Stern Gerlach Problems for spin-1/2 PhL 3.26.05

1. System in a linear combination of energy eigenstates: neutrino oscillations

Consider a state α in a 2-state system where 1 and 2 are energy eigenstates:

$|\alpha> = a\,|1> + b\,|2>$ // at time t=0

At some later time we have [apply $U = e^{-itH}$ in the Schrodinger picture]

$|\alpha(t)> = a\,e^{-iE_1t}\,|1> + b\,e^{-iE_2t}\,|2> = e^{-iE_1t}\,[\ a\,|1> + b\,e^{-i(E_2 - E_1)t}\,|2>\ ]$

This clearly says that the mixture of |1> and |2> that makes up state $|\alpha(t)>$ is *changing in time*. The state $|\alpha>$ is NOT an energy eigenstate. Nevertheless it is true that

$<\alpha(t)|\alpha(t)> = <\alpha(0)|e^{+itH}\ e^{-itH}\ |\alpha(0)> = <\alpha(0)|\alpha(0)> = |a|^2 + |b|^2 = 1$

Also

$<1|\alpha(t)> = a$ // does not change in time
$<2|\alpha(t)> = b$ // does not change in time

However, consider this:

$<\alpha(0)|\alpha(t)> = [\ a^*<1| + b^*<2|\ ]\ e^{-iE_1t}\,[\ a\,|1> + b\,e^{-i(E_2 - E_1)t}\,|2>\ ]$

$= |a|^2\ e^{-iE_1t} + |b|^2\ e^{-iE2t}$

$|<\alpha(0)|\alpha(t)>|^2 = [\ |a|^2\ e^{-iE_1t} + |b|^2\ e^{-iE2t}]\ [\ |a|^2\ e^{+iE_1t} + |b|^2\ e^{+iE2t}]$

$= |a|^4 + |b|^4 + |a|^2\,|b|^2\ \{\ e^{-iE_1t}\,e^{+iE2t} + c.c\ )$
$= |a|^4 + |b|^4 + 2\,|a|^2\,|b|^2\cos(-i\Delta Et\ )$

To be specific, suppose $a = b = 1/\sqrt{2}$. Then we have

$|<\alpha(0)|\alpha(t)>|^2 = (1/2)[\ 1 + \cos(-i\Delta Et\ )] = \cos^2(\Delta Et/2)$
$|<\beta(0)|\alpha(t)>|^2 = (1/2)[\ 1 - \cos(-i\Delta Et\ )] = \sin^2(\Delta Et/2)$

So, at time progresses, the system does "oscillate" between the states $|\alpha(0)>$ and $|\beta(0)>$, showing the effect of quantum mechanical interference. This is vaguely similar to Feynman's electron interference discussion, but there the variable is screen width x, not t as it is here (vol 2)

There is a famous place where this occurs -- neutrino oscillation.

http://en.wikipedia.org/wiki/Neutrino_oscillation

In this case, $|\alpha(0)\rangle$ and $|\beta(0)\rangle$ represent neutrinos of two flavors, μ and τ. In the theory, these are not energy eigenstates, but each is a linear combination of energy eigenstates which are two neutrinos of slightly different mass m_i. As the above web site shows, in the extreme relativistic limit in which neutrinos with small mass would be, you have $E_i = E + m_i^2/2E$ where E is roughly pc , the energy it would have if massless. In this case, $\Delta E = \Delta m_i^2/2E$ and you find that

$$|\langle\beta(0)|\alpha(t)\rangle|^2 = \sin^2(L/c \; c^4\Delta m_i^2/ 4E)$$

where t = L/c and L is the distance the neutrino propagates. This interference has been observed and has a long story to it. In QCD there are really three neutrinos e μ and τ so the interference is more complicated with several "mixing angles" but here we have the basic idea. By the way

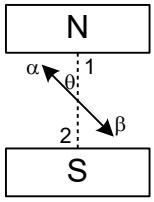
$$\frac{\Delta m^2\, c^3\, L}{4\hbar E} = \frac{\mathrm{GeV\,fm}}{4\hbar c} \times \frac{\Delta m^2}{\mathrm{eV}^2}\frac{L}{\mathrm{km}}\frac{\mathrm{GeV}}{E} \approx 1.267 \times \frac{\Delta m^2}{\mathrm{eV}^2}\frac{L}{\mathrm{km}}\frac{\mathrm{GeV}}{E},$$

so this says that if the mass difference is on the order of 1 eV and you deal with a 1 GeV neutrino, the scale of the oscillation distance is km, something you might actually measure.

2. Example: Stern Gerlach stuff

How Stern-Gerlach splits beams. What is the force on a magnetic moment **m** in a **B** field? Jackson page 149 tells us the answer that $\mathbf{F} = \nabla(\mathbf{m}\bullet\mathbf{B}) = (\mathbf{m}\bullet\nabla)\mathbf{B}$. Suppose $\mathbf{m} = m\hat{\mathbf{z}}$, then we get $\mathbf{F} = m\partial_z\mathbf{B}$. Suppose $\mathbf{B} = B_z(z)\hat{\mathbf{z}}$. Then we get $\mathbf{F} = m\partial_z B_z(z)\,\hat{\mathbf{z}}$. In a Stern-Gerlach beam splitter, as you move out of the end of the magnet rails, the $B_z(z)$ field becomes weaker so $\partial_z B_z(z)$ is some negative number. For m=+1, say, we will then have that $F_z = \partial_z B_z(z) < 0$ and such a moment is deflected downward. Forget the direction for now. Using several magnets, you can make a device which selects spin-up or spin-down electrons, see Feynman or the we (ref below). The output of the S-G simple experiment tells you that the B field direction selects the ups and downs in that direction, by the way, and you get two dots on an output screen, not some blur.

So, we could use a S-G filter to make a beam of up-electrons, and we could then send that to a second filter which was tilted at axial angle θ. Then relative to the B field in *that* filter, our electrons are in a mixed state! They are in state $|\alpha\rangle = \cos(\theta/2)\,|1\rangle + \sin(\theta/2)\,|2\rangle$, where θ is a fixed angle from the physical set-up. You can compute this fact by saying that $|\alpha\rangle = R_x(\theta/2)|1\rangle$ and then use the spin-1/2 rotation matrices given in Levitt for example. This then gives a real-world example of our α state.



end view of magnet rails

Here is a nice web site on S-G filters: (see also Feynman Vol III where he does it for spin 1 atoms).

http://www.upscale.utoronto.ca/GeneralInterest/Harrison/SternGerlach/SternGerlach.html

A few more details: (these 2 states are orthonormal)

$|\alpha\rangle = C_{\theta/2}\,|1\rangle + S_{\theta/2}\,|2\rangle \qquad\qquad |\beta\rangle = S_{\theta/2}\,|1\rangle - C_{\theta/2}\,|2\rangle$

Now what about the famous **precession**, how does that fit in here? Consider:

$\langle\alpha(t)|\mathbf{m}|\alpha(t)\rangle \sim \langle 1|\, R_x(-\theta/2)\, e^{+itH}\, \mathbf{I}\, e^{-itH}\, R_x(\theta/2)|1\rangle$

where $H = -\gamma B I_z$. Let's do these things as two separate rotations. Our general formula is this:

$\mathbf{Q} = \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;\mathbf{J}\;\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J}) = \mathbf{J}\cos\theta + \mathbf{J}\times\hat{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}(\hat{\mathbf{n}}\bullet\mathbf{J})(1-\cos\theta)$

and applying to the inner transformation we get exp argument as $-it(-\gamma B)I_z$ so $\beta = -\gamma tB$ and

$\mathbf{Q} = e^{+itH}\,\mathbf{I}\,e^{-itH} = \mathbf{I}\cos\beta + \mathbf{I}\times\hat{\mathbf{z}}\sin\theta + \hat{\mathbf{z}}\,I_z(1-\cos\beta)$

If we write $\mathbf{I} = \mathbf{I}_T + \mathbf{I}_L$ we get

$e^{+itH}\,\mathbf{I}_T\,e^{-itH} = \mathbf{I}_T\cos\beta + \mathbf{I}_T\times\hat{\mathbf{z}}\sin\beta$

$e^{+itH}\,\mathbf{I}_L\,e^{-itH} = \mathbf{I}_L$

and we know these describe our usual precession *of the spin operator*.

So start with S-G angle $\theta = 0$ so $|\alpha\rangle = |1\rangle$ then we get

$\langle 1|\mathbf{m}_T|1\rangle = \langle 1|\mathbf{I}_T\cos\beta + \mathbf{I}_T\times\hat{\mathbf{z}}\sin\beta\,|1\rangle =$

$= \langle 1|\,\mathbf{I}_T\,|1\rangle\cos\beta + \langle 1|\,\mathbf{I}_T\,|1\rangle\times\hat{\mathbf{z}}\sin\beta = 0$

The x and y spin matrices have no 11 component, so we get the RHS. As expected, a pure up state has no interesting $\langle\mathbf{m}_T\rangle$, and of course $\langle\mathbf{m}_L\rangle = 1/2\,\hat{\mathbf{z}}$. So we have no precession visible here.

Now let's try using a non-zero angle θ. In this case an easy way to get the result is this:

$$\langle\alpha(t)|\mathbf{m_T}|\alpha(t)\rangle \sim \langle\alpha(t)|\mathbf{I_T}|\alpha(t)\rangle \sim \langle\alpha(0)|e^{+itH}\,\mathbf{I_T}\,e^{-itH}|\alpha(0)\rangle$$

$$= [\,C_{\theta/2}\langle 1| + S_{\theta/2}\langle 2|\,]\,\mathbf{I_T}\cos\beta + \mathbf{I_T}\times\hat{\mathbf{z}}\sin\beta\,[\,C_{\theta/2}|1\rangle + S_{\theta/2}|2\rangle\,]$$

The diagonal matrix elements of $\mathbf{I}_T$ are zero as before, so we end up with

$$= \mathbf{A}\cos\beta + \mathbf{A}\times\hat{\mathbf{z}}\sin\beta \qquad \text{where} \quad \mathbf{A} = C_{\theta/2}S_{\theta/2}[\,\langle 1|\,\mathbf{I_T}|2\rangle + \langle 2|\,\mathbf{I_T}|1\rangle\,]$$

but only I_x survives in this last factor, and we have $\langle 1|\,\mathbf{I_T}|2\rangle + \langle 2|\,\mathbf{I_T}|1\rangle = 2\langle 1|\,I_x|2\rangle\hat{\mathbf{x}}$ which tells us that $\mathbf{A} = 2\,C_{\theta/2}S_{\theta/2}\langle 1|\,I_x|2\rangle\hat{\mathbf{x}} = \sin\theta\,\langle 1|\,I_x|2\rangle\hat{\mathbf{x}}$. So our result is now,

$$\langle\alpha(t)|\mathbf{m_T}|\alpha(t)\rangle = \sin\theta\,\langle 1|\,I_x|2\rangle\{\hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta\} = (1/2)\sin\theta\,\{\hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta\}$$

where $\beta = -\gamma tB$. So here finally we see our $|\alpha(t)\rangle$ precessing on its cone. We might add

$$\langle\alpha(t)|\mathbf{m_L}|\alpha(t)\rangle \sim \langle\alpha(0)e^{+itH}\,\mathbf{I_L}\,e^{-itH}\,|\alpha(0)\rangle = \langle\alpha(0)|\mathbf{I_L}|\alpha(0)\rangle =$$

$$[\,C_{\theta/2}\langle 1| + S_{\theta/2}\langle 2|\,]\,I_z\,\hat{\mathbf{z}}\,[\,C_{\theta/2}|1\rangle + S_{\theta/2}|2\rangle\,]$$

and now only the diagonal terms survive and we get

$$= \{\,C_{\theta/2}^2\langle 1|\,I_z|\,1\rangle + S_{\theta/2}^2\langle 2|\,I_z|\,2\rangle\,\}\hat{\mathbf{z}}$$

$$= \{\,C_{\theta/2}^2\,(1/2) + S_{\theta/2}^2(-1/2)\,\}\,\hat{\mathbf{z}} \quad = (1/2)\cos\theta\,\hat{\mathbf{z}}$$

which obviously fits our little precession model perfectly. To summarize:

$$\langle\alpha(t)|\mathbf{m}|\alpha(t)\rangle = (1/2)\,\gamma\,[\,\sin\theta\,(\hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta) + \cos\theta\,\hat{\mathbf{z}}\,]$$

where θ = our S-G fixed angle and $\beta = -\gamma tB$

which is exactly what I want the answer to be. The result for the $|\beta(t)\rangle$ are similar. In computing the $\mathbf{m}_T$ piece, we get **-A** in place of **A** so this part of the result changes sign. In computing the $\mathbf{m}_L$ piece we get this final result

$$\{\,S_{\theta/2}^2\,(1/2) + C_{\theta/2}^2(-1/2)\,\}\,\hat{\mathbf{z}} \qquad = -\,(1/2)\cos\theta\,\hat{\mathbf{z}}$$

so we get an overall minus sign, and our final precession answers are these:

$$\langle\alpha(t)|\mathbf{m}|\alpha(t)\rangle = (1/2)\,\gamma\,[\,\sin\theta\,(\hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta) + \cos\theta\,\hat{\mathbf{z}}\,]$$
$$\langle\beta(t)|\mathbf{m}|\beta(t)\rangle = -\,(1/2)\,\gamma\,[\,\sin\theta\,(\hat{\mathbf{x}}\cos\beta + \hat{\mathbf{y}}\sin\beta) + \cos\theta\,\hat{\mathbf{z}}\,]$$