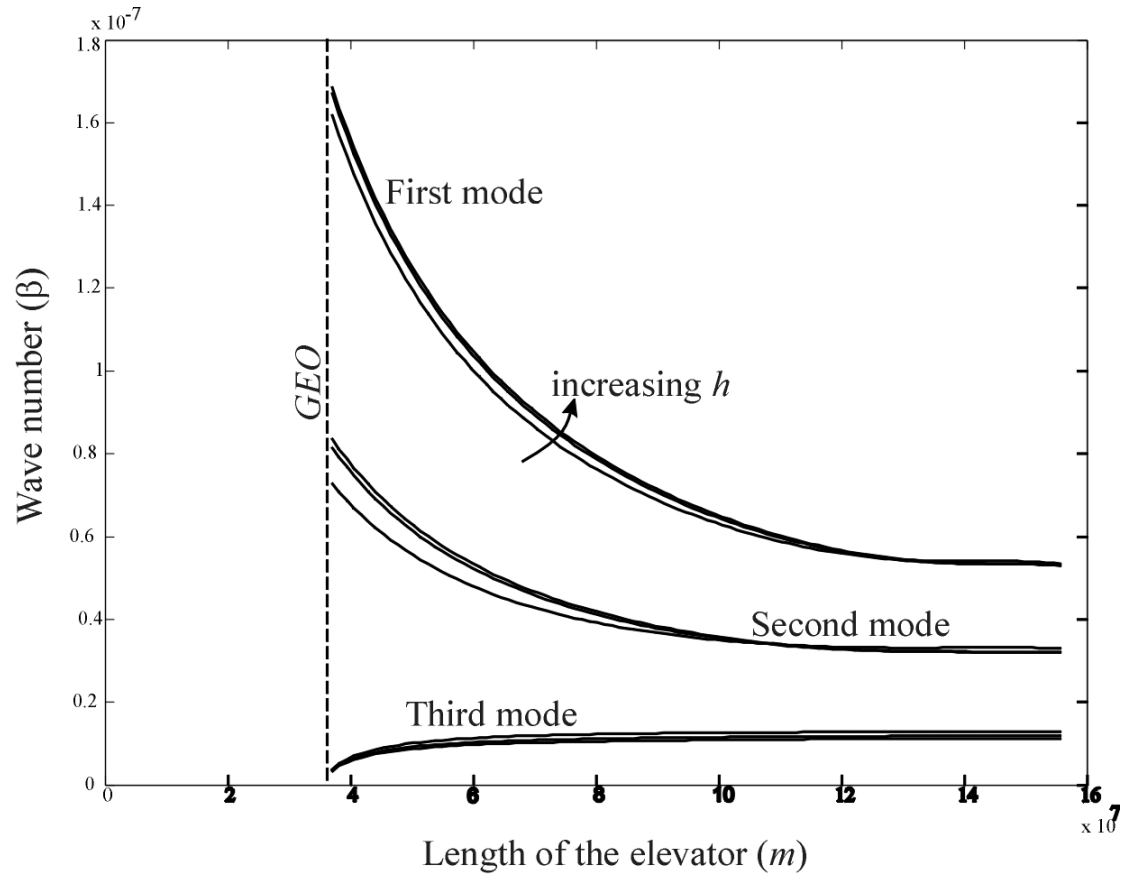


Figure 2.4: Variation of β with length of the elevator

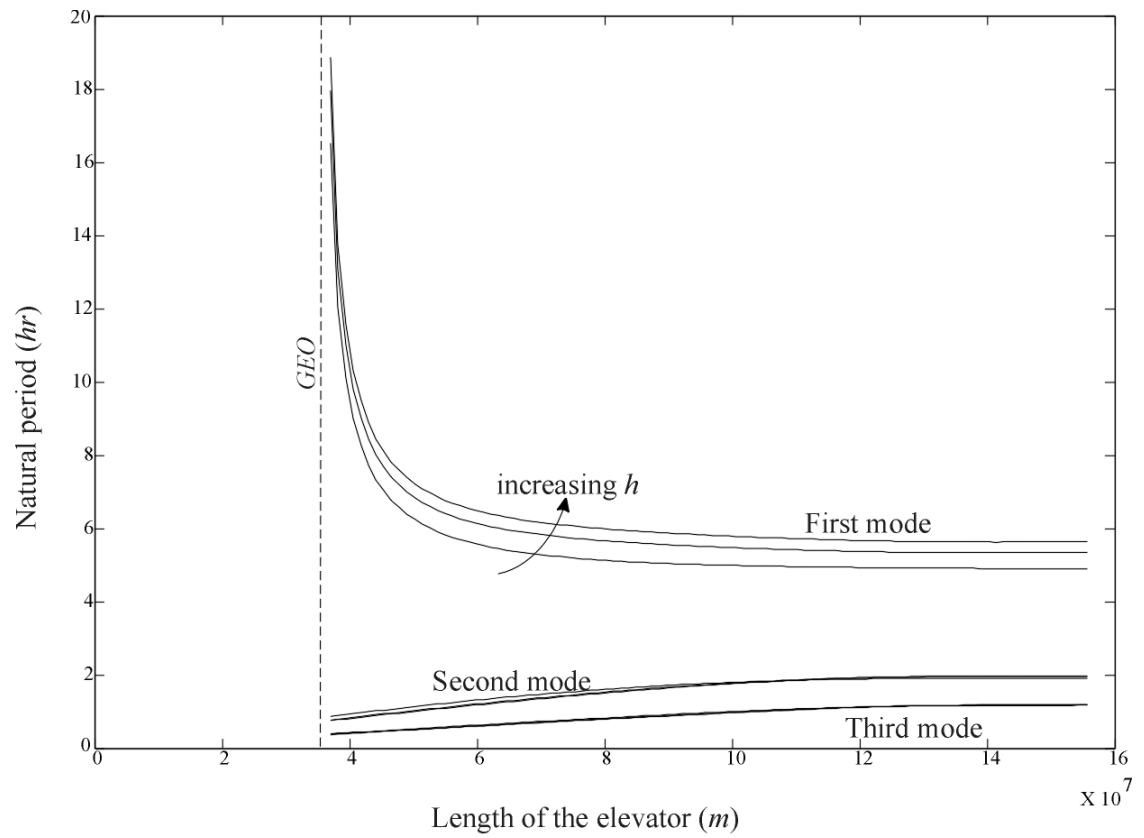


Figure 2.5: Natural periods of longitudinal vibration with length of the elevator

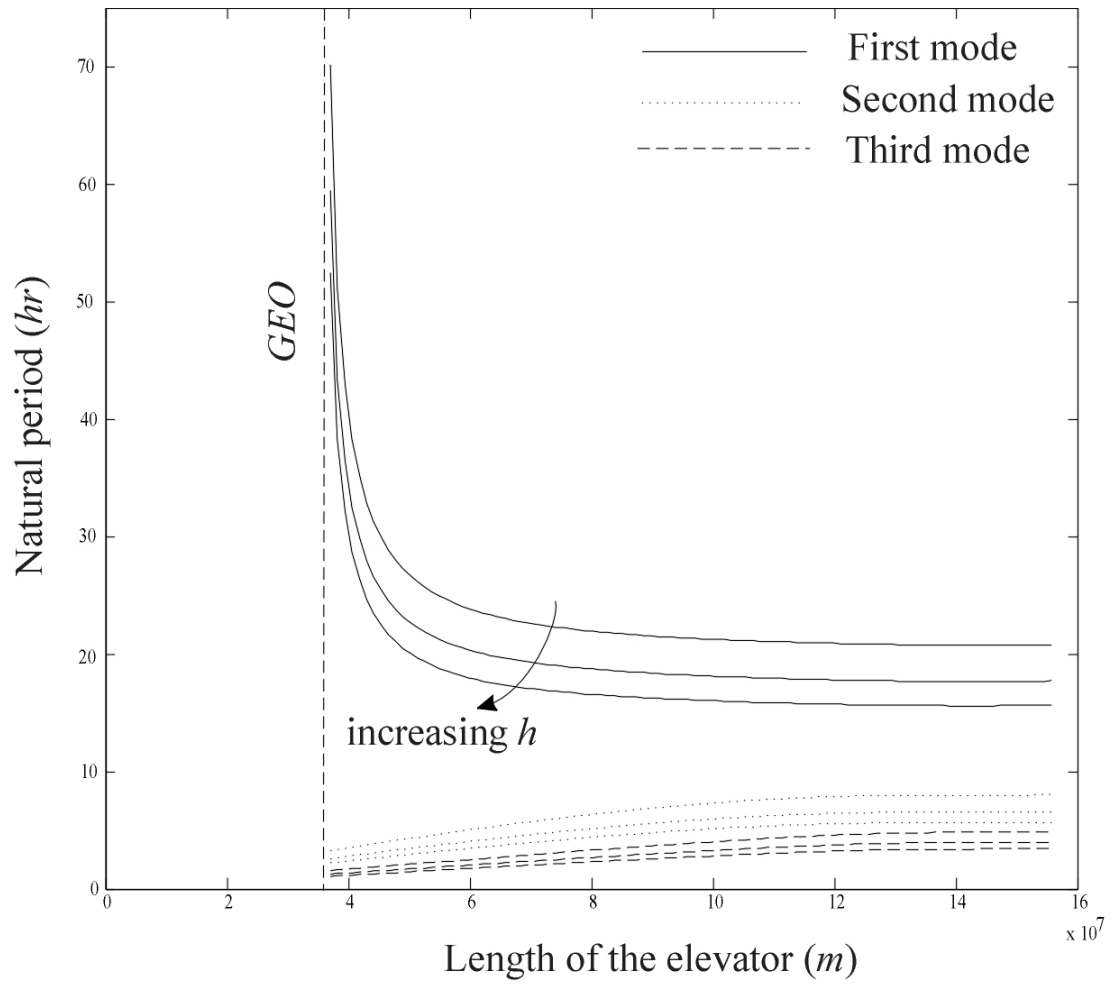


Figure 2.6: Natural periods of transverse vibration with length of the elevator

CHAPTER 3

NON-LINEAR MATHEMATICAL MODEL OF SPACE ELEVATOR

3.1 Introduction

In this chapter, the space elevator is modeled as a long elastic cable in three dimensions. The three displacements in the axial (u), meridional (v) and equatorial directions (w) of the elevator are functions of both the Lagrangian co-ordinate (X) along the elevator and time (t). The meridional direction points towards the North and equatorial direction points towards the West. The kinetic energy and potential energies of the elevator are obtained to form the Lagrangian. The three equations of motion are derived using Hamilton's principle. These three coupled and non-linear partial differential equations describe three-dimensional motion of the cable. Static solution of this model in the radial direction is considered and the cross-sectional area $A(X)$ and counter mass (M_p) are derived from that solution.

3.2 Equations of Motion

In this section, we will derive a set of non-linear equations of motion for a cable that is similar to that ones used by Perkins and Mote [14] and Han and Benaroya [15]. Let $\mathbf{r}$ be the position vector from the center of the Earth to an incremental element and is given by

$$\mathbf{r} = (r_0 + X + u)\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad (3.1)$$

Where

r_0 = Radius of the Earth,

X = Lagrangian coordinate,

xyz co-ordinate system is attached to the Earth and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the directions of x, y , and z respectively,

u is the displacement in the x direction and is called the radial displacement ,

v is the displacement in the y direction and is called the meridional displacement, when $\alpha = 0$ in Figure 3.1,

w is the displacement in the z direction and is called the equatorial displacement., when $\alpha = 0$ in Figure 3.1.

Figure 3.1 shows the xyz co-ordinate system and Figure 3.2 shows displacements along the three directions. In this case the elevator is at an angle of α with respect to the plane of the Earth's equator. The angular velocity of the Earth about its rotational axis is Ω , so the co-ordinate system rotates with the angular velocity Ω and vector form of the angular velocity is given by

$$\boldsymbol{\Omega} = \Omega \sin \alpha \mathbf{i} + \Omega \cos \alpha \mathbf{j}. \quad (3.2)$$

The kinetic energy of the elevator is given by

$$KE = \frac{1}{2} M_p \frac{\partial \mathbf{r}_p}{\partial t} \cdot \frac{\partial \mathbf{r}_p}{\partial t} + \int_0^L \frac{1}{2} m(X) \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial t} dX, \quad (3.3)$$

where

M_p is the mass of the counterweight,

$m(X) = \rho A(X)$ is the cable mass per unit length,

$\mathbf{r}_p$ is the position vector of the point mass $= (r_0 + L + u)\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, and

L is the undeformed length of the elevator.

The first term in Eqn 3.3 is kinetic energy due to the point mass and second term is kinetic energy due to the cable mass. The velocity vector, $\partial \mathbf{r} / \partial t$, is given by

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial t} &= \dot{u}\mathbf{i} + \dot{v}\mathbf{j} + \dot{w}\mathbf{k} + (\boldsymbol{\Omega} \times \mathbf{r}) \\ &= (\dot{u} + w\Omega \cos \alpha)\mathbf{i} + (\dot{v} - w\Omega \sin \alpha)\mathbf{j} \\ &\quad + (\dot{w} + v\Omega \sin \alpha - (r_0 + X + u)\Omega \cos \alpha)\mathbf{k}, \end{aligned} \quad (3.4)$$

and $\partial \mathbf{r}_p / \partial t = \partial \mathbf{r} / \partial t|_{L,t}$.

If the elevator is placed at the equator i.e. when $\alpha=0$, the velocity vector will be reduced to

$$\frac{\partial \mathbf{r}}{\partial t} = (\dot{u} + w\Omega) \mathbf{i} + \dot{v}\mathbf{j} + (\dot{w} - (r_0 + X + u)\Omega) \mathbf{k}. \quad (3.5)$$

The potential energy of the elevator can be attributed from gravitational and elastic forces acting on the elevator. The potential energy due to the elastic force is given by

$$PE_e = \int_V \left(\int \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} d\varepsilon_{ij} \right) dV. \quad (3.6)$$

Where ε_{ij} is the Green's strain given by

$$\varepsilon = u' + \frac{1}{2}(u'^2 + v'^2 + w'^2), \quad (3.7)$$

and σ_{ij} is the second Piola-Kirchoff stress.

The first integral in Eqn.3.6 refers to the integral over volume. Since the elastic potential energy acts in the x direction, the integrand of the second integral $\sigma_{ij} d\varepsilon_{ij}$ can be reduced to $\sigma_{11} d\varepsilon_{11}$. From Hook's law $\sigma_{11} = E \varepsilon_{11}$, so the elastic potential energy equation can be written as

$$PE_e = \int_V \frac{1}{2} E \varepsilon_{11}^2 dV. \quad (3.8)$$

The volume integral in Eqn 3.8 can be converted to line integral by writing the differential volume element as $dV = A(X) dX$. Now, Eqn 3.8 can be expressed as

$$PE_e = \int_0^L \frac{1}{2} E A(X) \varepsilon_{11}^2 dX, \quad (3.9)$$

The gravitational potential energy can be attributed from cable mass and point mass and it can be expressed as

$$PE_g = \int_0^L -\rho A(X) \frac{GM_e}{r} dX - \frac{GM_e M_p}{r_p}, \quad (3.10)$$

where M_e = Mass of the Earth,

r = magnitude of the position vector $\mathbf{r} = \sqrt{(r_0 + X + u)^2 + v^2 + w^2}$, and

r_p = magnitude of the position vector to the point mass
 $|\mathbf{r}_p| = \sqrt{(r_0 + L + u)^2 + v^2 + w^2}$.

The first and second terms in the above equation indicates the gravitational potential energy due to cable mass and the counter weight respectively. Total potential energy (PE) is

$$PE = PE_e + PE_g. \quad (3.11)$$

Lagrangian will be given by

$$\begin{aligned} L = KE - PE = \int_0^L \left(\frac{1}{2} \rho A(X) \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial t} - \frac{1}{2} EA \varepsilon^2 + \rho A(X) \frac{GM_e}{r} \right) dX \\ + \frac{1}{2} M_p \frac{\partial \mathbf{r}_p}{\partial t} \cdot \frac{\partial \mathbf{r}_p}{\partial t} + \frac{GM_e M_p}{r_p}. \end{aligned} \quad (3.12)$$

Hamilton's principle states that, the Hamilton's Integral (I), the variation of the Lagrangian L in equation 3.12 integrated over time should be equal to zero or

$$I = \delta \int_{t_1}^{t_2} L dt = 0. \quad (3.13)$$

Substituting the Lagrangian L from equation 3.12 in the Hamilton's Integral and simplifying gives

$$\begin{aligned}
 I &= \delta \int_{t_1}^{t_2} L \, dt \tag{3.14} \\
 &= \int_{t_1}^{t_2} \int_0^L \left\{ \left[\rho A(X)(-\ddot{u} - 2\Omega\dot{w} + (r_0 + X + u)\Omega^2) + (EA(X)\varepsilon(1 + u'))' \right. \right. \\
 &\quad \left. \left. - \rho A(X) \frac{gr_0^2(r_0 + X + u)}{r^3} \right] \delta u + \left[-\rho A(X)\ddot{v} + ([EA(X)\varepsilon v'])' - \rho A(X) \frac{gr_0^2 v}{r^3} \right] \delta v \right. \\
 &\quad \left. + \left[\rho A(X)(-\ddot{w} + 2\dot{u}\Omega + \Omega^2 w) + (EA(X)\varepsilon w')' - \rho A(X) \frac{gr_0^2 w}{r^3} \right] \delta w \right\} dX \, dt \\
 &\quad + \int_{t_1}^{t_2} \left\{ \left[-M_p \left(\ddot{u} + 2\Omega\dot{w} - (r_0 + L + u)\Omega^2 + \frac{gr_0^2(r_0 + L + u)}{r_p^3} \right) \right] \delta u(L, t) - \right. \\
 &\quad \left[M_p \left(\ddot{v} + \frac{gr_0^2 v}{r_p^3} \right) \right] \delta v(L, t) - \left[M_p \left(\ddot{w} - 2\Omega\dot{u} - \Omega^2 w - \frac{gr_0^2 w}{r_p^3} \right) \right] \delta w(L, t) \right\} dt . \\
 &\quad \left. - [EA(X)\varepsilon(1 + u')] \delta u \Big|_0^L - [EA(X)\varepsilon v'] \delta v \Big|_0^L - [EA(X)\varepsilon w'] \delta w \Big|_0^L \right\}
 \end{aligned}$$

From equation 3.14, we can deduce the three equations of motion and the boundary conditions for the space elevator system. The equations of motion are given by

$$\begin{aligned}
 \rho A(X)(\ddot{u} + 2\Omega\dot{w} - (r_0 + X + u)\Omega^2) \\
 = (EA(X)\varepsilon(1 + u'))' - \rho A(X) \frac{gr_0^2(r_0 + X + u)}{r^3}
 \end{aligned} \tag{3.15}$$

$$\rho A(X)\ddot{v} = (EA(X)\varepsilon v')' - \rho A(X) \frac{gr_0^2 v}{r^3} \tag{3.16}$$

$$\rho A(X)(\ddot{w} - 2\dot{u}\Omega - \Omega^2 w) = (EA(X)\varepsilon w')' - \rho A(X) \frac{gr_0^2 w}{r^3} \tag{3.17}$$

and the boundary conditions are given by

$$u(0, t) = 0, \quad v(0, t) = 0, \quad w(0, t) = 0 \tag{3.18}$$

$$\left[M_p(\ddot{u} + 2\Omega\dot{w} - (r_0 + L + u)\Omega^2) + EA(L)\varepsilon(1 + u') + M_p \frac{gr_0^2(r_0 + L + u)}{r_p^3} \right] \Big|_{L,t} = 0 \quad (3.19)$$

$$\left[M_p\ddot{v} + EA(L)\varepsilon v' + M_p \frac{gr_0^2 v}{r_p^3} \right] \Big|_{L,t} = 0 \quad (3.20)$$

$$M_p(\ddot{w} - 2\Omega\dot{u} - \Omega^2 w) + EA(X)\varepsilon w' + M_p \frac{gr_0^2 w}{r_p^3} \Big|_{L,t} = 0. \quad (3.21)$$

The three equations of motion are coupled through ε, r , centripetal acceleration, gravity and the Coriolis terms. Therefore, it is not possible to have motion in one direction without the motion in the other directions.

3.3 Static Solution in the Radial Direction

Let us find the static solution of Eqn 3.19 and derive the equations for the cross-sectional area and counter mass. From these equations, we will discuss the significance of the elongation factor 'C'.

3.3.1 Cross-sectional Area

Let us derive the cross sectional area of elevator based on the static configuration of elevator while taking the elongation factor into consideration. The elevator deforms from its original position because of the gravitational and centripetal forces acting on it. The co-ordinates of the elevator are defined from its base on the surface of the Earth. The initial co-ordinates of the elevator prior to its deformation are called Lagrangian co-ordinates, and are denoted as X . The co-ordinates of elevator after the deformation are called Euler co-ordinates, and are denoted as x . Elongation factor (C) depends on the tensile strength (σ) and Young's modulus (E), and is given by

$$C = \sqrt{1 + \frac{2\sigma}{E}} = \sqrt{1 + 2\varepsilon}. \quad (3.22)$$

To illustrate the difference between X and x , let's consider a portion of the space elevator as shown in figure 3.3. X is the initial position of the elevator and x is the position of the elevator after the deformation. The strain is given by

$$\varepsilon = \frac{1}{2} \frac{dx^2 - dX^2}{dX^2} \Rightarrow \frac{dx}{dX} = \sqrt{1 + 2\varepsilon}. \quad (3.23)$$

Form Eqns. 3.22 and 3.23, we can write that, $dx = C dX \Rightarrow x = C X$

Let us assume that the cross-sectional area of the elevator is a function of X . The cross-sectional area is derived by assuming that the static stress at any point X is constant and equal to σ . The static transverse displacements, V_{equil} and W_{equil} , are zeros and the radial static displacement, U_{equil} , is obtained by solving the following equation

$$\sigma = E \varepsilon_{equil}, \quad (3.24)$$

where ε_{equil} is given by
$$\varepsilon = U'_{equil} + \frac{1}{2} U'^2_{equil}. \quad (3.25)$$

Substituting Eqn 3.25 into Eqn 3.24 gives

$$\sigma = E \left(U'_{equil} + \frac{1}{2} U'^2_{equil} \right). \quad (3.26)$$

The above equation can be written as

$$U'^2_{equil} + 2U'_{equil} - \frac{2\sigma}{E} = 0, \quad (3.27)$$

and it can be solved with the initial condition $U_{equil}(X = 0) = 0$. The solution gives the static radial displacement and can be written as,

$$U_{equil} = (C - 1)X \quad (3.28)$$

The radial displacement at the free end is $(C-1)L$, this can be quite significant because L is a very large number.

If the elevator is in equilibrium and static stress is constant and equal to σ , Eqn 3.15 is reduced to

$$-\rho A(X) (r_0 + CX)\Omega^2 = C\sigma A'(X) - \rho A(X)g \frac{r_0^2}{(r_0 + CX)^2} \quad (3.29)$$

From the above equation, the ratio $A'(X)/A(X)$ can be written as,

$$\frac{A'(X)}{A(X)} = \frac{r_0^2}{hC} \left(\frac{1}{(r_0 + CX)^2} - \frac{(r_0 + CX)}{r_s^3} \right). \quad (3.30)$$

Integrating the above equation from X_s to X ,

$$[\log_e A(X)]|_{X_p}^X = \left[-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX)} + \frac{(r_0 + CX)^2}{2r_s^3} \right) \right] \Big|_{X_p}^X, \quad (3.31)$$

where X_p is the Lagrangian co-ordinate of the incremental element at any point along the elevator. The two convenient places are the GEO, $X_p = X_s$, and the surface of the Earth, $X_p = 0$. Eqn 3.31 can be solved for in terms of the cross-sectional area of the elevator as

$$A(X) = A_s \frac{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX)} + \frac{(r_0 + CX)^2}{2r_s^3} \right) \right)}}{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX_s)} + \frac{(r_0 + CX_s)^2}{2r_s^3} \right) \right)}} \text{ or } A_0 \frac{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX)} + \frac{(r_0 + CX)^2}{2r_s^3} \right) \right)}}{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{r_0} + \frac{r_0^2}{2r_s^3} \right) \right)}}. \quad (3.32)$$

The Eqn 3.32 is plotted in Figure 3.3 as a function of the current distance from Earth's surface, x , where the Eulerian co-ordinate is related to the Lagrangian coordinate by $x = CX$. The area is normalized by the area at the Earth's surface. This plot shows the variation in the cross-sectional area of the elevator, when the elongation factor is taken in

to account. From Figure 3.3, we can observe that the variation in the cross-sectional area due to elongation factor is quite significant.

3.3.2 Derivation of Counter mass

Let us consider an incremental element of mass dm as shown in Figure 3.5. The forces acting on this element are given by,

$$\uparrow \Sigma F_{radial} = dT - F_g = dm a_r \quad (3.33)$$

Where dT is the internal tension which varies along the elevator, F_g is the gravitational pull given by $F_g = G M_e dm/r^2$, and a_r is the radial acceleration given by $a_r = -v^2/r$. Eqn. 3.33 can be rewritten as

$$dT = \frac{G M_e dm}{r^2} - \frac{dm v^2}{r} = g r_0^2 \rho A(X) dX \left(\frac{1}{(r_0 + CX)^2} - \frac{(r_0 + CX)}{r_s^3} \right) \quad (3.34)$$

For the elevator to be in static equilibrium, the tension at the tip of the elevator must be zero or $T(L)=0$. For $C = 1$, we recover the value $L = 144000 \text{ km}$. For $C = 1.0583$, the undeformed length of the elevator is $L = 136000 \text{ km}$. The deformed length of the elevator is $l = CL = 144000 \text{ km}$. Note that the deformed length of the elevator is always 144000 km for any value of C .

When the point mass is included, the length of the elevator can be shortened. The free body diagram of forces acting on the point mass is shown in Figure 3.6. The summation of forces acting on the point mass is given by

$$\uparrow \Sigma F_{radial} = -T + F_g = M_p a_r,$$

which results in

$$M_p g r_0^2 \left(\frac{1}{(r_0 + CL)^2} - \frac{(r_0 + CL)}{r_s^3} \right) + T(L) = 0. \quad (3.35)$$

From Eqn 3.35, point mass can be written as,

$$M_p = \frac{T}{g r_0^2 \left(\frac{(r_0 + CL)}{r_s^3} - \frac{1}{(r_0 + CL)^2} \right)} \quad (3.36)$$

The tension acting on the point mass can be found by integrating Eqn. 3.34 or

$$\begin{aligned} T &= \int_0^L dT \\ &= \left(\frac{\rho A_s g r_0^2}{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX_s)} + \frac{(r_0 + CX_s)^2}{2r_s^3} \right) \right)}} \right) \int_0^L e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX)} + \frac{(r_0 + CX)^2}{2r_s^3} \right) \right)} \left(\frac{1}{(r_0 + CX)^2} \right. \\ &\quad \left. - \frac{(r_0 + CX)}{r_s^3} \right) dX \end{aligned} \quad (3.37)$$

Substituting Eqn 3.37 into Eqn 3.36 gives,

$$\begin{aligned} M_p &= \frac{A_s \sigma C}{\Omega^2 \left((r_0 + CL) - \frac{r_s^3}{(r_0 + CL)^2} \right)} \left(\frac{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CL)} + \frac{(r_0 + CL)^2}{2r_s^3} \right) \right)} - e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{r_0} + \frac{r_0^2}{2r_s^3} \right) \right)}}{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX_s)} + \frac{(r_0 + CX_s)^2}{2r_s^3} \right) \right)}} \right) \end{aligned} \quad (3.38)$$

The point mass depends on A_s , C , L , and h . The dependency on A_s can be eliminated if the point mass is normalized by the total mass of the elevator. The total mass of the elevator is given by

$$\begin{aligned} m &= \int_0^L \rho A(X) dX \\ &= \frac{\rho A_s}{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX_s)} + \frac{(r_0 + CX_s)^2}{2r_s^3} \right) \right)}} \int_0^L e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX)} + \frac{(r_0 + CX)^2}{2r_s^3} \right) \right)} dX \end{aligned} \quad (3.39)$$

The mass ratio is given by dividing Eqns. 3.38 and 3.39 or

$$\frac{M_p}{m} = \frac{h C}{r_0^2 \left(\frac{(r_0 + CL)}{r_s^3} - \frac{1}{(r_0 + CL)^2} \right)} \left(\frac{e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CL)} + \frac{(r_0 + CL)^2}{2r_s^3} \right) \right)} - e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{r_0} + \frac{r_0^2}{2r_s^3} \right) \right)}}{\int_0^L e^{\left(-\frac{r_0^2}{hC^2} \left(\frac{1}{(r_0 + CX)} + \frac{(r_0 + CX)^2}{2r_s^3} \right) \right)} dX} \right). \quad (3.40)$$

Now, M_p/m is a function of h , C , and L . This equation is plotted in Figure 3.7 for two values of C as a function of the elevator length, L . For this figure, h is set to 4709 km, and the elongation factors are $C = 1.0583$ and 1. From this figure, we observe that there is a significant deviation in the point mass to cable mass ratio, when elongation factor (C) is 1.0583.

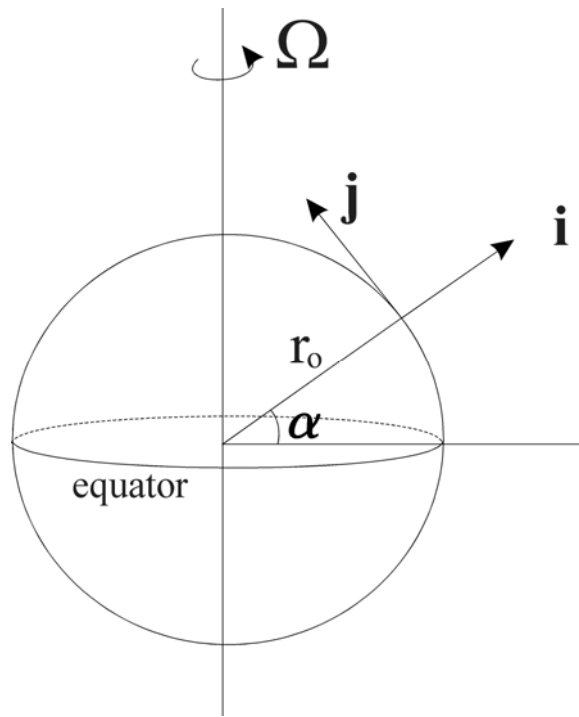


Figure 3.1: Co-ordinate system of the space elevator

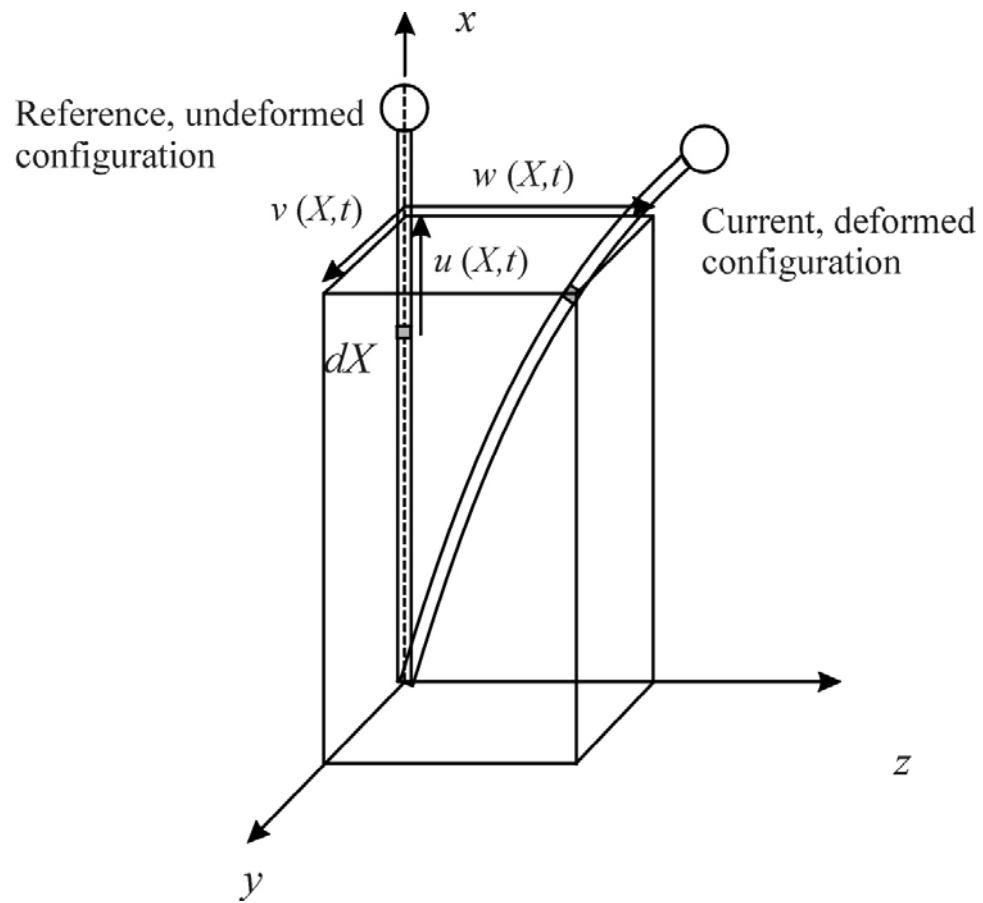


Figure 3.2: Displacements in non-linear model of the space elevator [15]

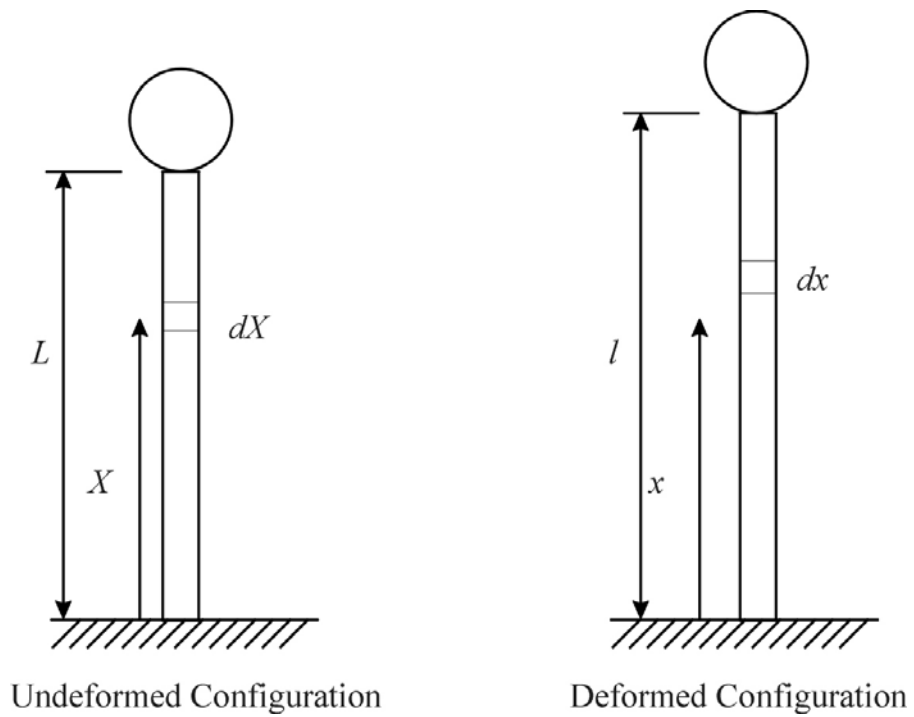


Figure 3.3: Difference between Euler and Lagrangian co-ordinates

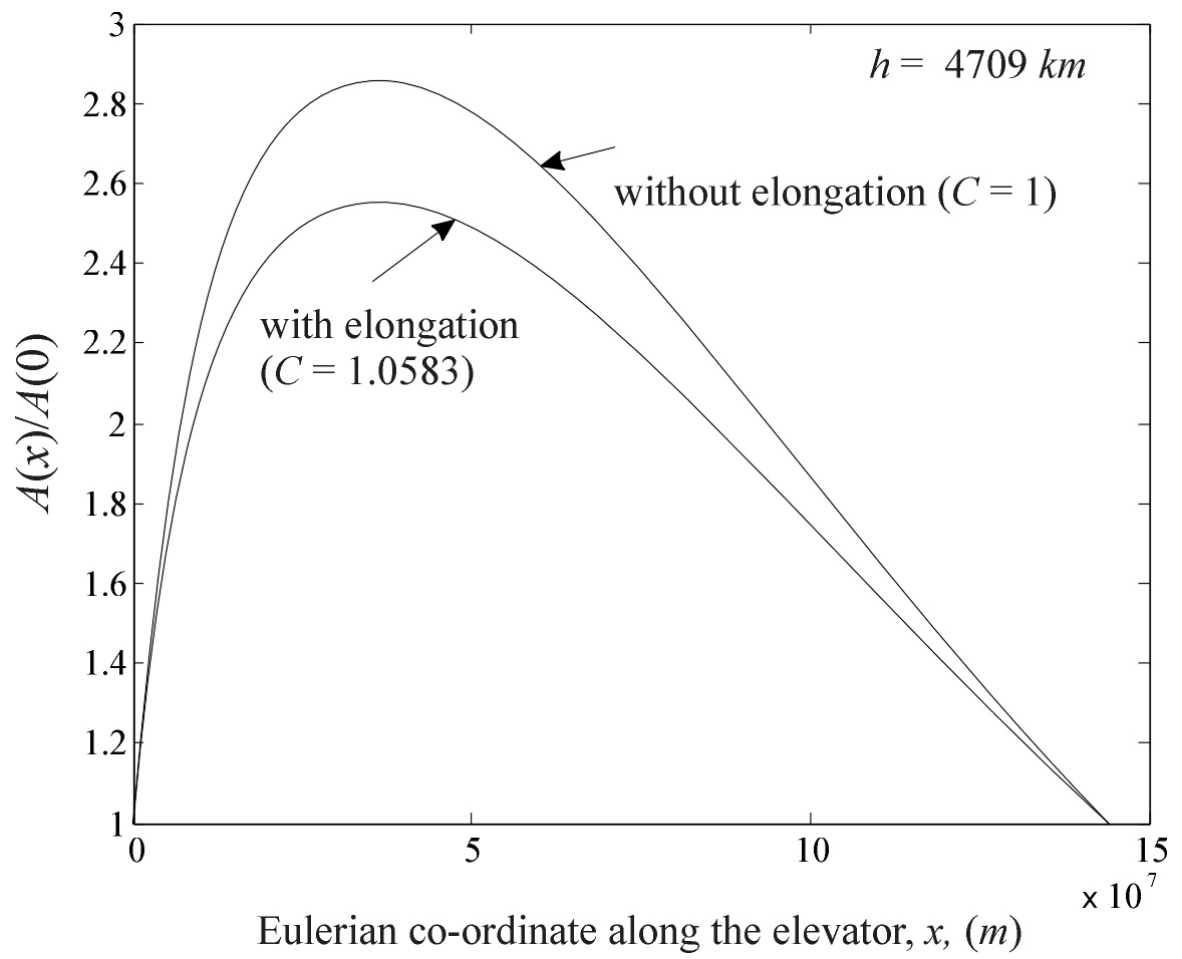


Figure 3.4: Variation in the cross-sectional area of the elevator with elongation factor

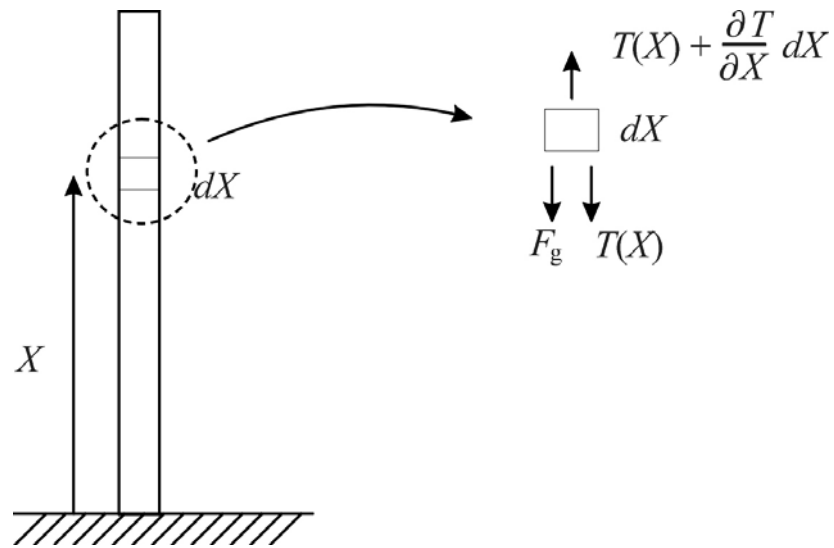


Figure 3.5: Forces acting on an incremental element

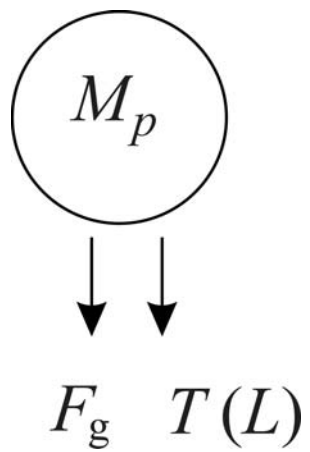


Figure 3.6: Forces acting on the counter mass

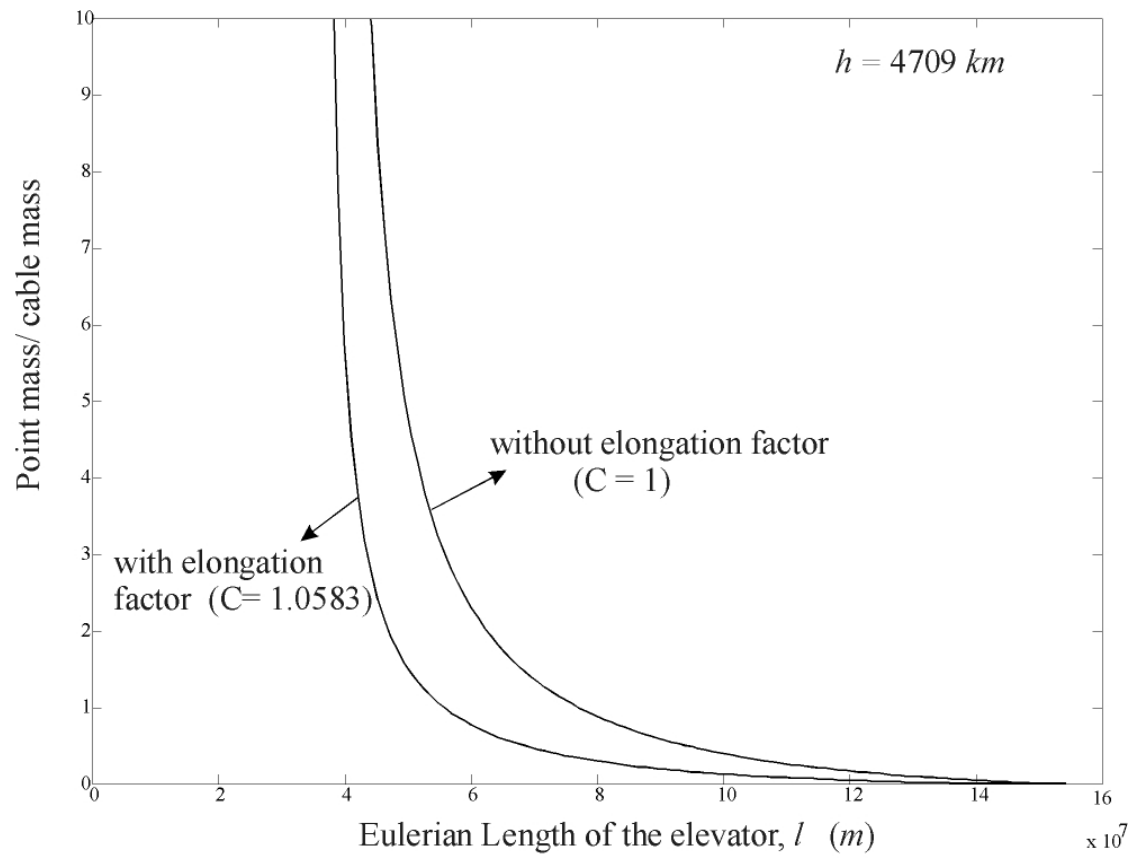


Figure 3.7: Variation in the point mass to cable mass ratio for different values of C

CHAPTER 4

TIDAL FORCES

4.1 Introduction

In this chapter, we will discuss how the tidal forces are formed because of differential gravitational pull on different parts of the Earth. The co-ordinates of the Earth-Moon and Earth-Sun systems are defined. Co-ordinate transformation matrices have been developed to convert the Earth-Moon and the Earth-Sun Co-ordinate systems into a co-ordinate system that rotates with the Earth. Equations of the tidal accelerations due to Moon and the Sun are derived, which are divided into three components acting along the axial, equatorial and meridional directions. These tidal accelerations have been plotted with the time for hundred hours. Forcing periods of the tidal accelerations have been found from their PSD plots.

4.2 Formation of Tidal Forces

According to Newton's law of gravitation, any two bodies exert gravitational force of attraction on each other and the force will act along the line joining the two bodies. The magnitude of this force is proportional to the product of the masses of two bodies and inversely proportional to the square of the distance between them. Considering the Earth-Moon system, the Earth exerts a gravitational force on the Moon and the Moon also exerts the equal gravitational force on the Earth. The equation for the gravitational force of attraction between the Earth and the Moon is

$$F = \frac{GM_e M_m}{r^2}, \quad (4.1)$$

where M_e = Mass of the Earth,

M_m = Mass of moon,

G = Newton's gravitational constant= $6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$, and

r = distance between center of the Earth to the center of moon.

The Earth exerts gravitational force on the Moon, which causes the moon to revolve around the Earth. In a similar way, the Moon also exerts gravitational force on the Earth causing it to revolve around the Moon. This combined effect causes the Earth and the Moon to revolve about a common point that is their center of mass. This point is also called Barycenter [16]. From Table 4.1, we can observe that the Earth's mass is almost 81 times greater than the mass of the Moon. Since the Earth is massive than the Moon, barycenter lies close to the Earth. In fact, it is inside the Earth [16].

The Earth is so large that the effect of gravitational force due to the Moon will be different at different locations on the Earth. The part of the Earth closer to the Moon will experience greater acceleration due to the Moon's gravitational pull than the one farther from the moon. This creates differential gravitational force across the Earth. As shown in Figure 4.1, point A experiences more gravitational pull from the Moon than point B. And point B experiences more gravitational pull than point C (The size of arrows indicates the magnitude of the acceleration; the figure is not up to scale). The shaded region in Figure 4.1 indicates the tidal bulge which occurs because of the difference in the gravitational pulls by the Moon at points A and C.

These forces will try to deform the shape of the Earth along its equator to make it into an egg shaped object. But these forces are very small compared to the internal forces (Mechanical and gravitational forces) of the Earth, so the Earth deforms only about few centimeters [17]. But water is in liquid state, so it experiences the effect of tidal forces. These ocean waters start flowing towards two opposite ends of the Earth, one near the moon and the other one away from the moon as shown in Figure 4.2. These are called Tidal bulges. Since the Earth takes one day to make complete rotation about its own axis

and tidal bulges move with respect to the Moon, a particular place on the Earth experiences two high tides and two low tides every day. But by the time the Earth returns to its original position after 24 hours, the Moon would have displaced by an angle 13.19° . Earth must turn approximately 53 minutes more after reaching its initial position to get to the new position of the tidal bulge. This happens because moon is orbiting the Earth with a period of 27.3 days.

In our solar system, the Sun is the next major source after the Moon that exerts considerable tidal gravitational force on the Earth. The Sun is massive than the Earth (Table 4.1), but the Moon is much closer to the Earth than the Sun (Table 4.2). According to Newton's law of gravitation, gravitational force is proportional to mass but inversely proportional to cube of distance between two objects. The Sun's share of the total tidal gravitational force acting on the Earth is only 30%.

The tidal effect increases when the Earth, the Moon and the Sun come in a straight line. This happens during New Moon and the Full moon. The ocean tides produced during this period are called Spring tides. The tidal effect on the Earth decreases when the gravitational force vectors from the Sun and the Moon are perpendicular to each other. This occurs during the first-quarter phase and third-quarter phases of the Moon. The tides produced during this period are called Neap tides. Formation of the Spring tides and Neap tides is explained in Figure 4.3.

4.3 The Earth-Moon Co-ordinate System

Let us first define coordinate systems as shown in Figure 4.4 in order to formulate the tidal forces due to the Moon. Let xyz frame be attached to the Earth so that the xyz frame rotates once every 24 hours, or with an angular velocity of $\Omega = 2\pi/(24 \times 60 \times 60) = 7.27 \times 10^{-5}(\text{rad/s})$. The unit vectors associated with the xyz frame are $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. The y axis is aligned with the North pole, x axis is normal to the Earth's surface

pointing directly at the incremental mass, and z axis is along the equator as shown in Figure 4.4. The position vector to the point mass can be written as $\mathbf{r} = r\mathbf{i}$. We can back out to the inertial frame abc , by rotating the xyz frame about the y axis by $-\Omega t$. The b axis of the inertial frame abc is still aligned with Earth's North Pole, and the plane of Earth's equator is on the ac plane. Note that the unit vectors associated with the abc frame are $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$. The relationship between unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is given by a matrix $[R_1]$ defined by

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & 0 & \sin(\Omega t) \\ 0 & 1 & 0 \\ -\sin(\Omega t) & 0 & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}. \quad (4.2)$$

The abc frame is rotated about the c axis by 28.65° , so that the new inertial frame, ABC , is now aligned with the orbital plane of the Moon about the Earth. Now, the Moon's orbital plane is on the AC plane, and the B axis is perpendicular to the orbital plane. The unit vectors associated with the ABC frame are $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. The relationship between the unit vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by a matrix $[R_2]$ defined by

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \cos(28.65) & \sin(28.65) & 0 \\ -\sin(28.65) & \cos(28.65) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}. \quad (4.3)$$

Finally, ABC frame is rotated by $\omega_{em}t$ about the B axis to obtain the XYZ frame that is attached to the Earth-Moon system. The unit vectors associated with the XYZ frame are $\mathbf{I}, \mathbf{J}$ and $\mathbf{K}$. The relationship between the unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by a matrix $[R_3]$ defined by

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix} = \begin{bmatrix} \cos(\omega_{em}t) & 0 & -\sin(\omega_{em}t) \\ 0 & 1 & 0 \\ \sin(\omega_{em}t) & 0 & \cos(\omega_{em}t) \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix}. \quad (4.4)$$

The relationship between different reference frames is shown in Figure 4.5 and relationship between the unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is given by,

$$\begin{bmatrix} I \\ J \\ K \end{bmatrix} = [R_3] [R_2] [R_1] \begin{bmatrix} i \\ j \\ k \end{bmatrix} \quad (4.5)$$

4.4 Equation of Tidal Forces due to the Moon

The co-ordinate system of the Earth, space elevator and the Moon is explained in Figure 4.6. Let us consider an incremental mass dm , which is at a distance r away from the center of the Earth and also on the plane of the Earth's equator. The Earth and the Moon rotate about their own axes and the Moon revolves around the Earth. These rotations are included by considering that

- The Earth rotates about its own axis by angular velocity Ω
- The rotation of the Earth about the Sun is neglected.
- The Earth and the Moon are orbiting around their center of mass (barycenter) with angular velocity ω_{em}

The center of mass of the Earth - Moon system is calculated as follows

$$M_e s_{em} = M_m (D_{em} - s_{em}),$$

$$s_{em} = \frac{M_m D_{em}}{M_m + M_e}, \quad (4.6)$$

where M_m = Mass of the Moon,

M_e = Mass of the Earth,

D_{em} = Distance between centers of the Earth and the Moon, and

s_{em} = Distance between center of the Earth and center of mass of the Earth- Moon system.

The angular velocity with which the Moon orbits the Earth is obtained by equating gravitational pull with the centripetal acceleration

$$\frac{GM_m M_e}{D_{em}^2} = M_e s_{em} \omega_{em}^2 = M_e (D_{em} - s_{em}) \omega_{em}^2,$$

$$\omega_{em}^2 = \frac{GM_m}{s_{em} D_{em}^2} = \frac{G(M_m + M_e)}{D_{em}^3}. \quad (4.7)$$

We have approximated the space elevator with a point mass at a distance r from center of the Earth. The forces acting on point mass are due to the Moon and rotation about the center of mass of Earth-Moon rotating system. The equation for the tidal forces acting on elevator per unit mass due to the Moon is given by

$$\frac{P_{moon}}{dm} = -s_{em} \omega_{em}^2 \mathbf{I} + \frac{GM_m}{R_{em}^3} (D_{em} \mathbf{I} - r \mathbf{i}), \quad (4.8)$$

Where ρ = Density of material used for space elevator,

A = Cross –sectional area of the elevator, and

P_{moon} = Tidal forces acting on the elevator due to the Moon.

P_{moon}/dm is called the differential acceleration felt by point mass due to the Moon.

After replacing the $\mathbf{I}$ in Eqn. 4.8 with the expression in Eqn. 4.5, the differential acceleration equation becomes

$$\begin{aligned}
 & \frac{P_{moon}}{dm} \\
 &= \left[\left(\left(\frac{GM_m D_{em}}{R_{em}^3} - s_{em} \omega_{em}^2 \right) (\cos(\omega_{em} t) \cos(\Omega t) \cos(28.65) + \sin(\omega_{em} t) \sin(\Omega t)) \right) \right. \\
 & \quad \left. - \left(\frac{GM_m r}{R_{em}^3} \right) \right] \mathbf{i} + \left[\left(\frac{GM_m D_{em}}{R_{em}^3} - s_{em} \omega_{em}^2 \right) (\cos(\omega_{em} t) \sin(28.65)) \right] \mathbf{j} \\
 & \quad + \left[\left(\frac{GM_m D_{em}}{R_{em}^3} - s_{em} \omega_{em}^2 \right) (\cos(\omega_{em} t) \sin(\Omega t) \cos(28.65) \right. \\
 & \quad \left. - \sin(\omega_{em} t) \cos(\Omega t)) \right] \mathbf{k}. \tag{4.9}
 \end{aligned}$$

P_{moon} is divided into three components P_u, P_v, P_w , which will act in the axial, equatorial and meridional directions respectively. So $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of the Eqn. 4.9 are the differential accelerations in axial, equatorial and meridional directions.

Figure 4.7 shows the differential accelerations felt by the elevator with time at different positions in different directions due to the Moon. We can clearly observe that as we move away from the Earth, differential accelerations are increasing. P_u, P_w curves are periodic with period of 12.5 *hr*. They are out of phase by angle 180° . The differential acceleration along constant longitude (P_v) is periodic with a period of 25 *hr*.

4.5 Equation of Tidal Forces due to the Sun

The Sun is the other main source for tidal forces apart from the Moon. The Sun contributes up to 30% of the total tidal forces acting on the Earth. The equation for the forces acting on the elevator due to the Sun can be derived in a similar way as we have done for the Moon. But the angle between the Earth's equator and orbital plane of the Earth is 23.5° as shown in figure 4.4. So the differential acceleration due to the Sun is given by

$$\frac{P_{sun}}{\rho A} = -s_{es}\omega_{es}^2 \mathbf{I} + \frac{GM_{sun}}{R_{es}^3} (D_{es}\mathbf{I} - r\mathbf{i}), \quad (4.10)$$

where P_{sun} = Forces acting on elevator due to the Sun,

M_{sun} = Mass of the Sun, and

D_{es} = Distance between centers of the Earth and the Sun.

The distance between center of the Earth and center of mass of the Earth- Sun system (s_{es}) is given by

$$s_{es} = \frac{M_s D_{es}}{M_s + M_e}. \quad (4.11)$$

The angular velocity (ω_{es}) of the Earth –Sun system about their center mass is given by

$$\omega_{es}^2 = \frac{GM_s}{s_{es}D_{es}^2} = \frac{G(M_s + M_e)}{D_{es}^3}. \quad (4.12)$$

For Earth-Sun system, the relation between reference frames **IJK** and **ijk** is given by

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix} = \begin{bmatrix} \cos(\omega_{es}t) & 0 & -\sin(\omega_{es}t) \\ 0 & 1 & 0 \\ \sin(\omega_{es}t) & 0 & \cos(\omega_{es}t) \end{bmatrix} \begin{bmatrix} \cos(23.5) & \sin(23.5) & 0 \\ -\sin(23.5) & \cos(23.5) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\Omega t) & 0 & \sin(\Omega t) \\ 0 & 1 & 0 \\ -\sin(\Omega t) & 0 & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \quad (4.13)$$

After replacing the **I** in Eqn 4.10 with the expression in Eqn. 4.13, the differential acceleration equation becomes

$$\begin{aligned}
 \frac{P_{sun}}{\rho A} = & \left[\left(\left(\frac{GM_s D_{es}}{R_{es}^3} - s_{es} \omega_{es}^2 \right) (\cos(\omega_{es} t) \cos(\Omega t) \cos(23.5) + \sin(\omega_{es} t) \sin(\Omega t)) \right) \right. \\
 & - \left(\frac{GM_s r}{R_{es}^3} \right) \mathbf{i} + \left[\left(\frac{GM_s D_{es}}{R_{es}^3} - s_{es} \omega_{es}^2 \right) (\cos(\omega_{es} t) \sin(23.5)) \right] \mathbf{j} \\
 & + \left[\left(\frac{GM_s D_{es}}{R_{es}^3} - s_{es} \omega_{es}^2 \right) (\cos(\omega_{es} t) \sin(\Omega t) \cos(23.5) \right. \\
 & \left. \left. - \sin(\omega_{es} t) \cos(\Omega t)) \right] \mathbf{k} . \right. \quad (4.14)
 \end{aligned}$$

P_{sun} is divided into three components P_u, P_v, P_w , which will act in the axial, equatorial and meridional directions respectively. So $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of the Eqn. 4.11 are the differential accelerations in axial, equatorial and meridional directions.

Figure 4.8 shows the differential accelerations felt by the elevator with time at different positions in different directions due to the Sun. We can clearly observe that as we move away from the Earth, differential accelerations are increasing. P_u, P_w curves are periodic with period of 12 *hr*. They are out of phase by angle 180°. The differential acceleration along constant longitude (P_v) is periodic with a period of 24 *hr*. The magnitudes of the differential accelerations caused by the Sun are less than half of those by the Moon.

4.6 PSD Plots of the Tidal Forces

Power spectral density (PSD) is used to describe the distribution of power contained in a signal over frequency [18]. PSD of a stationary random process x_n is mathematically related to the correlation sequence by the discrete-time Fourier transform and is given by

$$P_{xx}(\omega) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} R_{xx}(m) e^{-j\omega m} . \quad (4.15)$$

Since $\omega = 2\pi f / f_s$, the above equation can be expressed as a function of physical frequency f as

$$P_{xx}(\omega) = \frac{1}{f_s} \sum_{m=-\infty}^{\infty} R_{xx}(m) e^{-2\pi j f m / f_s} . \quad (4.16)$$

We are going to use the PSD plots to detect the forcing periods from the sinusoidal plot of differential accelerations due to the Moon and the Sun. Eqn. 4.9 consists of three sinusoidal functions $\cos(\omega_{em}t)$, $\cos(\Omega t)$, $\cos(28.65)$ indicating that it will have multiple periods. But from Figure 4.7 where the differential acceleration due to the Moon is plotted, we can observe a forcing period of 12.5 *hr* in longitudinal and meridional directions and 25 *hr* in the equatorial direction. From the PSD plot of differential accelerations due to the Moon in Figure 4.9, we can see that the differential accelerations have periods of 6.2 *hr*, 8.3 *hr*, 12.5 *hr* and 25 *hr*.

Eqn. 4.14 which explains the differential accelerations due to the Sun has three sinusoidal functions $\cos(\omega_{es}t)$, $\cos(\Omega t)$, $\cos(23.5)$. But from Figure 4.9 we cannot identify all the forcing periods. From the PSD plot of differential accelerations due to the Sun in Figure 4.9, we can see that the differential accelerations have periods of 6 *hr*, 8 *hr*, 12.5 *hr* and 25 *hr*.

Table 4.1: Mass of the Moon, the Earth & the Sun

Mass of the Moon	Mass of the Earth	Mass of the Sun
0.07349×10^{24}	5.9742×10^{24}	1.989×10^{30}

Table 4.2: Distance from the Earth to the Moon & the Sun

Distance between the Earth and the Moon	Distance between the Earth and the Sun
$3.84467 \times 10^8 \text{ m}$	1.496×10^{11}

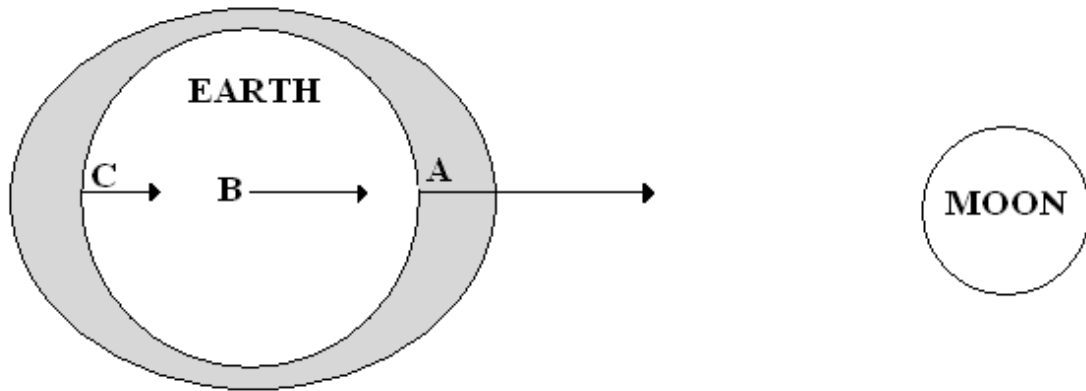


Figure 4.1: Gravitational pull on the Earth by the Moon causing tidal bulges [19]

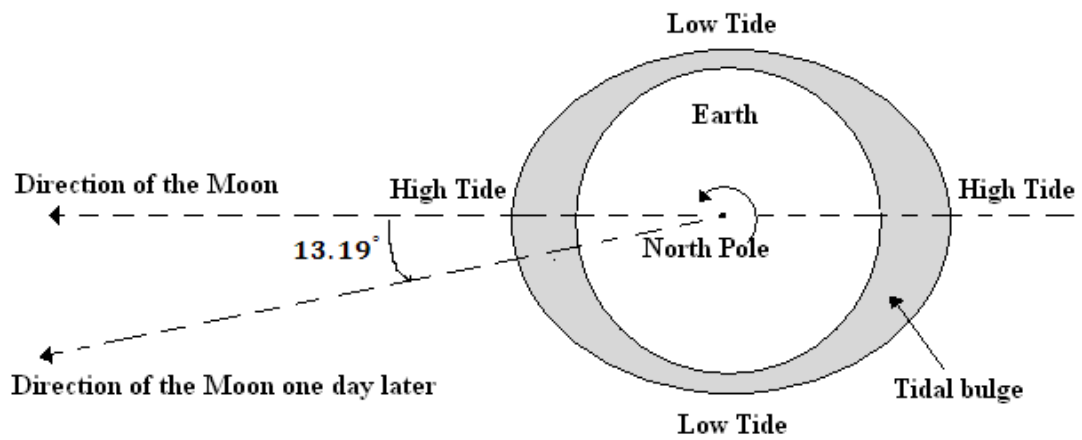


Figure 4.2: Displacement in the position of the Moon after 24 Hrs [16]

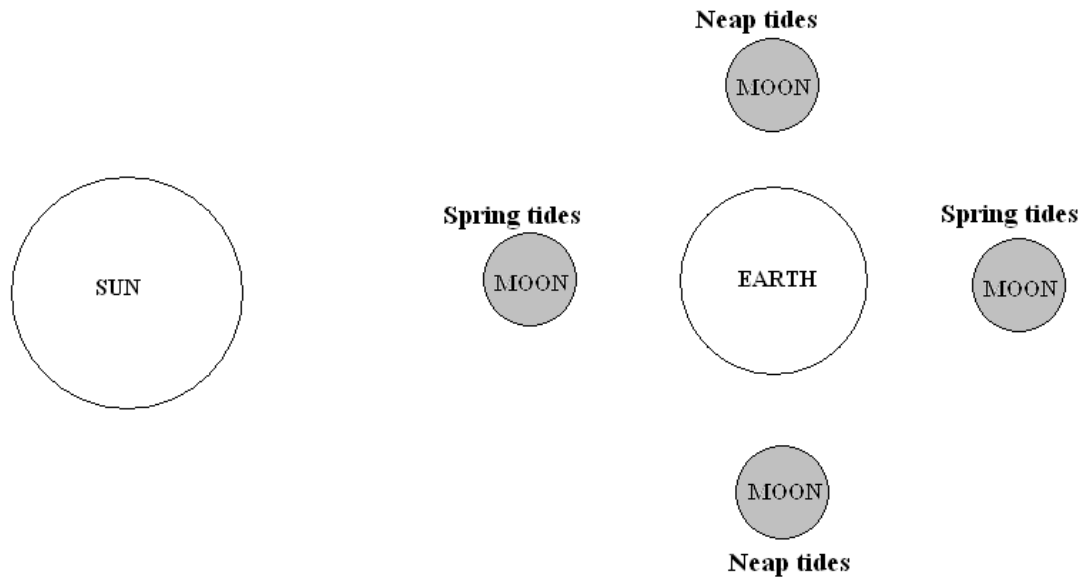


Figure 4.3: Spring and Neap tides [17]

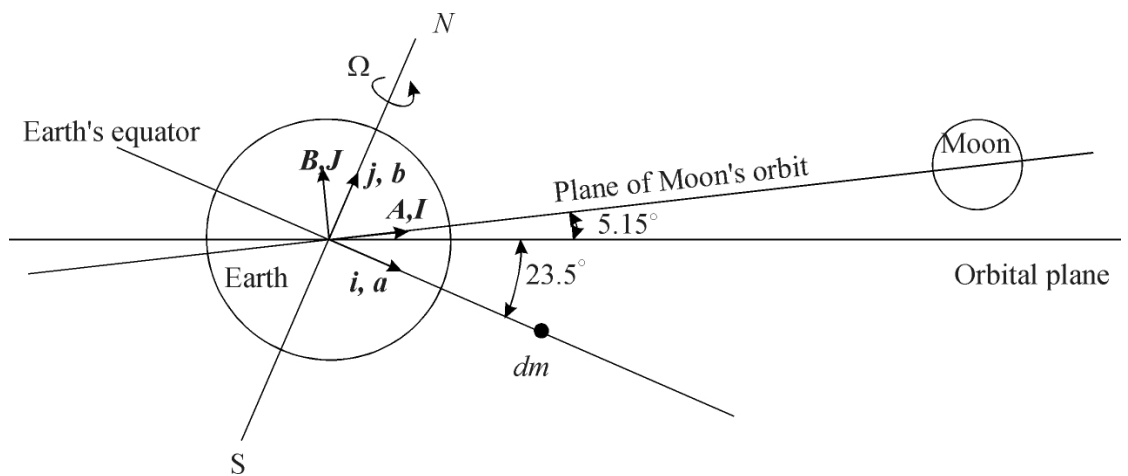


Figure 4.4: The Earth- Moon- Sun system

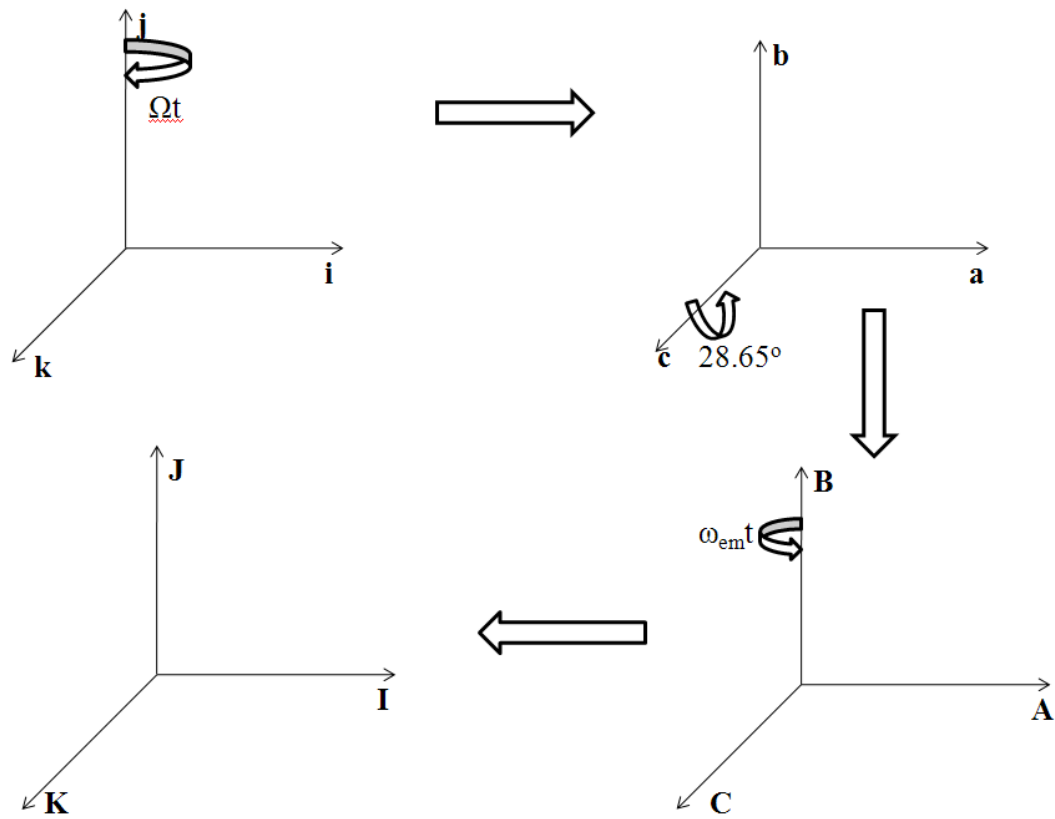


Figure 4.5: Relation between different reference frames

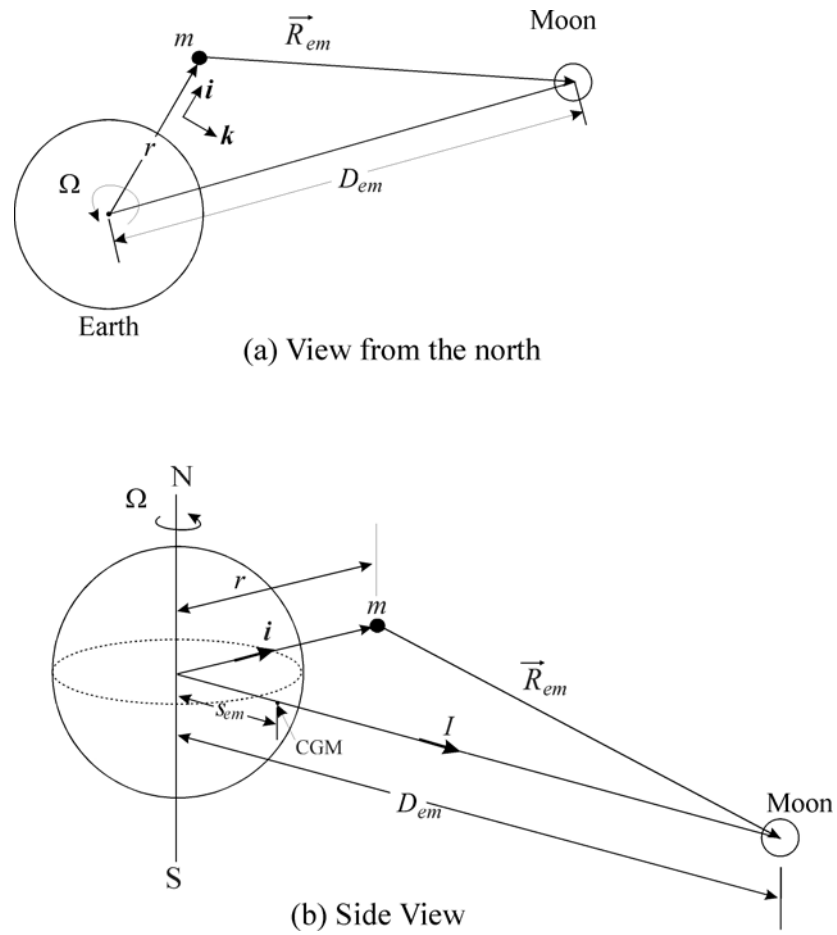


Figure 4.6: The Earth- Moon co-ordinate system including the space elevator

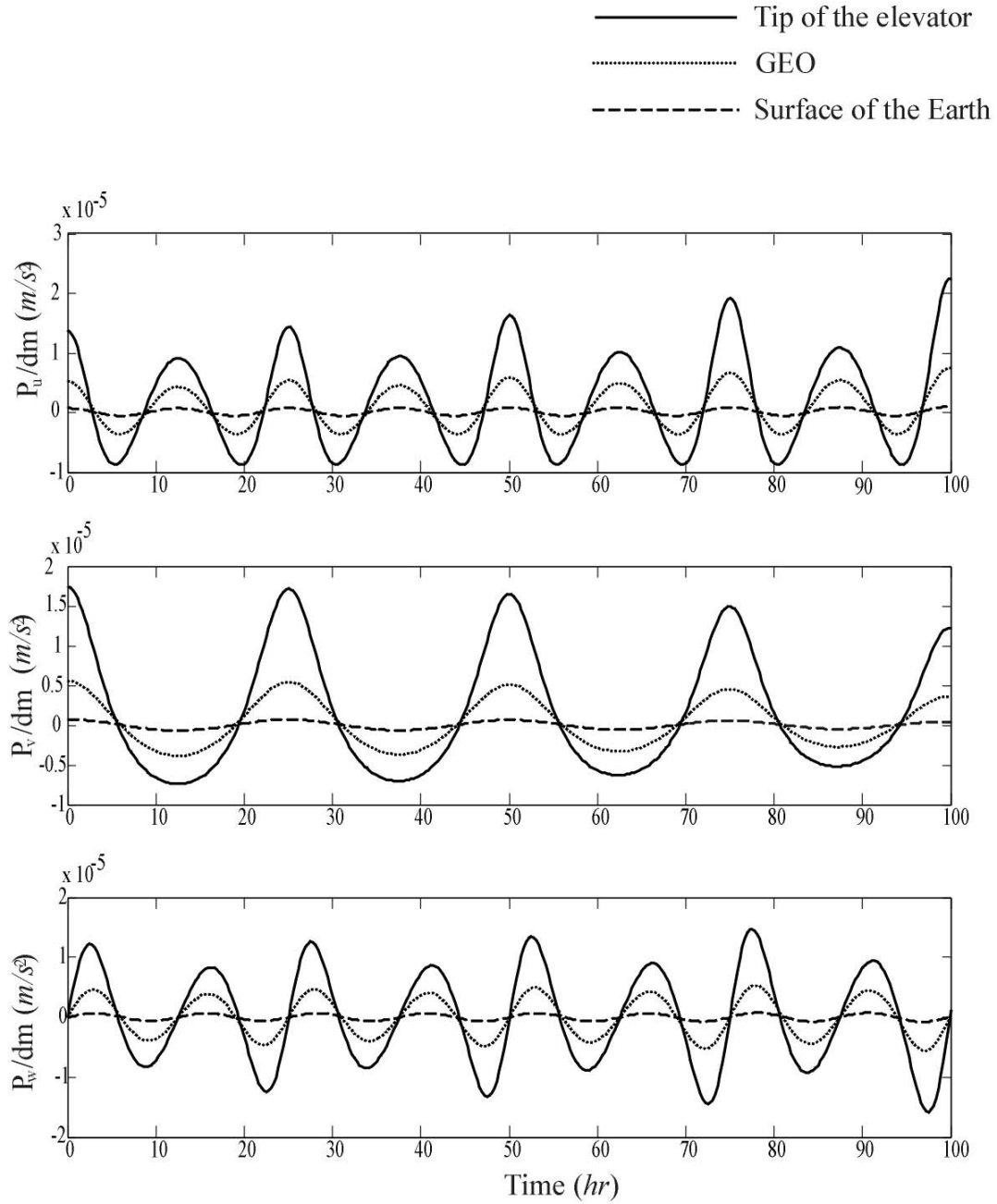


Figure 4.7: Differential accelerations at different positions of elevator due to the Moon

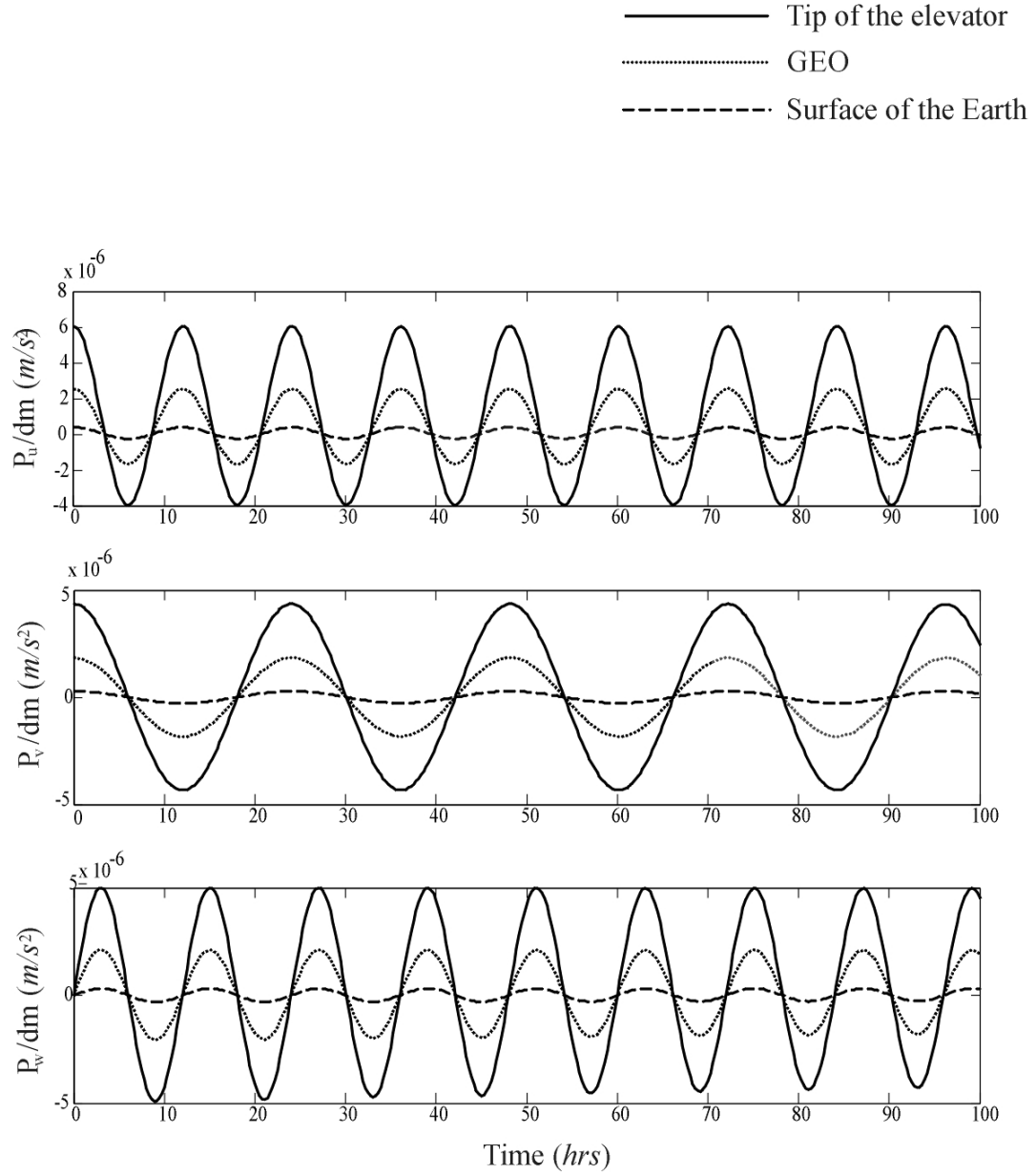


Figure 4.8: Differential accelerations at different positions of elevator due to the Sun

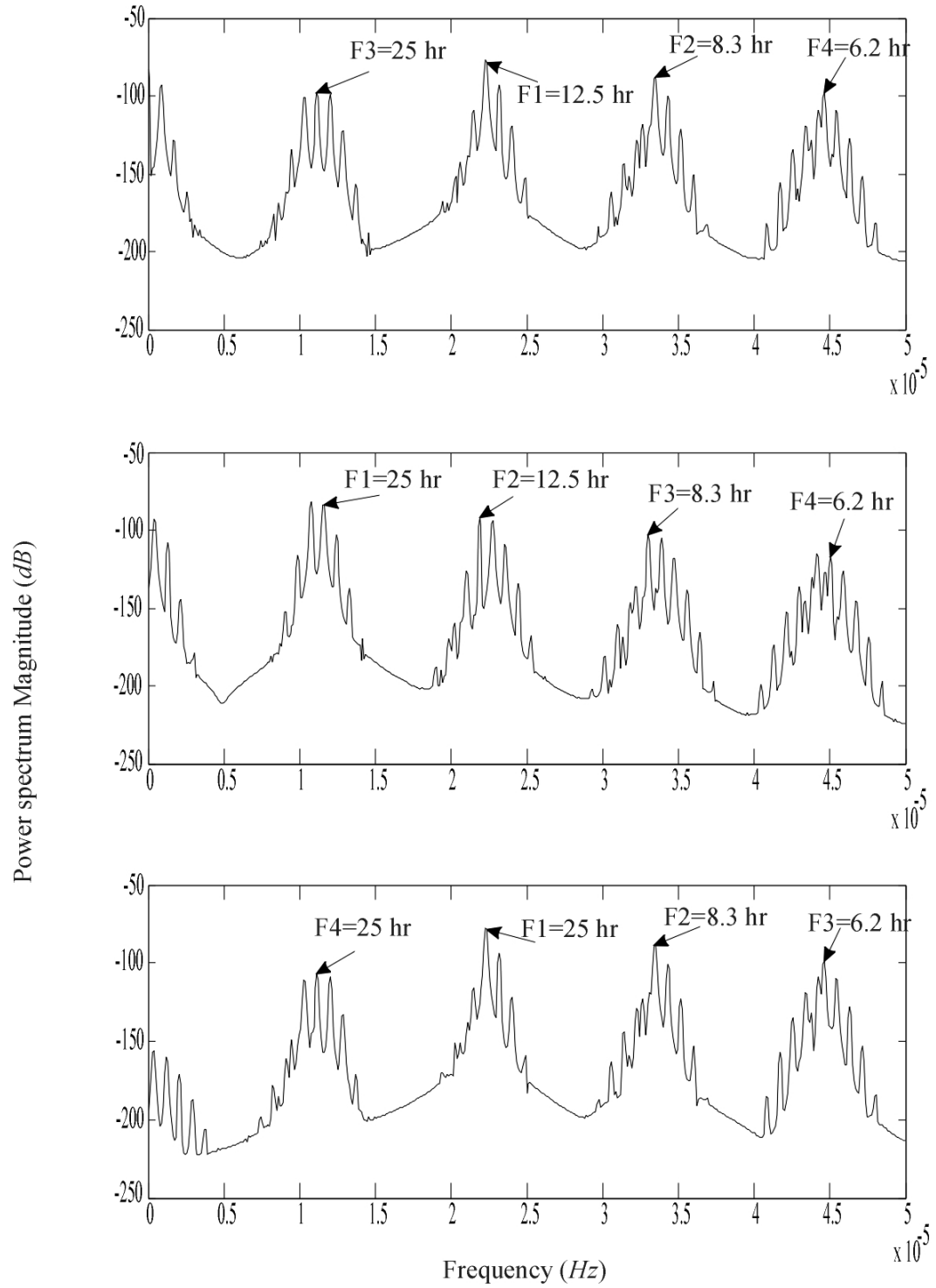


Figure 4.9: PSD plot of differential accelerations due to the Moon

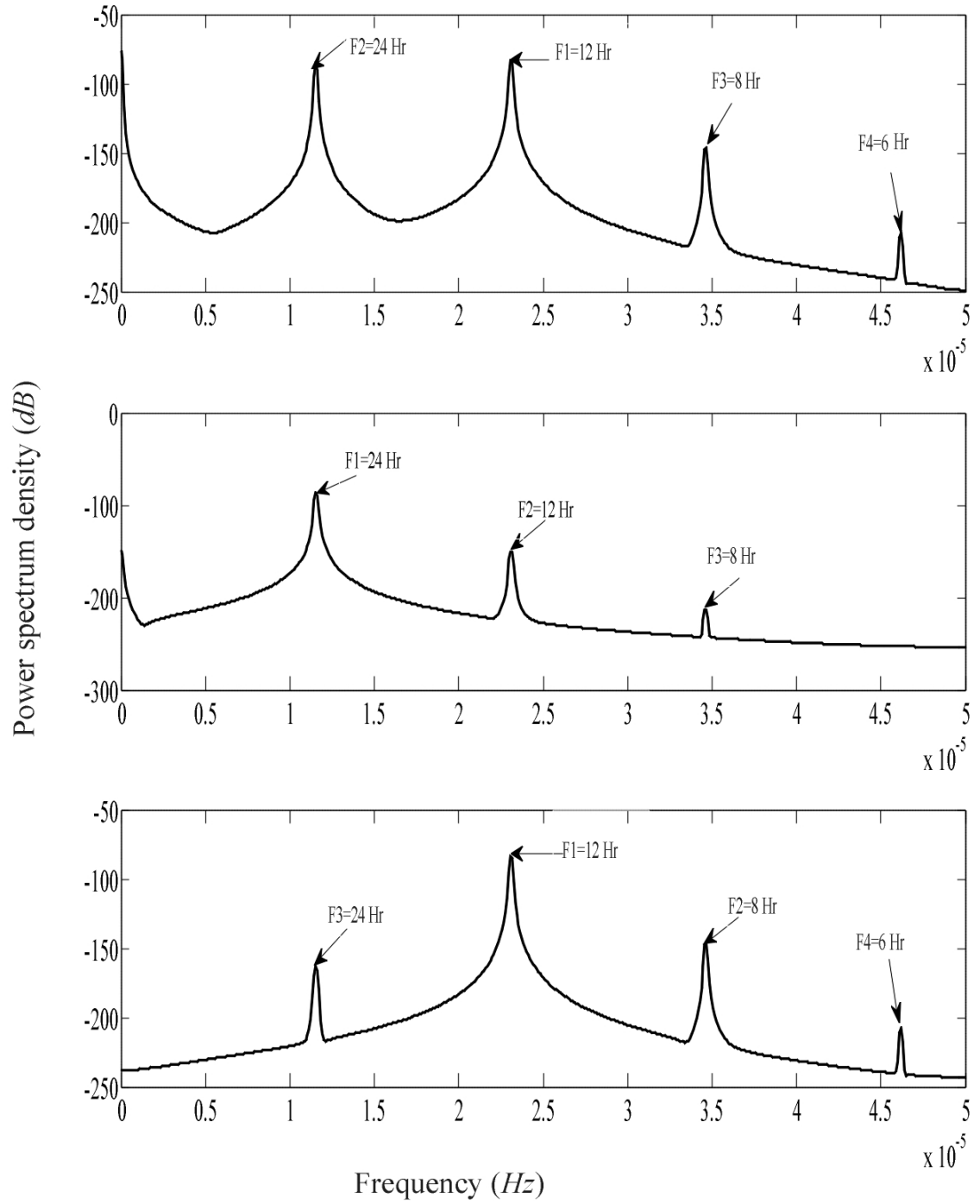


Figure 4.10: PSD plot of differential accelerations due to the Sun

CHAPTER 5

RESULTS AND DISCUSSION

5.1 Introduction

In this chapter, the response of the space elevator under the influence of tidal forces is calculated using linear and non-linear models. The equations of motion are given in Chapter 2 and 3. The finite difference method is used to calculate response of the space elevator under the influence of tidal forces. The detailed description of the method is given in Appendix B and D.

For the numerical results, we will use $E = 1 \text{ TPa}$ and $\rho = 1300 \text{ kg/m}^3$, where E is Young's modulus and ρ is the density. We will also use $\sigma = 60 \text{ GPa}$, which is within the experimentally measured tensile strength range for CNT [4, 5]. This value is significantly lower than its theoretical limit 300 GPa [4, 5]. This value results in $h = 4.7 \times 10^6 \text{ m}$. We will vary the length of the elevator for the numerical simulations.

5.2 Design of the Space Elevator

In this section, we will consider some design criterion for different lengths of the elevator. A particular range of length of the elevators that satisfy the particular design criterion will be suggested at the end of that criterion. In the following sections, we will develop some design criterions and come up with the optimum elevator design.

5.2.1 Maximum Amplitude Criterion

Let us consider responses at the point mass predicted by the linear models as first approximations. Recall that in this model the motion in the longitudinal and transverse directions are decoupled and the Coriolis and the centripetal accelerations are neglected. Figure 5.1 and Figure 5.2 shows the longitudinal and transverse displacements of the tip of the elevator as a function of time for different elevator lengths. Figure 5.3 explains displacements of the elevator in three directions for the nonlinear coupled model of the

elevator. Note that the natural frequencies of the system vary with elevator lengths as shown in Figure 5.4 and Figure 5.5. In general, the response amplitude will grow as one of the natural frequencies of the system approach one of the forcing frequencies. When the natural frequency exactly matches the forcing frequency we have resonance where the response grows linearly with time without a bound. However, in this case, it is not easy to predict the amplitude increase because one of the natural frequency may be approaching one of the forcing frequency while another natural frequency may be getting further away from some other forcing frequency. It is cumbersome to numerically simulate the response for every elevator length to obtain the response amplitudes. The upper bound on the response amplitudes can be obtained analytically using modal analysis.

The mathematical details of the modal analysis are given in Appendix C. The modal analysis method is used to find the maximum displacements for different elevator lengths. The maximum displacement is given by the sum of the absolute values of the amplitudes of the modal displacements. This can only be the upper bound because as we know that the amplitude of $\sin(t) + 2\sin(2t)$ is less than the sum of the amplitudes or 3.

Figure 5.6 represents the maximum displacements of elevator for different lengths of the elevator. Longitudinal displacements for 53,050 *km* and 56,750 *km* are very high as the natural periods are close to the forcing periods of 6.2 *hr* and 6 *hr*. Similarly, there is a sudden raise in transverse displacements for elevators of length 54,750 *km* and 59150 *km*, as their natural periods are close to the forcing periods of 25 *hr* and 24 *hr*. The second transverse natural period coincides with the tidal frequencies at 6 *hr* and 6.2 *hr* when the lengths of the elevator are 74,400 *km* and 78500 *km*, respectively. The resonance at these elevator lengths are shown in Figure 5.6 for transverse displacement. Note that the resonance at the higher mode has sharper peaks affecting smaller range of *L*. That is, it is easier to avoid the resonance at the higher mode by adjusting the elevator length slightly. It is interesting to note that both the longitudinal and the transverse amplitudes increase with increasing length. This agrees with the trend shown in the response plots in Figure

5.1 and 5.2. Once we decide the acceptable response amplitudes, we can deduce the range of acceptable length of elevator for the design purposes.

5.2.2 Point Mass Criterion

Figure 5.7 shows the point mass required for different lengths of the elevator so that the elevator will be in tension. The point mass is used to shorten the length of the elevator. The point mass to the cable mass ratio decreases with the increase in the length of the elevator.

- The point mass required for lengths of the elevator from 35,786 *km* to 53,000 *km* is more than the mass of the cable. During the construction of the elevator, sending a counter mass weighing more than the mass of the cable to an altitude of more than 35,000 *km* is not possible. Therefore, we are not going to consider elevator lengths of below 53,000 *km* in our further designs.
- Even though point mass required for elevators of length more than 100,000 *km* is very low, constructing elevators of such a height is not practically feasible. The rate of decrease in the counter mass required for elevator lengths beyond 100,000 *km* is very small, so increasing the length of the elevator beyond that point is not recommended.

From point mass criterion, we have eliminated lengths of the elevator below 53,000 *km* and beyond 100,000 *km* from our design criterion. Therefore, we will consider lengths of the elevator between 53,000 *km* and 100,000 *km* in our next criterion.

5.2.3 Resonance

The forcing periods of the tidal forces due to the Moon and the Sun were calculated in Chapter 4. Tidal periods of the Moon are 6.2 *hr*, 8.3 *hr*, 12.5 *hr* and 25 *hr*. Tidal periods of the Sun are 6 *hr*, 8 *hr*, 12 *hr* and 24 *hr*.

- The transverse natural periods of first, second and third modes of vibration of the elevator are plotted in Figure 5.5 for different lengths of the elevator between 53,000 *km* and 100,000 *km*. Resonance will occur when the natural periods of the elevator are equal or close to the forcing periods of the tidal forces. The transverse natural periods of first mode of vibration for elevators of length between 53,000 *km* and 63,000 *km* are around 24 *hr* and 25 *hr*, which are the first forcing periods of tidal forces due to the Sun and the Moon respectively.
- The transverse natural periods of second mode of vibration for elevators of length between 68,000 *km* and 85,000 *km* are close to the forcing periods of tidal forces due to the Sun (6 *hr*) and the Moon (6.2 *hr*). These regions are indicated by a rectangle in the figure.
- The longitudinal natural periods of first mode of vibration are close to the forcing periods of the tidal forces due to the Sun and the Moon for elevator lengths between 53,000 *km* and 60,000 *km* as shown in Figure 5.4

From above discussions, we can eliminate lengths of the elevator between 53,000 *km* and 63,000 *km* and 68,000 *km* and 85,000 *km* from our design criterion. Therefore, we will consider lengths of the elevator between 63,000 *km* and 68,000 *km* and 85,000 *km* and 100,000 *km* in our next criterion.

To illustrate the difference between linear and nonlinear models of the elevator, let us consider an elevator of length 82,000 *km* and plot the PSD of their responses. From the PSD plots of response in the axial directions for linear and nonlinear models shown in Figure 5.8, we can observe that both of them have peaks in all the forcing frequency regions. From the nonlinear PSD plot, we can see that there is a peak at 4.63 *hr* indicating that it is the natural period of axial vibration. But the natural period for linear model is 5.3 *hr*, so there is a shift in the natural period of the nonlinear model from the linear model. Figure 5.9 and Figure 5.10 are linear and nonlinear PSD comparisons for the meridional and equatorial responses. From these plots we can observe that the natural

periods calculated from the linear model are consistent with those obtained from nonlinear model.

5.2.4 Dynamic Response of the Elevator

In this section we will consider two different design cases of the elevator such that they represent the different regions of lengths of the elevator mentioned in the last section and plot their responses and velocities. The two design cases are 66,000 *km* and 91,000 *km*

Case 1: Length of the Elevator is 66,000 *km*

Figure 5.11 and 5.12 show the position and velocity plots for the two-dimensional model of the elevator at the point mass. The maximum displacement of the point mass in the longitudinal direction is 120 *m* and maximum axial velocity of the point mass is 0.05 *m/s*. The maximum displacement of the point mass in the transverse direction is 6 *km* and the transverse velocity is 0.6 *m/s*.

Figure 5.13 and 5.14 shows the position and velocity plots predicted by the three-dimensional model of the elevator respectively. The maximum displacements of the elevator in axial, equatorial and meridional directions are 110 *m*, 10 *km* and 3 *km*, respectively. Maximum velocities of the elevator in axial, equatorial and meridional directions are 0.02 *m/s*, 0.5 *m/s* and 0.04 *m/s*, respectively. When we compare the responses obtained using the two models, we find that both models predict similar amplitudes for the longitudinal motion. However, the transverse amplitude differs by a factor of 10/6. This shows that the two-dimensional linear models which do not take Coriolis acceleration, centripetal acceleration, stretch in the elevator, and nonlinear coupling between the longitudinal and the transverse motions are not adequate in predicting the response of the space elevator. In order to accurately predict the transverse motion, the three-dimensional nonlinear modeling is inevitable.

Case 2: Length of the Elevator is 91,000 *km*

Figure 5.15 and 5.16 are the response and velocity plots for the two-dimensional model of the elevator. Maximum displacement of the elevator in the longitudinal direction is 230 m and maximum velocity of the elevator is 0.05 *m/s*. The maximum displacement of the elevator in the transverse direction is 8 *km* and the velocity of the elevator is 0.6 *m/s*.

Figure 5.17 and 5.18 shows the response and velocity plots for nonlinearly coupled three-dimensional model of the elevator, respectively. The maximum displacements of the elevator in axial, equatorial and meridional directions are 230 *m*, 30 *km* and 6 *km*, respectively. The maximum velocities of the elevator in axial, equatorial and meridional directions are 0.05 *m/s*, 1.5*m/s* and 0.1 *m/s*, respectively. The transverse amplitude of 30 *km* may seem large. However, this is very small compared to the total length of the cable. In addition, the maximum velocity that corresponds to this motion is small enough to permit necessary operations on the elevator.

Again, we note that the transverse amplitudes, after taking into account the Coriolis acceleration, centripetal acceleration, nonlinear coupling between the transverse and the longitudinal motion, the stretch in the elevator, is about 30/8 times the one predicted by the simple linear model. Therefore, the linear model should not be used when more accurate dynamic results are required.

5.2.5 Accessible Destinations

A space elevator is used to reach the destinations within the Earth orbits as well as the solar orbits. Destinations within the Earth orbits are easily reachable as they are thousands of kilometers away from the Earth. Space elevator acts as a sling to launch payloads to destinations which are trillions of kilometers away. Edwards [1] found a relation between the length of the elevator and the distance up to which it can launch the payloads. Figure 5.19 explains the relationship between solar orbits accessible and

elevator length. As the length of the elevator increases, the range of accessible solar orbits is increasing.

From the discussion in Section 5.1.4, we found that the displacements and velocities of the elevator for design case 1 are half of those for case 2 and length of the elevator in case 2 is 27,000 *km* longer than the one in the case 1. From Figure 5.19 we can see that the elevator in case 1 can only be used for transportation of payloads within the Earth orbits where as the elevator in case 2 can launch payloads up to solar orbits. From Figure 5.19, we see that the elevator in case 2 (91,000 *km* long) can deliver payloads to Venus, Mars and Jupiter. Even though case 2 has higher displacements, its accessibility in space makes it a very useful design. Therefore, depending on the type of destinations that should be reached either of the designs which satisfy that particular requirement should be selected.

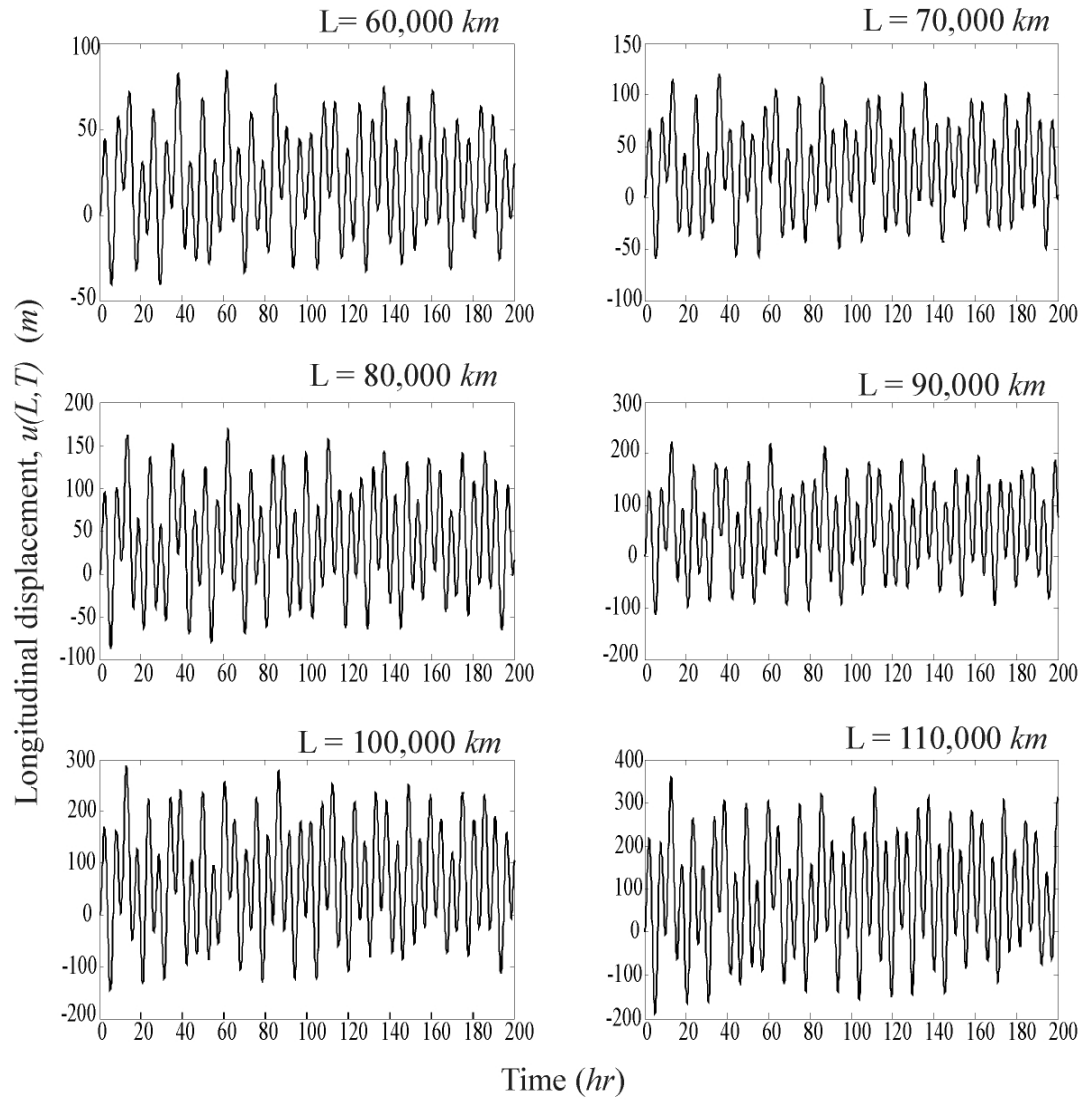


Figure 5.1: Longitudinal displacements of the pointmass for various lengths of the elevator for linear model

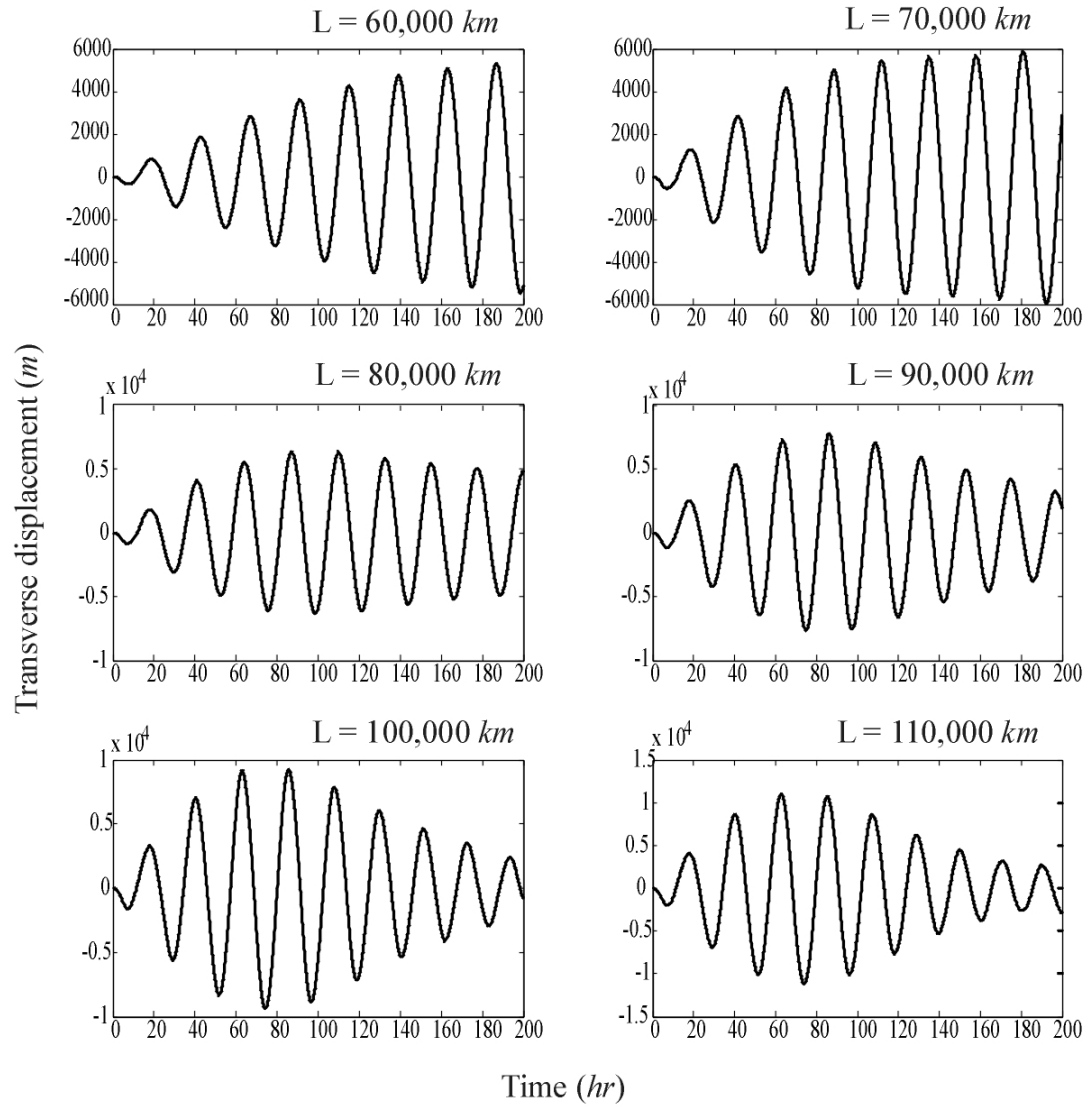


Figure 5.2: Transverse displacements of the poinmass for various lengths of the elevator for linear model

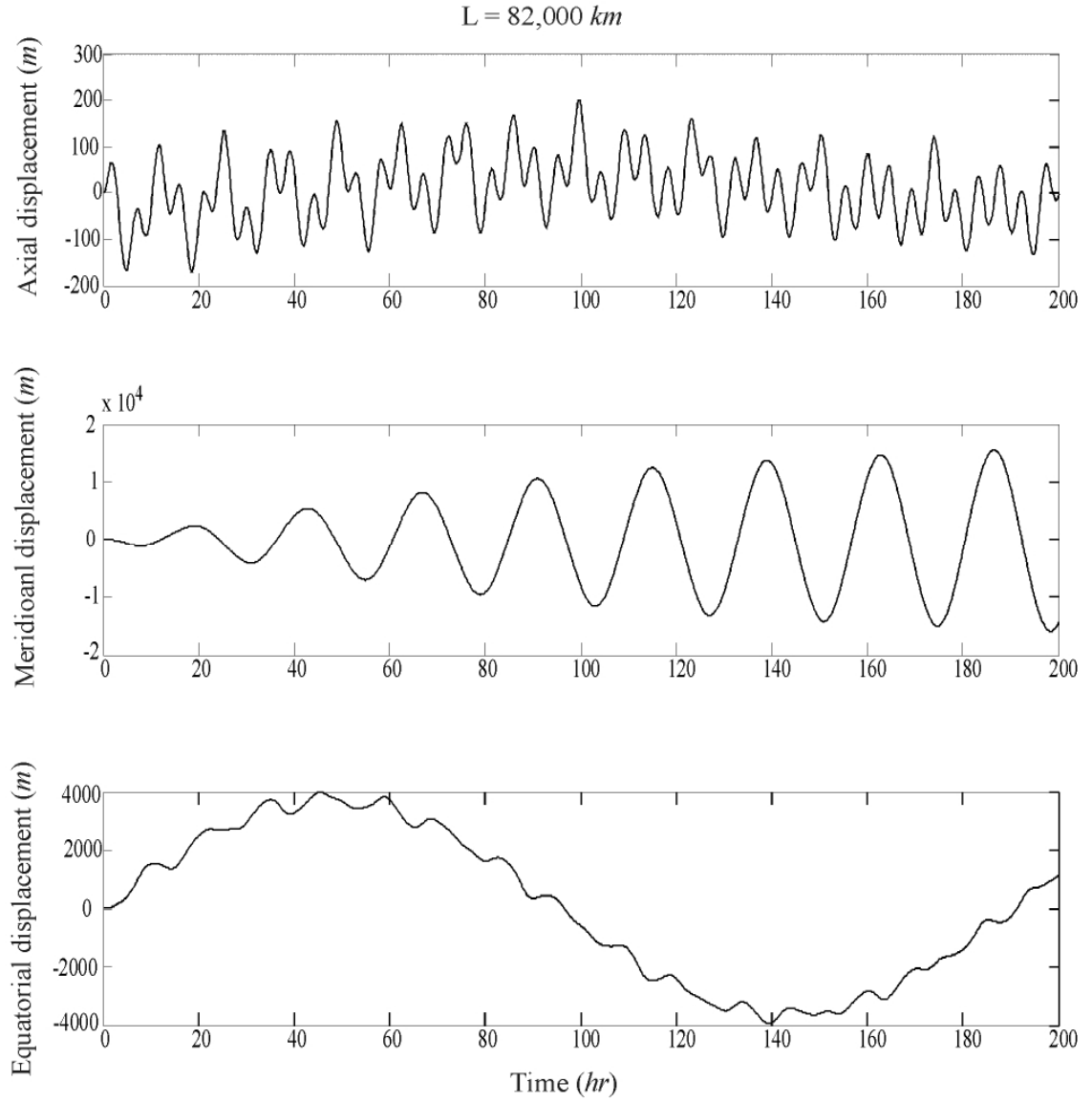


Figure 5.3: Displacements in the axial, meridional, and equatorial directions of the pointmass for nonlinear model

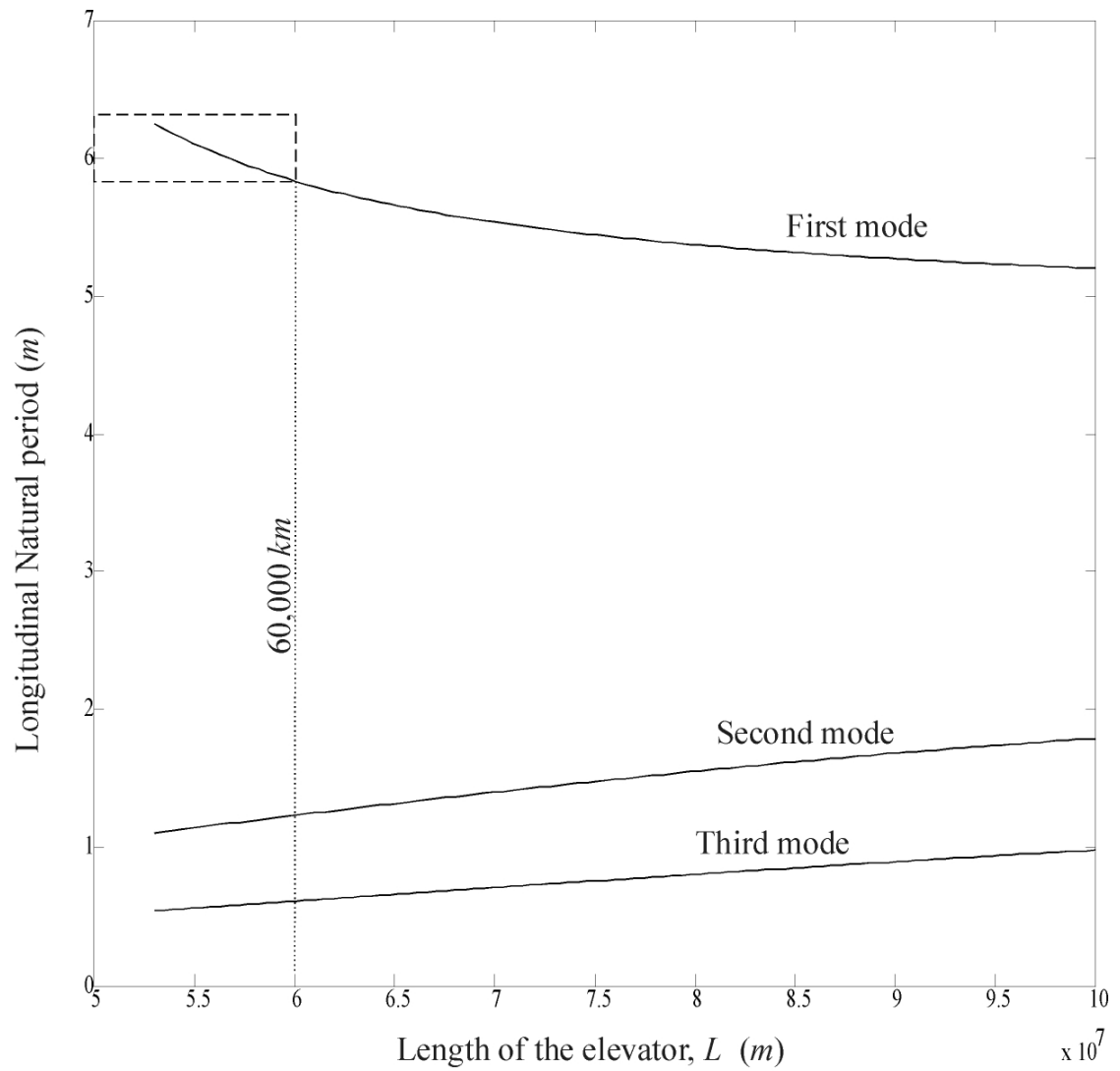


Figure 5.4: Longitudinal natural periods of first, second and third modes of vibration predicted by the linear models for different lengths of the elevator indicating the regions of possible resonance

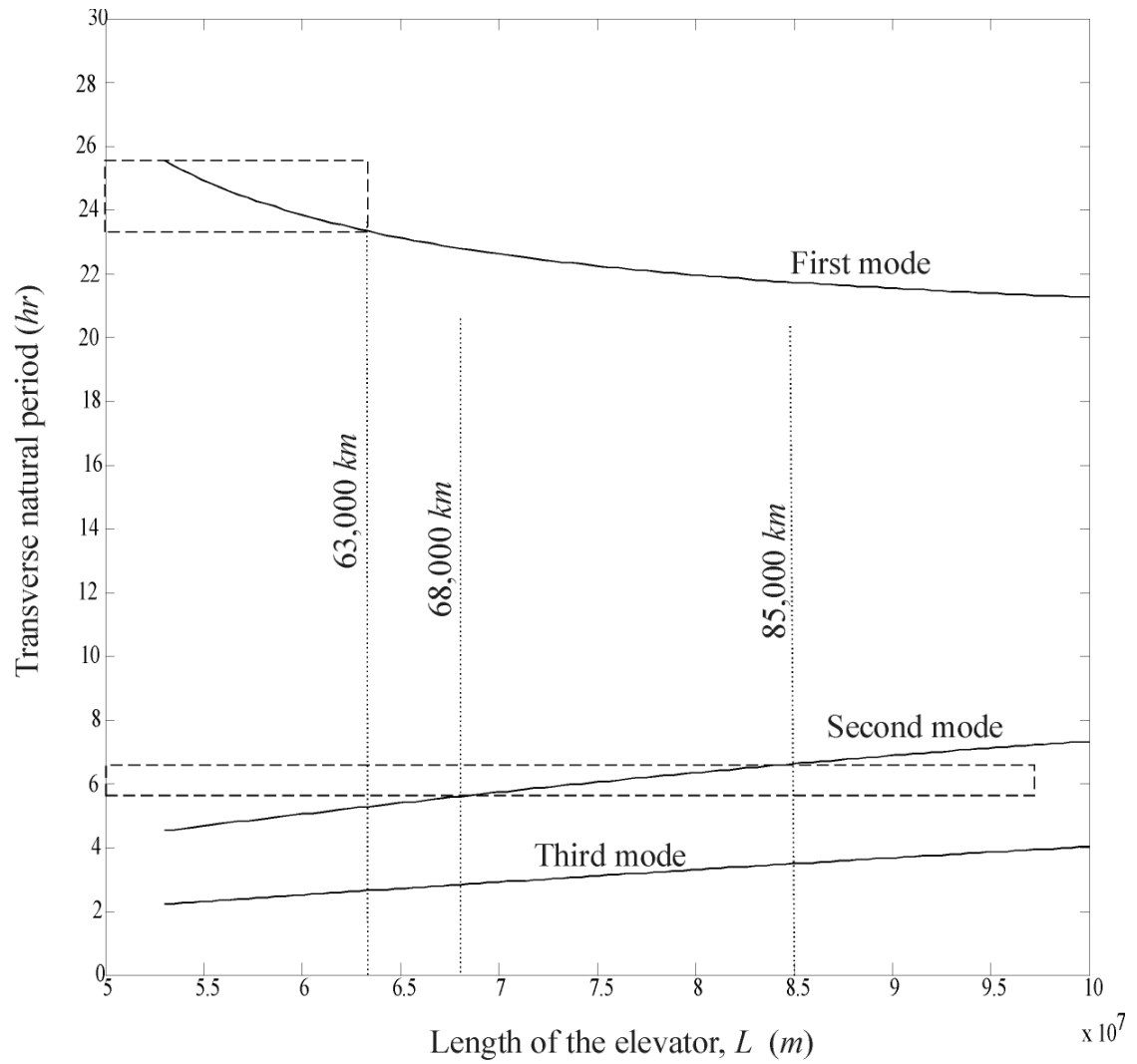


Figure 5.5: Transverse natural periods of first, second and third modes of vibration predicted by the linear models for different lengths of the elevator indicating the regions of possible resonance

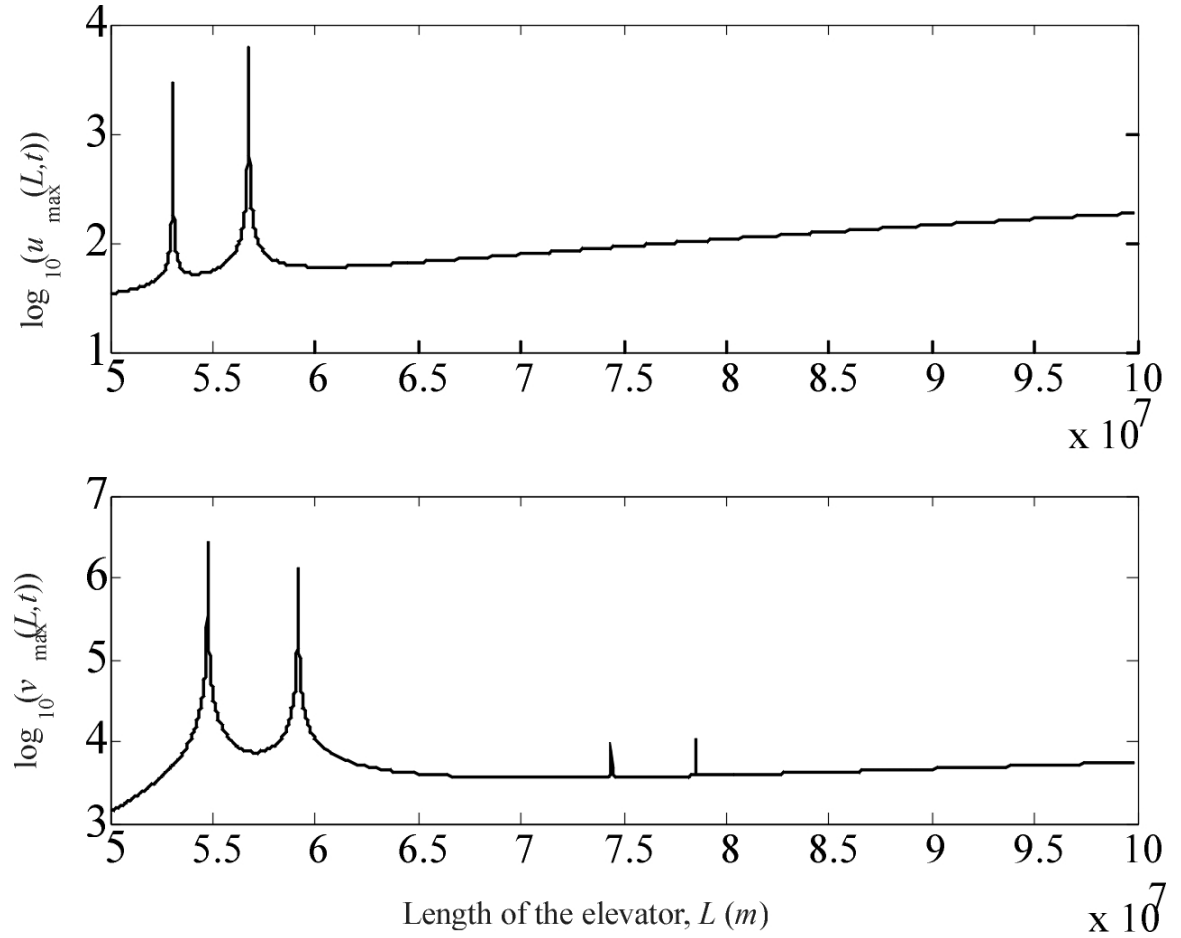


Figure 5.6: Maximum displacements of the pointmass in the longitudinal and transverse directions for various lengths of the elevator

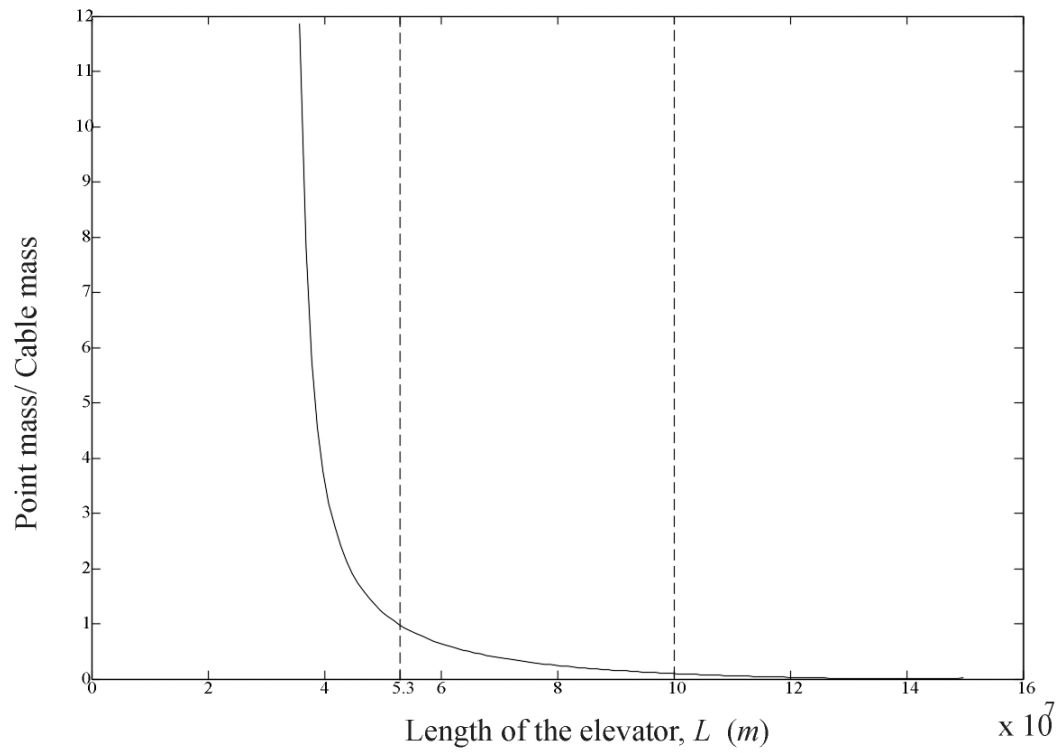


Figure 5.7: Point mass over cable mass ratio for various lengths of the elevator

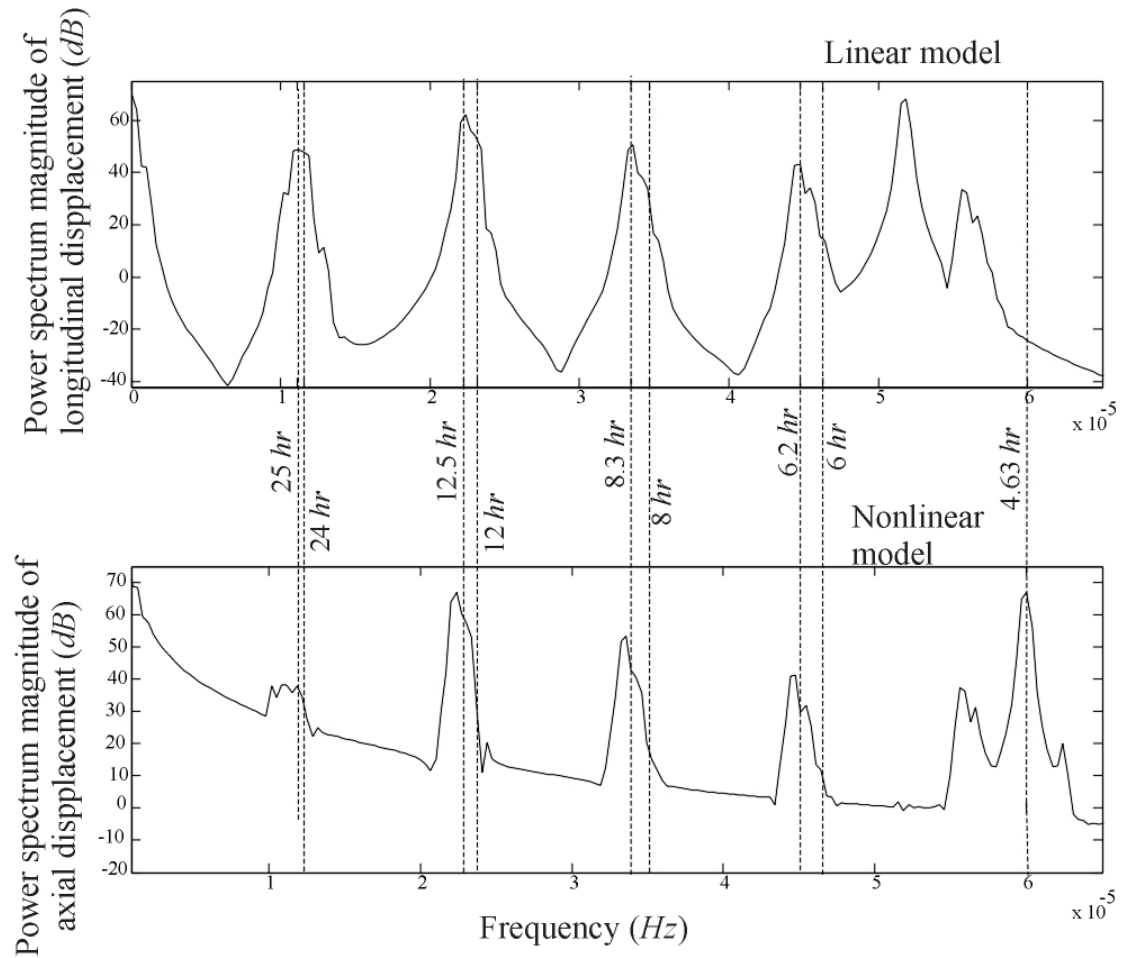


Figure 5.8: PSD plots for the longitudinal response of the pointmass predicted by the linear and nonlinear models

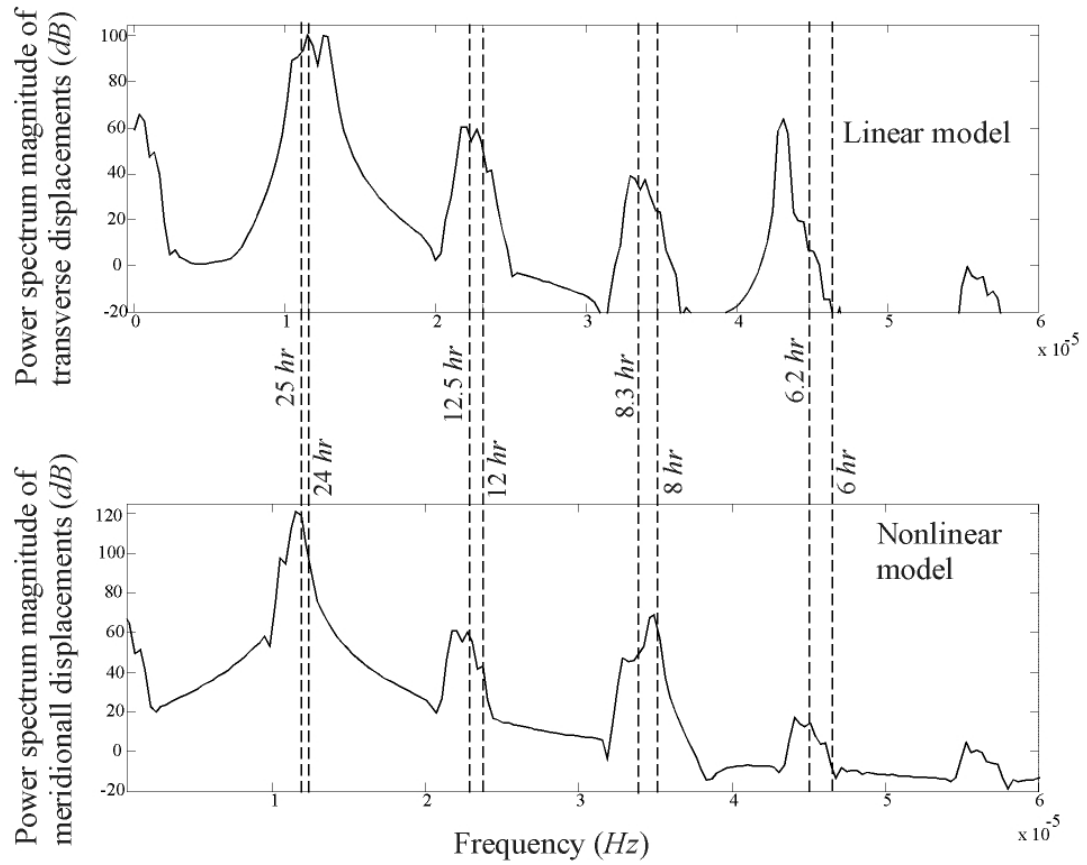


Figure 5.9: PSD plots for the transverse response of the pointmass predicted by the linear and nonlinear models

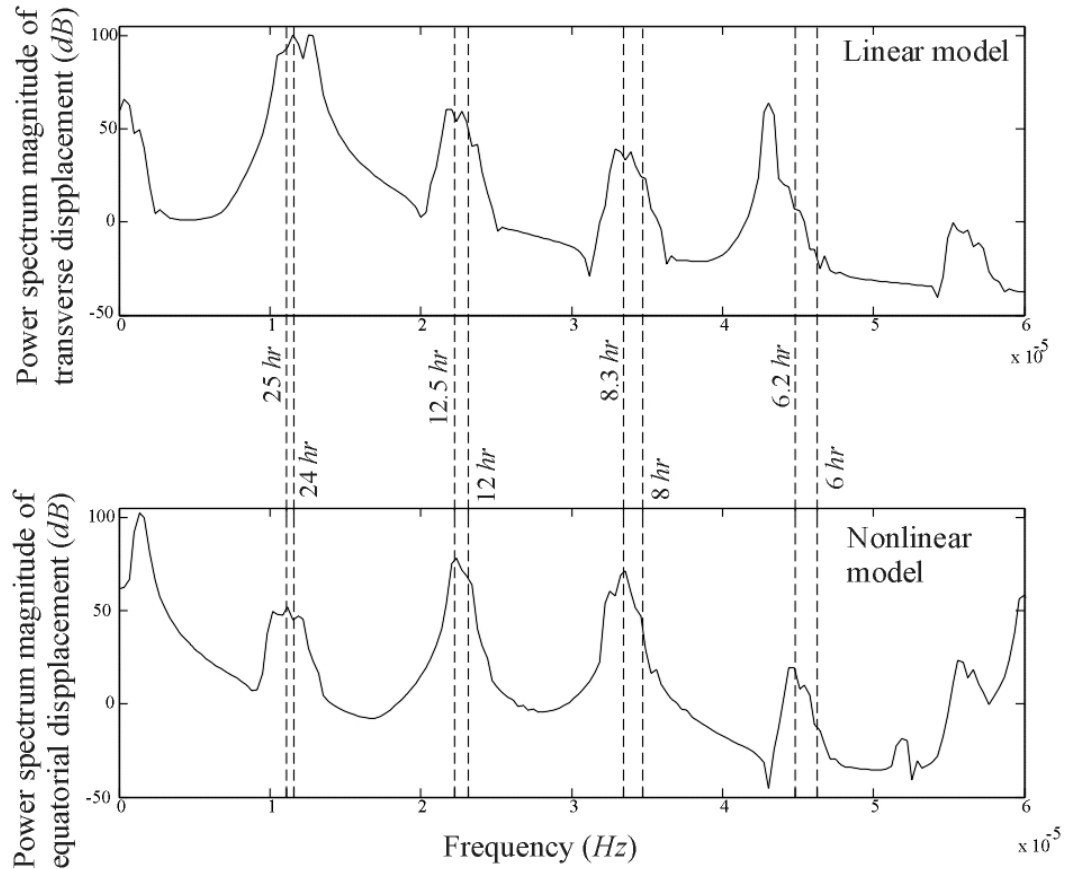


Figure 5.10: PSD plots for the transverse (equatorial) response of the pointmass predicted by the linear and nonlinear models

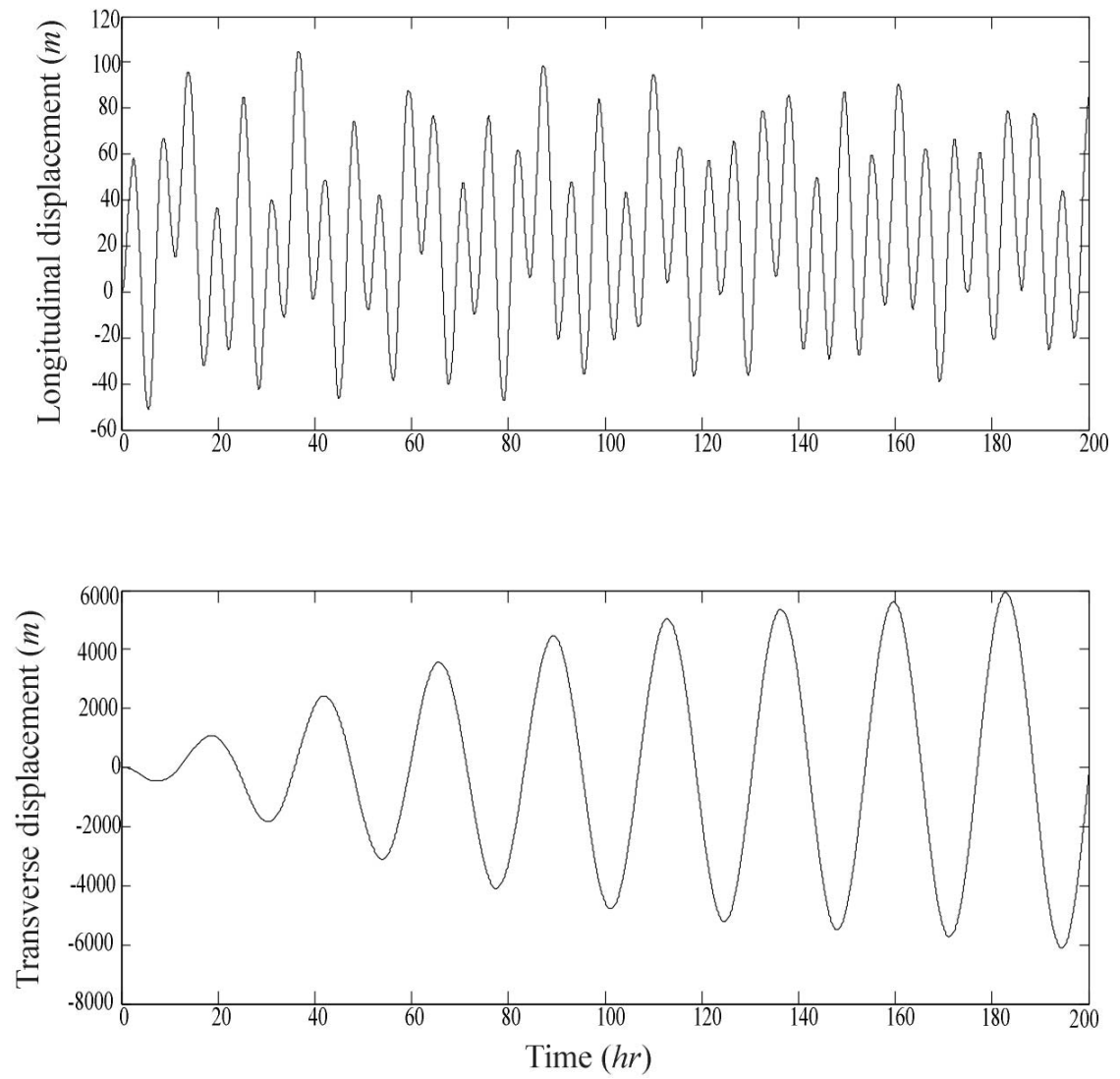


Figure 5.11: Displacements of the pointmass in the longitudinal and transverse directions for $L = 66,000 \text{ km}$ predicted by the linear model

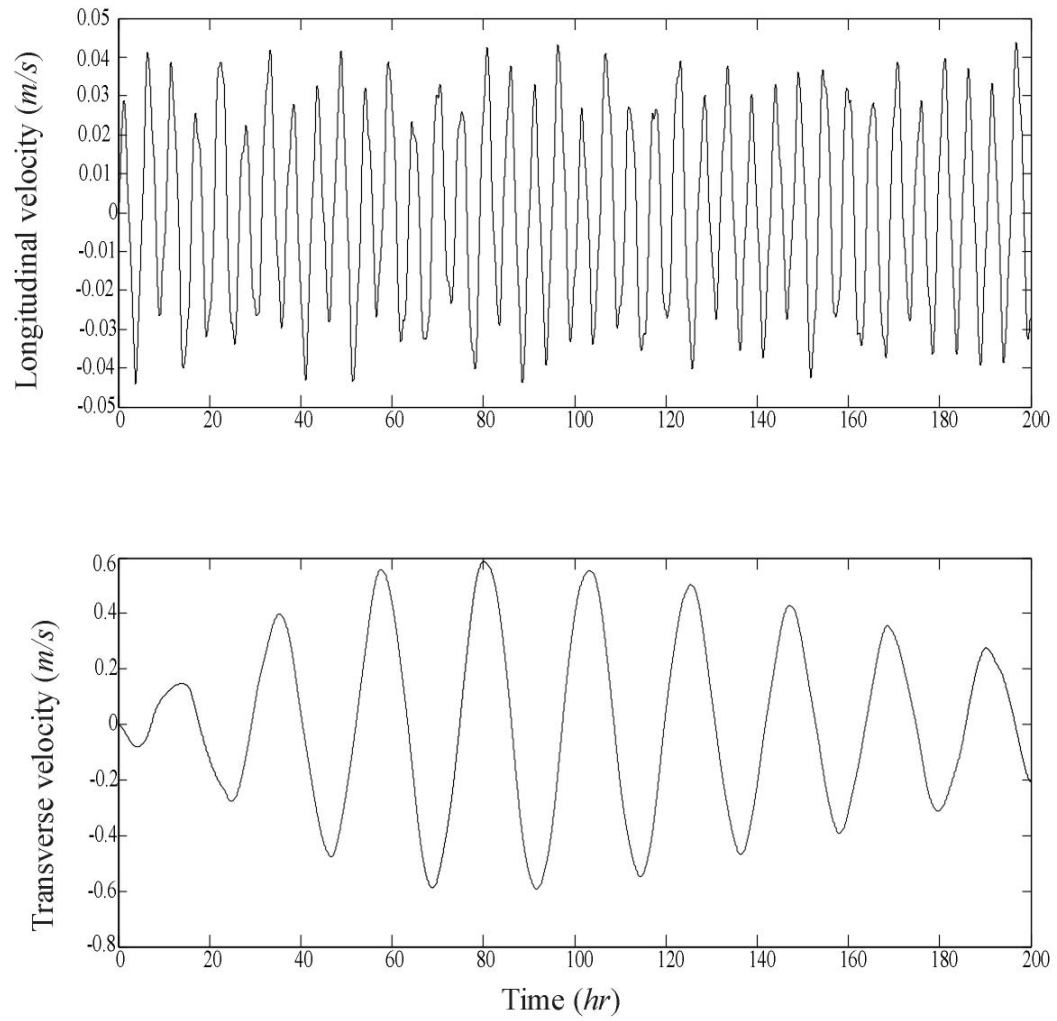


Figure 5.12: Velocities of the pointmass in the longitudinal and transverse directions for $L = 66,000 \text{ km}$ predicted by the linear model

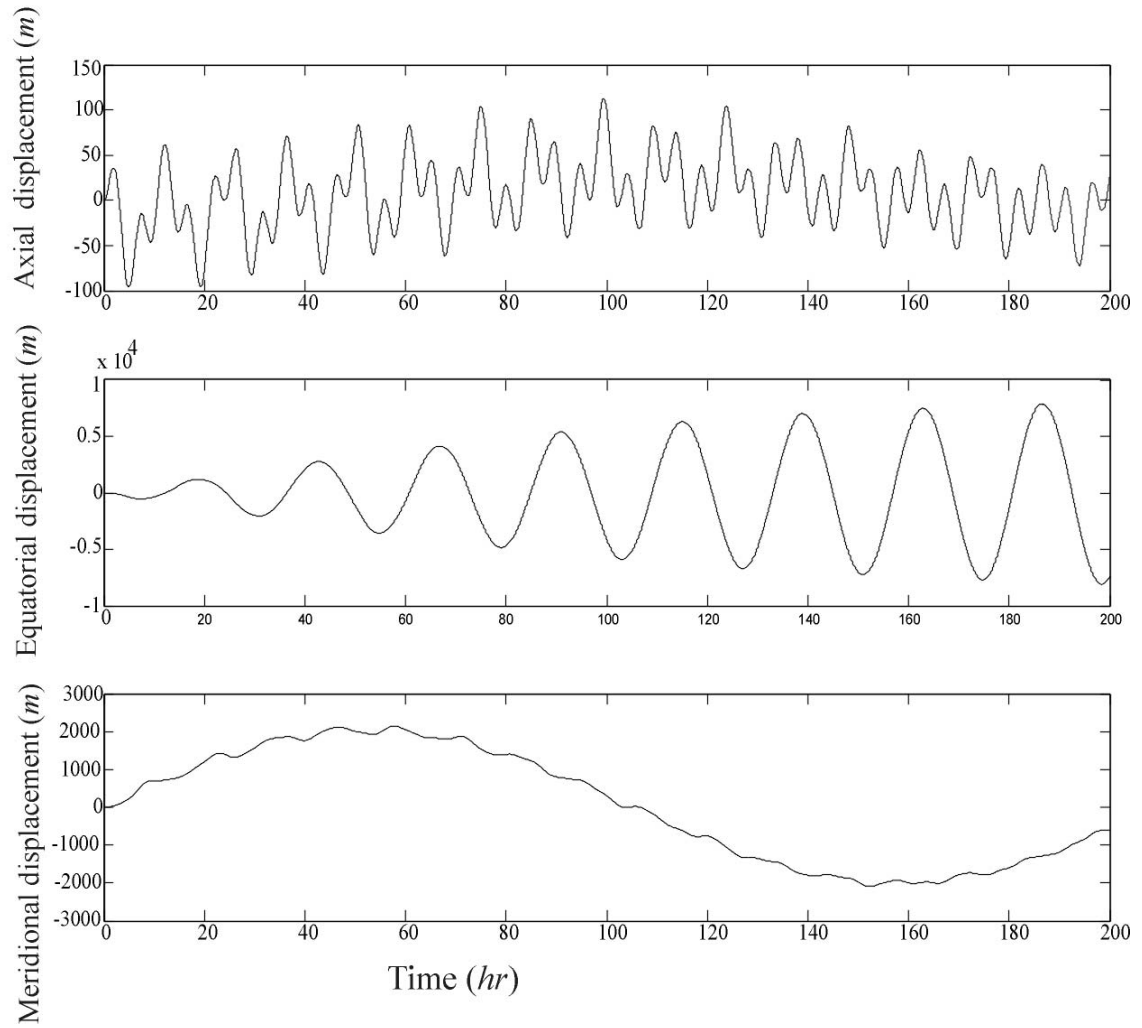


Figure 5.13: Displacements of the pointmass in the axial, equatorial and meridional directions for $L = 66,000 \text{ km}$ predicted by the nonlinear model

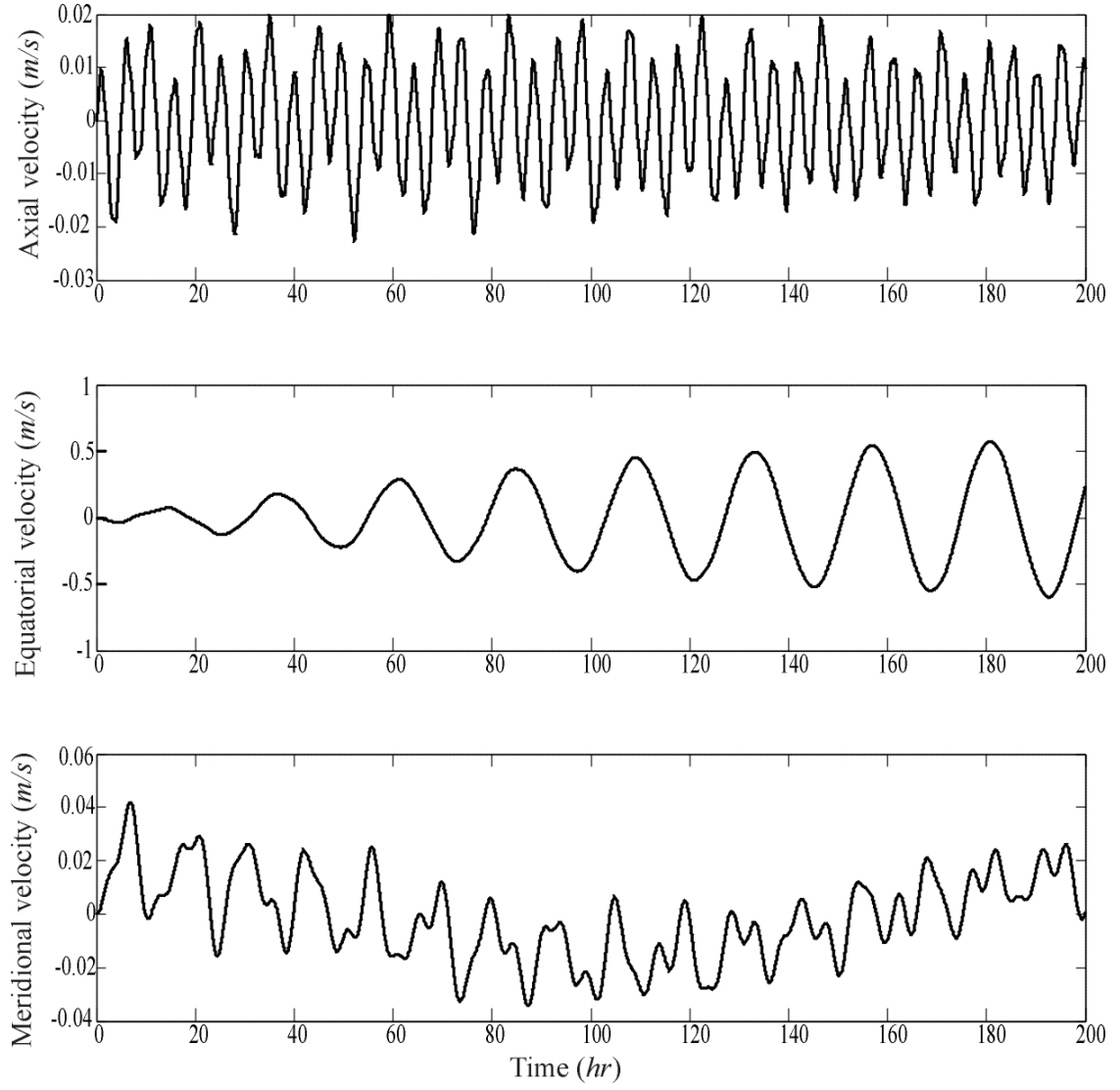


Figure 5.14: Velocities of the pointmass in the axial, equatorial and meridional directions for $L = 66,000 \text{ km}$ predicted by the nonlinear model

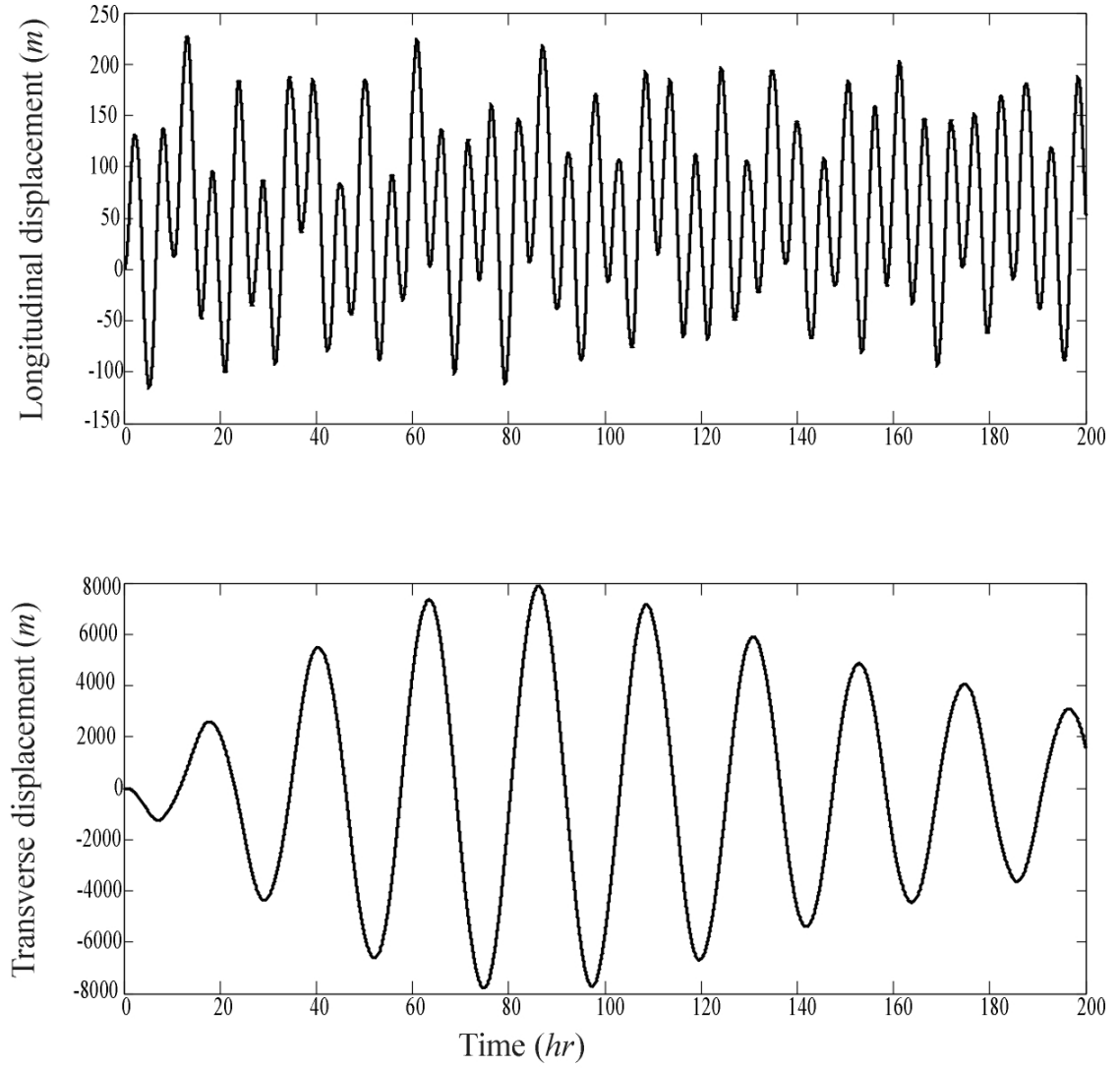


Figure 5.15: Displacements of the pointmass in the longitudinal and transverse directions for $L = 91,000 \text{ km}$ predicted by the linear model

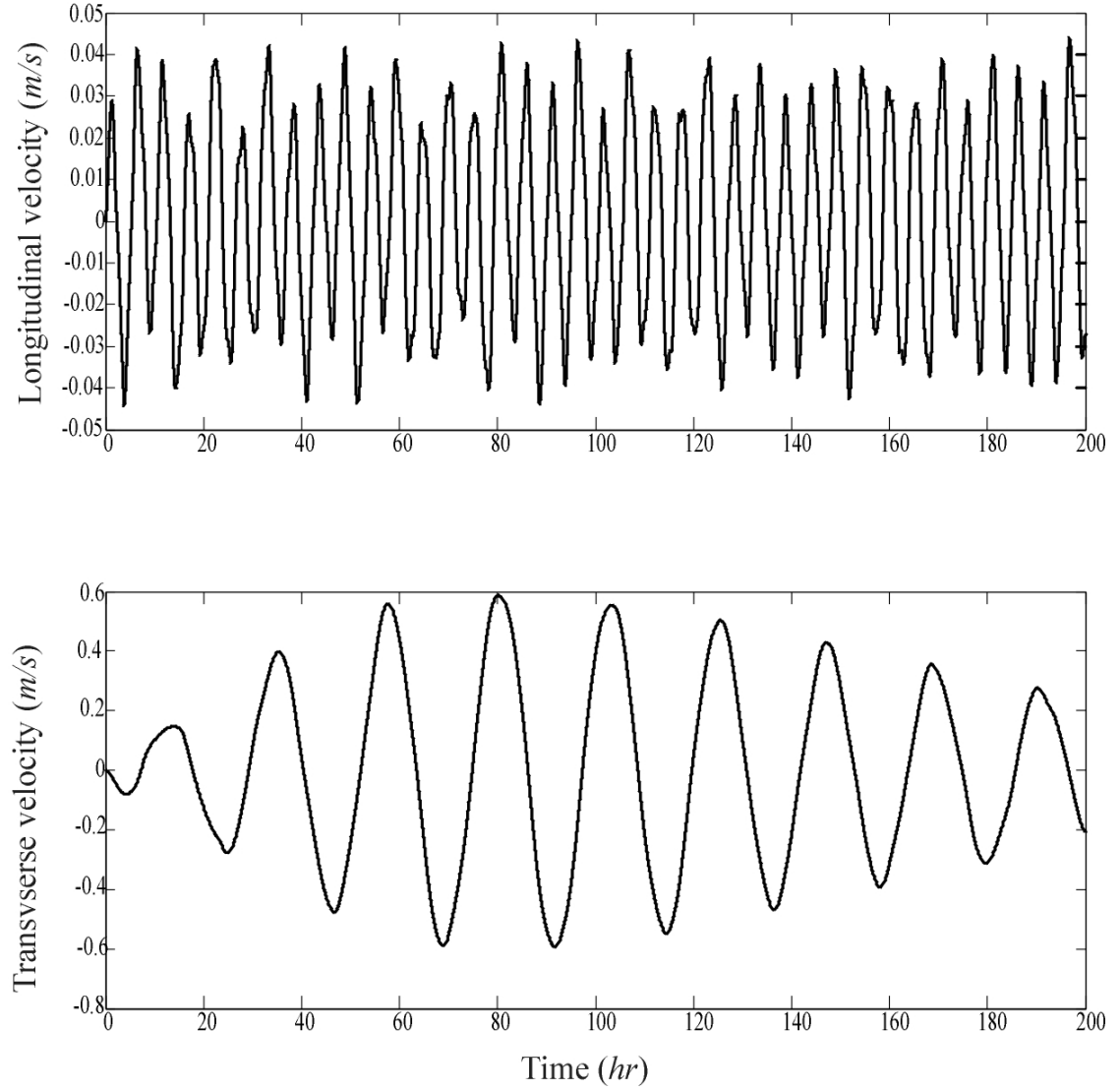


Figure 5.16: Velocities of the pointmass in the longitudinal and transverse directions for $L = 91,000 \text{ km}$ predicted by the linear model

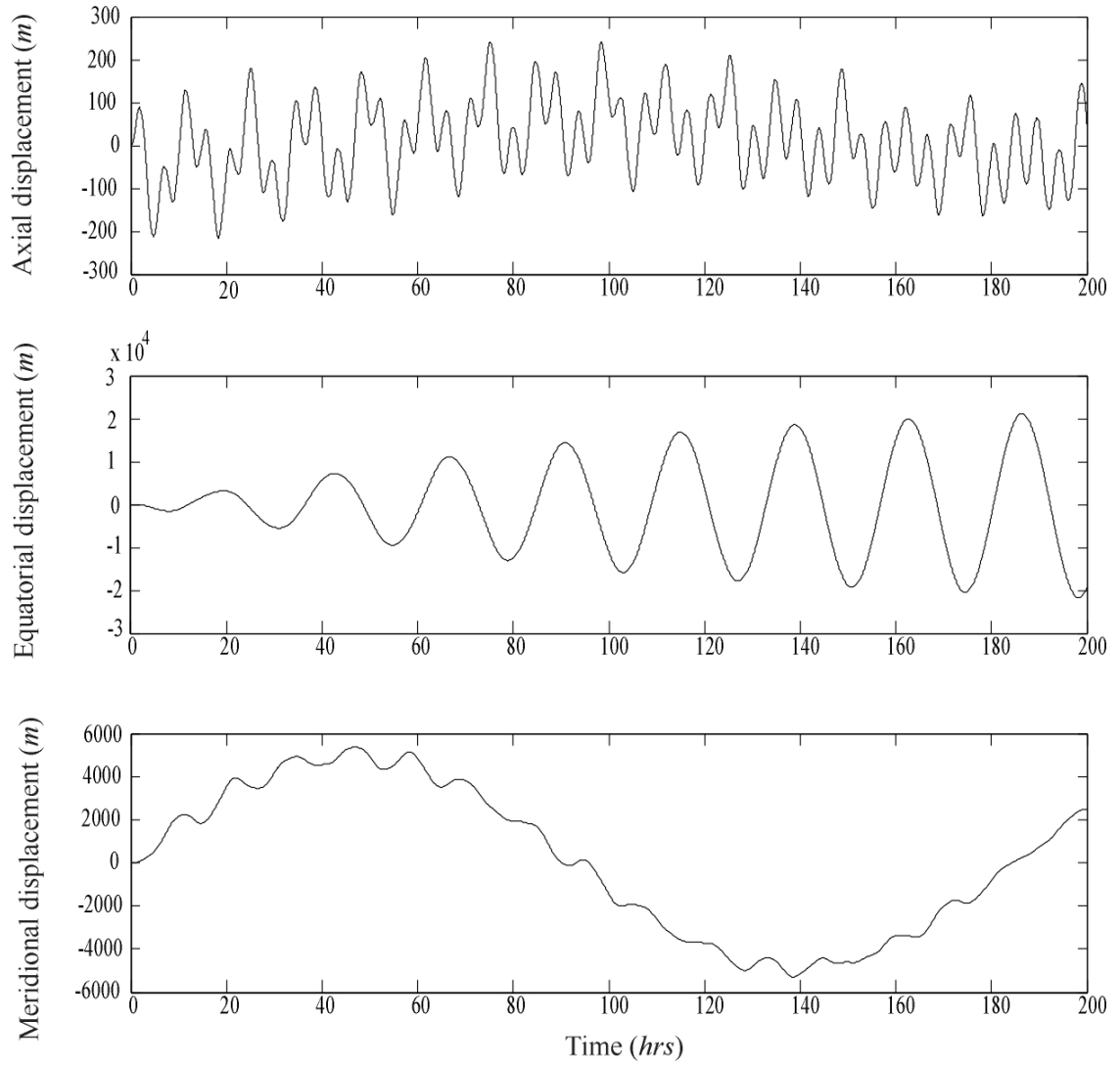


Figure 5.17: Displacements of the pointmass in the axial, equatorial and meridional directions for $L = 91,000 \text{ km}$ predicted by the nonlinear model

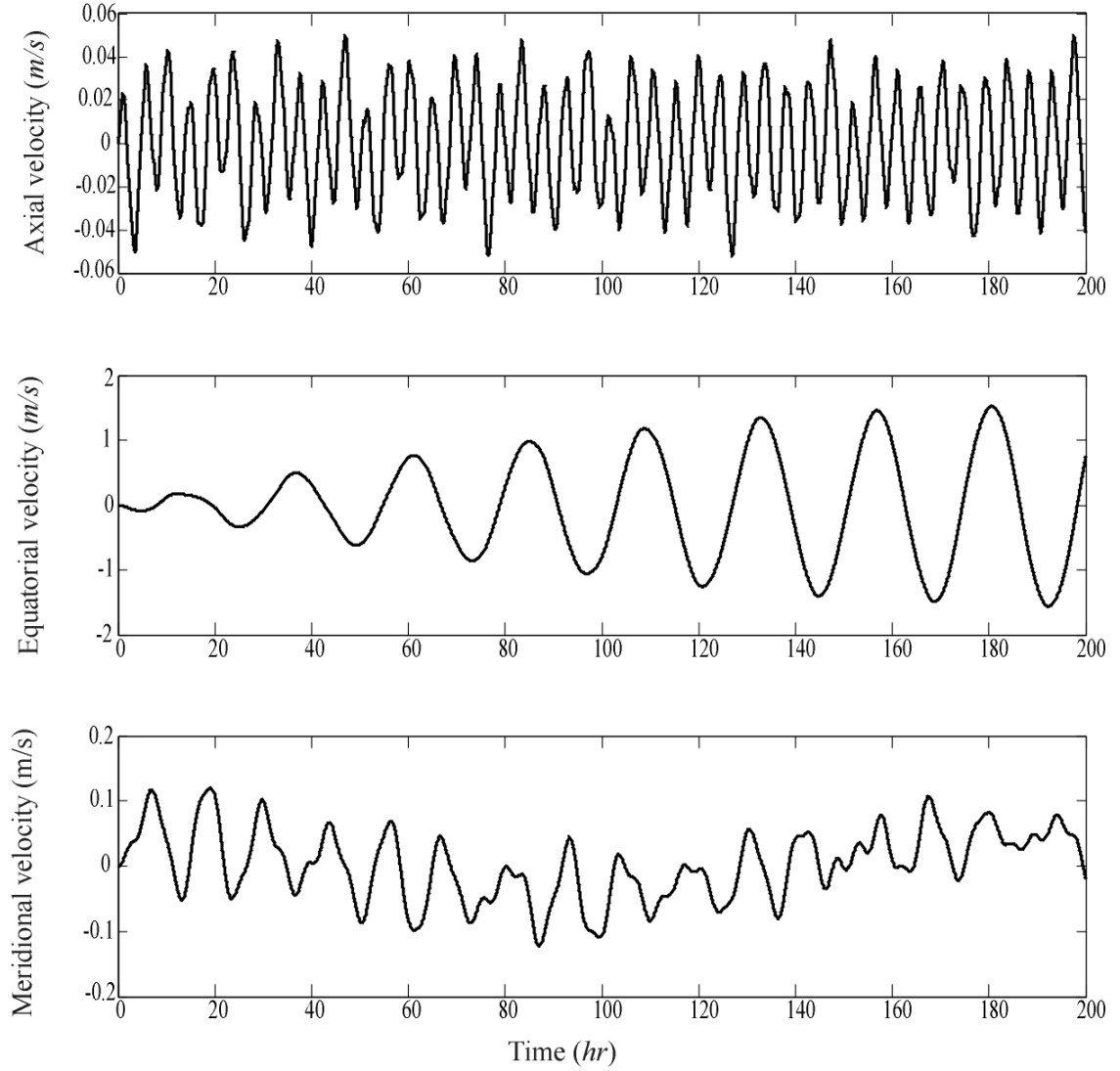


Figure 5.18: Velocities of the pointmass in the axial, equatorial and meridional directions for $L = 91,000 \text{ km}$ predicted by the nonlinear model

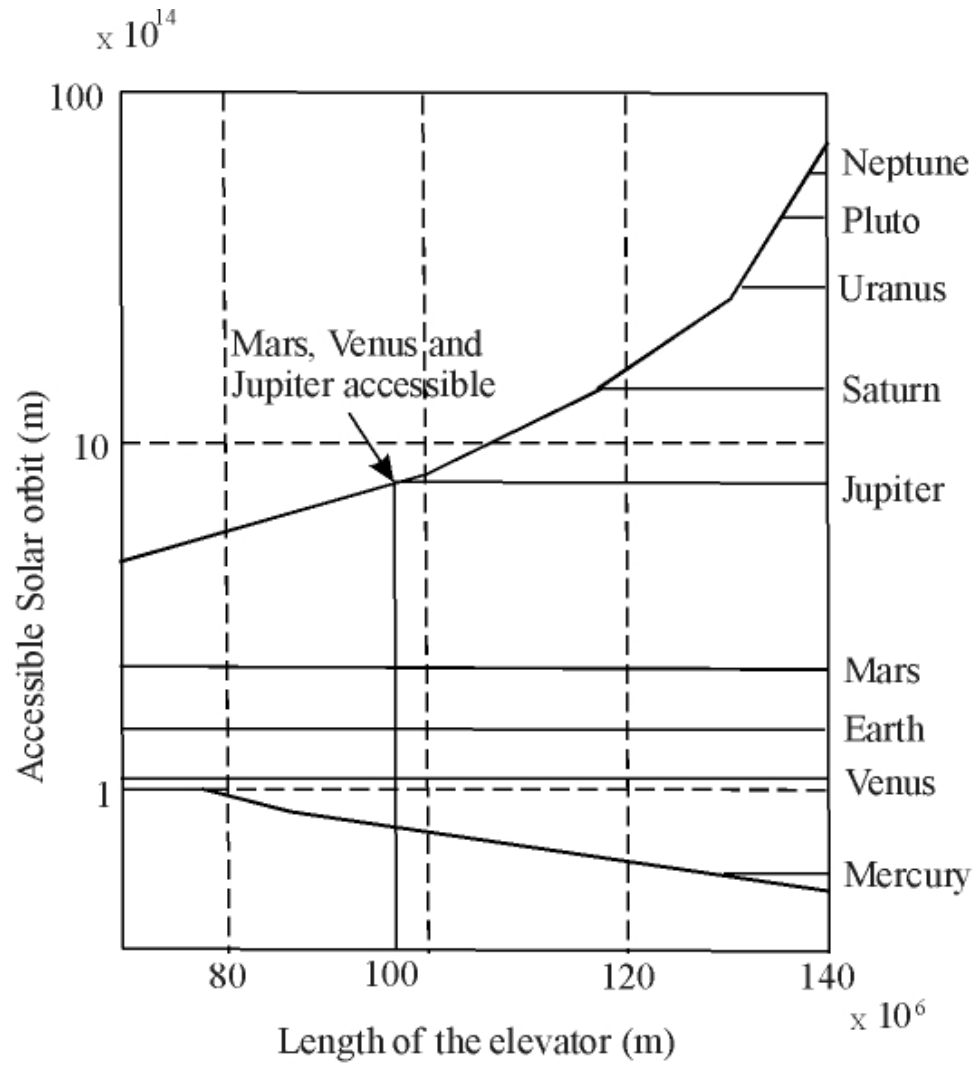


Figure 5.19: Solar orbits accessible for different lengths of the elevator [1]

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

In this thesis, we have introduced dynamic analysis into the design of the space elevator. After going through different design criterion, we have rounded upon two design cases. These two designs are serving two different purposes.

The first design case which has length of the elevator as 66,000 *km* will be primarily useful for delivering payloads up to thousands of kilometers away in space. The maximum displacement and velocity of the elevator in this case are 10 *km* and 0.6 *m/s* respectively. The point mass required at the tip of the elevator for this case is 0.47 times the mass of the cable. Sending this point mass to an altitude of 66,000 *km* will not be a problem during the deployment of the elevator.

The second design case which has length of the elevator as 91,000 *km* will be useful for delivering payloads within the Earth orbits as wells as to solar orbits. This elevator can reach destinations in Venus, Mars and Jupiter. The maximum displacement and velocity of the elevator in this case are 30 *km* and 1.5 *m/s* respectively. The point mass required at the tip of the elevator for this case is 0.15 times the mass of the cable. Sending this point mass to an altitude of 91,000 *km* will not be a problem during the deployment of the elevator.

Linear model results are accurate when examining the natural frequencies. For the actual response, the nonlinear coupled 3D modeling is necessary because the linear 2D model severely underestimates the amplitude of the transverse response.

Resonance due to the parked elevator cars can be considered into the design of the space elevator as these cars can change the fundamental periods of the elevator. Transient response due to the accelerating elevator car can be investigated as this should be very small to ensure safe operation. Stability of the final design of the elevator in axial, equatorial and meridional directions can be checked.

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APPENDIX A

MATLAB CODES

Appendix A contains MATLAB codes used for obtaining different plots in previous Chapters

A.1 taperedcrosssection.m

```
clear;
ro=6378e3;
den=1300;
g=9.81;
rs=42164e3;
strs=60e9:30e9:150e9;
rf=152164e3;
dr=(rf-ro)/100;
for i=1:length(strs)
r=[ro:dr:rf]';
h(i)=strs(i)/(den*g);
tr=exp(ro/h(i)+ro^4/2/h(i)/rs^3-ro^2./r/h(i)-r.^2/2/h(i)/rs^3*ro^2)
plot(r-ro,tr);
hold on
end
```

A.2 mpovercablem.m

```
clear;
ro=6378e3;
den=1300;
omega=(2*pi)/(24*60*60);
GM=398600e9;
g=9.81;
h=[6e6 13e6 25e6]';
rs=42164e3;
dR=1000e3;
dr=100e3;
Rf=[42164e3:dR:152164e3]';
L=Rf-ro;

for j=1:length(h)
```

```

strs(j)=h(j)*den*g;

for i=1:length(Rf)
rf=Rf(i);
r=[6378e3:dr:rf]';
A = exp((((ro^2)./(r*h(j)))-
((ro^2*r.^2)/(2*h(j)*rs^3)))+(ro/h(j))+((ro^4)/(2*h(j)*rs^3))));
A1 = exp((((ro^2)./(r*h(j)))-
((ro^2*r.^2)/(2*h(j)*rs^3)))+(ro/h(j))+((ro^4)/(2*h(j)*rs^3))))./(r.^2);
A2= r.*exp((((ro^2)./(r*h(j)))-
((ro^2*r.^2)/(2*h(j)*rs^3)))+(ro/h(j))+((ro^4)/(2*h(j)*rs^3))));
cm(i)=den*trapz(A)*dr;
Af1(i)=den*trapz(A1)*dr;
Af2(i)=den*trapz(A2)*dr;
mp(i)=(((omega^2*Af2(i))-(GM*Af1(i)))/((GM/rf^2)-(rf*(omega^2))));
mpoverm(i)=abs(mp(i))/cm(i);
end
figure(2)
plot(L,mpoverm);
hold on
end

```

A.3 natperiodtolength.m

```

clear;
counter=1;
cn=1;
ro=6378e3;
rs=35786e3+ro;
rho=1300;
g=9.81;
h=[2e6 8e6 20e6]';
GM=398600e9;
E=1e12;
omega=(2*pi)/(24*60*60);
Rff=162164e3;
dR=(Rff-rs)/100;
Rf=[rs+dR:dR:Rff]';
L=Rf-ro;
n2=length(Rf);
n1=length(h);

```

```

for i=1:n1
sig(i)=h(i)*rho*g

for j=1:n2
rf=Rf(j);
dr=(rf-ro)/50;
r=[ro+dr:dr:rf]';
tr(i)=exp(0.776*ro/h(i));
c=length(r);
As=tr(i)*5e-6;
A=As*exp(-ro^2/h(i)*(1./r+r.^2/2/rs^3))./exp(-ro^2/h(i)*(1./rs+rs.^2/2/rs^3));
e=ones(c,1);
Afn=spdiags([A],0,c,c);
Afn2=Afn;
Aprime=((ro^2)/(h(i)*r.^2))-((r.*ro^2)/(h(i)*rs^3));

Q=spdiags([e -2*e e],-1:1,c,c)./(dr^2);
Q(c,c-1)=2/dr^2;
k1=Afn*Q;
Q2=spdiags([-e e],[-1,1],c,c)./(2*dr);
Q2(c,c-1)=0;
Aprime2=spdiags([Aprime],0,c,c);
Aprimefn=Aprime2*Afn;
k2=Aprimefn*Q2;

r1=[ro:dr:rf]';
A1=As*exp(-ro^2/h(i)*(1./r1+r1.^2/2/rs^3))./exp(-ro^2/h(i)*(1./rs+rs.^2/2/rs^3));
f=(1./r1.^2-r1/rs^3);
M(i)=rho*trapz(A1)*dr;
mp(i)=rho*trapz(A1.*f)*dr*(rs^3*rf^2)/(rf^3-rs^3);
mratio=mp(i)/M(i);

m=Afn*eye(c);
m(c,c)=((Afn(c,c)))+(2*mp(i)/(rho*dr))+(mp(i)*Aprime(c)/rho));
k=-[k1+k2]*eye(c)*E/rho;

[ev,w2]=eig(k,m);
temp=[(diag(w2)) ev'];
sd=sortrows(temp);
fr=sort(sqrt(diag(w2)));

```

```
tu1(j)=2*pi/(fr(1)*3600);
tu2(j)=2*pi/(fr(2)*3600);
tu3(j)=2*pi/(fr(3)*3600);

beta1(j)=sqrt(fr(1)^2*rho/E);
beta2(j)=sqrt(fr(2)^2*rho/E);
beta3(j)=sqrt(fr(3)^2*rho/E);

kV=-[k1+k2]*eye(c)*sig(i)/rho;
[Vev,Vw2]=eig(kV,m);
Vtemp=[(diag(Vw2)) Vev'];
Vsd=sortrows(Vtemp);
Vfr=sort(sqrt(diag(Vw2)));

tV1(j)=2*pi/(Vfr(1)*3600);
tV2(j)=2*pi/(Vfr(2)*3600);
tV3(j)=2*pi/(Vfr(3)*3600);
end
figure(1)
plot(L,beta1,L,beta2,L,beta3);
hold on

figure(2)
plot(L,tu1,L,tu2,L,tu3);
hold on

figure(3)
plot(L,tV1,L,tV2,L,tV3);
hold on
end
```

A.4 crosssection.m

```
clear;
ro=6378e3;
den=1300;
g=9.81;
rs=42164e3;
E=1e12;
strs=60e9;
L=144000e3;
```

```

rf=L+ro;
dr=(rf-ro)/100;
for j=1:length(strs)
h(j)=strs(j)/(den*g);
B(j)=sqrt(1+2*strs(j)/E);
r=[ro:dr:rf]';
el=B(j)*r;
As=exp(ro^2/h(j)/B(j)^2*(1./(ro)+(ro).^2/2/rs^3))./exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(rs-ro)))+(ro+B(j)*(rs-ro))^2/2/rs^3));
A = As*exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(rs-ro)))+(ro+B(j)*(rs-ro))^2/2/rs^3))./exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(r-ro)))+(ro+B(j)*(r-ro))^2/2/rs^3));

Aor=exp(ro/h(j)+ro^4/2/h(j)/rs^3-ro^2./r/h(j)-r.^2/2/h(j)/rs^3*ro^2);
plot(el,A,el,Aor);
hold on
end

```

A.5 pointmass.m

```

clear;
ro=6378e3;
den=1300;
E=1e12;
g=9.81;
GM=398600e9;
strs=60e9;
omega=(2*pi)/(24*60*60);
rs=42164e3;
dR=1000e3;
dr=100e3;
Rf=[42164e3:dR:152164e3]';
L=Rf-ro;

for j=1:length(strs)
h(j)=strs(j)/(den*g);
B(j)=sqrt(1+2*strs(j)/E);
l=B(j)*L;

for i=1:length(Rf)

```

```

rf=Rf(i);
r=[6378e3:dr:rf]';
As=exp(ro^2/h(j)/B(j)^2*(1./(ro)+(ro).^2/2/rs^3))./exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(rs-
ro)))+(ro+B(j)*(rs-ro))^2/2/rs^3));
A = As*exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(rs-ro)))+(ro+B(j)*(rs-
ro))^2/2/rs^3))./exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(r-ro)))+(ro+B(j)*(r-ro)).^2/2/rs^3));
cm(i)=den*trapz(A)*dr;
Mp(i)=B(j)*strs(j)/omega^2/((ro+B(j)*L(i))-rs^3/(ro+B(j)*L(i))^2)*...
exp(ro^2/h(j)/B(j)^2*(1./(ro+B(j)*(rs-ro)))+(ro+B(j)*(rs-ro))^2/2/rs^3))*...
(exp(-ro^2/h(j)/B(j)^2*(1./(ro+B(j)*L(i)))+(ro+B(j)*L(i))^2/2/rs^3))-exp(-
ro^2/h(j)/B(j)^2*(1/ro+ro^2/2/rs^3)));

Mpoverm(i)=abs(Mp(i))/cm(i);

Aor = exp((((ro^2)./(r*h(j)))-
((ro^2*r.^2)/(2*h(j)*rs^3)))+(ro/h(j))+((ro^4)/(2*h(j)*rs^3))));
A1 = exp((((ro^2)./(r*h(j)))-
((ro^2*r.^2)/(2*h(j)*rs^3)))+(ro/h(j))+((ro^4)/(2*h(j)*rs^3))))./(r.^2);
A2= r.*exp((((ro^2)./(r*h(j)))-
((ro^2*r.^2)/(2*h(j)*rs^3)))+(ro/h(j))+((ro^4)/(2*h(j)*rs^3))));
cm(i)=den*trapz(Aor)*dr;
Af1(i)=den*trapz(A1)*dr;
Af2(i)=den*trapz(A2)*dr;
mp(i)=(((omega^2*Af2(i))-(GM*Af1(i)))/((GM/rf^2)-(rf*(omega^2))));
mpoverm(i)=abs(mp(i))/cm(i);
end
figure(1)
plot(l,Mpoverm,l,mpoverm)
hold on
end

```

A.6 accelpromoon.m

```

clear;
ro=6378e3;
rs=35786e3+ro;
G=6.67e-11;
Mm=0.07349e24;
Me=5.9742e24;
Dem=3.84467e8;
Sem=(Mm*Dem)/(Me+Mm);

```

```

omega2=sqrt(G*(Mm+Me)/Dem^3);
omega=(2*pi)/(24*60*60);
double(omega);
double(omega2);
t=0:1000:36e4;
r1=ro;
r2=rs;
r3=100e6;
a1=(Dem*((cos(28.65*pi/180)*cos(t.*omega).*cos(t.*omega2))+(sin(t.*omega2).*sin(t.*omega))))-r1).^2;
b1=Dem^2*(cos(t.*omega2).*sin(28.65*pi/180)).^2;
c1=Dem^2*((cos(28.65*pi/180)*cos(t.*omega2).*sin(t.*omega))-(sin(t.*omega2).*cos(t.*omega))).^2;
Rem1=sqrt((a1+b1+c1));
pu11=((cos(28.65*pi/180)*cos(t.*omega).*cos(t.*omega2))+(sin(t.*omega2).*sin(t.*omega))).*((G*Mm*Dem./Rem1.^3)-(Sem*omega2^2));
pu12=(G*Mm*r1./Rem1.^3);
pu1=pu11-pu12;
a2=(Dem*((cos(28.65*pi/180)*cos(t.*omega).*cos(t.*omega2))+(sin(t.*omega2).*sin(t.*omega))))-r2).^2;
b2=Dem^2*(cos(t.*omega2).*sin(28.65*pi/180)).^2;
c2=Dem^2*((cos(28.65*pi/180)*cos(t.*omega2).*sin(t.*omega))-(sin(t.*omega2).*cos(t.*omega))).^2;
Rem2=sqrt((a2+b2+c2));
pu21=((cos(28.65*pi/180)*cos(t.*omega).*cos(t.*omega2))+(sin(t.*omega2).*sin(t.*omega))).*((G*Mm*Dem./Rem2.^3)-(Sem*omega2^2));
pu22=(G*Mm*r2./Rem2.^3);
pu2=pu21-pu22;
a3=(Dem*((cos(28.65*pi/180)*cos(t.*omega).*cos(t.*omega2))+(sin(t.*omega2).*sin(t.*omega))))-r3).^2;
b3=Dem^2*(cos(t.*omega2).*sin(28.65*pi/180)).^2;
c3=Dem^2*((cos(28.65*pi/180)*cos(t.*omega2).*sin(t.*omega))-(sin(t.*omega2).*cos(t.*omega))).^2;
Rem3=sqrt((a3+b3+c3));
pu31=((cos(28.65*pi/180)*cos(t.*omega).*cos(t.*omega2))+(sin(t.*omega2).*sin(t.*omega))).*((G*Mm*Dem./Rem3.^3)-(Sem*omega2^2));
pu32=(G*Mm*r3./Rem3.^3);
pu3=pu31-pu32;
pv1=(sin(28.65*pi/180)*cos(t.*omega2)).*((G*Mm*Dem./Rem1.^3)-(Sem*omega2^2));
pv2=(sin(28.65*pi/180)*cos(t.*omega2)).*((G*Mm*Dem./Rem2.^3)-(Sem*omega2^2));
pv3=(sin(28.65*pi/180)*cos(t.*omega2)).*((G*Mm*Dem./Rem3.^3)-(Sem*omega2^2));

```

```

pw1=((cos(28.65*pi/180)*sin(t.*omega).*cos(t.*omega2))-
(sin(t.*omega2).*cos(t.*omega))).*((G*Mm*Dem./Rem1.^3)-(Sem*omega2^2));
pw2=((cos(28.65*pi/180)*sin(t.*omega).*cos(t.*omega2))-
(sin(t.*omega2).*cos(t.*omega))).*((G*Mm*Dem./Rem2.^3)-(Sem*omega2^2));
pw3=((cos(28.65*pi/180)*sin(t.*omega).*cos(t.*omega2))-
(sin(t.*omega2).*cos(t.*omega))).*((G*Mm*Dem./Rem3.^3)-(Sem*omega2^2));
subplot(3,1,1);plot(t/3600,pu1,t/3600,pu2,t/3600,pu3);

subplot(3,1,2);plot(t/3600,pv1,t/3600,pv2,t/3600,pv3);

subplot(3,1,3);plot(t/3600,pw1,t/3600,pw2,t/3600,pw3);

```

A.7 sunpsd.m

```

clear;
t=0:10000:3e7;
r=100e6;

%properties of the earth
ro=6378e3;
rs=35786e3+ro;
G=6.67e-11;%universal gravitational constant
Me=5.9742e24;%mass of earth
Ms=1.989e30;%mass of sun
Des=1.496e11;%distance from moon to earth centre

omega=(2*pi)/(24*60*60);%rotational rate of earth
thetas=23.5*pi/180;%the angle between plane of earths equator and earth sun orbital
plane
omegae=sqrt(G*(Ms+Me)/Des^3);%angular velocity of earth sun system
Ses=(Ms*Des)/(Me+Ms);%distance from earth to centre of mass

d1=(cos(thetas).*cos(omega*t).*cos(omegae*t))+(sin(omegae*t).*sin(omega*t));%ith
component of vector drawn from earth to sun centre
d2=(cos(omegae*t).*sin(thetas));%jth component of vector drawn from earth to sun
centre
d3=((cos(thetas)*cos(omegae*t).*sin(omega*t))-(sin(omegae*t).*cos(omega*t)));%kth
component of vector drawn from earth to sun centre

Res=sqrt((Des*d1-(r)).^2+Des^2*d2.^2+Des^2*d3.^2);%distance from elevator to sun
centre

```

```
pus=((G*Ms*Des./Res.^3)-(Ses*omegaes^2)-(G*Ms*(r)./Res.^3)).*d1;%differential
tidal accelerations due to sun in radial direction
pvs=((G*Ms*Des./Res.^3)-(Ses*omegaes^2)).*d2;%differential tidal accelerations due to
sun in meridional direction
pws=((G*Ms*Des./Res.^3)-(Ses*omegaes^2)).*d3;%differential tidal accelerations due
to sun in equitorial direction
```

```
N=2^10
```

```
subplot(3,1,1);
psd(pus,N,1/(t(2)-t(1)));
```

```
subplot(3,1,2);
psd(pvs,N,1/(t(2)-t(1)));
```

```
subplot(3,1,3);
psd(pws,N,1/(t(2)-t(1)));
```

A.8 newresponse3.m

```
clear;
global counter cn
counter=1;
cn=1;
ro=6378e3;
rs=35786e3+ro;
rho=1300;
g=9.81;
sig=60e9;
GM=398600e9;
E=1e12;
omega=(2*pi)/(24*60*60);
Rff=11.6378e7;
Ri=56.378e6;
dR=(Rff-Ri)/6;
Rf=[Ri+dR:dR:Rff]';
n2=length(Rf);
n1=length(sig);
```

```

for i=1
h(i)=sig(i)/(rho*g);

for j=1:n2
rf=Rf(j);
dr=(rf-ro)/20;
r=[ro+dr:dr:rf]';
tr(i)=exp(0.776*ro/h(i));
c=length(r);
As=tr(i)*5e-8;
A=As*exp(-ro^2/h(i)*(1./r+r.^2/2/rs^3))./exp(-ro^2/h(i)*(1./rs+rs.^2/2/rs^3));%As=1
e=ones(c,1);
Afn=spdiags([A],0,c,c);
Afn2=Afn;
Aprime=((ro^2)/(h(i)*r.^2))-((r.*ro^2)/(h(i)*rs^3)));

Q=spdiags([e -2*e e],[-1:1,c,c])./(dr^2);
Q(c,c-1)=2/dr^2;
k1=Afn*Q;
Q2=spdiags([-e e],[-1,1],c,c)./(2*dr);
Q2(c,c-1)=0;
Aprime2=spdiags([Aprime],0,c,c);
Aprimefn=Aprime2*Afn;
k2=Aprimefn*Q2;

r1=[ro:dr:rf]';
A1=As*exp(-ro^2/h(i)*(1./r1+r1.^2/2/rs^3))./exp(-ro^2/h(i)*(1./rs+rs.^2/2/rs^3));
f=(1./r1.^2-r1/rs^3);
M(i)=rho*trapz(A1)*dr;
mp(i)=rho*trapz(A1.*f)*dr*(rs^3*rf^2)/(rf^3-rs^3);
mratio=mp(i)/M(i);

m=Afn*eye(c);
m(c,c)=((Afn(c,c))+(2*mp(i)/(rho*dr))+(mp(i)*Aprime(c)/rho));
k=-[k1+k2]*eye(c)*E/rho;

[ev,w2]=eig(k,m);
temp=[(diag(w2)) ev'];
sd=sortrows(temp);
fr=sort(sqrt(diag(w2)));
tu1(i,j)=2*pi/(fr(1)*3600); %Longitudinal

```

```

tu2(i,j)=2*pi/(fr(2)*3600);
tu3(i,j)=2*pi/(fr(3)*3600);
D=[zeros(c,c) eye(c);inv(-m)*k zeros(c,c)];
t=[0:0.1:200]*3600;
Yo=zeros(2*c,1);
option=odeset('RelTol', 1e-8, 'AbsTOL', 1e-8);
[t,U]=ode45(@taperufun4,t,Yo,[],D,m,c,r,rho,A);
utip=U(:,c);
figure(3)
subplot(3,2,j);
plot(t/3600,U(:,end));

figure(1)
subplot(3,2,j);
plot(t/3600,utip);

kV=-[k1+k2]*eye(c)*sig(i)/rho;
ApAseon=diag(Aprime);

Kseon=sig(i)/rho*(ApAseon*Q2+Q);
Gseon=[zeros(c,c) eye(c);Kseon zeros(c,c)];
[Vev,Vw2]=eig(kV,m);
Vtemp=[(diag(Vw2)) Vev'];
Vsd=sortrows(Vtemp);
Vfr=sort(sqrt(diag(Vw2)));
tV1(i,j)=2*pi/(Vfr(1)*3600); %Transverse
tV2(i,j)=2*pi/(Vfr(2)*3600);
tV3(i,j)=2*pi/(Vfr(3)*3600);
G=[zeros(c,c) eye(c);inv(-m)*kV zeros(c,c)];
t=[0:0.1:200]*3600;
Yo=zeros(2*c,1);
[t,V]=ode45(@tapervfun2,t,Yo,[],G,m,c,r,rho,A);
Vtip=V(:,c);
%eval(['save data',num2str(i),num2str(j),' t r V Vtip;'])
figure(4)
subplot(3,2,j);
plot(t/3600,V(:,end));

figure(2)
subplot(3,2,j);
plot(t/3600,Vtip);
end

```

end

A.9 taperufun4.m

```
function dot=taperufun4(t,Y,D,m,c,r,rho,A)
global counter cn
[pu pv pw]=tidalf(r,t);

dot=(D*Y)+[zeros(c,1);inv(m)*pu.*A];

counter=counter+1;
if counter==1000*cn;
    t/3600
    cn=cn+1;
end
```

A.10 tapervfun2.m

```
function dot=tapervfun2(t,Y,G,m,c,r,rho,A)
global counter cn
[pu pv pw]=tidalf(r,t);

dot=(G*Y)+[zeros(c,1);inv(m)*pv.*A];

counter=counter+1;
if counter==1000*cn;
    t/3600
    cn=cn+1;
end
```

A.11 threedresponse.m

```
clear
global counter cn N h Q1 Q2
global go ro rs X L omega m Mp sigma E rho hh Ap Al Mo Mm Ms G rmoon rsun force
global wfu wfv wfw

N=20; %number of nodes

%properties of the Earth
omega=7.272205216643040e-005; facs=1;
G=6.672000000000000e-011;
```

```

Mo=5.9736000000000001e+024;
Mm=0.07349e24; Ms=1.989e30;
ro=6378000;
go=G*Mo/ro^2; rs=(G*Mo/omega^2)^(1/3);
rmoon=3.84467e8; rsun=1.496e11;

%properties of the cable
E=1e12; rho=1300; sigma=60e9/facs;
L=8.2e7;

%spatial dimension
h=L/N;
X=[h:h:L]';
rs=(G*Mo/omega^2)^(1/3); rl=ro+L;
hh=sigma/rho/go;
B=sqrt(1+2*sigma/E);

As=0.115*1e-6;
Mp=As*B*sigma/omega^2/((ro+B*L)-rs^3/(ro+B*L)^2)*...
    exp(ro^2/hh/B^2*(1./(ro+B*(rs-ro))+(ro+B*(rs-ro))^2/2/rs^3))*...
    (exp(-ro^2/hh/B^2*(1./(ro+B*L)+(ro+B*L)^2/2/rs^3))-exp(-
ro^2/hh/B^2*(1/ro+ro^2/2/rs^3)));

A=As*exp(-ro^2/hh/B^2*(1./(ro+B*X)+(ro+B*X).^2/2/rs^3))...
/exp(-ro^2/hh/B^2*(1./(ro+B*(rs-ro))+(ro+B*(rs-ro))^2/2/rs^3));

Ao=As*exp(-ro^2/hh/B^2*(1/ro+ro^2/2/rs^3))...
/exp(-ro^2/hh/B^2*(1./(ro+B*(rs-ro))+(ro+B*(rs-ro))^2/2/rs^3));

%cross-sectional area at the surface

taper=max(A)/Ao;
Al=A(end); %area at the end of the cable
Ap=ApA(X); % A'/A along the cable

%necessary matrices
e=ones(N,1);
Q1=spdiags([-e e],[-1 1], N,N)./(2*h);
Q2=spdiags([e -2*e e],[-1:1,N,N])./(h^2);

```

```

force=1;

%time integration
tf=3600*200; %final time
tint=3600*0.1;% time at which the values are recorded
qn=1; %number of times that the data is saved to disk

%initial displacement
vc=0; wc=0; %
Vo=vc*X; Wo=wc*X; %
Vop=vc; Wop=wc; %
Vopp=zeros(size(X)); Wopp=zeros(size(X)); %
%STATIC
%static configuration
Qu1=Q1; Qu1(N,N-1)=0;
Qu2=Q2; Qu2(N,N-1)=2/h^2; Qu2(N,N)=-2/h^2;
uepl=Mp/E/Al*((ro+L)*omega^2-go*ro^2/(ro+L)^2);
r=ro+X;
ue=inv(Qu2+diag(ApA(X))*Qu1)*(rho/E*(go*ro^2./r.^2-r.*omega^2)...
-[zeros(N-1,1);2/h*uepl]-[zeros(N-1,1);ApA(L)*uepl]);

for guess=1:30
    tmp=Mp/E/Al*((ro+L+ue(end))*omega^2-go*ro^2/(ro+L+ue(end))^2);
    r=ro+X+ue;
    Root=roots([0.5 1.5 1 -tmp]);
    I=find(Root>0);
    uepl=Root(I);
    uep=Q1*ue; uep(end)=uepl;
    ue=inv(Qu2)*((rho/E*go*ro^2*(1./r.^2-r/rs^3)...
    -ApA(X).*uep.*(1+1.5*uep+0.5*uep.^2))./(1+3*uep+1.5*uep.^2)...
    -[zeros(N-1,1);2/h*uepl]);
end

IC(:,1)=[ue;zeros(N,1);Vo;zeros(N,1);Wo;Vo];

Yo=IC(:,1);

%DYNAMIC
Ys=Y0';

```

```

ts=0;
save prac ts Yo Ys
options=odeset('RelTol',1e-10,'AbsTol', 1e-10);
int=tf/qn;
PLOT=1;
counter=0; cn=1;

for q=1:qn;
    tspan=[int*(q-1):tint:int*q]';
    load prac
    [t,Y]=ode45('mainfuns',tspan,Yo); % dynamic displacements
    ts=[ts;t(2:size(t,1))];
    NN=size(t,1); tmpY=Y(2:size(Y,1),:);
    Yo=Y(NN,:); Ys=[Ys;tmpY];
    save prac ts Yo Ys
    if PLOT==1;
        figure(1)
        subplot(3,1,1)
        plot(ts/3600,Ys(:,N)-ue(N),'b')
        grid
        subplot(3,1,2)
        plot(ts/3600,Ys(:,3*N),'b')
        grid
        subplot(3,1,3)
        plot(ts/3600,Ys(:,5*N),'b')
        grid

        figure(2)
        subplot(3,1,1)
        plot(ts/3600,Ys(:,2*N))
        grid
        subplot(3,1,2)
        plot(ts/3600,Ys(:,4*N))
        grid
        subplot(3,1,3)
        plot(ts/3600,Ys(:,6*N))
        grid

        pause(1)
    end
end
end

```

save 3Dcase2 ts Yo Ys

A.12 mainfuns.m

```
function dot=mainfuns(t,Y)
global counter cn N h Q1 Q2 go L ro E X omega rho Mp Al Ap G Mm Ms Mo rmoon
rsun force
%force due to sun and moon
pu=zeros(N,1); pv=zeros(N,1); pw=zeros(N,1);

if force==1
w=2*pi/24/3600;
omm=sqrt(G*(Mm+Mo)/rmoon.^3);
thetam=28.65*pi/180;
sm=Mm*rmoon/(Mo+Mm);
c1=cos(thetam).*cos(omm*t).*cos(w*t)+sin(omm*t).*sin(w*t);
c2=-sin(thetam).*cos(omm*t);
c3=cos(thetam).*cos(omm*t).*sin(w*t)-sin(omm*t).*cos(w*t);
Rm=sqrt((rmoon*c1-(ro+X)).^2+rmoon^2*c2.^2+rmoon^2*c3.^2); %

pum=-G*Mm./Rm.^3.*(ro+X)+(G*Mm./Rm.^3*rmoon-omm^2*sm).*c1;
pvm=(G*Mm./Rm.^3*rmoon-omm^2*sm).*c2;
pwm=(G*Mm./Rm.^3*rmoon-omm^2*sm).*c3;

thetas=23.5*pi/180;
oms=sqrt(G*(Ms+Mo)/rsun.^3);
ss=Ms*rsun/(Mo+Ms);
d1=cos(thetas).*cos(oms*t).*cos(w*t)+sin(oms*t).*sin(w*t);
d2=-sin(thetas).*cos(oms*t);
d3=cos(thetas).*cos(oms*t).*sin(w*t)-sin(oms*t).*cos(w*t);
Rs=sqrt((rsun*d1-(ro+X)).^2+rsun^2*d2.^2+rsun^2*d3.^2);
pus=-G*Ms./Rs.^3.*(ro+X)+(G*Ms./Rs.^3*rsun-oms^2*ss).*d1;
pvs=(G*Ms./Rs.^3*rsun-oms^2*ss).*d2;
pws=(G*Ms./Rs.^3*rsun-oms^2*ss).*d3;

pv=pvm+pvs; pu=pum+pus; pw=pwm+pws
end

u=Y(1:N); v=Y(2*N+1:3*N); w=Y(4*N+1:5*N);
ud=Y(N+1:2*N); vd=Y(3*N+1:4*N); wd=Y(5*N+1:6*N);
up=Q1*u; upp=Q2*u;
vp=Q1*v; vpp=Q2*v; wp=Q1*w; wpp=Q2*w;
r=sqrt((ro+X+u).^2+v.^2+w.^2);%
```

```

ru=ro+X+u; %
Apl=ApA(L);
strain=up+0.5*(up.^2+vp.^2+wp.^2); %
strainp=upp+up.*upp+vp.*vpp+wp.*wpp; %
udd=-2*omega*wd+ru*omega^2-
go*ro^2*ru./r.^3+E/rho*(strainp.*(1+up)+strain.*(upp+Ap.*(1+up)))+pu;%
vdd=-go*ro^2*v./r.^3+E/rho*(strainp.*vp+strain.*(vpp+Ap.*vp))+pv;
wdd=2*omega*ud+w*omega^2-
go*ro^2*w./r.^3+E/rho.*(strainp.*wp+strain.*(wpp+Ap.*wp))+pw;

ul=u(end); vl=v(end); wl=w(end);
udl=ud(end); vdl=vd(end); wdl=wd(end);
ulm=u(end-1); vlm=v(end-1); wlm=w(end-1);
rl=r(end); %
rul=ro+L+u(end); %
last=[udd(end-1); vdd(end-1); wdd(end-1)];
F=1;
index=1;
for JJJ=1:5
    uddl=last(1); vddl=last(2); wddl=last(3);
    TMP1=-Mp/E/Al*(uddl+2*wdl*omega-rul*omega^2+go*ro^2*rul/rl^3);
    TMP2=-Mp/E/Al*(vddl+go*ro^2*vl/rl^3);
    TMP3=-Mp/E/Al*(wddl-2*udl*omega-wl*omega^2+go*ro^2*wl/rl^3);
    Root=roots([2 1 0 -(TMP1^2+TMP2^2+TMP3^2)]);
    I=find(Root>0);
    epsilon=Root(I);
    upl=TMP1/epsilon-1;
    uppl=2*(-ul+ulm)/h^2+2/h*upl;
    vpl=TMP2/epsilon;
    vppl=2*(-vl+vlm)/h^2+2/h*vpl;
    wpl=TMP3/epsilon;
    wppl=2*(-wl+wlm)/h^2+2/h*wpl;
    epsilonp=uppl+upl*uppl+vpl*vppl+wpl*wppl;

    F=[uddl+2*wdl*omega-rul*omega^2-
    E/rho*(epsilonp*(1+upl)+epsilon*(uppl+Apl*(1+upl)))+go*ro^2*rul/rl^3-pu(end);
    vddl-E/rho*(epsilonp*vpl+epsilon*(vppl+Apl*vpl))+go*ro^2*vl/rl^3-pv(end);
    wddl-2*udl*omega-wl*omega^2-
    E/rho*(epsilonp*wpl+epsilon*(wppl+Apl*wpl))+go*ro^2*wl/rl^3-pw(end)];

    dEdu=-2*Mp/E/Al*TMP1/(6*epsilon^2+2*epsilon);
    dEdv=-2*Mp/E/Al*TMP2/(6*epsilon^2+2*epsilon);

```

$$dEdw = -2 * Mp / E / A1 * TMP3 / (6 * \epsilon^2 + 2 * \epsilon);$$

$$tmp = (-Mp / E / A1 / \epsilon);$$

$$\begin{aligned} dupdu &= tmp - TMP1 / \epsilon^2 * dEdu; & duppdu &= 2 / h * dupdu; \\ dvpdu &= -TMP2 / \epsilon^2 * dEdu; & dvppdu &= 2 / h * dvpdu; \\ dwpdu &= -TMP3 / \epsilon^2 * dEdu; & dwppdu &= 2 / h * dwpdu; \end{aligned}$$

$$\begin{aligned} dupdv &= -TMP1 / \epsilon^2 * dEdv; & duppdv &= 2 / h * dupdv; \\ dvpdv &= tmp - TMP2 / \epsilon^2 * dEdv; & dvppdv &= 2 / h * dvpdv; \\ dwpdv &= -TMP3 / \epsilon^2 * dEdv; & dwppdv &= 2 / h * dwpdv; \end{aligned}$$

$$\begin{aligned} dupdw &= -TMP1 / \epsilon^2 * dEdw; & duppdw &= 2 / h * dupdw; \\ dvpdw &= -TMP2 / \epsilon^2 * dEdw; & dvppdw &= 2 / h * dvpdw; \\ dwpdw &= tmp - TMP3 / \epsilon^2 * dEdw; & dwppdw &= 2 / h * dwpdw; \end{aligned}$$

$$dEpdu = duppdu + dupdu * upl + duppdu * upl + dvpdu * vppl + dvppdu * vpl + dwpdu * wppl + dwppdu * wpl;$$

$$dEpdv = duppdv + dupdv * upl + duppdv * upl + dvpdv * vppl + dvppdv * vpl + dwpdv * wppl + dwppdv * wpl;$$

$$dEpdw = duppdw + dupdw * upl + duppdw * upl + dvpdw * vppl + dvppdw * vpl + dwpdw * wppl + dwppdw * wpl;$$

$$J = [1 - E / \rho * (dEpdu * (1 + upl) + \epsilon n p * dupdu + dEdu * (upl + Apl * (1 + upl)) + \epsilon * (duppdu + Apl * dupdu)) \dots$$

$$- E / \rho * (dEpdv * (1 + upl) + \epsilon n p * dupdv + dEdv * (upl + Apl * (1 + upl)) + \epsilon * (duppdv + Apl * dupdv)) \dots$$

$$- E / \rho * (dEpdw * (1 + upl) + \epsilon n p * dupdw + dEdw * (upl + Apl * (1 + upl)) + \epsilon * (duppdw + Apl * dupdw));$$

$$- E / \rho * (dEpdu * vpl + \epsilon n p * dvpdu + dEdu * (vppl + Apl * vpl) + \epsilon * (dvppdu + Apl * dvpdu)) \dots$$

$$1 - E / \rho * (dEpdv * vpl + \epsilon n p * dvpdv + dEdv * (vppl + Apl * vpl) + \epsilon * (dvppdv + Apl * dvpdv)) \dots$$

```

-
E/rho*(dEpdw*vpl+epsilon*p*dvpdw+dEdw*(vppl+Apl*vpl)+epsilon*(dvppdw+Apl*dv
pdw));
-
E/rho*(dEpdu*wpl+epsilon*p*dw pdu+dEdu*(wppl+Apl*wpl)+epsilon*(dwppdu+Apl*d
wpdu)) ...
-
E/rho*(dEpdv*wpl+epsilon*p*dw pdv+dEdv*(wppl+Apl*wpl)+epsilon*(dwppdv+Apl*d
wpdv)) ...
1-
E/rho*(dEpdw*wpl+epsilon*p*dw pdw+dEdw*(wppl+Apl*wpl)+epsilon*(dwppdw+Apl*
dw pdw));
last=last-inv(J)*F;
index=index+1;
end
udd(end)=last(1);
vdd(end)=last(2);
wdd(end)=last(3);
dot=[ud; udd; vd; vdd; wd; wdd];

counter=counter+1;
if counter==1000*cn;
    t/3600
    cn=cn+1;
end

```

A.13 APA.m

```

function out=ApA(X)
global ro rs hh
global sigma E
B=sqrt(1+2*sigma/E);
out=ro^2/hh/B*(1./(ro+B*X).^2-(ro+B*X)/rs^3);

```

A.14 modalboth.m

```

clear;
tic
r0=6378e3;
rs=35786e3+r0;
rho=1300;
g=9.81;
sig=60*1e9;

```

```

h=sig/rho/g;
GM=398600e9;
E=1e12;
omega=(2*pi)/(24*60*60);
Ri=5e7+r0;
Rff=10e7+r0;
dR=(Rff-Ri)/5000;
Rf=[Ri:20*dR:7e7+r0] [7.4e7+r0+dR:dR:7.9e7+r0] [7.9e7+r0+dR:20*dR: Rff]';
length(Rf)

n2=length(Rf);
n3=8; %number of modes
nq=10; %number of fourier expansion for Q
for j=1:n2
    %j=1;

    rf=Rf(j);
    L(j)=rf-r0;

    c=20;
    dr=(rf-r0)/c;
    r=[r0+dr:dr:rf]';
    tr=exp(0.776*r0/h);
    A=exp((((r0^2)/(r*h))-((r0^2*r.^2)/(2*h*rs^3)))+(r0/h)+((r0^4)/(2*h*rs^3))));
    A=A/max(A);
    e=ones(c,1);
    Afn=spdiags([A],0,c,c);
    Aprime=((r0^2)/(h*r.^2))-((r.*r0^2)/(h*rs^3)); %A'/A
    Q=spdiags([e -2*e e],-1:1,c,c)./(dr^2);
    Q(c,c-1)=2/dr^2;

    k1=Afn*Q;
    Q2=spdiags([-e e],[-1,1],c,c)./(2*dr);
    Q2(c,c-1)=0;
    Aprime2=spdiags([Aprime],0,c,c);
    Aprimefn=Aprime2*Afn;
    k2=Aprimefn*Q2;
    As=1;

    r1=[r0:dr:rf]';
    A1=As*exp(-r0^2/h*(1./r1+r1.^2/2/rs^3))./exp(-r0^2/h*(1./rs+rs.^2/2/rs^3));

```

```

f=(1./r1.^2-r1/rs^3);
M=rho*trapz(A1)*dr;
mp=rho*trapz(A1.*f)*dr*(rs^3*rf^2)/(rf^3-rs^3); %
mratio=mp/M;
Aratio=trapz(A1)/A1(end);

m=Afn*eye(c);
m(c,c)=((Afn(c,c))+(2*mp/(rho*dr))+(mp*Aprime(c)/rho));
k=-[k1+k2]*eye(c)*E/rho;
KU=k;
mU=m;
for p=1:c-1
    fac=KU(p,p+1)/KU(p+1, p);
    mU(p+1, :)=mU(p+1,:)*fac;
    KU(p+1, :)=KU(p+1,:)*fac;
end
KU=round(1e12*KU)/1e12;

[V,D]=eig(KU,mU);
temp=[(diag(D)) V'];
sd=sortrows(temp);
fr=sort(sqrt(diag(D)));
periodu(j,1:n3)=2*pi./(fr(1:n3)*3600);
wn(1:n3)=fr(1:n3);
U=sd(1:n3,2:end)';

kV=-[k1+k2]*eye(c)*sig/rho;
ApAseon=diag(Aprime);
KV=kV;
mV=m;
for p=1:c-1
    fac=KV(p,p+1)/KV(p+1, p);
    mV(p+1, :)=mV(p+1,:)*fac;
    KV(p+1, :)=KV(p+1,:)*fac;
end
KV=round(1e12*KV)/1e12;

[Vev,Vw2]=eig(KV,mV);
Vtemp=[(diag(Vw2)) Vev'];
Vsd=sortrows(Vtemp);
Vfr=sort(sqrt(diag(Vw2)));
periodv(j,1:n3)=2*pi./(Vfr(1:n3)*3600); %Transverse

```

```

wVn(1:n3)=Vfr(1:n3);
Y(:,1:n3)=Vsd(1:n3,2:end)'; %these are same as U1, U2 and U3

tf=25;
dt=0.1;
x=r-r0;
t=[0:dt:tf]*3600;

[pum pvm pwm pus pvs pws]=tidalfnew(x,t);
dx=x(2)-x(1);

%Calculating maximum displacement
AFN=repmat(diag(Afn)', size(pum,1),1);
Qum=(pum.*AFN)*Y; Qus=(pus.*AFN)*Y;
Qvm=(pvm.*AFN)*Y; Qvs=(pvs.*AFN)*Y;

wm=2*pi/(25*3600);
ws=2*pi/(24*3600);

fourierQum=zeros(n3,size(t,1)); fourierQus=zeros(n3,size(t,1));
fourierQvm=zeros(n3,size(t,1)); fourierQvs=zeros(n3,size(t,1));
Amu=zeros(n3, nq); Asu=zeros(n3, nq);

for k=1:nq %ith mode, k number of fourier expansions in forcing terms

    Amu(:,k)=(trapz(Qum.*repmat(cos((k-1)*wm*t),1,n3)))/trapz(cos((k-1)*wm*t).^2);
    Asu(:,k)=(trapz(Qus.*repmat(cos((k-1)*ws*t),1,n3)))/trapz(cos((k-1)*ws*t).^2);

    Amv(:,k)=(trapz(Qvm.*repmat(cos((k-1)*wm*t),1,n3)))/trapz(cos((k-1)*wm*t).^2);
    Asv(:,k)=(trapz(Qvs.*repmat(cos((k-1)*ws*t),1,n3)))/trapz(cos((k-1)*ws*t).^2);

    fourierQum=fourierQum+Amu(:,k)*cos((k-1)*wm*t');
    fourierQus=fourierQus+Asu(:,k)*cos((k-1)*ws*t');
    fourierQvm=fourierQvm+Amv(:,k)*cos((k-1)*wm*t');
    fourierQvs=fourierQvs+Asv(:,k)*cos((k-1)*ws*t');
end

wmk=[0:nq-1]*wm;
wsk=[0:nq-1]*ws;
[WMK WN]=meshgrid(wmk, wn);
[WSK WN]=meshgrid(wsk, wn);

```

```
%[WMK WVN]=meshgrid(wmk, wVn);
[WSK WVN]=meshgrid(wsk, wVn);

betau=Amu./(WN.^2-WMK.^2);
alphau=Asu./(WN.^2-WSK.^2);

betav=Amv./(WVN.^2-WMK.^2);
alphav=Asv./(WVN.^2-WSK.^2);

ampetau=(sum(abs(betau'))+sum(abs(alphau')))'
ampetav=(sum(abs(betav'))+sum(abs(alphav')))'
maxuL(j)=abs(Y(end,:))*ampetau;
maxvL(j)=abs(Y(end,:))*ampetav;

if rem(j,100)==1
    j
    length(Rf)
end

end

figure(1)
subplot(2,1,1)
plot(L,log10(maxuL));
subplot(2,1,2)
plot(L,log10(maxvL));

toc
```

APPENDIX B

FINITE DIFFERENCE METHOD

Let us find the discrete form of the continuous model for the elevator equation given in Eqn 2.18. Let's assume that the elevator is divided into n number of elements as shown in figure B.1. The equation of motion of space elevator at i^{th} node is given by,

$$\rho A_i \ddot{y}_i = (H A(x) y_i')' + f_i(t) \quad (i = 1 \text{ to } n) \quad (\text{B. 1})$$

$$\text{with boundary conditions} \quad u_1(t) = 0, M_p \ddot{y}_n + H A_n y_n' = 0,$$

$f_i(t)$ indicates the tidal forces acting per unit length of the elevator.

Eqn. B.1 can be written as,

$$\ddot{y}_i = \frac{H A_i'}{\rho A_i} y_i' + \frac{H}{\rho} y_i'' + p_i(t), \quad (\text{B. 2})$$

where $p_i(t)$ is the tidal force per unit mass. For longitudinal motion, $p_i(t)$ will be replaced by the tidal acceleration in the radial direction of the elevator $p_{ui}(t)$ and for transverse motion $p_i(t)$ will be replaced by the tidal acceleration in the equatorial direction of the elevator $p_{ui}(t)$.

We can write y_i' and y_i'' in terms of y_i, y_{i+1}, y_{i-1} by using the finite difference formulae as,

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2\Delta}, \quad (\text{B. 3})$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta^2},$$

where Δ is the length of element i.e. $(r_{i+1} - r_i)$.

Substituting Eqn B.3 in boundary condition at the tip of the elevator in Eqn B.1 gives,

$$M_p \ddot{y}_n + H A_n \left(\frac{y_{n+1} - y_{n-1}}{2\Delta} \right) = 0, \quad (\text{B. 4})$$

From Eqn 2.23, y_{n+1} can be written as,

$$y_{n+1} = y_{n-1} - \frac{2 M_p \ddot{y}_n \Delta}{H A_n} \quad (\text{B. 5})$$

The last equation from the set of equations in B.2, i.e. the EOM at the n^{th} node, can be written as

$$\ddot{y}_n = \frac{H A'_n}{\rho A_n} y'_n + \frac{H}{\rho} y''_n + p_n(t) \quad (\text{B. 6})$$

Substituting Eqn. B.3 in Eqn. B.6 gives,

$$\ddot{y}_n = \frac{H A'_n}{\rho A_n} \left(\frac{y_{n+1} - y_{n-1}}{2\Delta} \right) + \frac{H}{\rho} \left(\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta^2} \right) + p_n(t). \quad (\text{B. 7})$$

Substituting y_{n+1} value from Eqn 2.24 into Eqn 2.26 gives,

$$\left(1 + \frac{2M_p}{\rho A_n} \frac{1}{\Delta} + \frac{M_p}{\rho A_n} \frac{A'_n}{A_n} \right) \ddot{y}_n = \frac{2H}{\rho \Delta^2} (y_{n-1} - y_n) + p_n(t). \quad (\text{B. 8})$$

The boundary conditions are applied to the EOM by the Eqn.B.8. The n no. of equations of motion at n no. of nodes of the elevator can be represented in a matrix form as,

$$\begin{aligned}
 \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_n \end{Bmatrix} &= \left(\frac{H}{\rho} \frac{1}{\Delta^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & 2 \\ & & & & -2 \end{bmatrix} \right. \\
 &+ \frac{H}{\rho} \frac{1}{2\Delta} \begin{bmatrix} \frac{A'_1(r)}{A_1(r)} & & & & \\ & \frac{A'_2(r)}{A_2(r)} & & & \\ & & \ddots & & \\ & & & \frac{A'_n(r)}{A_n(r)} & \\ & & & & \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \\
 &+ \begin{Bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{Bmatrix} \Bigg) \quad (B.9)
 \end{aligned}$$

Where X_n in the last element of mass matrix (i.e. coefficient matrix of $\{\ddot{u}_i\}$) is defined as

$$X_n = 1 + \frac{2M_p}{\rho A_n} \frac{1}{\Delta} + \frac{M_p}{\rho A_n} \frac{A'_n}{A_n} \quad (B.10)$$

Eqn 2.28 is in the form of a multi degree of freedom spring- mass system $[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$. And the $[M]$ and $[K]$ matrices are the coefficient matrices of $\{\ddot{u}\}$ and $\{u\}$ respectively.

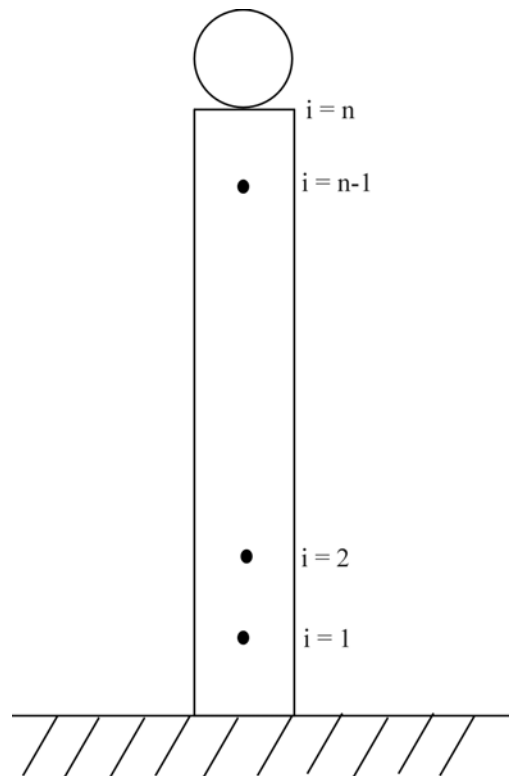


Figure B.1: Descretization of elevator into n number of elements

APPENDIX C

NUMERICAL ANALYSIS OF LINEAR EQUATIONS OF MOTION

C.1. Modal Analysis

Modal analysis is used to transform the equations of motion from physical coordinate system to the principal coordinate system using the orthogonality properties of the modes of vibration. The advantage of transforming to principal coordinate system is that we can decouple the equations of motion and each of the decoupled equations can be solved as a system rather than solving n ordinary differential equations [15].

The equation of motion of elevator for the longitudinal motion including the tidal forces can be written from Appendix B as,

$$[M]\{\ddot{u}\} + [K]\{u\} = \{p_u(t)\}. \quad (C. 1)$$

$\{p_u(t)\}$ is a column vector defined as

$\{p_u(t)\} = [p_u(x_1, t) \ p_u(x_2, t) \ \dots \dots p_u(x_n, t)]^T$. Where $p_u(x, t)$ is the sum of tidal forces due to moon and sun acting in the axial direction on the elevator at a distance x away from the surface of earth.

$[M]$ and $[K]$ matrices are given by Eqn B.9 and can be written as,

$$\begin{aligned}
 [K] = & - \left(\frac{E}{\rho \Delta^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & 2 & -2 \end{bmatrix} \right. \\
 & + \frac{E}{\rho 2\Delta} \begin{bmatrix} \frac{A'_1(r)}{A_1(r)} & & & & \\ & \frac{A'_2(r)}{A_2(r)} & & & \\ & & \ddots & & \\ & & & \frac{A'_n(r)}{A_n(r)} & \\ & & & & X_n \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & & \ddots & \\ & & & 0 & 0 \end{bmatrix} \left. \right) \\
 [M] = & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & X_n \end{bmatrix}, \tag{C.2}
 \end{aligned}$$

Where X_n is given by Eqn B.10.

By transforming $\{u(t)\}$ to modal or principal coordinate system $\{\eta(t)\}$ in terms of eigen vector matrix $[U]$

$$\{u(t)\} = [U] \{\eta(t)\} \tag{C.3}$$

Where $[U] = [\{U_1\} \{U_2\} \dots \{U_n\}]$ and $\{U_1\}, \{U_2\}, \dots, \{U_n\}$ are the eigen vectors corresponding to the first, second n^{th} modes of vibration.

After substituting Eqn C.3 in Eqn C.1, Eqn C.1 can be written as,

$$[M][U] \{\ddot{\eta}(t)\} + [K][U] \{\eta(t)\} = \{p_u(t)\} \tag{C.4}$$

Multiplying Eqn C.4 by $[U]^T$ gives,

$$[U]^T [M] [U] \{\ddot{\eta}(t)\} + [U]^T [K] [U] \{\eta(t)\} = [U]^T \{p_u(t)\}. \quad (C. 5)$$

From modal orthogonality and normalization properties we can write,

$$[U]^T [M] [U] = [I] \quad (C. 6)$$

$$[U]^T [K] [U] = [L]$$

$$[U]^T \{p_u(t)\} = \{Q(t)\}$$

Where $[I]$ is $n \times n$ identity matrix, $[L]$ is a diagonal matrix with the diagonal elements as the square of natural frequencies of n no. of modes of vibration and $\{Q(t)\}$ is the tidal force vector.

Using Eqn C.6, Eqn C.5 is given by,

$$\{\ddot{\eta}(t)\} + [L] \{\eta(t)\} = \{Q(t)\}. \quad (C. 7)$$

Let's consider the Eqn C.7 at i^{th} node and is given by,

$$\ddot{\eta}_i(t) + \omega_i^2 \eta_i(t) = Q_i(t). \quad (C. 8)$$

$Q_i(t)$ in the above equation is the sum of tidal forces due to moon and sun i.e. $Q_i(t) = Q_{mi}(t) + Q_{si}(t)$. $Q_i(t)$ can be expanded in terms of fourier series as

$$Q_i(t) = A_0 + \sum_{j=1}^{\infty} A_j \cos(\omega_{mj}t) + B_0 + \sum_{j=1}^{\infty} B_j \cos(\omega_{sj}t), \quad (C. 9)$$

Where $\omega_{mj} = j * \omega_m$, $\omega_{sj} = j * \omega_s$, $\omega_m = \frac{2\pi}{25 \times 3600}$ and $\omega_s = \frac{2\pi}{24 \times 3600}$.

The Fourier coefficients in Eqn. C.9 are given by,

$$A_0 = \frac{\int_0^t Q_{mi}(t) dt}{t}, \quad A_j = \frac{\int_0^t Q_{mi}(t) \cos(\omega_{mj}t) dt}{\int_0^t \cos(\omega_{mj}t)^2 dt}, \quad (C. 10)$$

$$B_0 = \frac{\int_0^t Q_{si}(t) dt}{t} , \quad B_j = \frac{\int_0^t Q_{si}(t) \cos(\omega_{sj}t) dt}{\int_0^t \cos(\omega_{sj}t)^2 dt} .$$

The solution of Eqn. C.8 is given by, $\eta_i(t) = \eta_{hi}(t) + \eta_{pi}(t)$. The particular solution ($\eta_{pi}(t)$) and homogeneous solutions ($\eta_{hi}(t)$) are given by,

$$\eta_{pi}(t) = \frac{A_0}{\omega_i^2} + \sum_{j=1}^{\infty} \frac{A_j}{\omega_i^2 - \omega_{mj}^2} \cos(\omega_{mj}t) + \frac{B_0}{\omega_i^2} + \sum_{j=1}^{\infty} \frac{B_j}{\omega_i^2 - \omega_{sj}^2} \cos(\omega_{sj}t), \quad (C.11)$$

$$\eta_{hi}(t) = - \left(\frac{A_0}{\omega_i^2} + \sum_{j=1}^{\infty} \frac{A_j}{\omega_i^2 - \omega_{mj}^2} + \frac{B_0}{\omega_i^2} + \sum_{j=1}^{\infty} \frac{B_j}{\omega_i^2 - \omega_{sj}^2} \right) \cos(\omega_i t) .$$

The total response of the elevator is $u(t) = [U] \eta(t)$.

APPENDIX D

NUMERICAL ANALYSIS OF NONLINEAR EQUATIONS OF MOTION

Based on the nonlinear coupled equations of motion derived in Chapter 3, the equations of motion of the i th node are given by

$$\ddot{u}_i + 2\Omega\dot{w}_i - (r_0 + L + u_i)\Omega^2 - \frac{E}{\rho} \left(\varepsilon'_i(1 + u'_i) + \varepsilon_i \left(u''_i + \frac{A'(L)}{A(L)}(1 + u'_i) \right) \right) + \frac{gr_0^2(r_0 + L + u_i)}{r^3} - \frac{P_u}{\rho A(X)} = 0 \quad (D.1)$$

$$\ddot{v}_i - \frac{E}{\rho} \left(\varepsilon'_i v'_i + \varepsilon_i \left(u''_i + \frac{A'(L)}{A(L)} v'_i \right) \right) + \frac{gr_0^2 v_i}{r^3} - \frac{P_v}{\rho A(X)} = 0 \quad (D.2)$$

$$\ddot{w}_i - 2\dot{u}_i\Omega - \Omega^2 w_i - \frac{E}{\rho} \left(\varepsilon'_N w'_i + \varepsilon_i \left(w''_i + \frac{A'(L)}{A(L)} w'_i \right) \right) + \frac{gr_0^2 w_i}{r^3} - \frac{P_v}{\rho A(X)} = 0 \quad (D.3)$$

which is valid for $i=2$ to $n-1$. The spatial derivatives are replaced by

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2\Delta}, \quad (D.4)$$

$$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta^2}.$$

where y can be replaced by u, v or w depending on the direction of motion under consideration. For $i=1$, we use $u_0 = v_0 = 0$ to solve the above equations of motion.

For $i=n$, we need to make use of the boundary conditions at the point mass. If the elevator is divided in to N number of nodes, the boundary conditions at the end of the elevator are given by

$$\left[M_p(\ddot{u}_N + 2\Omega\dot{w}_N - (r_0 + L + u_N)\Omega^2) + EA(L)\varepsilon_N(1 + u'_N) + M_p \frac{gr_0^2(r_0 + L + u_N)}{r_p^3} \right] \Big|_{L,t} = 0 \quad (\text{D.5})$$

$$\left[M_p\ddot{v}_N + EA(L)\varepsilon_N v'_N + M_p \frac{gr_0^2 v_N}{r_p^3} \right] \Big|_{L,t} = 0 \quad (\text{D.6})$$

$$M_p(\ddot{w}_N - 2\Omega\dot{u}_N - \Omega^2 w_N) + EA(L)\varepsilon_N w'_N + M_p \frac{gr_0^2 w_N}{r_p^3} \Big|_{L,t} = 0 \quad (\text{D.7})$$

Eqns. E.5, E.6 and E.7 will be solved to find ε_N , which is a function of $\ddot{u}_N, \ddot{v}_N, \ddot{w}_N$.

From Finite difference formulae, we can write

$$u''_N = \frac{-2u_N + 2u_{N-1}}{\Delta^2} + \frac{2}{\Delta} u'_N \quad (\text{D.8})$$

$$v''_N = \frac{-2v_N + 2v_{N-1}}{\Delta^2} + \frac{2}{\Delta} v'_N \quad (\text{D.9})$$

$$w''_N = \frac{-2w_N + 2w_{N-1}}{\Delta^2} + \frac{2}{\Delta} w'_N \quad (\text{D.10})$$

Since u'_N, v'_N, w'_N are functions of $\ddot{u}_N, \ddot{v}_N, \ddot{w}_N$ respectively, u''_N, v''_N, w''_N are also functions of $\ddot{u}_N, \ddot{v}_N, \ddot{w}_N$.

ε'_N can be written as,

$$\varepsilon'_N = u''_N + u'_N u''_N + v'_N v''_N + w'_N w''_N \quad (\text{D.11})$$

Equations of motion at the Nth can be expressed in terms of three functions F, G and H as,

$$F = \ddot{u}_N + 2\Omega\dot{w}_N - (r_0 + L + u_N)\Omega^2 - \frac{E}{\rho} \left(\varepsilon'_N(1 + u'_N) + \varepsilon_N \left(u''_N + \frac{A'(L)}{A(L)}(1 + u'_N) \right) \right) + \frac{gr_0^2(r_0 + L + u_N)}{r^3} - \frac{P_u}{\rho A(X)} = 0 \quad (\text{D. 12})$$

$$G = \ddot{v}_N - \frac{E}{\rho} \left(\varepsilon'_N v'_N + \varepsilon_N \left(u''_N + \frac{A'(L)}{A(L)} v'_N \right) \right) + \frac{gr_0^2 v_N}{r^3} - \frac{P_v}{\rho A(X)} = 0 \quad (\text{D. 13})$$

$$H = \ddot{w}_N - 2\dot{u}_N\Omega - \Omega^2 w_N - \frac{E}{\rho} \left(\varepsilon'_N w'_N + \varepsilon_N \left(w''_N + \frac{A'(L)}{A(L)} w'_N \right) \right) + \frac{gr_0^2 w_N}{r^3} - \frac{P_v}{\rho A(X)} = 0 \quad (\text{D. 14})$$

Newton-Raphson method is used to solve the non-linear equations E.12, E.13 and E.14 for the accelerations $\ddot{u}$, $\ddot{v}$ and $\ddot{w}$ as

$$\begin{Bmatrix} \ddot{u}_N \\ \ddot{v}_N \\ \ddot{w}_N \end{Bmatrix}_{i+1} = \begin{Bmatrix} \ddot{u}_N \\ \ddot{v}_N \\ \ddot{w}_N \end{Bmatrix}_i - \begin{bmatrix} \frac{\partial f}{\partial \ddot{u}_N} & \frac{\partial f}{\partial \ddot{v}_N} & \frac{\partial f}{\partial \ddot{w}_N} \\ \frac{\partial g}{\partial \ddot{u}_N} & \frac{\partial g}{\partial \ddot{v}_N} & \frac{\partial g}{\partial \ddot{w}_N} \\ \frac{\partial h}{\partial \ddot{u}_N} & \frac{\partial h}{\partial \ddot{v}_N} & \frac{\partial h}{\partial \ddot{w}_N} \end{bmatrix}_i^{-1} \begin{Bmatrix} f(\ddot{u}_N) \\ g(\ddot{v}_N) \\ h(\ddot{w}_N) \end{Bmatrix}_i \quad (\text{D. 15})$$

Displacements of the elevator in axial, meridional and equatorial directions u , v and w are calculated numerically from the acceleration terms.

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