Messiah Scattering Theory Notes, Chaps 10 and 11 PhL 12.24.08

I just finished Schiff's first foray into **scattering theory** (Chapter 5 Schiff), and thought I would *peruse* Messiah on this subject, since I just perused Saxon yesterday. The same results are obtained, but Messiah has different angles on things, different words, different emphasis, different notation, and much more detail on computation of the phase shifts which I did not really follow because I am not planning on doing that soon.

Messiah's book is **1958** (his preface date) so has exactly a half-century of aging, and there is now a $20 (street) Dover single volume edition.

ERRATA for Vol I on the back of the Vol 1 preface page. No errata for Vol II exists.

These notes concern Messiah's Chapters 10 (scattering theory) and 11 (Coulomb scattering *and* bound state theory) These sections fall within Part II : Simple Systems. His Chapter 19 which I have not read concerns approximation methods used for scattering problems, such as the Born Approximation.

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The organization of Messiah's book

It is a bit complicated. There are four levels of hierarchy. At the highest level we have Parts and that structure is this

Part I : **Formalism** // these 2 parts make Volume I
Chapters I through VIII.
Part II : **Simple Systems**
Chapters IX through XII

Part III : **Symmetries** // these 3 parts make Volume II
Chapters XIII to XV
Part IV : **Approximation**
Chapters XVI to XIX
Part V: **Relativistic**
Chapters XX and XXI

Within a Part, there are three levels of structure which I might call Chapter, Topic, Section. As you see above, both the Parts and the Chapters bear Roman numerals, but the Chapter numbers continue upwards through the entire two volumes, they don't restart with each part, so there are 21 chapters total. Equations always bear the Chapter number, so an equation in Chapter V might be (V.112).

Within each Chapter, the topics are I,II,III ...., while the Sections are 1,2,3... . In each Chapter the topic numbers start afresh with I,II, ... and so do the Section numbers. These numbers continue upward through the chapter and don't restart with each topic. So, to uniquely locate something, you need specify only a Chapter number and a Section number. Here is an example of this structure.

Part I : Formalism

Chapter I Origins of Quantum Theory
Section 1. Introduction
Topic I The End of the Classical Period
Section 2.
Section 3.
Topic II The End of the Classical Period
Section 4.
Section 5.
Section 6. etc
etc

Chapter II Matter Waves and the SE
Section 1. Introduction
Topic I whatever
Section 2.
etc

About Albert Messiah

Messiah has *very* thin web presence. He never wrote another book. Messiah taught courses at Saclay starting in 1953 Centre d'Études Nucléaires de Saclay (now CEA-Saclay). One site says he fought in the French Free Army in Africa under Leclerc in WWII (circa 1940-43).

Saclay is a commune in the southwestern suburbs of Paris, France. It is located 18.7 km (11.6 mi) from the center of Paris. It is best known for the large scientific facility CEA Saclay, mostly dealing with nuclear and particle physics.

Here is an interesting hit from 1986:

Albert Messiah
Inst. de Res. Fondamentale, Commissariat a l'Energie Atomique, C.E.N. de Saclay, Gif-sur-Yvette, France

Professor ALBERT MESSIAH, Sackler Visiting Scholar 1985/1986. Directeur de Recherches, Institut de Recherche Fondamentale, Commissariat a l'Energie Atomique, C.E.N. de Saclay, Gif-sur-Yvette, France.

http://www.cea.fr/recherche_fondamentale , but search yields nothing on this site.

Saclay was probably like a French version of Los Alamos and Messiah was a physics guy there.

I think his papers all say A. M. Messiah or A. M. L. Messiah. The APS archive

http://prola.aps.org/search/field/author/Messiah_A_M_L

shows a string of papers 1951-1966 with other people, all with the 2 initials.

Aha! I searched using Directeur de Recherches and found this

www.uarga.org/downloads/Documentation/Lettre_J.Chirac_Pr.Pellerin.pdf

which is a letter dated **2005** with Albert as a signer. It is in French and refers to our boy as an ancien Director which might mean "former" in this context. In 1972 Messiah wrote a bio for Bloch who died.

This letter above relates to Pellerin who was a French AEC type guy who downplayed Cherobyl and then in a certain area 500 people got cancer, so in 2005 he was feeling heat, Chirac as well.

Conclusion: This guy probably stopped writing papers in 1966 and was made the Research Director so got buried in management stuff. It seems he was still Director in 1986 so that would explain the lack of papers. He must have been born ≤ 1920 to be in African warfare in 1940, so he would be maybe 90 years old in 2009. So maybe he retired at age 70 in 1989. When he does die, surely someone will put up a bio. He may have personal reasons to stay off the web, who knows.

Chapter X: Scattering Problems. Central Potential and Phase Shift Method

1. Introduction What you can measure is a cross section, and this is related to asymptotic forms of wave functions. In this chapter, we are going to do central potential only and look at the phase shift idea. This is all non-relativistic QM using the SE.

I. Cross Sections and Scattering Amplitudes .

2. Definition of the Cross Section. Uses the nice word monoërgic, not sure what language this is from. Develops the idea of a cross section in terms of flux and outgoing dΩ angle. Mentions that beam particles don't interact with each other, and target particles are randomly spaced perhaps and don't interact so as if you just had a sum of experiments on individual targets with individual beam particles. This is the kind of detail you always see in Messiah. Lots of good descriptive words. So we get the differential and total cross sections defined. A cross section that was 1A x 1A would be 10^{-20} m^2 = 10^{-16} cm^2 would be 10^8 barns. A target that was 10^{-15}m x 10^{-15}m = 10^{-30} m^2 = 10^{-26} cm^2 = 10 millibarns. Electron radius is maybe 3 times 10^{-15} m so get maybe 100 millibarn. We are only talking elastic scattering which he defines as meaning the internal quantum state of the beam particle does not change.

3. Stationary Scattering Wave. We get the TISE and the usual asymptotic assumed form for scattering with f(Ω) called "the scattering amplitude". Flux ratio shows that f^2 = cross section.

4. Wave Packets used to represent a scattering experiment. This and the next two sections form a group. The text is more closely spaced to indicate this is sort of optional material on a special topic, though the half-page preface says nothing about this formatting method. Albert says he has taken this discussion from some writings of Chew and Low, but no reference is given. The subject is how you fit the notion of actual "particles" being scattered with the TISE solution (the stationary scattering state) we are talking about. You imagine a particle being a relatively large wave packet that varies smoothly with a narrow **k** band, and in particular is much larger than the interaction region. We are always in the non-Coulomb case in these sections. He comes up with a series of inequalities that have to be met for the stationary solution to be applicable. Four dimensions are identified as being important: λ of the beam particles, D is distance away of detectors, and d and l are the transverse and longitudinal dimensions of the incoming packets. Pictures are drawn of the experiment. One obvious rule is that a << d,l where a has its usual meaning as the range of the potential. And D cannot be so large that your packets spread way out getting to the detector, which causes $\sqrt{D\lambda}$ << d and l. And you have to keep the detector (counter he

calls it) away from the incident beam. In a footnote, he gives typical numbers and says these things are always well satisfied.

5. Scattering of a Wave Packet by a Potential. A spatial shape $\chi(\boldsymbol{\rho})$ is assumed for the incoming packet, be careful because he is using $\boldsymbol{\rho}$ for $\mathbf{r}$, and using $\boldsymbol{\kappa}$ for $\mathbf{k}$, but his $\boldsymbol{\kappa}$ looks a lot like an $\mathbf{x}$ and makes you think space instead of momentum. So $A(\boldsymbol{\kappa})$ is the momentum profile. At time 0 the packet is called $\Phi_{\mathbf{b}}(\mathbf{r},0)$ and has the local shape χ, but is positioned at some $\mathbf{r} = \mathbf{b}$ location. Notice that the combination of this location $\mathbf{b}$ and the beam direction $\hat{\mathbf{k}}$ implies an impact parameter "b", but this is not quite made clear. In accordance with our known effect of translation on a wave function, we just pick up a phase and hence $\Phi_{\mathbf{b}}(r,0) = e^{i\mathbf{k}\bullet(\mathbf{r}-\mathbf{b})}\chi(\mathbf{r}-\mathbf{b})$. We then know how to pull this out to time t in terms of the A(k) values and he centers his dk' integral on the value k which is the strong peak of A(k). Roughly the incoming wave is then X.13. I then stop following the details, but the final result is X.20 where we see that our scattered wave portion has the expected form, but is multiplied by our assumed packet shape centered at some time-dependent location, where $\mathbf{s}$ is the gradient of f in k-space which seems to be an adjustment to the packet's position.

6. Calculation of Cross Sections. This section then completes the discussion and shows that if our inequalities are met, what our experiment actually measures is the stationary f^2, and our correction factor integrates away to unity. So Messiah has attacked this issue which every reader of the canonical description of scattering must wonder about. Basically, during the time the packet is overlapping the scattering region, you have the scattering happening and that stationary wave is everywhere, you are burning the target. Before and after this possible overlap, the packet is just a packet. You just decompose the packet as the usual superposition of things (in and out going) and the components do the simple thing, and then the packets do particle scattering, and you are able to actually measure f^2 in this way.

7. Collision of two particles: LAB and CMS. This is very well done, similar to Schiff but better. The whole change in cross section is just $d\Omega/d\Omega_1$ where 1 means lab. He comes up with the famous $\tan\theta$ formula which is converted to a $\cos\theta$ formula in X.24. He uses τ for what Schiff calls the ratio γ, and here it is just the mass ratio since only elastic is being considered. I guess elastic means that the entire "relative particle" does not change internal quantum state, and this includes the beam and the target particle. He then considers the two ranges of γ, just as Schiff did. All highly organized, and we get X.26 as the relation between the two cross sections.

II. Scattering by a Central Potential: Phase Shifts

8. Decomposition into Partial Waves: Phase Shift Method. In the usual manner, M finds the form of the $f(\theta)$ partial wave amplitude he calls f_ℓ , but he fails to put it into the form I like of $e^{2i\delta} - 1$ which shows that the scattering part is what picks up the phase shift, though he shows this bottom of page 386. He then replicates Schiff's formulas for $d\sigma/d\Omega$ and σ_{tot} . All is well. Uses the same method of balancing the in and outgoing parts. Oddly, he never mentions that the solution of the SE that is regular at r=0 is $j_\ell(kr)$, he just calls it y_ℓ . He does not explain very well why X.30 is true, but I know why and it is true you don't need to know anything about j_ℓ . A few more words here might have been good.

9. Semiclassical Representation of the Collision. Impact Parameters. This is the Schiff page 121 idea that only the lower partial waves up to some ℓ_{max} contribute at a given energy k. M argues that the

effective impact parameter for a partial wave is b = ℓ/k. He then looks at the lowest several Bessel functions and we see how there will be no interaction if a is small, and so on. I drew my own pictures of these things in my Schiff notes.

III. Potential of Finite Range

10. Relation between the phase shift and the log derivative. This is Schiff page 121 bottom, but I don't follow all of the M details. Schiff said you want to match the ratio of value to slope of the radial function at the boundary between the core region and the intermediate region where centrifugal still operates but V does not and this gives you a way to compute phase shifts from the potential. This R ratio is that log derivative M is talking about which he then calls q_ℓ . I am going to skip the details here for now, mention of a penetration factor and breakdown of $\delta_\ell = q_\ell + \tau_\ell$ where the τ_ℓ does not depend on V(r).

Let's try harder on this because the result keeps being used. X.36 defines q_ℓ in terms of the interior potential-dependent radial function y_ℓ at some $r = r_o$, fine. This thing must be the γ_ℓ of Schiff on page 121 equation A. Outside this radius, we have some lincomb of j and n that is determined by δ. He shows this in A, but this disagrees by a sign from Schiff 19.7 perhaps due to how n is defined. Yes, Messiah confirms this on page 489 in a footnote, fine. M then shows asymptotic forms for his radial function but expressed in terms of this rescaled Hankel thing $u^\pm$. Having re-expressed his y_ℓ in terms of these $u^\pm$ functions, the calculation of the q_ℓ is then as in XC.40. But this is all "on the outside" where we have no potential so yes, it is a function of δ_ℓ since we assumed this in X.37 for example. This gives q_ℓ as a function of δ_ℓ, something he was "seeking".

Next, he evaluates various pieces of X.40 at the boundary $r = r_o$. He then defines a certain v_ℓ in X.41 as related to the abs value of the $u^\pm$ at the boundary, and v_ℓ is the "penetration factor". He then defines the objects $q_\ell^\pm$ to be the log derivatives of the u function itself (at the boundary), not y as in X.36. How he ends up with E is not clear, he must know that there is some phase τ_ℓ which is unknown but makes this an equality. The right part of E (the ratio) must be some other phase, and he calls that ratio $e^{2i\rho_\ell}$, so I guess I have to agree with X.42 with X.43.

Well, not totally happy here but I did a little more. By the way, he does make use of the Wronskian he throws out, the use is a few equations later.

11. Limit of the phase shift at very small k. This is now Schiff page 122 stuff, only the ℓ=0 survives in this limit, scattering is isotropic, a scattering length a is defined which is just $\delta_0 = -a/k$ and in terms of this the cross section is $4\pi a^2$. This is more than I want to know, but glad M has it all in writing here. He notes that there could be a resonance in this $\ell = 0$ wave.

12. Convergence of the partial wave series. Yes, it converges, and this is more Schiff page 122.

13. Scattering by a Hard Sphere. He gets the Schiff result for cross section in X.50, and then he examines the low and high energy limits of this thing At low we have only ℓ=0 and we get $4\pi a^2$. He ends up with some mysterious shadow comments relating to the high k limit. This stuff has a similar haziness to Schiff's discussion, I guess I am equally confused by both presentations.

IV. Resonances

14. Scattering by a deep square well. This is not as hard as the hard sphere above. OK, he is going to treat this situation and show that there is a resonance whenever a well bound state appears just below the top. I am skipping this because I am just not yet into his set of variables.

15. Study of a Scattering Resonance: Metastable states. Here, your f(θ) is dominated by a resonance in some partial wave ℓ as shown in 64 (as function of energy of course), and the claim is that for the deep well above, we find that at resonance, the wave function is penetrating into the well whereas normally this does not happen. This is just that 1D situation we saw in those "movies".

Now from a particle point of view, it you have a metastable "particle" in the s-channel, your scattering amplitude will show perhaps a peak in a partial wave amplitude for ℓ that corresponds to the spin of the particle. Perhaps this is what the rho resonance does, just dimly remembering.

16. Observation of the Lifetime of Metastable States. (401). This is one of these close-lined optional sections I think. He points out here that ordinarily a cross section experiment won't be able to see a resonance due to the packet spread in energy. So somehow he comes up with a way to see it after all, and that is what this section is about.

IV. Various Formulae and Properties

The problem is computing the phase shifts from the potential, and I think all this stuff regards methods by which you might do that.

17. Integral Representations of the Phase Shifts. **Just scanned this, OK.**

18. Dependence on the potential: sign of the phase shifts.

19. The Born approximation. He is jumping ahead of me in Schiff now, so I hold on this. At the end Chapter 5, Schiff has not yet powered up the whole integral equation scattering formalism yet where Born applies. Schiff was doing scattering just with differential equations and asymptotic behavior of their solutions.

20. Effective Range Theory: The Bethe Theory. Save for a rainy day.

Chapter XI: The Coulomb Interaction

1. Introduction. The reduced mass Hamiltonian is clearly stated, and we are going to first do the bound state problem (H atom), and then the collision problem. So he has gathered these two subjects into the same book unit, an interesting idea. In Goldstein, the elliptical and hyperbolic orbits are treated uniformly in the same book section. Schiff chose to separate these two developments into their respective chapters.

I. The Hydrogen Atom

2. The SE. He takes us to what Schiff (and therefore I) call the χ equation with no first derivatives. For scattering, $\varepsilon = k^2$, but for bound state we use $\kappa = ik$ as in X1.6. Comments on the hydrogen-like atomic situations.

3. Order of Magnitude Estimate of ground state energy. This is hard to resist. If confined to some radius a, then $\Delta p\ a = \hbar$ and then $K = \Delta p^2/2\mu$ and you then want to minimize $K+V = (\hbar/a)^2/2\mu - e^2/a$. It just happens that this minimizes at a = Bohr and E = -13.5 eV, the exactly correct non-rel answer! One was only hoping to get ball park here, a "mere coincidence" M points out. But a great teaching fact.

4. Solution in Sphericals Changes functional form to extract the power and expo and ends up with ODE XI.12 which is confluent and the F solution is stated with $c = 2\ell+2$ and $a = \ell+1-\lambda$ where $\lambda = n$ it will turn out. This will become "i n" I think in the scattering problem. The confluent series must terminate and we thereby get that n' must be integer, and the principle n is defined in the usual way. Appendix B has properties of interest

5. The Spectrum. It is stated as $E = -K/n^2$ as usual. M comments that as $E \to 0$ from below, states approach the continuum and are denumerable for this 1/r class potential. For faster potentials, the bound state spectrum abruptly becomes continuous at E=0, having been finitely discrete for E<0. In order to truncate the series, we must have n' = 0,1,2,3.... If we just define the PQN $n = n' + \ell + 1$, then if you classify by n, then of course n = 1,2,3.. as well. Then for given n, you have $n' = n - (\ell+1) = 1,2,3$ so that means the maximum allowed ℓ is n-1, since we have $n' = (n - 1) - \ell$. I keep getting confused on this simple fact. Degeneracy of n is n^2 when you add them in the usual way. I think in parabolics, each separated Laguerre has n, so again you get n^2. Nice picture of the spectrum is shown. We have yet to add relativity and spin. The relativity effects are only order 10^{-4} or smaller.

6. The Eigenfunctions. Stated in terms of associated Laguerres for those confluents. He notes that you need to use that generating function thing to do computations and this gives the normalizer as shown. He then shows that

$$\langle r \rangle_{n'\ell} = (a/2)\ [\ 3n'^2 + (\ell+1)(2n' + 2\ell + 3)\ ]$$

where I did a little pencil conversion. He notes correctly that n' is the number of notes in the radial function which starts with n'=0 and no nodes at all.

Now what about the classical limit and the correspondence principle? It turns out that the solutions with maximal ℓ become, for large n (and thus large ℓ) the Bohr circular orbits. Perhaps the other values of ℓ give some of the elliptical orbits of "the old quantum theory", plots are on page 38. This is based on that Sommerfeld idea of quantized action. The way M shows that we get the circular orbit idea is that the standard deviation of r does this: $\sigma(r)/\langle r \rangle \to 0$ as $n \to \infty$ and this means the distribution for probability is going to a shell at $\langle r \rangle$, and you pick a space direction and get a circular orbit I guess. Really need to include the angles as well to do this right, but we get the point. This might be one of those cases considered in Schrodinger's paper on correspondence examples (Google books only)

And so we conclude on this very famous and significant problem.

II. Coulomb Scattering

7. The Coulomb Scattering Wave. M writes our good old SE, then makes the famous ansatz XI.24 and arrives at the confluent equation of variable ik(r-z) with c = 1 and a = -iγ. Messiah uses γ where Schiff uses "n". M never mentions the word "parabolic", there is no need really, we have a complete exact solution bang! He uses W_1 and W_2 for the two Bateman Ψ functions which appear in Φ = Ψ + Ψ' as keep abbreviating this equation. We take the large-arg limits using the (now famous to me) Appendix B and we end up with the Coulomb variation of the e^{ikz} + f(θ)e^{ikr}/r format.

8. The Rutherford Formula. M He makes my argument about dropping 1/r terms in flux and so we get the same relation between cross section and f. He says that the fact that we get the exact classical Rutherford is "an accident", but I suspect there is a deeper reason for this somewhere. As with Rutherford, the cross section is the same whether Coulomb force is attractive or repulsive because only n^2 appears in the result. The θ dependence is the same for all k, and the total cross section is infinite due to the forward direction, and in practice this is never an issue.

9. Partial Waves. Now we are doing spherical coordinates again, so we start with our χ radial equation in 37 and do the usual functional change and get 39 which is our confluent again, and it is solved by the famous F_ℓ function, probably similar to the A&S one. So all we've said in this section is that F_ℓ is the regular solution and it is F in argument ξ = -2ikr and roughly he then refers to this solution F_ℓ as the imaginary part of a complex version of the same function .

10. Partial Waves. Now we have a messy section with lots of math the goal of which is to obtain the large-r limit shown in XI.50 which I think is showing my fact that the entire left term "plane wave" is inbound, while the scattered stuff with the $e^{2i\sigma_\ell}$ phase shift is outbound. It seems strange that M is not producing a partial wave expansion for $f_c(\theta)$ as in Schiff 21.22, but it is of course there implicitly. Somewhere he points out that this partial wave expansion does not converge very fast.

110. Modifications due to another short range force OK, here we finally end up with the f(θ) result in a form like 21.27. The real answer here comes from my simple notes and is this:

$$f(\theta) = (1/2ik) \Sigma_\ell (2\ell+1) P_\ell(z) \; e^{+2i(\sigma_\ell+\varphi_\ell)}$$

and then you can rewrite it as in Schiff 21.27 to separate the coulomb and non-coulomb parts,

$$f(\theta) = (1/2ik) \Sigma_\ell (2\ell+1) P_\ell(z) \, 2 \, e^{+2i\sigma_\ell} (e^{+2i\varphi_\ell} - 1) + (1/2ik) \Sigma_\ell (2\ell+1) P_\ell(z) \; e^{+2i\sigma_\ell}$$

$$= f_{mod_short_range}(\theta) + f_c(\theta)$$

and if you have only a few partial waves, this let's you detangle the coulomb part of the cross section with enough effort. M claims that with both these forces active, the partial wave series converges fast again.