BD V1 Chap 10 : Nonelectromagnetic Interactions PhL 11.19.08

10.1 Introduction (210) 1
10.2 Strong Interactions (211). 2
A. A Crude Theory. 2
B. Applications of this Crude Theory 3
Explain 10.5. 3
Explain 10.6. 4
Explain 10.7. 4
Explain 10.14 6
10.3 Isospin (222) 8
How to we convert these things to "Feynman Rules" 9
Explain 10.7. 9
10.4 Conserved currents (226) 11
A. Preliminaries and the Proton Current 11
B. The Pion Current and sum conserved 12
C. Repeat this all in isospin notation 14
Electromagnetic current. 14
Nucleonic current. 14
Isospin current. 15
10.5 Approximate Calculations: Nucleon-nucleon scattering. (227) N-N scattering 17
Derive 10.46 then E and F 17
Explain 10.47 through page 229A: how n-n, p-n and p-p are all "equal" 19
Non-relativistic limit and effective potential 21
Derive 10.51 21
Derive the angular average result page 230 C. 25
Derive 10.52 28
10.6 Meson Nucleon Scattering. (231) π-N scattering 29

This chapter provides a whirlwind tour basically of all of particle physics other than QED and gravitation. It starts with hadronic stuff, then ends up with weak interactions. Everything here is based on mimicking QED, they are just trying to show how the QED ideas might explain everything. Interestingly, this is more or less how the standard model exists about 44 years later (1964 to 2008).

10.1 Introduction (210)

The four known interactions are mentioned with their coupling constants:

strong	1	
EM	10^{-2}	1/137
weak	10^{-6}	
grav	10^{-40}	

It is mentioned that the theory of weak interactions has renormalization problems that as of 1964 were not resolved. And how to you do perturbation theory if g = 1?

10.2 Strong Interactions (211).

A. A Crude Theory.

We just plain imagine a graph like that on page 211 to "explain" nucleon scattering (especially neutron scattering where there is no Coulomb force), in analogy with QED. The question is this: can we use our QED tools to at least learn something?

It seems reasonable to assume some sort of neutral mediator particle like the photon which we assume has some mass μ. For low energy nucleon scattering we get a general factor akin to 10.1 and then 10.2. If you do the Fourier transform (I did this in pencil in Mandl margin page 21), you get the famous Yukawa potential as in A, all very approx.

BD claim without proof that the strong force has a range of 10^{-13} cm at most, and they say this is about 1/3 of the classical electron radius. This leads us (page 212A) to expect a mediator particle in the range of 200 MeV, and the pion is 140 MeV, not too far away. Yukawa made his conjecture in 1935, the pion was found in 1947.

Now on page 212-213 BD give an explanation of why the three pions have odd intrinsic parity, whatever this means (I have not reviewed that subject yet). I accept this as true for now, and I accept the claim made on page 213 that strong interactions tend to preserve parity as a symmetry. So the claim is that this means that our boson mediator field here called φ_0 is a pseudoscalar field or wavefunction. We put this fact on hold for a moment.

Now our basic "theory" will be encompassed by 10.3 and 10.4 which we must comment on. We need to start with 6.52 on page 96 which says this:

$$(i\partial\!\!\!/ - m)\psi = e A\!\!\!/ \psi = e\, A^{\mu}\, \gamma_{\mu}\, \psi \qquad \rightarrow \qquad (i\partial\!\!\!/ - m)\psi = g\, \varphi\, \Gamma\, \psi$$

This says that the Dirac equation is "driven by" something on the right side. What you see there got there via the "minimal prescription" for adding EM interactions to the Dirac theory. But in the abstract, you see the product of four items: a coupling constant e, a gamma matrix structure, the photon field A, and the Dirac field ψ. We want to concoct some reasonable thing to put here for strong interaction which drives the Dirac equation. The coupling constant will be "g" with the idea that g ~ 1. The fields will be ψ and φ, and the gamma structure is some Γ. We don't have $\gamma_{\mu}A^{\mu}\,\psi$, we just have some $\Gamma\,\varphi\,\psi$. So at this point, we have a crude guess to use for the RHS of equation 10.3.

Now 10.4 makes a claim for what "drives" the KG equation for the intermediary boson field. To make a guess here, we go to page 109 7.27 which says

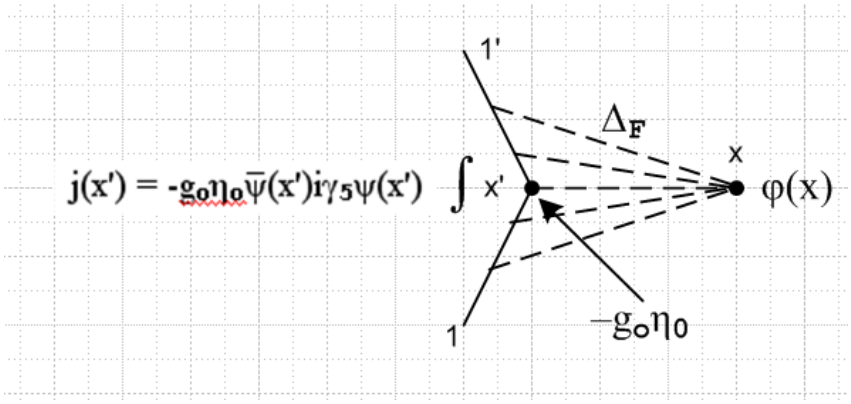
$$\Box A^{\mu} = J^{\mu} = e\bar{\psi}\,\gamma^{\mu}\psi \qquad \rightarrow \qquad (\Box + \mu^2)\,\varphi = g\,\bar{\psi}\,\Gamma\,\psi \;\; * (-\eta)$$

So since γ^{μ} appears here, we imagine that Γ should appear in the strong case, just matching the previous 10.3 situation where it seems roughly that we had $e\,\gamma_{\mu} \rightarrow g\,\Gamma$. The extra factor $-\eta$ is ± 1 and will be determined later. So this is the shabby basis then of 10.4.

If we then look at 10.4, it seems that if the pion is a pseudoscalar, the LHS out to transform as a pseudoscalar. To get the RHS to work that way as well, you need $\Gamma = \gamma_5$. I don't understand the argument that it should be $i\gamma_5$ which is a purely imaginary matrix. It seems exactly backwards. I will back-burner this issue, it is only a phase. The main point is that we need the γ_5 to get the right parity property of 10.4. But this same $\Gamma = i\gamma_5$ then must appear in 10.3 by our above arm waving arguments.

B. Applications of this Crude Theory

Explain 10.5. Let's examine some of these claimed equations. First, look at 10.4 which "says" that the "source" on the RHS is *generating* a pion field φ(x). Notice that we have $-g_o\eta_0$ on the RHS. We can solve this differential equation using the Green's function method, and the solution is exactly 10.5 ! And here is my picture of this solution:



I show multiple dotted lines to suggest the integration over x' shown in 10.5. The dotted line is of course the propagator Δ_F for the pion. Each sourcing point x' makes a contribution to the pion field created (from nothing!) at point x. Next, we need to verify the exact constant factor of +i $(-g_o\eta_0)$ we see in 10.5. The $-g_o\eta_0$ of course just comes from the "source" current as we see above, but where does the +i come from?

Go back to page 187 and first note how Δ_F is defined in 9.9 (with minus sign). We basically have this equation where I abbreviate the RHS of 10.4 as "j":

$$(\Box_x + \mu^2)\,\varphi(x) = j(x) \qquad (\Box_x + \mu^2)\Delta_F(x-x') = -\,\delta^4(x-x') \qquad \text{// 9.9}$$

Let's *conjecture* that the solution of our ODE is this

$$\varphi(x) = -\int d^4x'\,\Delta_F(x-x')\,j(x')$$

Apply $(\Box_x + \mu^2)$ to both sides. On the left we get j(x). On the right we get j(x) as well, because $(\Box_x + \mu^2)$ hits $\Delta_F(x-x')$ and creates $-\,\delta^4(x-x')$ which then produces + j(x).

Therefore, *my* version of 10.4 is this:

$$\varphi(x) = -\int d^4x'\,\Delta_F(x-x')\,j(x') = -\,(-\,g_o\eta_o)\int d^4x'\,\Delta_F(x-x')[\,\bar\psi(x')i\gamma_5\psi(x')]$$

$$= -i\,g_o\int d^4x'\,i\Delta_F(x-x')[\,\bar\psi(x')i\gamma_5\psi(x')]\,\eta_o$$

where in the last line I have added (-i)*i and grouped the i with Δ_F. So I have just derived 10.5 exactly, nothing is fuzzy, all factors are correct.

Notice in this derivation and from our picture that we are "creating" the pion field φ out of nothing, there is no pion "leg" anywhere. So we don't talk about it as Δφ, just φ.

Explain 10.6.

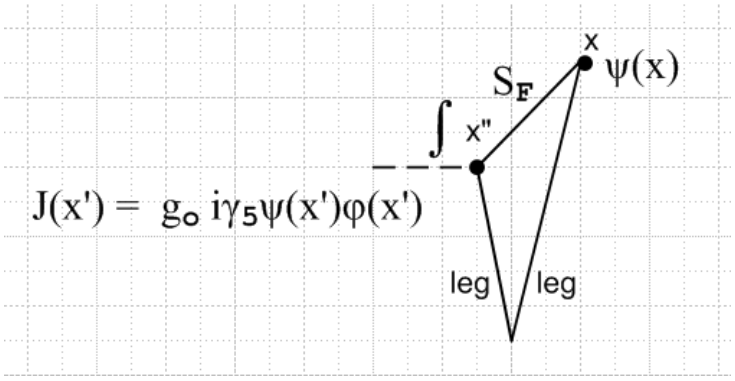
Here we repeat the above derivation. In this case we have $J(x') = g_o\, i\gamma_5\psi(x')\varphi(x')$:

$$(i\partial\!\!\!/ - m)\,\psi(x) = J(x) \qquad (i\partial\!\!\!/ - m)S_F(x-x') = +\,\delta^4(x-x') \qquad // \text{ 6.40}$$

Let's *conjecture* that the solution of our ODE is this

$$\psi(x) = +\int d^4x'\, S_F(x-x')\, J(x')$$

In this way we would develop equation 10.6 without the Δ on the LHS. The sign is + instead of -, and we don't add the (-i)*i. In this case, we argue that the current J creates a *change* in the ψ field because the wavefunction is already present as a "leg". Here is my picture of this situation



The direct leg to point x is the nucleon without the effect of the pion. The indirect leg through point x" creates a change in ψ at point x, and this change is due to the pion field φ(x) at point x" (this point was called x in our previous picture). So I think 10.6 is fully explained.

Comments: The gamma five current from the left nucleon creates a pion field, and this field then scatters the nucleon on the right. Here we think of the "field" in the same sense as the EM field $A^\mu(x)$, it is not a QFT field yet.

Explain 10.7.

As they suggest, go back to 6.56, but make this replacement:

$$e\, A_\mu(x')\gamma^\mu \;\rightarrow\; g_o\, \varphi(x')\;(i\gamma_5)$$

which is sort of our J(x') shown above. Then 6.56 says

$$S_{fi} = -i \int d^4x'' \, \bar{\psi}(x'') \; g_o \; (i\gamma_5) \, \psi(x'') \, \varphi(x'')$$

then insert 10.5 for φ(x''),

$$\varphi(x'') = -i \, g_o \int d^4x' \, i\Delta_F(x''-x')[\, \bar{\psi}(x')i\gamma_5\psi(x')] \, \eta_o$$

and we then have

$$S_{fi} = -i \int d^4x'' \, \bar{\psi}(x'') \; g_o \; (i\gamma_5) \, \psi(x'') \, \{ -i \, g_o \int d^4x' \, i\Delta_F(x''-x')[\, \bar{\psi}(x')i\gamma_5\psi(x')] \, \eta_o \}$$

$$= (-ig_o)^2 \int d^4x' \int d^4x'' \, [\bar{\psi}(x'') \, (i\gamma_5) \, \psi(x'') \,] \, \{ \, i\Delta_F(x''-x') \, \eta_o\} \, [\, \bar{\psi}(x')i\gamma_5\psi(x')]$$

$$= - \int d^4x' \int d^4x'' \, [\bar{\psi}(x'') \, (i \, g_o \, \gamma_5) \, \psi(x'') \,] \, \{ \, i\Delta_F(x''-x') \, \eta_o\} \, [\, \bar{\psi}(x')i \, g_o \, \gamma_5\psi(x')]$$

where we label on ψ with 1, and the other with 2. Either way should give the same result! In any event, we have now exactly derived 10.7. If of course has the same current-propagator-current form that we saw in 7.32 page 109, but here we have a "gamma five current".

Notice that BD have chosen to associate η_0 with the propagator. They do this because there is only one such propagator, but two vertices, so we can't put η_0 with the vertices. Now S_{fi} is where the Feynman rules come from, and we see our rules:

- put $(i \, g_o \, \gamma_5)$ at each vertex
- put $i\Delta_F \, \eta_o$ for each pion propagator

So, 10.7 is for the graph on page 211 where p_1 becomes p_1'. There is a cross graph page 214 that we add in with a relative minus sign, getting thereby 10.8 where p_1 becomes p_2' .

So far it seems we have been talking **p-p scattering** and that is why we see ψ_p everywhere. Now in equations 10.9 and 10.10 we are going to rewrite 10.3 and 10.4, but for **n-n scattering**! Now 10.9 says that the neutron Dirac equation is driven by a similar factor to what we had in 10.3 but we add a possible sign factor $-\varepsilon_0$. Then equation 10.4 needs to account for both the proton and neutron gamma five current, so we just add the two currents as shown with this new phase $-\varepsilon_0$. We assume the SAME coupling constant for neutrons to our pion as for the protons just based on experimental evidence. So now we have a miniature theory for nucleon-π^0 interactions, we have added the neutron as a third player on the fermion side of the ledger.

OK, now what about **n-p scattering?** The graphs would be those on page 216, but now we have to have a charged pion intermediary boson, so our theory is going to have more added in. No problem, we just posit a KG driven equation as in 10.11 for the new charged field, in direct analogy with 10.4. We allow the charged pion to have a different mass and a different coupling constant g_+, and a different sign phase η_+. And of course do CC to get the KG equation for the π^- pion wavefunction and this gives 10.12. The various new vertices are shown on pages 216 and 217.

We come now to 10.13 where we are back to the driven Dirac equations. The general rule here is that you take the KG driving current, change the sign and remove the η, and that is what goes in 10.13. This correctly yields all four terms you see there, except an additional ε_+ has been added for the last term in B.

You need to think of the three fields appearing in 10.13 (for each of the RHS terms) as a vertex of the theory. Thus, the terms are these (everything is in reverse order)

$\pi^0 + p \rightarrow p$	first term in 10.13A	$ig_o\gamma_5$
$\pi^+ + n \rightarrow p$	second term in 10.13A	$ig_+{}^*\gamma_5$
$\pi^0 + n \rightarrow n$	first term in 10.13B	$-\varepsilon_o ig_o\gamma_5$
$\pi^- + p \rightarrow n$	second term in 10.13B	$i\varepsilon_+ g_+\gamma_5$

These things would be clearer I suspect in a QFT version of this whole discussion. Comparing the four terms on the RHS of 10.13, I think the Feynman rules for each kind of vertex should be as I show in the right column above. They are sort of normalized against the first vertex rule which we actually directly found above. Notice that no η's appear in any of these vertex rules, so in my various pencil notes in the book perhaps I should now have written the η's since they really go with the propagator.

In any event, I think now with 10.13A and B and also 10.10, we have all the ingredients for our pion-nucleon scattering soup. I think the isospin symmetry soon will determine all these relative signs and coupling constants, but we are not there yet.

Explain 10.14

This is supposed to be the sum of the graphs shown in 10.3 for n-p scattering, so we can try out our full set of Feynman rules above. For the left graph we should have:

$ig_o\gamma_5$ (p side) and $(-\varepsilon_o)ig_o\gamma_5$ (n side) with $i\Delta_F\eta_0$ for the π^0

This agrees with the first term in 10.14 if we associate an overall extra – sign as usual with S_{fi}.

What about the second graph? I need to add a time reversal rule to the above list where I think we need to star the g constant relative to the other direction, so:

$\pi^+ + n \leftarrow p$	second term in 10.13A	$ig_+\gamma_5$

Then here is what I think the second term should have:

$ig_+{}^*\gamma_5$ (left side $\pi^+ + n \rightarrow p$) and $ig_+\gamma_5$ (right side $\pi^+ + n \leftarrow p$) with $i\Delta_F\eta_+$ for the π^+

and this agrees in the same sense with the second term in 10.14.

But suppose we interpret this same graph with a π^- going to the right! Then we would get

$i\varepsilon_+ g_+{}^*\gamma_5$ (left side $\pi^- + p \leftarrow n$) and $ig_+\gamma_5$ (right side $\pi^+ + n \leftarrow p$) with $i\Delta_F\eta_+$ for the π^+

STOP. I am doing something wrong here. If I use the two π^- rules, I will get ε_+^2 but they get only one, so I would have to spend another 5 hours digging deeper into this whole situation. For now, let's accept 10.15 as the right conclusion which is how I would have set it from the start.

The next step, begun under my pencil line on page 217 is to figure out how g_0 and g_+ are related, and to settle a value for η_+ and ε_o. We are now supposed to "imagine" that n and p are the same for the time being, then for n-p scattering we are going to have all 4 graphs shown in 218 top. This whole approach makes very little sense to me, but I will play along. Graphs (a) and (c) are the same except for the assumed relative Fermi minus sign. And the same for graphs (b) and (d). Notice the $|g_+|^2$ appearing which seems to sort of agree with my time reversal rule or something similar.

If we regard (a) and (d) as the "non-exchange" graphs and take their minus sign out, we get the first term in 10.16. The second term then has the overall minus and is the exchange term. Fine.

Now here is their point. Equation 10.16 for S_{fi} is for our imagined n-p scattering including all the graphs. On the other hand, 10.7 + 10.8 is our result for p-p scattering, again including the exchange graph. If we force n-p and p-p scatterings to have equal amplitudes, then we must have

$g_o^2\eta_0$ // common factor in 10.7 and 10.8

= – [bracket factor in 10.16] // giving 10.17

We then reject the first solution 10.18 because we know there is a charged pion coupling, and we end up with the second solution which says:

$g_+ = \sqrt{2}\, g_0$ $\varepsilon_0 = +1$ $\eta_0 = -\eta_+$

I have verified that the above does satisfy 10.17.

Now having done this, they are going to go back to $\eta_0 = +\eta_+$ and add a new Feynman rule to make a minus sign whenever an odd number of pions are exchanged, fine. Their big issue seems to be how to deal with Fermi antisymmetry when you have n and p. Are n and p identical or not? For the strong force they are, so you have to handle this antisymmetry!

But they then admit on page 221 top that the antisym for n-p is just a convention, not the real thing. They are doing all this impossible stuff to get set up for the isospin stuff to come.

The final act of this messy section (reminds me a bit of the early propagator chapter discussion) is that if you choose $\eta_+ = +1$, then you find that the sum of proton and pion electric currents is conserved. I have not computed the rightmost expressions in 10.22, but supposedly this can be done using the appropriate "wave equations". For the first, you would use 10.13A to replace $\partial\psi_p$ somehow, and for the second you would do something with 10.12 I think. { I have now done this in detail below in Section 10.4 where the subject of conserved currents has come up again. }

I have skipped in my notes here the whole awful discussion on page 219 which relates to the pictures on page 220. All this is related to the antisymmetry discussion and finding the relative signs of higher order graphs. But I know that we don't really have a perturbation theory for pion exchange, so I am not that interested in higher order graphs! But at least I am still on the wagon here.

10.3 Isospin (222)

First, we gather up all our previous equations in 10.26 and 10.27, using the values like $\varepsilon_o = +1$ etc which we determined in Section 10.2. We then form an isospin I = ½ spinor as in 10.28 for the nucleon. Really this is a direct product situation between isospin space and Dirac space, so they just imagine that ψ_n inside 10.28 is the 4 component Dirac thing. Fine. The isospin generators are going to be $I_i = ½ \tau_i$ and we define raising and lowering as in 10.31, though the reason for the name "charge raising" is not clear at this point and the terms seem backwards.

Next, we define an isospin I = 1 vector for the pions as in 10.32.

When the dust settles, all our messy equations in 10.26 are now reduced to these:

$(i\partial\!\!\!/ - M)\Psi = g_0 i\gamma_5(\boldsymbol{\tau} \bullet \boldsymbol{\varphi})\Psi$ // spinor = spinor

$(\Box + \mu^2)\,\boldsymbol{\varphi} = -\,g_0\,\bar{\Psi}\,i\gamma_5\,\boldsymbol{\tau}\,\Psi$ // vector = vector

and it is noted that we are now in an *approximation* where we say $M_n = M_p$ and the three pions have the same mass. Perhaps this approximation would become exact if you could "turn off" E&M. That is, perhaps E&M totally accounts for these mass differences.

$\pi^{\pm}$

$I^G(J^P) = 1^-(0^-)$

Mass $m = 139.57018 \pm 0.00035$ MeV (S = 1.2)
Mean life $\tau = (2.6033 \pm 0.0005) \times 10^{-8}$ s (S = 1.2)
$c\tau = 7.8045$ m

π^0

$I^G(J^{PC}) = 1^-(0^{-+})$

Mass $m = 134.9766 \pm 0.0006$ MeV (S = 1.1)
$m_{\pi^\pm} - m_{\pi^0} = 4.5936 \pm 0.0005$ MeV
Mean life $\tau = (8.4 \pm 0.6) \times 10^{-17}$ s (S = 3.0)
$c\tau = 25.1$ nm

The pion masses differ by about 4.5/140 ~ 3%

p

$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

Mass $m = 1.00727646688 \pm 0.00000000013$ u
Mass $m = 938.27203 \pm 0.00008$ MeV [a]

n

$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

Mass $m = 1.0086649156 \pm 0.0000000006$ u
Mass $m = 939.56536 \pm 0.00008$ MeV [a]

The nucleon masses differ by about 1.2/938 ~ 1%.

Now, back to our two equations from above:

$(i\partial\!\!\!/ - M)\Psi = g_0 i\gamma_5(\boldsymbol{\tau} \bullet \boldsymbol{\varphi})\Psi$ // spinor = spinor

$(\Box + \mu^2)\,\boldsymbol{\varphi} = -\,g_0\,\bar{\Psi}\, i\gamma_5\,\boldsymbol{\tau}\,\Psi$ // vector = vector

How to we convert these things to "Feynman Rules" ? My inclination would be to repeat the logic we went through for equations 10.5,6,7. The first two are Green's solutions of the first two equations above. We only really need the solution to the second equation above. This is just a vector equation so treat as three separate equations, and 10.5 becomes

$$\boldsymbol{\varphi}(x) = -ig_0 \int d^4x'\; i\,\Delta_F(x-x')\,[\,\bar{\Psi}\; i\gamma_5\,\boldsymbol{\tau}\,\Psi\,]$$

Now I will just quote my "Explain 10.7" notes from above, making the appropriate changes:

Explain 10.7.

As they suggest, go back to 6.56, but make this replacement:

$e\,A_\mu(x')\gamma^\mu \;\rightarrow\; g_o\,\boldsymbol{\varphi}(x')\;(i\gamma_5)$ // now an isospin vector

which is sort of our J(x') shown above. Then 6.56 says

$S_{fi} = -i \int d^4x''\,[\,\bar{\Psi}(x'')\;g_o\;(i\gamma_5)\,\boldsymbol{\tau}\,\Psi(x'')\,] \bullet \boldsymbol{\varphi}(x'')$ // vector • vector = isoscalar

Notice now that our Dirac "gamma five current" which is a Lorentz pseudoscalar is now made as well into an isospin *vector* by the addition of the $\boldsymbol{\tau}$ vector inside the bilinear expression.

then insert 10.5 for φ(x'),

$$\boldsymbol{\varphi}(x'') = -ig_0 \int d^4x'\; i\,\Delta_F(x''-x')\,[\,\bar{\Psi}(x')\,i\gamma_5\,\boldsymbol{\tau}\,\Psi(x')\,]$$ // vectorized in isospace

and we then have

$$S_{fi} = -i \int d^4x''\,]\,[\,\bar{\Psi}(x'')\;g_o\;(i\gamma_5)\,\boldsymbol{\tau}\,\Psi(x'')\,] \bullet \{\,-ig_0 \int d^4x'\; i\,\Delta_F(x''-x')\,[\,\bar{\Psi}(x')\,i\gamma_5\,\boldsymbol{\tau}\,\Psi(x')\,]\,\}$$

$$= (-ig_o)^2 \int d^4x' \int d^4x''\;[\,\bar{\Psi}(x'')\;(i\gamma_5)\,\boldsymbol{\tau}\,\Psi(x'')\,]\;\{\,i\Delta_F(x''-x')\}\;\bullet\,[\,\bar{\Psi}(x')\,i\gamma_5\,\boldsymbol{\tau}\,\Psi(x')\,]$$

The meson propagator has no isospin sense to it, but to make notation nice, we can associate an isospin unit matrix with the propagator to write this in a sort of dyadic form

$$= (-ig_o)^2 \int d^4x' \int d^4x'' \; [\, \bar{\Psi}(x'') \; (i\gamma_5)\, \boldsymbol{\tau}\, \Psi(x'') \,] \bullet \; \{ i\boldsymbol{\Delta}_F(x''-x') \} \bullet [\, \bar{\Psi}(x')\, i\gamma_5\, \boldsymbol{\tau}\, \Psi(x') \,]$$

which we then write out in isospin components like this:

$$= (-ig_o)^2 \int d^4x' \int d^4x'' \; [\, \bar{\Psi}(x'') \; (i\gamma_5)\, \boldsymbol{\tau}\, \Psi(x'') \,]_\alpha \; \{ i\Delta_F(x''-x')\, \delta_{\alpha\beta} \} \; [\, \bar{\Psi}(x')\, i\gamma_5\, \boldsymbol{\tau}\, \Psi(x') \,]_\beta$$

We have changed nothing of course in these last cosmetic steps. Now of course the α index can be moved to the τ matrix, etc, and we can move the coupling constant factors inside to get

$$S_{fi} \;=\; + \int d^4x' \int d^4x'' \; [\, \bar{\Psi}(x'')\, (-ig_o)\, (i\gamma_5)\, \boldsymbol{\tau}_\alpha\, \Psi(x'') \,] \; \{ i\Delta_F(x''-x')\, \delta_{\alpha\beta} \} \; [\, \bar{\Psi}(x')\, (-ig_o)\, i\gamma_5\, \boldsymbol{\tau}_\beta\, \Psi(x') \,]$$

This is S_{fi} for the basic pion exchange graph on page 211, and from it we can read off the Feynman rules.

Rule 2 on page 224 bottom is now very clear, staring at our form above.

Rule 3 includes the $\delta_{\alpha\beta}$ sense of the pion propagator as shown above. If we look at the new isospin version of 10.6 which is the solution of $(i\partial\!\!\!/ - M)\Psi = g_o i\gamma_5(\boldsymbol{\tau} \bullet \boldsymbol{\varphi})\Psi$, we would write

$$\Delta\Psi(x) = \int d^4x''\, S_F(x-x'')\, [\, g_o\, (i\gamma_5)\, \boldsymbol{\tau}\, \Psi(x'') \,] \bullet \boldsymbol{\varphi}(x'') \qquad \text{// new 10.6}$$

and we could write this showing the I=1/2 indices which we might call m

$$\Delta\Psi(x)_m = \int d^4x'' \; \{ S_F(x-x'')\, \delta_{mm'} \} \; [\, g_o\, (i\gamma_5)\, \boldsymbol{\tau}\, \Psi(x'')_m \,] \bullet \boldsymbol{\varphi}(x'')$$

so here we are "associating" a unit 2x2 matrix with S_F and this then gives the rest of the Rule 3 page 225.

Rule 4 concerns the external legs.

For the nucleons, we use the obvious things like neutron = (0,1) = χ_n, and we could throw in an isospin projector as illustrated in 10.35 where $P_n = (1 - \tau_3)/2$ has a 1 in the lower right corner only and so projects out your (0,1) part. The reason this is mentioned is that we recall the use of regular spin projectors like Σ which let us evaluate cross sections in terms of traces, and perhaps we will end up with some isospin space traces a few pages from now.

For the pions, we go back to our "rules" of the last section. For example, look at the $\tau_+\varphi_+$ term in 10.30. I associate this term with the process $\pi^+ + N \rightarrow N$ where N means nucleon. So, at a vertex where an external π^+ is absorbed, we want a factor $\tau_+\varphi_+$ for the pion external leg (times its usual norm factor which is not shown here). We can write $\tau_+\varphi_+ = \tau \bullet \hat{\varphi}^+$ given the definitions in the first two lines of 10.36. So I think I am happy with all of Rule 4 for external legs, but I need to see some examples soon.

Rule 5 concerns the antisymmetry extended rule we discussed in an earlier section, as well as fermion loops. Since we are never going to do fancy graphs like this, I am not concerned now about this rule.

Now finally comes "the great admission". Whereas $\alpha = e^2/4\pi = 1/137$ is small, $\alpha_{strong} = g^2/4\pi = 14$ (I guess based on experiment which perhaps is coming in next sections), so perturbation theory is

impossible, it diverges right off the bat. This is the first time I have seen this number "14", interesting. They admit here they have just been mimicking QED with all these rules and graphs, as if there were a perturbation theory.

10.4 Conserved currents (226)

A. Preliminaries and the Proton Current

First, I need to backtrack and review what the Dirac equation looks like "both ways" where $p^\mu = +i\partial^\mu$

$(i\partial\!\!\!/ - m)\,\psi = 0$ $\qquad$ $(p\!\!\!/ - m)\,\psi = 0$

$\psi^\dagger(i\overleftarrow{\partial\!\!\!/} - m)^\dagger = 0$ $\qquad$ $\psi^\dagger(\overleftarrow{p\!\!\!/} - m)^\dagger = 0$

$\psi^\dagger(-i\overleftarrow{\partial\!\!\!/}^\dagger - m) = 0$ $\qquad$ $\psi^\dagger(\overleftarrow{p\!\!\!/}^\dagger - m) = 0$

$\bar\psi\,(-i\gamma^0\overleftarrow{\partial\!\!\!/}^\dagger\gamma^0 - m) = 0$ $\qquad$ $\bar\psi\,(\gamma^0\overleftarrow{p\!\!\!/}^\dagger\gamma^0 - m) = 0$

$\bar\psi\,(-i\overleftarrow{\partial\!\!\!/} - m) = 0$ $\qquad$ $\bar\psi\,(\gamma^0\overleftarrow{p\!\!\!/}^\dagger\gamma^0 - m) = \bar\psi\,(\overleftarrow{p}_\mu^{\,*}\,\gamma^\mu - m) = 0$ $\qquad$ // MS: $\bar\psi(p\!\!\!/ - m) = 0$

In the above, I am always thinking of p and ∂ as differential operators, so when they act "to the left", you have to show this with the backwards over-arrow. We used $\gamma^0\gamma^0 = 1$ and $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$ in the above. If ψ were in momentum space (MS), *then* we could treat $p\!\!\!/$ as a real 4-vector of numbers against γ^μ . In this case, we would not need or want the over arrow, and we would have $p^* = p$ so we get the final form on the right above which is very simple. Notice in the diff op case that we cannot say $\overleftarrow{p}_\mu^{\,*}\,\gamma^\mu = \overleftarrow{p\!\!\!/}^{\,*}$, we have to write this thing out. So for diff ops, the ∂ notation is clearer. And notice the -i in the bottom left form above.

Now let's start over assuming some sort of RHS driving term "a". We have

$(i\partial\!\!\!/ - m)\,\psi = a$ $\quad\Rightarrow\quad$ $(i\partial\!\!\!/)\,\psi = a + m\psi$

$\psi^\dagger(i\overleftarrow{\partial\!\!\!/} - m)^\dagger = a^\dagger$

$\bar\psi\,(-i\overleftarrow{\partial\!\!\!/} - m) = \bar a$ $\quad\Rightarrow\quad$ $\bar\psi(i\overleftarrow{\partial\!\!\!/}) = -(\bar a + m\bar\psi)$

where $\bar a$ occurs because we have to mult both sides by γ^0 from the right. Then on the right we show the result we have been looking for, the action from the right of the $i\partial\!\!\!/$ operator.

Now think of equation 10.13A as $(i\partial\!\!\!/ - m)\,\psi_p = a$ so a = the RHS of 10.13A. Suppose then we want to know the divergence of the proton current. We calculate:

$$i\partial_\mu j^\mu = i\partial_\mu (\bar\psi\gamma^\mu\psi) = \bar\psi\,(i\overleftarrow{\partial}\,)\,\psi + \bar\psi(\,i\partial)\,\psi = -(\,\bar a + m\bar\psi)\,\psi + \bar\psi\,(a + m\psi) = -\,\bar a\,\psi + \bar\psi\,a$$

In our case we have

$$a = g_o\, i\gamma_5\,\psi_p\varphi_0 + g_+^*\, i\gamma_5\,\psi_n\varphi_+$$

$$a^\dagger = -\,g_o\,\psi_p^\dagger i\gamma_5\,\varphi_0 - \,g_+\,\psi_n^\dagger\, i\gamma_5\,\varphi_+^*$$

$$\bar a = +\,g_o\,\bar\psi_p\; i\gamma_5\,\varphi_0 + \;g_+\;\bar\psi_n\, i\gamma_5\,\varphi_+^* \qquad\qquad \text{// since } \gamma^0\gamma_5\gamma^0 = -\gamma_5$$

So our result should then be

$$i\partial_\mu j_p^\mu = -\,\bar a\,\psi_p + \bar\psi_p\,a$$

$$= -\{\,g_o\,\bar\psi_p\; i\gamma_5\,\varphi_0 + \;g_+\bar\psi_n\, i\gamma_5\,\varphi_+^*\,\}\,\psi_p \;+\bar\psi_p\{\,g_o\, i\gamma_5\,\psi_p\varphi_0 + g_+^*\, i\gamma_5\,\psi_n\varphi_+\,\}$$

$$= -\,g_o\,\bar\psi_p\; i\gamma_5\,\psi_p\,\varphi_0 - \;g_+\,\bar\psi_n\, i\gamma_5\,\psi_p\,\varphi_+^* \;+\, g_o\bar\psi_p\, i\gamma_5\,\psi_p\varphi_0 + g_+^*\bar\psi_p\, i\gamma_5\,\psi_n\varphi_+$$

$$= -\,g_+\bar\psi_n\, i\gamma_5\,\psi_p\,\varphi_+^* \;+\, g_+^*\,\bar\psi_p\, i\gamma_5\,\psi_n\varphi_+$$

and this agrees with the RHS of 10.22A if we remember that $\varphi_+^* = \varphi_-$ as in 10.12. So, this is the divergence of the proton current, and this current is NOT conserved.

B. The Pion Current and sum conserved

From our KG theory of the last chapter we know that

$$j^\mu = \varphi_+^*\, i\overleftrightarrow{\partial}^\mu\varphi_+ = i\;\varphi_+^*\,[\partial^\mu - \overleftarrow{\partial}^\mu]\,\varphi_+ = i\,\varphi_+^*\,(\partial^\mu\,\varphi_+) - i\,(\partial^\mu\,\varphi_+^*)\,\varphi_+$$

and this is what appears in the square bracket in 10.22B. Our driven KG equation is this:

$$(\partial_\mu\partial^\mu + m^2)\,\varphi_+ = c \qquad\Rightarrow\qquad \partial_\mu\partial^\mu\,\varphi_+ = c - m^2\,\varphi_+$$

where I assume some arbitrary source term "c", and now m is the pion mass. So from above we have

$$-\,i\partial_\mu j_n^\mu = i\partial_\mu\,\{\,i\,(\partial^\mu\,\varphi_+^*)\,\varphi_+ - i\,\varphi_+^*\,(\partial^\mu\,\varphi_+)\} = \partial_\mu\,\{-\,(\partial^\mu\,\varphi_+^*)\,\varphi_+ + \varphi_+^*\,(\partial^\mu\,\varphi_+)\}$$

$$= -(\,\partial_\mu\partial^\mu\,\varphi_+^*)\,\varphi_+ - (\partial^\mu\,\varphi_+^*)(\partial_\mu\,\varphi_+) + (\partial_\mu\,\varphi_+^*)(\partial^\mu\,\varphi_+) + \varphi_+^*(\partial_\mu\,\partial^\mu\,\varphi_+)$$

$$= -(\,\partial_\mu\partial^\mu\,\varphi_+^*)\,\varphi_+ + \varphi_+^*(\partial_\mu\,\partial^\mu\,\varphi_+) \qquad\qquad \text{// two middle terms cancel}$$

$$= -\,(c - m^2\,\varphi_+)^*\,\varphi_+ + \varphi_+^*\,(c - m^2\,\varphi_+)$$

$$= -(c^* - m^2 \varphi_+^*)\varphi_+ + \varphi_+^* (c - m^2 \varphi_+)$$

$$= -c^* \varphi_+ + m^2 \varphi_+^* \varphi_+ + \varphi_+^* c - m^2 \varphi_+^* \varphi_+$$

$$= -c^* \varphi_+ + \varphi_+^* c \qquad \text{// two } m^2 \text{ terms cancel}$$

Now in our situation, we have from (10.11) that

$$c = -g_+ \eta_+ \bar{\psi}_n i\gamma_5 \psi_p$$

$$c^* = -g_+^* \eta_+ [\bar{\psi}_n i\gamma_5 \psi_p]^*$$

Now we digress to observe this little theorem:

$$x = a^T B c \text{ = a scalar number}$$

$$x^T = x = c^T B^T a$$

Applying this, we find that

$$[\bar{\psi}_n i\gamma_5 \psi_p]^* = \psi_n^T [\gamma^0(-i)\gamma_5] \psi_p^* = \psi_p^\dagger \gamma_5 (-i) \gamma^0 \psi_n = \psi_p^\dagger \gamma^0 \gamma_5 (+i) \psi_n = \bar{\psi}_p i\gamma_5 \psi_n$$

Therefore we know that

$$c = -g_+ \eta_+ \bar{\psi}_n i\gamma_5 \psi_p$$
$$c^* = -g_+^* \eta_+ [\bar{\psi}_n i\gamma_5 \psi_p]^* = -g_+^* \eta_+ [\bar{\psi}_p i\gamma_5 \psi_n]$$

So we get

$$i\partial_\mu j_{n+}^\mu = +c^* \varphi_+ - \varphi_+^* c = +\{-g_+^* \eta_+ [\bar{\psi}_p i\gamma_5 \psi_n]\} \varphi_+ - \varphi_+^* \{-g_+ \eta_+ \bar{\psi}_n i\gamma_5 \psi_p\}$$

$$= -g_+^* \eta_+ [\bar{\psi}_p i\gamma_5 \psi_n] \varphi_+ + g_+ \eta_+ [\bar{\psi}_n i\gamma_5 \psi_p] \varphi_+^*$$

and this agrees exactly with 10.22. So, setting $\eta_+ = 1$, here are our two results:

$$i\partial_\mu j_p^\mu = -g_+ \bar{\psi}_n i\gamma_5 \psi_p \varphi_+^* + g_+^* \bar{\psi}_p i\gamma_5 \psi_n \varphi_+$$

$$i\partial_\mu j_{n+}^\mu = g_+ [\bar{\psi}_n i\gamma_5 \psi_p] \varphi_+^* - g_+^* [\bar{\psi}_p i\gamma_5 \psi_n] \varphi_+$$

and therefore:

$\partial_\mu[\, j_p{}^\mu + j_{\pi+}{}^\mu] = 0$

C. Repeat this all in isospin notation

Electromagnetic current. The first term in 10.37 is the proton current $\bar\psi_p\gamma_\mu\psi_p$. The projector kills off the potential $\bar\psi_n\gamma_\mu\psi_n$ term that is lurking in $\bar\Psi\,\gamma_\mu\Psi$. You need to write it out like this:

$$(\psi_p\ \psi_n)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}\gamma_\mu\psi_p\\ \gamma_\mu\psi_n\end{pmatrix} = (\psi_p\ \psi_n)\begin{pmatrix}1&0\\0&0\end{pmatrix}\gamma_\mu\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} = (\psi_p\ \psi_n)\begin{pmatrix}\gamma_\mu&0\\0&0\end{pmatrix}\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} = \text{etc etc}$$

You could put the projector on either side of the γ_μ because they act in different spaces.

The pion current in 10.37 is this:

$$\varepsilon_{3jk}\,\varphi_j\ \partial_\mu\varphi_k = \varphi_1\partial_\mu\varphi_2 - \varphi_2\partial_\mu\varphi_1$$

$$= \tfrac{1}{2}\, i\,(\varphi_+ + \varphi_-)\,(\partial_\mu\,\varphi_+ - \partial_\mu\,\varphi_-) \;-\; \tfrac{1}{2}\, i\,(\varphi_+ - \varphi_-)\,(\partial_\mu\,\varphi_+ + \partial_\mu\,\varphi_-)$$

$$= i\,\varphi_-\ \partial_\mu\,\varphi_+ \;- i\,\varphi_+\ \partial_\mu\,\varphi_- \qquad = \text{the pion current shown in 10.22 square bracket}$$

So the notation is pretty compact:

$$j_\mu = \ \bar\Psi\,\gamma_\mu\Psi\ + [\ \boldsymbol{\varphi}\ \text{x}\ \partial_\mu\boldsymbol{\varphi}\ ]_3 \qquad \text{// the total electric current and we showed } \partial^\mu j_\mu = 0.$$

and then 10.38 gives the electric charge Q, fine and dandy.

Nucleonic current. Now something rather new : a "nucleonic current" which is just what I was talking about above:

$$J^N{}_\mu \equiv\ \bar\Psi\,\gamma_\mu\Psi \qquad\qquad \text{// } = \bar\psi_p\gamma_\mu\psi_p + \bar\psi_n\gamma_\mu\psi_n$$

Now let's quote our work in section A above which was this:

$$(i\partial\!\!\!/ - m)\,\Psi = A \qquad \Rightarrow \qquad (\,i\partial\!\!\!/)\,\Psi \ = \ A + m\,\Psi$$

$$\bar\Psi\,(-i\overleftarrow{\partial\!\!\!/} - m) = \bar A \qquad \Rightarrow \qquad \bar\Psi\,(i\overleftarrow{\partial\!\!\!/}\,) \ = -\,(\bar A + m\bar\Psi)$$

where now we endow ψ with isospin spinorhood to be Ψ and the equations of course are the same except A must also then be a spinor. From 10.33 we will have $A = g_o i\,\gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\,\Psi$, but first do this:

$$i\,\partial^\mu\,J^N{}_\mu \ = i\,\partial^\mu\,[\bar\Psi\,\gamma_\mu\Psi] = \bar\Psi\,(\,i\partial\!\!\!/)\,\Psi + \bar\Psi\,(i\overleftarrow{\partial\!\!\!/}\,)\,\Psi$$

$$= \bar\Psi\,(A + m\,\Psi) - \ (\bar A + m\bar\Psi)\,\Psi \ = \ \bar\Psi\,A \ - \bar A\,\Psi$$

Meanwhile we have

$$A = g_o i\,\gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\,\Psi$$

$$A^\dagger = g_o(-i)\,\Psi^\dagger(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})^\dagger\gamma_5 = g_o(-i)\,\Psi^\dagger(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\gamma_5 \qquad \text{// Pauli's are Hermitian and } \boldsymbol{\varphi} = \text{real}$$

$$\bar{A} = g_o\, i\,\bar{\Psi}\,\gamma_5\,(\boldsymbol{\tau}\bullet\boldsymbol{\varphi}) \qquad \text{// gamma 0 and 5 anticommute}$$

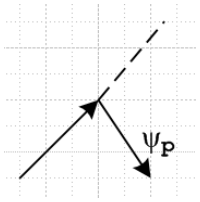
so we end up with

$$i\,\partial^\mu J^N_\mu = \bar{\Psi}\,A - \bar{A}\,\Psi = \bar{\Psi}\{\, g_o i\,\gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\,\Psi\,\} - \{\, g_o\, i\,\bar{\Psi}\,\gamma_5\,(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\}\,\Psi = 0 \;\;!!$$

so we have shown that our "nucleonic current" is conserved

$$\partial^\mu J^N_\mu = 0$$

The "charge" of the nucleonic current is called N and is presented in 10.40 which is obviously correct. The interpretation is glossed over, however. If you have only positive energy Dirac states for ψ_p and ψ_n then N is the total number of nucleons. But if you have a negative energy solution say for ψ_p, then I guess you would say that $\int d^3x\ \psi_p^\dagger\psi_p$ was the number of "holes", but if you have 1 hole going backwards in time, you have to count it as -1 antiparticles going forward in time. As an example, suppose you could have a proton and anti-proton annihilate into a neutral pion like so:



Clearly the number of nucleons after the annihilation is 0, so this must be the number before. We could the down arrow negative energy thing as a -1 in the nucleon count for the reason just stated. BD completely gloss over this subject, they just make the statement that N is total nucleons minus total anti-nucleons.

Isospin current. And now to round things out, we introduce yet another current which is the **"isospin current"** and the claim of 10.43 is that you write it like so

$$\mathbf{J}_\mu = \tfrac{1}{2}\,\bar{\Psi}\,\gamma_\mu\,\boldsymbol{\tau}\,\Psi + \boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}$$

which is in fact an isospin vector of currents! The claim is that this third current is also conserved in our theory, but this is another thing I will have to prove:

$$i\,\partial^\mu\mathbf{J}_\mu = \tfrac{1}{2}\,(i\,\partial^\mu\bar{\Psi})\gamma_\mu\,\boldsymbol{\tau}\,\Psi + \tfrac{1}{2}\,\bar{\Psi}\,\gamma_\mu\,\boldsymbol{\tau}\,(i\,\partial^\mu\Psi) + (i\,\partial^\mu\boldsymbol{\varphi})\times\partial_\mu\boldsymbol{\varphi} + i\,\boldsymbol{\varphi}\times(\partial^\mu\partial_\mu\boldsymbol{\varphi})$$

$$= \tfrac{1}{2}\,\bar{\Psi}\ (i\overleftarrow{\partial}\,)\,\boldsymbol{\tau}\,\Psi\ + \tfrac{1}{2}\,\bar{\Psi}\,\boldsymbol{\tau}\,(\,i\partial)\,\Psi)\ + (i\,\partial^{\mu}\,\boldsymbol{\varphi})\ x\ \partial_{\mu}\boldsymbol{\varphi}\ \ + i\,\boldsymbol{\varphi}\ x\ (\partial^{\mu}\partial_{\mu}\boldsymbol{\varphi})$$

Exactly as above we use these facts on the nucleon side of things (10.33)

$$(i\partial - m)\,\Psi = A \qquad \Rightarrow \qquad (\,i\partial)\,\Psi \ = \ A + m\,\Psi \qquad\qquad A = g_o i\,\gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\,\Psi$$

$$\bar{\Psi}\,(-i\overleftarrow{\partial}\ - m) = \bar{A} \qquad \Rightarrow \qquad \bar{\Psi}\,(i\overleftarrow{\partial}\,) \ = -\,(\bar{A} + m\bar{\Psi}) \qquad\qquad \bar{A} = g_o\ i\ \bar{\Psi}\ \gamma_5\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})$$

Consider the third term in our divergence above:

$$\partial^{\mu}\boldsymbol{\varphi}\ \mathbf{x}\ \partial_{\mu}\boldsymbol{\varphi}\ = \partial_0\boldsymbol{\varphi}\ \mathbf{x}\ \partial_0\boldsymbol{\varphi}\ - \partial_1\boldsymbol{\varphi}\ \mathbf{x}\ \partial_1\boldsymbol{\varphi} - \partial_2\boldsymbol{\varphi}\ \mathbf{x}\ \partial_2\boldsymbol{\varphi} - \partial_3\boldsymbol{\varphi}\ \mathbf{x}\ \partial_3\boldsymbol{\varphi}$$

and each individual term is of the form **v** x **v** and so each term vanishes.

The last term calls for our driven KG equation (10.34),

$$(\partial^{\mu}\partial_{\mu} + \mu^2)\boldsymbol{\varphi}\ = -\,g_0\bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\tau}\ \Psi$$

$$\Rightarrow\ \partial^{\mu}\partial_{\mu}\ \boldsymbol{\varphi}\ = -\,\mu^2\ \boldsymbol{\varphi} - g_0\bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\tau}\ \Psi$$

and then our result from above is

$$i\ \partial^{\mu}\ \mathbf{J}_{\mu}\ =\ =\ \tfrac{1}{2}\,\bar{\Psi}\ (i\overleftarrow{\partial}\,)\,\boldsymbol{\tau}\,\Psi\ + \tfrac{1}{2}\,\bar{\Psi}\,\boldsymbol{\tau}\,(\,i\partial)\,\Psi)\ + (i\,\partial^{\mu}\,\boldsymbol{\varphi})\ x\ \partial_{\mu}\boldsymbol{\varphi}\ \ + i\,\boldsymbol{\varphi}\ x\ (\partial^{\mu}\partial_{\mu}\boldsymbol{\varphi})$$

$$= -\,\tfrac{1}{2}(\bar{A} + m\bar{\Psi})\,\boldsymbol{\tau}\,\Psi\ + \tfrac{1}{2}\,\bar{\Psi}\,\boldsymbol{\tau}\,(A + m\,\Psi)\ + 0 + i\,\boldsymbol{\varphi}\ x\ \{\,-\,\mu^2\,\boldsymbol{\varphi} - g_0\bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\tau}\ \Psi\ \}$$

The $\boldsymbol{\varphi}$ x $\boldsymbol{\varphi}$ term is 0, so we are left with

$$i\ \partial^{\mu}\ \mathbf{J}_{\mu}\ = -\,\tfrac{1}{2}(\{\ g_o\ i\ \bar{\Psi}\ \gamma_5\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\}\ + m\bar{\Psi})\ \boldsymbol{\tau}\ \Psi\ \ + \tfrac{1}{2}\,\bar{\Psi}\,\boldsymbol{\tau}\,(\{g_o i\,\gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\,\Psi\ \} + m\,\Psi)\ - ig_0\,\boldsymbol{\varphi}\ x\ \bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\tau}\ \Psi$$

$$= -\,\tfrac{1}{2}\,g_o\bar{\Psi}\ i\ \gamma_5\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \boldsymbol{\tau}\ \Psi\ \ - \tfrac{1}{2}\,m\bar{\Psi}\ \boldsymbol{\tau}\ \Psi\ + \tfrac{1}{2}\,g_o\bar{\Psi}\ \boldsymbol{\tau}\ i\ \gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \Psi + \tfrac{1}{2}\,m\bar{\Psi}\ \boldsymbol{\tau}\ \Psi\ - ig_0\,\boldsymbol{\varphi}\ x\ \bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\tau}\ \Psi$$

$$= -\,\tfrac{1}{2}\,g_o\bar{\Psi}\ i\ \gamma_5\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \boldsymbol{\tau}\ \Psi\ + \tfrac{1}{2}\,g_o\bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\tau}\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \Psi - ig_0\ \bar{\Psi}\ i\ \gamma_5\ \boldsymbol{\varphi}\ x\ \boldsymbol{\tau}\ \Psi$$

$$= \ -\,\tfrac{1}{2}\,g_o\bar{\Psi}\ i\ \gamma_5\ \{\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \boldsymbol{\tau}\ - \boldsymbol{\tau}\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ + 2i\ \boldsymbol{\varphi}\ x\ \boldsymbol{\tau}\ \ \}\ \Psi$$

where the two $m\bar{\Psi}\ \boldsymbol{\tau}\ \Psi$ terms cancelled early on, and we now want to show that {...} = 0. Here we go:

$$\{\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \boldsymbol{\tau}\ - \boldsymbol{\tau}\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ + 2i\ \boldsymbol{\varphi}\ x\ \boldsymbol{\tau}\}_i = \{\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ \boldsymbol{\tau}_i\ - \boldsymbol{\tau}_i\ (\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\ + 2i\ \varepsilon_{ijk}\boldsymbol{\varphi}_j\boldsymbol{\tau}_k\}$$

$$= \boldsymbol{\varphi}_n\boldsymbol{\tau}_n\ \boldsymbol{\tau}_i\ -\ \boldsymbol{\tau}_i\boldsymbol{\tau}_n\boldsymbol{\varphi}_n\ +\ \ 2i\ \varepsilon_{ijk}\boldsymbol{\varphi}_j\boldsymbol{\tau}_k\ = \boldsymbol{\varphi}_j\ [\boldsymbol{\tau}_j,\boldsymbol{\tau}_i] + 2i\ \varepsilon_{ijk}\boldsymbol{\varphi}_j\boldsymbol{\tau}_k$$

$= \varphi_j \{ \ [\tau_j, \tau_i] + 2i\, \varepsilon_{ijk}\tau_k \ \} = = \varphi_j \{ \ 2i\varepsilon_{jik}\tau_k + 2i\, \varepsilon_{ijk}\tau_k \ \}$

$= \varphi_j \{ \ -2i\varepsilon_{ijk}\tau_k + 2i\, \varepsilon_{ijk}\tau_k \ \} = 0$

Thus, we have shown that the isospin current shown in 10.43 is in fact conserved. In BD2 p 100 this is shown to be the Noether current of an isospin rotation on the Lagrangian

So at this point I want to verify 10.41 which is supposed to relate our three different kinds of "charge" from our three different currents. You can see that the τ_3 term in 10.38 gives I_3, and then the remaining term gives ½ of 10.40, so indeed we have shown that

$Q = N/2 + I_3$ // so we have proven 10.41

Comments: in our toy theory for pion-nucleon interactions, the three currents are exactly conserved, because I proved it for each one above. In the real world, pion and nucleon masses are not equal, and our theory is only approximately applicable. But nevertheless, since we know that Q and N are rigorously conserved, it must be that I_3 is rigorously conserved, even though all of I is not.

But I might argue that equation 10.41 itself is not rigorously valid if the toy theory is wrong. Of course if you define I_3 by 10.41, then it will be "rigorously conserved". All the equations here really are theory-dependent.

I had forgotten this relationship 10.41 in a theory like this.

We are continuing blindly down our path of having at least some kind of approximate theory to predict in some crude sense the nature of real world pion-nucleon scattering.

Unresolved question: why is it $i\gamma_5$ instead of just γ_5 ? I did not follow the argument presented early on for this.

10.5 Approximate Calculations: Nucleon-nucleon scattering. (227) N-N scattering

We first consider our simple pion-exchange graphs for N+N→N+N and S_{fi} is as shown in 10.45 All the factors are right, and you can just compare this to 7.82 page 136 on the kinematics. Relative minus sign between the terms (as noted, even if not identical nucleons).

Notice here how the **isospin structure** is that shown in 10.46. They did not bother to put $\delta_{\alpha\beta}$ on the propagator, we just have the general idea of [....$\boldsymbol{\tau}$...] • [....$\boldsymbol{\tau}$...] going right through the propagator.

Derive 10.46 then E and F

For the p-p case, the structure is the LHS of 10.46 and we can compute this right off the bat, but they are not telling us HOW to do this little computation. It is best done, I think, like the Dirac space trace theorem stuff. We can write, for example:

$$(\chi_p)_\alpha(\chi_p^\dagger)_\beta = (\,1\ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}(1 + \tau_3) = P_p = \text{projection operator.}$$

Then the LHS of 10.46 is this: (summed on i and all other repeated indices)

$(\chi_p^\dagger)_\alpha\,(\tau_i)_{\alpha\beta}\;(\chi_p)_\beta(\chi_p^\dagger)_a\,(\tau_i)_{ab}\;(\chi_p)_b$

$= [(\chi_p)_b(\chi_p^\dagger)_\alpha]\,(\tau_i)_{\alpha\beta}\;[(\chi_p)_\beta(\chi_p^\dagger)_a\,]\,(\tau_i)_{ab}$

$= \mathrm{tr}(\,P_p\,\tau_i\,P_p\,\tau_i) = \tfrac{1}{4}\,\mathrm{tr}\,[(1+\tau_3)\,\tau_i\,(1+\tau_3)\,\tau_i\,]$

In the terms where i=1,2, we get zero for this reason:

$\mathrm{tr}\,[(1+\tau_3)\,\tau_i\,(1+\tau_3)\,\tau_i\,] = \mathrm{tr}\,[(1+\tau_3)\,\tau_i\,\tau_i\,(1-\tau_3)\,] = \mathrm{tr}\,[(1+\tau_3)\;(1-\tau_3)\,]$
$= \mathrm{tr}[\,1-\tau_3^2] = \mathrm{tr}(1\text{-}1) = \mathrm{tr}(0) = 0.$

Only the i=3 term survives, and we get for that one term.

$= \tfrac{1}{4}\,\mathrm{tr}\,[(1+\tau_3)\,\tau_3\,(1+\tau_3)\,\tau_3\,] = \tfrac{1}{4}\,\mathrm{tr}\,[(1+\tau_3)\,(1+\tau_3)\,] = \tfrac{1}{4}\,\mathrm{tr}\,[2\,] = \tfrac{1}{4} * 4 = 1$

and we have thus verified 10.46.

For n-n scattering, we would replace in the above calculation P_p by $P_n = \tfrac{1}{2}\,(1-\tau_3)$. The i=1,2 terms are still zero, and the i=3 term is still 1, so we get exactly the same result.

We then come to n-p scattering. Here, unfortunately, we have a notational inconsistency between page 228 and the graphs on page 216 top. In the graphs, we associate 1 = n and 2 = p for the input states. But on page 228 for p-n scattering (as they call it), we shall use 1 = p and 2 = n and this is the same for both terms in S_{fi} in 10.45. You might say that the graphs are for n-p scattering, but we are talking p-n scattering. So this just adds to our confusion, but we can handle it. I have added pencil labels to 10.45 showing which states are n and which are p for the first terms in 10.48 and 10.49, to be discussed below. To add to the confusion, the authors have the n and p labels wrong in both second terms in 48 and 49 (or they have the 1 and 2 labels wrong, take your choice).

In the first S_{fi} term, the isospin structure is for the left graph top of page 216 (except n↔p as just discussed), so we get the isospin structure in equation E. How do we do this calculation?

$(\chi_n)_b\,(\chi_p^\dagger)_\alpha\,(\tau_i)_{\alpha\beta}\;(\chi_p)_\beta(\chi_n^\dagger)_a\,(\tau_i)_{ab}$

$(\chi_n)_\alpha(\chi_p^\dagger)_\beta = (\,0\;1)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix} = \tfrac{1}{2}\,(\tau_1 + i\tau_2)$

$(\chi_p)_\alpha(\chi_n^\dagger)_\beta = (\,1\;0)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix} = \tfrac{1}{2}\,(\tau_1 - i\tau_2)$

We then have E being

$E = \tfrac{1}{4}\,\mathrm{tr}[(\tau_1 + i\tau_2)\,\tau_i(\tau_1 - i\tau_2)\,\tau_i]$

Let's just do each term one at a time:

i = 1: $\tfrac{1}{4}\,\mathrm{tr}[(\tau_1 + i\tau_2)\,\tau_1(\tau_1 - i\tau_2)\,\tau_1] = \tfrac{1}{4}\,\mathrm{tr}[(\tau_1 + i\tau_2)\,\tau_1^2(\tau_1 + i\tau_2)\,] = \tfrac{1}{4}\,\mathrm{tr}[(\tau_1 + i\tau_2)\,(\tau_1 + i\tau_2)\,]$
$= \tfrac{1}{4}\,\mathrm{tr}[\,1 - 1] = 0$

i = 2: $\frac{1}{4}\,tr[(\tau_1 + i\tau_2)\,\tau_2(\tau_1 - i\tau_2)\,\tau_2] = \frac{1}{4}\,tr[(\tau_1 + i\tau_2)\,\tau_2^2(-\tau_1 + i\tau_2)] = \frac{1}{4}\,tr[(\tau_1 + i\tau_2)\,(-\tau_1 + i\tau_2)]$
$= \frac{1}{4}\,tr[-1 - 1] = -\frac{1}{4}\,tr(2) = -1$

i = 3: $\frac{1}{4}\,tr[(\tau_1 + i\tau_2)\,\tau_3(\tau_1 - i\tau_2)\,\tau_3] = \frac{1}{4}\,tr[(\tau_1 + i\tau_2)\,\tau_3^2(-\tau_1 - i\tau_2)] = -\frac{1}{4}\,tr[(\tau_1 + i\tau_2)\,(\tau_1 + i\tau_2)] = 0$

We conclude then that E = -1 which agrees with the text.

In the second S_{fi} term for n-p scattering, we get the structure shown in F from the right graph on page 216 top. Our machinery above can handle this as is:

$F = tr(P_p\,\tau_i\,P_n\,\tau_i) = \frac{1}{4}\,tr[(1 + \tau_3)\,\tau_i\,(1 - \tau_3)\,\tau_i]$

i = 1 or 2: $\frac{1}{4}\,tr[(1 + \tau_3)\,\tau_i\,(1 - \tau_3)\,\tau_i] = \frac{1}{4}\,tr[(1 + \tau_3)\,\tau_i^2\,(1 + \tau_3)] = \frac{1}{4}\,tr[(1 + \tau_3)\,(1 + \tau_3)]$
$= \frac{1}{4}\,tr[(2)] = 1$

i=3: $\frac{1}{4}\,tr[(1 + \tau_3)\,\tau_3\,(1 - \tau_3)\,\tau_3] = \frac{1}{4}\,tr[(1 + \tau_3)\,\tau_3^2\,(1 - \tau_3)] = \frac{1}{4}\,tr[(1 + \tau_3)\,(1 - \tau_3)] = 0$

Thus, the sum is 2 from the i=1 and i=3 terms, and this confirms that F = 2 as shown.

Now, since the isospin factors were the same for n-n and p-p, the S_{fi} is exactly the same for both, so the cross section will be exactly the same. But for n-p, we see that the isospin factor weight for the two terms is different, so we have to ponder this a bit!

Explain 10.47 through page 229A: how n-n, p-n and pp are all "equal"

I agree that 10.47 is the correct state for I=1 $I_3 = 0$ -- it has the symmetric sum deal, old hat. The other I_3 terms are of course trivially symmetric being $|\,\frac{1}{2},\frac{1}{2}\,\rangle$ and $|\,-\frac{1}{2}\;-\frac{1}{2}\,\rangle$.

Suppose then that we use 10.47 as the isospin structure for the FINAL state in the S_{fi} scattering. Then consider just the first term in S_{fi}. We have to break this first term into two terms, 1A and 1B

term 1A has $\chi_{1'} = \chi_p$ and $\chi_{2'} = \chi_n$ but still has $\chi_1 = \chi_p$ and $\chi_2 = \chi_n$
term 1B has $\chi_{1'} = \chi_n$ and $\chi_{2'} = \chi_p$ but still has $\chi_1 = \chi_p$ and $\chi_2 = \chi_n$

we then add these two terms and divide by $\sqrt{2}$. The resulting isospin structure will be:

$1A + 1B = p\,\tau\,p \bullet n\,\tau\,n + n\,\tau\,p \bullet p\,\tau\,n$ = LHS of 10.48

What we are really doing is saying that, for this first term, the final state is not just $\chi_{1'} = \chi_p$ and $\chi_{2'} = \chi_n$ as shown in pencil . We want a linear combination for the final state.

What happens with the second term in S_{fi}? We have to break this second term into two terms, 2A and 2B

term 2A has $\chi_{2'} = \chi_n$ and $\chi_{1'} = \chi_p$ but still has $\chi_1 = \chi_p$ and $\chi_2 = \chi_n$
term 2B has $\chi_{2'} = \chi_p$ and $\chi_{1'} = \chi_n$ but still has $\chi_1 = \chi_p$ and $\chi_2 = \chi_n$

we then add these two terms and divide by $\sqrt{2}$. The resulting isospin structure will be:

$2A + 2B = n\,\tau\, p \bullet\ p\,\tau\, n + p\,\tau\, p \bullet\ n\,\tau\, n$ = LHS of 10.48, just has terms swapped now

So both terms in S_{fi} thus have exactly the same isospin structure, namely, LHS of 10.48. We have already computed each term in E and F, so we get the RHS of 10.48 which is $1/\sqrt{2}$. Thus, for scattering into this final channel, S_{fi} (n-p) = $1/\sqrt{2}$ S_{fi}(p-p).

Before explaining this $1/\sqrt{2}$ factor, BD digress with **comments about symmetry** which are very good. In the case we just considered, obviously the isospin part of the wavefunction is symmetric. We find that each term in 10.45 has isospin weight $1/\sqrt{2}$ so the two terms still retain their relative minus sign. We can see that if we swap the two final momenta in S_{fi} we get a minus sign, and this would be true also back in coordinate space which we have not talked much about. We swap the position of the two final particles between our two terms. Therefore, we have symmetric in isospin and antisymmetric in "space" so the point is that we are overall antisymmetric as Fermi statistics requires.

But what about the spin wavefunction? The four spin states s_1 and $s_{1'}$ and s_2 and $s_{2'}$ are still hiding in S_{fi} in 10.45. When we talk about swapping final particles which swaps the two terms in S_{fi}, the spin labels are also swapped. That is, we swap for example 1' ↔ 2' for both p_k and s_k. So I guess we are then antisymmetric under this combined label swap. To really deal with all this, I need a more powerful notation, but let's not dive off into that now.

Now lets consider the I=0 state shown in 10.49. What does this minus sign do to all our algebra above concerning 1A, 1B, 2A, 2B ?? The way I have described things above, I think the 1B and 2B terms both have a new minus sign. So I will then have:

$1A + 1B = p\,\tau\, p \bullet\ n\,\tau\, n - n\,\tau\, p \bullet\ p\,\tau\, n$ = LHS of page 229 equation A $= -3/\sqrt{2}$

$2A + 2B = n\,\tau\, p \bullet\ p\,\tau\, n - p\,\tau\, p \bullet\ n\,\tau\, n$ = the negative of the above! $= +3/\sqrt{2}$

What happens now is that the two terms in S_{fi} have a relative plus sign after the isospin stuff is done, and this means we have space-spin symmetric combined with isospin antisymmetric, and overall we still get our required Fermi antisymmetry. So this is the big point they are making here. For pp and nn scattering, the isospin state is symmetric being | ½ ,½ > and | -½ -½ > as noted above, and the space part is antisymmetric, so we see how in all three cases p-p, n-n and p-n scattering, the symmetry works out correctly.

Now finally we come back to the issue that we thought all three cross sections should be the same, that was the basis of our setting up the theory. In this regard, we must only consider the I = 1 channel for the p-n since the pp and nn are also in the I=1 channel and have no cross section at all in the I=0 channel. But we have this $1/\sqrt{2}$ sitting there in 10.48 which makes us think that the cross section for p-n will be reduced by a factor of ½ relative to, say, p-p scattering. But as they point out, in p-p you have identical particles for real in the final state, and you have to add that overcounting correction ½ as in 7.81 page 135 which was for the example of two final identical photons. This applies to TOTAL cross section, and then we find that the I=0 total cross section will be identical for n-n, p-p, and n-p scattering in our nice little toy theory with perfect symmetry. We are now mid page 229 and I draw another light pencil line to indicate a change of topic.

Non-relativistic limit and effective potential

Non rel limit. To derive 10.50, look at page 30 3.7 where we have for example,

$$\psi = \begin{pmatrix} u \\ \boldsymbol{\sigma}\bullet\mathbf{p}\ u / D \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad E = E+m \approx 2M \text{ non rel}$$

We know that $\gamma_0\gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and therefore we put pieces together to get (recall $\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$)

$$\psi'^\dagger = \begin{pmatrix} u' \\ \boldsymbol{\sigma}\bullet\mathbf{p}'\ u' / D' \end{pmatrix}^\dagger = (u'^\dagger\ [\boldsymbol{\sigma}\bullet\mathbf{p}'\ u'/D']^\dagger) = (u'^\dagger\ u'^\dagger\ \boldsymbol{\sigma}\bullet\mathbf{p}'/D')$$

$$\bar\psi'\gamma_5\psi = \psi'^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi = (u'^\dagger\ u'^\dagger\ \boldsymbol{\sigma}\bullet\mathbf{p}'/D') \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \boldsymbol{\sigma}\bullet\mathbf{p}\ u / D \end{pmatrix} = (u'^\dagger\ u'^\dagger\ \boldsymbol{\sigma}\bullet\mathbf{p}'/D') \begin{pmatrix} \boldsymbol{\sigma}\bullet\mathbf{p}\ u/ D \\ -u \end{pmatrix}$$

$$= u'^\dagger\ \boldsymbol{\sigma}\bullet\mathbf{p}\ u/D - u'^\dagger\ \boldsymbol{\sigma}\bullet\mathbf{p}'/D'\ u$$

In the non-rel limit, D = 2M and we end up then with

$$= u'^\dagger\ \boldsymbol{\sigma}\bullet(\mathbf{p}-\mathbf{p}')\ u / 2M$$

where u and u' at this point are up or down spinors. Now maybe 10.50 is meant to be true for other u spinors, but I will just keep it as is for up and down spinors unless they use it more generally.

Derive 10.51. We need to write out S_{fi} in coordinate space, not momentum space as in 10.45. Normally momentum space is much simpler, so we don't see very many prototypes for coordinate space except for more complex graphs. We can use the example in 7.32 as a model where we will refer to the two spacetime points as x and y to start, then later we can convert them to x_1 and x_2 which will cause the appearance of $\mathbf{r}_1$ and $\mathbf{r}_2$. So here is our starting point for the coordinate space version of just the first term in 10.45:

$$S_{fi} = -i \int d^4x \int d^4y\ [\text{ left side } 1\to 1']_a\ i\,\Delta_F(x-y)\delta_{ab}\ [\text{ right side } 2\to 2']_b$$

where the subscripts are vector isospin indices. For the four external "legs", we use forms like 10.35. So:

$$[\text{ left side } 1\to 1']_a = \bar\psi_{1'}(x)\ \{ ig_o\gamma^5\tau_a \}\ \psi_1(x)$$
$$[\text{ right side } 2\to 2']_b = \bar\psi_{2'}(x)\ \{ ig_o\gamma^5\tau_b \}\ \psi_2(x)$$

where for example we have this combination iso 2-spinor Dirac 4-spinor:

$$\psi_1(x) = N * e^{-ip1\cdot x}\ u_1\ \chi_1 \qquad N = \text{norm factor as shown in 10.35, don't care right now}$$

Let's insert the full FT for Δ_F to get: (see 9.10 page 187)

$$\Delta_F(x\text{-}y) = \int d^4q/(2\pi)^4\, e^{-iq\cdot(x-y)}\, 1/(q^2 - m)^2 \qquad q = p_1 - p_{1'}$$

Now use the approx shown in p 229 B to get

$$\Delta_F(x\text{-}y) = -\int d^4q/(2\pi)^4\, e^{-iq\cdot(x-y)}\, 1/(\mathbf{q}^2 + m)^2 \qquad q = p_1 - p_{1'}$$

where I am just marking what value we know q will soon be taking. At this point, we can do both integrations dx^0 and dy^0 because the only dependence on these variables is in exponentials. The dx^0 integral will yield $2\pi\delta(q^0 - [p_1{}^0 + p_{1'}{}^0])$ which will then remove our dq^0 integration setting q^0 as shown. Then the dy^0 integration will give our overall energy conservation factor $2\pi\delta([p_2{}^0 + p_{2'}{}^0] - [p_1{}^0 + p_{1'}{}^0])$. At this point we have: (where we remove the central δ_{ab} and arranged the 2π factors a bit)

$$S_{fi} = 2\pi\delta(\text{energy}) *$$
$$-i \int d^3x \int d^3y\ [\text{ left side } 1\rightarrow 1']_a\ i \{- \int d^3q/(2\pi)^3\, e^{+iq\bullet(x-y)}\, 1/(\mathbf{q}^2 + m)^2\}\ [\text{ right side } 2\rightarrow 2']_a$$

Now we use our Mandl page 21 result which is this

$$\int d^3q/(2\pi)^3\, e^{+iq\bullet(x-y)}\, 1/(\mathbf{q}^2 + m)^2 = e^{-m|x-y|}/4\pi\,|\mathbf{x}\text{-}\mathbf{y}|$$

so we then have

$$S_{fi} = 2\pi\delta(\text{energy}) *$$
$$-i \int d^3x \int d^3y\ [\text{ left side } 1\rightarrow 1']_a\ i \{- e^{-m|x-y|}/4\pi\,|\mathbf{x}\text{-}\mathbf{y}|\ \}\ [\text{ right side } 2\rightarrow 2']_a$$

where it is now understood that the energy phase portions of the exponentials in the left and right side brackets are now gone (they were used up in our $dx^0 dy^0$ integrations). So we have for example

$$\psi_1(x) = N * e^{-ip1\bullet x}\, u_1\ \chi_1$$

$$[\text{ left side } 1\rightarrow 1']_a = \bar{\psi}_{1'}(x)\ \{ ig_o\gamma^5\tau_a \}\ \psi_1(x) = N_1 N_{1'} e^{+ip1'\bullet x}\, e^{-ip1\bullet x}\, \bar{u}_{1'} \chi_{1'}{}^{\dagger}\ \{ ig_o\gamma^5\tau_a \}\ u_1\chi_1$$

and similarly for the right side factor. At this point we can use 10.50 and combine the exponentials:

$$[\text{ left side } 1\rightarrow 1']_a = N_1 N_{1'}/2M\, e^{-i(p1-p1')\bullet x}\, \chi_{1'}{}^{\dagger}\, u_1{}^{\dagger}\, \boldsymbol{\sigma}\bullet(\mathbf{p}_1 - \mathbf{p}_{1'})\ \{ig_o\tau_a\}\ u_1\chi_1$$

where now the u's are just 2-spinors. Similarly we now have

$$[\text{ right side } 2\rightarrow 2']_a = N_2 N_{2'}/2M\, e^{-i(p2-p2')\bullet y}\, \chi_{2'}{}^{\dagger}\, u_2{}^{\dagger}\, \boldsymbol{\sigma}\bullet(\mathbf{p}_2 - \mathbf{p}_{2'})\ \{ig_o\tau_a\}\ u_2\chi_2$$

We can of course write

$e^{-ik\bullet x}\,\mathbf{k} = i\,\nabla_x\, e^{-ik\bullet x}$ so the above then become

$$[\text{ left side } 1\to 1']_a = i\, N_1 N_{1'}/2M\, \chi_{1'}^{\dagger}\, u_{1'}^{\dagger}\, \boldsymbol{\sigma}\bullet\nabla_x \;\{ig_o\tau_a\}\, u_1\chi_1\; e^{-i(p1-p1')\bullet x}$$

$$[\text{ right side } 2\to 2']_a = i\; N_2 N_{2'}/2M\, \chi_{2'}^{\dagger}\, u_2^{\dagger}\, \boldsymbol{\sigma}\bullet\nabla_y \,\{ig_o\tau_a\}\, u_2\chi_2\; e^{-i(p2-p2')\bullet y}$$

Now we could show all isospin and Pauli spin indices explicitly, but a shortcut notation is to put a 1 and 2 label on these $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ matrix vectors so we can remember which one goes with which spinors. Thus, we rewrite the above as

$$[\text{ left side } 1\to 1']_a = i\, N_1 N_{1'}/2M\, \chi_{1'}^{\dagger}\, u_{1'}^{\dagger}\, \boldsymbol{\sigma}^1\bullet\nabla_x \;\{ig_o\tau^1{}_a\}\, u_1\chi_1\; e^{-i(p1-p1')\bullet x}$$

$$[\text{ right side } 2\to 2']_a = i\; N_2 N_{2'}/2M\, \chi_{2'}^{\dagger}\, u_2^{\dagger}\, \boldsymbol{\sigma}^2\bullet\nabla_y \,\{ig_o\tau^2{}_a\}\, u_2\chi_2\; e^{-i(p2-p2')\bullet y}$$

We can then write the product of the above left and right sides in this way

$$[]^*[] = i\, N_1 N_{1'}/2M * i\; N_2 N_{2'}/2M \; * (ig_o)^2 * e^{-i(p1-p1')\bullet x} * e^{-i(p2-p2')\bullet y} *$$

$$\chi_{1'}^{\dagger} u_{1'}^{\dagger} \chi_{2'}^{\dagger} u_2^{\dagger}\, \{\, \boldsymbol{\tau}^1\bullet\boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1\bullet\nabla_x \;\; \boldsymbol{\sigma}^2\bullet\nabla_y \,\}\, u_1\chi_1\, u_2\chi_2$$

and now we assemble all the pieces to get

$$S_{fi} = 2\pi\delta(\text{energy}) *$$

$$-i \int d^3x \int d^3y \;[\text{ left side } 1\to 1']_a \; i\,\{-\, e^{-m|x-y|}/4\pi\,|\mathbf{x}\text{-}\mathbf{y}|\,\}\;[\text{ right side } 2\to 2']_a$$

$$= +\, 2\pi\delta(\text{energy})^*\, N_1 N_{1'} N_2 N_{2'} (2M)^{-2}\, g_0^2 \int d^3x \int d^3y \; * \{ - e^{-m|x-y|}/4\pi\,|\mathbf{x}\text{-}\mathbf{y}|\,\}$$

$$\chi_{1'}^{\dagger} u_{1'}^{\dagger} \chi_{2'}^{\dagger} u_2^{\dagger}\, \{\, \boldsymbol{\tau}^1\bullet\boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1\bullet\nabla_x \;\; \boldsymbol{\sigma}^2\bullet\nabla_y \,\}\, u_1\chi_1\, u_2\chi_2 \;\; e^{-i(p1-p1')\bullet x}\, e^{-i(p2-p2')\bullet y}$$

The normalizing factors are $N_1 = 1$ from 10.35 in the non-rel limit, so away they go. So write one more time:

$$= +\, 2\pi\delta(\text{energy})^*\;\; g_0^2/[4\pi(2M)^2\,] \;\; * \int d^3x \int d^3y \;\; * \{ - e^{-m|x-y|}/\,|\mathbf{x}\text{-}\mathbf{y}|\,\}$$

$$\chi_{1'}^{\dagger} u_{1'}^{\dagger} \chi_{2'}^{\dagger} u_2^{\dagger}\, \{\, \boldsymbol{\tau}^1\bullet\boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1\bullet\nabla_x \;\; \boldsymbol{\sigma}^2\bullet\nabla_y \,\}\, u_1\chi_1\, u_2\chi_2 \;\; e^{-i(p1-p1')\bullet x}\, e^{-i(p2-p2')\bullet y}$$

Now we combine the constant with the {...} factor. Also, I think we can do parts integration to cast the gradients over onto the e^{-mr}/r factor, the two minus signs will cancel, and then we have

$$= 2\pi\delta(\text{energy})^* \; g_0^2/[4\pi(2M)^2] \; * \int d^3x \int d^3y \; * e^{-i(p1-p1')\bullet x} e^{-i(p2-p2')\bullet y}$$

$$\chi_1'^{\dagger} u_1'^{\dagger} \chi_2'^{\dagger} u_2^{\dagger} \{ \boldsymbol{\tau}^1 \bullet \boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1 \bullet \nabla_{\mathbf{x}} \;\; \boldsymbol{\sigma}^2 \bullet \nabla_{\mathbf{y}} \} \{ - e^{-m|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}| \} \;\; u_1\chi_1 \, u_2\chi_2$$

Now we can replace $\nabla_{\mathbf{y}} f(\mathbf{x}-\mathbf{y}) = - \nabla_{\mathbf{x}} f(\mathbf{x}-\mathbf{y})$ *and* we can use our knowledge that there must be overall 3-momentum conservation (though it does not pop out here) to get our final result for our first term

$$S_{fi} = 2\pi\delta(\text{energy})^* \; * \int d^3x \int d^3y \; * e^{-i(p1-p1')\bullet (x-y)}$$

$$\chi_1'^{\dagger} u_1'^{\dagger} \chi_2'^{\dagger} u_2^{\dagger} \{ g_0^2/[4\pi(2M)^2] \;\; \boldsymbol{\tau}^1 \bullet \boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1 \bullet \nabla_{\mathbf{x}} \;\; \boldsymbol{\sigma}^2 \bullet \nabla_{\mathbf{x}} \} \{ e^{-m|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}| \} \;\; u_1\chi_1 \, u_2\chi_2$$

Now, if we regard our labels on the σ's and τ's to match the labels of the *initial* states, we can then obtain the second term by making the interchange 1' ↔ 2' and adding an overall minus sign. Now we know that the expo factor means the only large contribution to the integral comes when $m|\mathbf{x}-\mathbf{y}| << 1$, say, and in this range, we can say $q \bullet (x-y) << |\mathbf{q}|/m$ (m = pion mass). So, if we assume we are in a non-rel limit where the 3-momentum transfer is much smaller than m, we can ignore the first exponential, and thus we can do a sort of Born approximation. In this case, then, the swap 1' ↔ 2'only affects the large lower factor, and we can then write

$$S_{fi} = 2\pi\delta(\text{energy})^* \; * \int d^3x \int d^3y$$

$$\chi_1'^{\dagger} u_1'^{\dagger} \chi_2'^{\dagger} u_2^{\dagger} \{ [g_0^2/[4\pi(2M)^2] \;\; \boldsymbol{\tau}^1 \bullet \boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1 \bullet \nabla_{\mathbf{x}} \;\; \boldsymbol{\sigma}^2 \bullet \nabla_{\mathbf{x}}] - (1' \leftrightarrow 2') \} \{ e^{-m|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}| \} \;\; u_1\chi_1 \, u_2\chi_2$$

At this point, we isolate this entire central factor and call it $V(\mathbf{x},\mathbf{y})$ so get

$$V(\mathbf{x},\mathbf{y}) = [g_0^2/[4\pi(2M)^2] \;\; \boldsymbol{\tau}^1 \bullet \boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1 \bullet \nabla_{\mathbf{x}} \;\; \boldsymbol{\sigma}^2 \bullet \nabla_{\mathbf{x}}] \, e^{-m|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}| - (1' \leftrightarrow 2') \}$$

But the meaning here is a bit subtle. We have to understand what we really mean by (1' ↔ 2') and this has to do with the index linkages on the τ and σ matrices. In the exchange term, for example, $\boldsymbol{\tau}^1$ links to the $u_1\chi_1$ state on the right, but links to the $\chi_2'^{\dagger} u_2^{\dagger}$state on the left. With this understood, we can write our final form

$$S_{fi} = 2\pi\delta(\text{energy})^* \int d^3x \int d^3y \;\; \chi_1'^{\dagger} u_1'^{\dagger} \chi_2'^{\dagger} u_2^{\dagger} \, V(\mathbf{x},\mathbf{y}) \;\; u_1\chi_1 \, u_2\chi_2$$

$$V(\mathbf{x},\mathbf{y}) = g_0^2/[4\pi(2M)^2] \, (1-P_{1' \leftrightarrow 2'}) \, \boldsymbol{\tau}^1 \bullet \boldsymbol{\tau}^2 \;\; \boldsymbol{\sigma}^1 \bullet \nabla_{\mathbf{x}} \;\; \boldsymbol{\sigma}^2 \bullet \nabla_{\mathbf{x}} \;\; \{ e^{-m|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}| \}$$

and so finally **we have derived 10.51 exactly**. They write the constant as f^2/m^2 where f^2 is a dimensionless parameter $f^2 = g_o^2/4\pi * (m/2M)^2 \sim 14 * (135/2000)^2 \sim 14 / (14.8)^2 \sim 1/15$, so just maybe we can do some perturbation theory here??

Subtlety #1: We need to think about those gradients a bit. Suppose we had:

$\partial_{\mathbf{x}}\,\partial_{\mathbf{y}}\,f(x,y)$

We think of the gradients as acting separately, and in either order. But suppose we have the special case

$\partial_{\mathbf{x}}\,\partial_{\mathbf{y}}\,f(x-y)$

and suppose we replace $\partial_{\mathbf{y}}\,f(x-y) = -\,\partial_{\mathbf{x}}\,f(x-y)$. Then we end up with

$$-\,\partial_{\mathbf{x}}\;\partial_{\mathbf{x}}\,f(x-y) \;= -\,\partial_{\mathbf{x}}^{2}\,f(x-y) \;=\; -\,\partial_{\mathbf{x}}\,[\;\partial_{\mathbf{x}}\,f(x-y)\;]$$

so the point is that you really to have a second derivative here, and the first $\partial_{\mathbf{x}}$ acts on "everything to the right". We shall refer to this issue a little ways below.

Derive the angular average result page 230 C.

Now on page 230 we are supposed to do an "angular average" of $V(\mathbf{x},\mathbf{y})$. What does this even mean? First of all, I think that V(x,y) is really only a function of $\mathbf{r} = \mathbf{x}\text{-}\mathbf{y}$. I can write the above as

$$V(\mathbf{r}) = g_0^{2}/[4\pi(2M)^{2}]\;(1-P_{1'\leftrightarrow 2'})\;\boldsymbol{\tau}^{1}\bullet\boldsymbol{\tau}^{2}\;\;\boldsymbol{\sigma}^{1}\bullet\nabla_{\mathbf{r}}\;\boldsymbol{\sigma}^{2}\bullet\nabla_{\mathbf{r}}\;\{\,e^{-\mathbf{m}\mathbf{r}}/r\,\}$$

so here is something I could take an "angular average" of, and the **core part** is this

$$\int d\Omega/4\pi\;\;\boldsymbol{\sigma}^{1}\bullet\nabla\;\;\boldsymbol{\sigma}^{2}\bullet\nabla\;\;\{\,e^{-\mathbf{m}\mathbf{r}}/r\,\}$$

where $\nabla = \nabla_{\mathbf{r}}$ is the usual gradient, nothing special. Let's expose the vector indices above so

$$=(\boldsymbol{\sigma}^{1})_{\mathbf{i}}(\boldsymbol{\sigma}^{2})_{\mathbf{j}}\int d\Omega/4\pi\;\partial_{\mathbf{i}}\partial_{\mathbf{j}}\;\{\,e^{-\mathbf{m}\mathbf{r}}/r\,\}\;=(\boldsymbol{\sigma}^{1})_{\mathbf{i}}(\boldsymbol{\sigma}^{2})_{\mathbf{j}}\;\;I_{\mathbf{i}\mathbf{j}}$$

I want to argue that $I_{\mathbf{ij}}$ must be proportional to $\delta_{\mathbf{ij}}$ because it is a rank 2 tensor, and there are no other tensor forms available. But let's first do a little work as follows:

$$\partial_{\mathbf{j}}f(r) = \partial r/\partial r_{\mathbf{j}}\;\partial f/\partial r \;=\; (\partial_{\mathbf{i}}r)\;\partial f/\partial r \;\;=(r_{\mathbf{j}}/r)\;\;\partial f/\partial r\;\;.$$

$$\partial_{\mathbf{i}}[\partial_{\mathbf{j}}f(r)] = \;\partial_{\mathbf{i}}[\;(r_{\mathbf{j}}/r)\;\;\partial f/\partial r\;]$$

$$=(r_{\mathbf{j}}/r)\;\partial_{\mathbf{i}}(\partial f/\partial r)+\partial_{\mathbf{i}}(r_{\mathbf{j}}/r)\;\partial f/\partial r\;\;=r_{\mathbf{j}}/r\;r_{\mathbf{i}}/r\;\partial^{2}f/\partial r^{2}+[\;\delta_{\mathbf{ij}}/r-r_{\mathbf{j}}/r^{2}(\partial_{\mathbf{i}}r)\;]\;\partial f/\partial r$$

$$=r_{\mathbf{j}}/r\;r_{\mathbf{i}}/r\;\partial^{2}f/\partial r^{2}+\;[\;\delta_{\mathbf{ij}}/r-r_{\mathbf{j}}/r^{2}r_{\mathbf{i}}/r\;]\;\partial f/\partial r$$

$$=r_{\mathbf{i}}r_{\mathbf{j}}/r^{2}\;[\partial^{2}f/\partial r^{2}-\;(1/r)\partial f/\partial r]\;+\delta_{\mathbf{ij}}/r\;\partial f/\partial r$$

Now let's sidetrack for a moment to examine the Laplacian in spherical coordinates. From our TK sheet 6 we have, when acting on only a f(r), that

$$\nabla^2 f(r) = (1/r^2)\partial_r [\, r^2 \partial_r] f(r) = (1/r^2) \{ 2r\, \partial_r + r^2 \partial^2_r \} f(r)$$

$$= \{ (2/r)\, \partial_r + \partial^2_r \} f(r)$$

which may be useful at some point. Now back to our integral,

$$I_{ij} = \int d\Omega/4\pi\, \partial_i \partial_j \{ e^{-mr}/r \} = \int d\Omega/4\pi\, (r_i r_j/r^2) [\partial^2_r - (1/r)\partial_r] + \delta_{ij}/r\, \partial_r] \{ e^{-mr}/r \}$$

$$= \int d\Omega/4\pi (r_i r_j/r^2) [\partial^2_r - (1/r)\partial_r] \{ e^{-mr}/r \} + \delta_{ij}/r\, \partial_r \{ e^{-mr}/r \}$$

where we have done the angular average of the second term (there is nothing there to average!). Now shuffle the first term a bit

$$= \{ \int d\Omega/4\pi\, r_i r_j \} (1/r)^2 [\partial^2_r - (1/r)\partial_r] \{ e^{-mr}/r \} + \delta_{ij}/r\, \partial_r \{ e^{-mr}/r \}$$

Now we need to compute this angular average:

$$\int d\Omega/4\pi\, r_i r_j = F_{ij}$$

which I argue is proportional to δ_{ij}. To find the coefficient, consider

$$F_{33} = \int d\Omega/4\pi\, r_3 r_3 = r^2 \int d\Omega/4\pi\, \cos^2\theta = \tfrac{1}{2} r^2 \int_{-1}^{1} dx\, x^2 = \tfrac{1}{2} r^2\, 2/3 = r^2/3$$

This I conclude that

$$\int d\Omega/4\pi\, r_i r_j = F_{ij} = r^2/3\, \delta_{ij}$$

and we then have from above

$$I_{ij} = \{ \int d\Omega/4\pi\, r_i r_j \} (1/r)^2 [\partial^2_r - (1/r)\partial_r] \{ e^{-mr}/r \} + \delta_{ij}/r\, \partial_r \{ e^{-mr}/r \}$$

$$= \{ r^2/3\, \delta_{ij} \} (1/r)^2 [\partial^2_r - (1/r)\partial_r] \{ e^{-mr}/r \} + \delta_{ij}/r\, \partial_r \{ e^{-mr}/r \}$$

$$= \delta_{ij} [r^2/3\, (1/r)^2 [\partial^2_r - (1/r)\partial_r] + 1/r\, \partial_r] \{ e^{-mr}/r \}$$

$$= \delta_{ij} [1/3 [\partial^2_r - (1/r)\partial_r] + 1/r\, \partial_r] \{ e^{-mr}/r \}$$

$$= \delta_{ij}\ \ 1/3\ [\partial^2_r - \ (1/r)\partial_r + 3/r\ \partial_r\ ]\ \{\ e^{-mr}/r\ \}$$

$$= \delta_{ij}\ \ 1/3\ [\partial^2_r + 2/r\ \partial_r\ ]\ \{\ e^{-mr}/r\ \}$$

$$= \delta_{ij}\ \ 1/3\ \nabla^2\{\ e^{-mr}/r\ \}$$

where we have now used the Laplacian result anticipated above. So next use this vector identity from TK

$$\nabla^2(fg) = f\,\nabla^2 g\ + g\nabla^2 f\ + 2\ \nabla f \bullet \nabla g$$

where $f = e^{-mr}$ and $g = 1/r$. Then we have

$$\nabla f\ = \hat{\mathbf{r}}\ \partial_r f = -m\ e^{-mr}\ \hat{\mathbf{r}}$$

$$\nabla g = \ \hat{\mathbf{r}}\ \partial_r g\ = \hat{\mathbf{r}}\ \partial_r(1/r) = -\ 1/r^2\ \hat{\mathbf{r}}$$

$$\Rightarrow\ 2\ \nabla f \bullet \nabla g\ = +2m/r^2\ e^{-mr}$$

Continuing

$$\nabla^2 g\ = \nabla^2(1/r) = -4\pi\ \delta^3(\mathbf{r}) \qquad \text{// our famous point charge result}$$

$$\nabla^2 f\ = \nabla^2\ e^{-mr}\ = 1/r^2\partial_r(\ r^2\partial_r)\ e^{-mr}\ = -m/r^2\partial_r(\ r^2\ e^{-mr})$$

$$= (-m/r^2)\ [\ 2r - m\ r^2]\ e^{-mr}\ = -m[\ 2/r - m]\ e^{-mr}\ = [\ m^2 - 2m/r\ ]\ e^{-mr}$$

Now assemble all these pieces:

$$\nabla^2\{\ e^{-mr}/r\ \}\ = \nabla^2(fg) = f\,\nabla^2 g\ + g\nabla^2 f\ + 2\ \nabla f \bullet \nabla g$$

$$= e^{-mr}\ [-4\pi\ \delta^3(\mathbf{r})] + 1/r\ [\ m^2 - 2m/r\ ]\ e^{-mr} + 2m/r^2\ e^{-mr}$$

$$= [\ -4\pi\ \delta^3(\mathbf{r})\ + m^2/r\ ]\ e^{-mr}$$

so that our integral above must be

$$I_{ij} = \delta_{ij}\ \ 1/3\ \nabla^2\{\ e^{-mr}/r\ \} = \delta_{ij}\ \ 1/3\ \ [\ -4\pi\ \delta^3(\mathbf{r})\ + m^2/r\ ]\ e^{-mr}$$

and we then have for our "core part" ,

$$\int d\Omega/4\pi\ \ \boldsymbol{\sigma}^1 \bullet \nabla\ \ \boldsymbol{\sigma}^2 \bullet \nabla\ \ \{\ e^{-mr}/r\ \} = \ (\boldsymbol{\sigma}^1)_i(\boldsymbol{\sigma}^2)_j\ \ I_{ij}$$

$$= \boldsymbol{\sigma}^1 \bullet \boldsymbol{\sigma}^2\ \ \ 1/3\ \ [\ -4\pi\ \delta^3(\mathbf{r})\ + m^2/r\ ]\ e^{-mr}$$

Now in the δ^3 term which forces r= 0, we can set the expo to 1, so have

$$= \boldsymbol{\sigma}^1 \bullet \boldsymbol{\sigma}^2 \quad 1/3 \ [-4\pi \, \delta^3(\mathbf{r}) \ + m^2 \, e^{-mr}/r \ ]$$

and we conclude finally that

$$< V(\mathbf{r}) >_{\Omega} = g_0^2/[4\pi(2M)^2] \, (1\text{-}P_{1'\leftrightarrow 2'}) \, \boldsymbol{\tau}^1 \bullet \boldsymbol{\tau}^2 \, \boldsymbol{\sigma}^1 \bullet \boldsymbol{\sigma}^2 \quad 1/3 \ [-4\pi \, \delta^3(\mathbf{r}) \ + m^2 \, e^{-mr}/r \ ]$$

This was an amazingly complicated calculation, probably there is a simpler way, but at least **we have now derived result C on page 230.**

Derive 10.52. The first equality is trivial since either vector $\boldsymbol{\sigma}^2 = 3$. In other words, we have

$$1/12 \ [\ (\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2)^2 - 6] \ [\ (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^2 - 6] \ = 1/12 \ [\ 2 \ \boldsymbol{\tau}_1 \bullet \boldsymbol{\tau}_2 \ ] \ [\ 2 \ \boldsymbol{\sigma}_1 \bullet \boldsymbol{\sigma}_2 \ ] \ = 1/3 \ \boldsymbol{\tau}_1 \bullet \boldsymbol{\tau}_2 \ \boldsymbol{\sigma}_1 \bullet \boldsymbol{\sigma}_2$$

Then we can talk about total isospin **T** and total spin **S** as operators, eg, $\mathbf{T} = \boldsymbol{\tau}_1 \oplus \boldsymbol{\tau}_2$ with eigenvalue T(T+1) for $\mathbf{T}^2$. Now finally they talk about all three symmetries: isospin, spin and space. If we have sym in space, then one *or* the other of spin and isospin has to be antisym and the other sym. In terms of CG composition, this means that either T=1 and S=0 or T=0 and S=1. In either of these cases, we get -1 for the factor shown in 10.52, nothing fancy there. I think the <..> in 10.52 refer to expectation value in an assumed state, so we would have to make various linear combinations of our states like $| \ u_1\chi_1 \ u_2\chi_2 >$, which in itself is a good reason for our operator abstraction as shown in 10.51 where we can attach whatever states we want on both sides. That is, we have to make 2-particle eigenstates of **T** and **S**. When this is done, you then get the result E which, according to our either/or argument above, evaluates to -1.

So, assuming proper states of either kind that allow a non-zero S_{fi}, we can talk about S-wave action in nucleon scattering. The S wave is symmetrical as an orbital in the reduced coordinate we have been talking about (**r**), and so our comments about T and S then apply.

Now what about that angular average we computed above? It is a function of just r, not **r**, so we can think of this average as some kind of effective adjusted Coulomb-like potential that is spherically symmetrical in **r**-space and I guess if you have a spherically symmetrical S state, you can replace the messy potential with this averaged potential and maybe the replacement is exact or, if not, very close.

The upshot then is that this effective scalar potential is that given in 10.53. Unlike E&M, "like charges attract" in our π-N world with g_0 coupling constant. We don't have one nucleon with $+g_0$ and the other with $-g_0$. So the Yukawa potential is attractive! And you see now why they defined the dimensionless f constant because we then get just a dimensionless multiple of the Yukawa potential plus a point core $\delta^3(\mathbf{r})$ force that is repulsive since it has a plus sign.

I guess I never thought about whether the pion as mediator creates an attractive or repulsive force! The claim and all the math here say it is attractive except for the delta part.

Does this explain observed scattering data? The claim is that it does not for the lower partial waves where the two nucleons get very close together, because then higher order diagrams swamp the lowest one we did. BUT, for higher partial waves the claim is that this model more or less does explain things. I don't have any books with this detail in them, but we are given a 1961 review paper which I could go read if I wanted to learn more.

I presume quarks are attracted to each other as are nucleons in this discussion, with gluons in place of pions as the mediator.

The Sakurai book has some discussion of N-N scattering and quotes some of the BD results.

Digression. Now what about that i in the $i\gamma_5$, I have still not resolved this issue. If the i were not there, I suspect our force would be repulsive instead of attractive. Yes indeedy. THAT is the real reason this i is inserted here, otherwise you get Coulomb-like repulsion, but everybody knows that the nuclear force holds things together! So the BD argument with the word "real" on page 213 is in fact totally bogus. The point is very obvious, you need (coupling constant)^2 < 0 for attraction between objects of the same hadronic charge, so that constant is set to ig_0 where g_0 is real. Good.

10.6 Meson Nucleon Scattering. (231) π-N scattering

More comments on Feynman Rules: Let's review what we really found in the last section:

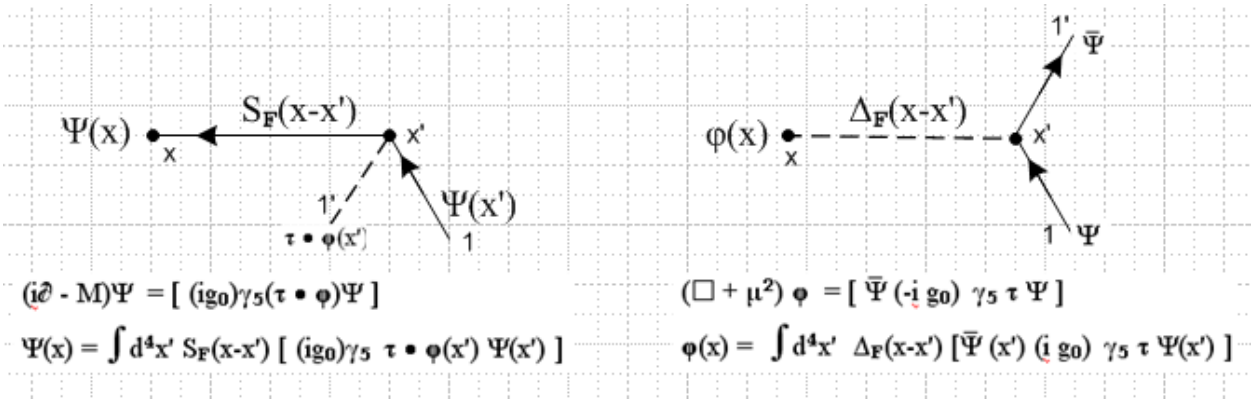
$$(\Box + \mu^2)\,\varphi = [\,\bar{\Psi}\,(-i\,g_0)\;\gamma_5\,\boldsymbol{\tau}\,\Psi\,] \qquad (\Box_x + \mu^2)\Delta_F(x-x') = -\,\delta^4(x-x')$$

$$\varphi(x) = \int d^4x'\;\Delta_F(x-x')\,[\bar{\Psi}\,(x')\,(i\,g_0)\;\gamma_5\,\boldsymbol{\tau}\,\Psi(x')\,]$$

$$(i\partial\!\!\!/ - M)\Psi = [\,(ig_0)\gamma_5(\boldsymbol{\tau}\bullet\boldsymbol{\varphi})\Psi\,] \qquad (i\partial\!\!\!/ - m)S_F(x-x') = +\,\delta^4(x-x')$$

$$\Psi(x) = \int d^4x'\;S_F(x-x')\,[\,(ig_0)\gamma_5\;\boldsymbol{\tau}\bullet\boldsymbol{\varphi}(x')\,\Psi(x')\,]$$

In each case, you can verify the integral form by applying the operator on both sides. The minus sign on the RHS of the ODE for $\boldsymbol{\varphi}$ is offset by the minus sign in the definition of Δ_F, so both integral forms have an overall plus sign. Here I repeat the above information with some corresponding graphs:



In the left picture, I have put both the pion and nucleon as inbound because φ and ψ appear in [...]. On the right, I put the $\bar{\psi}$ nucleon as outbound since $\psi^\dagger$, while the ψ nucleon is inbound.

Now I claim that in the integral form, the thing in the [...] tells you the "Feynman Rule". The way I have done things, the rules seem to be this:

1. Put Δ_F or S_F for propagators without any factors of i

2. Put this factor at each vertex: $+ig_o\gamma_5\,\boldsymbol{\tau}$

3. Put the usual $\bar{\Psi}$ and Ψ for external nucleon legs.
4. Put $\boldsymbol{\varphi}$ for external pion leg. It will dot with the $\boldsymbol{\tau}$ at the vertex.
5. If you rotate a pion leg around to the other sense, it becomes $\boldsymbol{\varphi}^*$.
6. We already know how to rotate nucleon legs around

My rules are a little different from those quoted on page 224 and 225. I could probably make things agree, but let's let it ride for now. My Main Point here: the "Feynman Rule" for a vertex is basically determined by the driving term in the equation of motion, give or take a sign depending on how the propagator is defined.

The BD rule is to put $(-ig_o)\ i\gamma_5\,\boldsymbol{\tau}$ at each vertex, and to put +i on all propagators. You see this rule in effect in 10.45. I suppose you have to think about whether your rules are for coordinate space or momentum space.

We see this same rule also in **10.54 for π-N scattering**! Look at the upper graph on page 231. The inbound pion is labeled 1 and we see the factors $\boldsymbol{\varphi}_1\ u_1\chi_1$ associated with this, in agreement with my picture on the left above. At the second vertex the pion there is outbound so is φ*. Everything is correct in 10.54. The pions have different leg normalizers as shown in 9.5: the have $1/\sqrt{2\omega}$ while the nucleons have $\sqrt{M/E}$, so the pions are more like photons in this regard. The $(2\pi)^6$ is also correct, each leg contributes the same amount.

So I am completely happy with 10.54 and also I agree with 10.55, and I know that crossing is true for all orders for general Chew-like reasons.

At this point, I have to go off and learn a little basic low energy scattering theory because BD are assuming the reader know about this stuff. I will have to hunt around a bit tomorrow, signing off for 11.23.08. That Kallen book has lots of theory of π-N I notice. So things are going to slow down again.