# Chapter 7: Applications (100)

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**Overview:** This is the meat and potatoes section of this book so far, 47 jam-packed pages. Many of the basic QED cross sections are computed at least in part. The Dirac theory was 1928, and you can see that all these calculations were first done in the 1929-1935 time frame, very long ago from 2008, but still pretty new to me. I suspect computations were done differently and less conveniently in earlier times, now we always use the trace method even for a polarized cross sections, making use of the energy and spin projectors. Even so, the computations still take a lot of work. Probably computer algebra (Clifford) programs can do the traces now, and I downloaded some stuff on the subject but don't want to get sidetracked too much. In all these calculations, the photon is treated as a "particle" (really a plane wave $A^\mu$) when a leg of a process, but otherwise as just a classical field even though it gets into propagators via the Moller potential idea. So it is pretty nice to see all these calculations done with only the Dirac theory of the electron (but with covariant E&M inclusion) and with no quantum field theory at all. Some day I will learn more about this history here, the names and geography are getting quite familiar now. As noted above, each of the major cross sections now has a name or names associated with it, which helps people remember what the process is.

### 7.1 Coulomb Scattering of Electrons (100) ( Mott 1929)

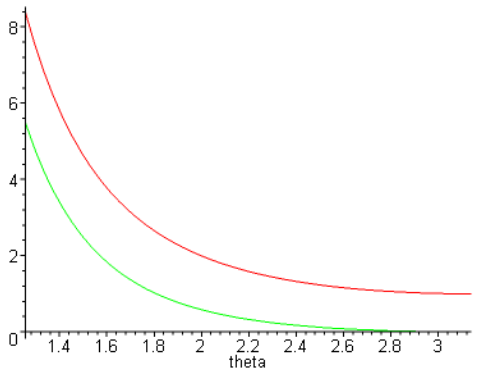
**Preliminary Comments: Compare to Rutherford result, and discuss BD units.**

Certainly this is *the* prototype application, the single vertex graph with regular positive-energy electrons, and a V = Zq/r potential. Conveniently, I recently read Goldstein's writeup on Rutherford scattering and here is the result he got on page 84: (lab frame, quote is from my G notes)

$\sigma(\Theta) = 1/4\ (ZZ'e^2/2E)^2\ \csc^4(\Theta/2)$ $\Theta$ defined as deviation from the initial direction

"This is the Rutherford classical result as derived in my Livesey book page 154 where E = ½ $mv^2$. Rutherford did the theory in 1911, Geiger and Muller did the experiments in 1913, and this showed the existence of a very small nucleus and the planetary model of the atom (Bohr model) was born. Most beam charges go straight through, only those that come very close to a nucleus get deflected a lot, hence the $1/\sin^4(\Theta/2)$ factor. It showed that the $1/r^2$ force law was accurate down to something like $10^{-14}$ m. The beam was alpha particles created by a radon source."

Now let's look ahead at the final result of our present calculation which is Mott equation 7.22. It is a relativistically correct result and has an extra $\beta^2$ term which Goldstein's derivation does not of course produce. For large $\beta$ the cross section is even more strongly peaked in the forward direction than the non-rel formula says. For $\beta \sim 1$, the numerator is $\cos^2\theta/2$ and this completely kills off backward direction scattering which is at $\theta = \pi$. Here are some Maple plots of the angular factors with $\beta = 0$ and $\beta = 1$:




where I just show the results for $\theta \geq .4\pi$ to suppress the huge peak near small $\theta$.

```
plot([(csc(theta/2))^4,(csc(theta/2))^4*(cos(theta/2))^2],theta=.4*Pi..1.*Pi);
```

**Units.** Now let's look at the non-angular part of the formula and figure out what "units" we are using. BD give us a hint of their unit choice by saying V = -Ze/4πr. This tells me that $k_1 = 1/4\pi$ and I added some notes to my "units" document. The BD system is called the **Heaviside-Lorentz System** (see Jackson p 616 also) and we have these things being true in such a system:

$1/137 = \alpha = k_1 e^2/(\hbar c) = e_{HL}^2/(4\pi\hbar c)$ where $e_{HL}^2 = 4\pi\, e_{esu}^2$

where $e_{esu}$ is the electron charge in a system where $k_1 = 1$, which is what Goldstein uses. Therefore, if we want to convert Goldstein's result above to BD units, we would $e_{esu}^2 = e_{HL}^2/4\pi$ to get

$\sigma(\Theta) = 1/4\ (ZZ'e_{HL}^2/2E)^2\ \csc^4(\Theta/2) / 4\pi$

We could then make the replacement $e_{HL}^2/4\pi = \hbar c\alpha$ to get

$$\sigma(\Theta) = 1/4\ (ZZ'\hbar c\alpha/2E)^2 \csc^4(\Theta/2) = \tfrac{1}{4}\ Z^2 Z'^2 \alpha^2 (\hbar c)^2 (1/2E)^2 \csc^4(\Theta/2)$$

But then $2E = mv^2$ and we would then have

$$\sigma(\Theta) = \tfrac{1}{4}\ Z^2 Z'^2 \alpha^2 (\hbar c)^2 (1/mv^2)^2 \csc^4(\Theta/2)$$

We can compare this with BD 7.22 taken to the NR limit where we have

$$p\beta = \gamma mv*v/c \rightarrow mv^2/c$$

so we could then write once again our Goldstein result as

$$\sigma(\Theta) = \tfrac{1}{4}\ Z^2 Z'^2 \alpha^2 (\hbar c)^2 (1/cp\beta)^2 \csc^4(\Theta/2) = 1/4\ Z^2 Z'^2 \alpha^2 \hbar^2 (1/p\beta)^2 \csc^4(\Theta/2)$$

$$= 1/(4p^2\beta^2) * Z^2 Z'^2 \alpha^2 \hbar^2 / \sin^4(\Theta/2)$$

and this agrees with BD if we set $\hbar = 1$ and set $Z' = 1$ since BD's projectile is an electron.

So I think we understand the BD system of units now. In this system we have the following facts:

| | | |
|---|---|---|
| div**E** = ρ | curl **E** = - (1/c) ∂**B**/∂t | **F** = q (**E** + (**v**/c) x **B** ) |
| div **B** = 0 | curl **B** = (1/c) **J** + (1/c) ∂**E**/∂t | 1/137 = α = $e^2/(4\pi\hbar c)$ |
| **E** = - (1/c) ∂**A**/∂t - ∇φ | **B** = ∇ x **A** | |

and if we go ahead and set $\hbar = c = 1$, things are even simpler:

| | | |
|---|---|---|
| div**E** = ρ | curl **E** = - ∂**B**/∂t | **F** = q (**E** + **v**x **B** ) |
| div **B** = 0 | curl **B** = **J** + ∂**E**/∂t | 1/137 = α = $e^2/(4\pi)$ |
| **E** = - ∂**A**/∂t - ∇φ | **B** = ∇ x **A** | |

The formula is credited to **Neville Francis Mott** (1905-1996) who cranked this out in 1929 pretty soon after the Dirac theory came out in 1928. Amazingly, Mott's paper is online here in a book on Mott! He got a Nobel in 1977 for other stuff he did.

http://books.google.com/books?id=Nudd4Tg2zl0C

and he says in his intro that he is doing this using the new Dirac version of the Schrodinger equation. His method is somewhat different from what we are doing here and uses traditional scattering methods.

Now, 7.1 is our 6.56 so we are happy with it.

7.2? Let's check the normalization here. We would have

$\bar{\psi}\,\psi = (m/EV)\,\bar{u}\,u = (m/EV)\,\bar{w}^1 w^1 = (m/EV)$ // according to 3.9b

But probability is really this:

$\psi^\dagger\psi = (m/EV)\,u^\dagger u = (m/EV)\,w^{\dagger 1} w^1 = (m/EV)(E/m) = V$ // according to 3.11

So when we then get $\int d^3x\ \psi^\dagger\psi = 1$, so happy with 7.2 and 7.3, and note that we have $\varepsilon = +1$ since regular electrons.

A: As discussed above, this is the potential $\varphi$ in the Heaviside-Lorentz System. The scattering center has Z, and $e < 0$ so we are scattering off a positive potential.

7.4: The extra $\gamma^0$ comes from the $\not{A}$, all other factors are verified. The $d^3x$ spatial integral should be familiar to me, but it is not! Easy to do, however:

$$\int d^3x\ e^{-q\cdot r}/r = 4\pi \int r dr\, e^{-iqr} = 4\pi\,(i\partial_q) \int dr\, e^{-iqr} = 4\pi\,(i\partial_q)[\, i/q\,(0 - 1) \qquad = -4\pi * -1/q^2 = 4\pi/q^2$$

We are then left with $\int dx^0 \exp(i[E_f - E_i]) = 2\pi\delta(E_f - E_i)$ and this then clinches 7.5. By the way, notice that in doing this integral, we suddenly assumed that $V = \infty$ . So we really need to regard everything in this limit. There is another normalization method I recall where you can do this from the start, but OK.

**Phase Space:** Now comes this traditional confusion. We are scattering into a final momentum $\mathbf{p_f}$ and we want to know how many states are associated with the volume $d^3p_f$ ? Suppose we had a one dimensional problem and asked about dp in a box of length L. I just solved this problem quickly. Imagine a 1D box going from x=0 to x=L. Plane wave is $e^{ipx}$. Impose "periodic boundary conditions" so $e^{ipL} = 1$, This then says that cos(pL)=1 or that pL = 0,2π,4π.... = 2πn so that $p = p_n = (2\pi/L)n$ and we have discrete values of $p_n$ in our plane wave in a box. Thus we have $\Delta p = (2\pi/L)\Delta n$ so $\Delta n = (L/2\pi)\,\Delta p$. That is , suppose as we go from p to p+Δp and we find (for some L) that n goes from 1000 to 1013. Then Δn = 13 and that is how many states are associated with the range Δp. If we do this whole process in 3D we would end up with this: $\Delta n_i = (L_i/2\pi)\,\Delta p_i$ for i = 1,2,3. Then in a range $d^3p = \Delta p_1\,\Delta p_2\,\Delta p_3$ we have $\Delta n_1\,\Delta n_2\,\Delta n_3 = \Delta n$, and we say that $\Delta n = d^3p\,(L_1\,L_2\,L_3)/(2\pi)^3 = d^3p\,V/(2\pi)^3$ and this verifies page 101 equation A.

7.6: Recall that $S_{fi}$ from page 89 is the "quantum amplitude" to go from an initial state i to a final state f. To get a probability, we square this and multiply by the above phase space. That is what appears in 7.6 LHS. The RHS of 7.6 I have checked, factor by factor. The HL replacement $\alpha = e_{HL}^2/(4\pi\hbar c) = e^2/4\pi$ makes α appear, which we like since no confusion about its units! It is dimensionless 1/137. Notice that the spinor stuff got squared. I am happy to set the second $2\pi\delta(0) = T$ since it came from the time integral. Both explanations BD give for this are fine, the heuristic second one is easier and quite obvious. This puts a T on the RHS, and then we can get the scattering RATE shown in page 102 A. We pick up a 4 by combining various numerical factors, so page 102 A is fine.

Now what do we mean by a "rate" ? It is probability of "transition" from initial to final state per unit time. So this brings up our next potential source of confusion. How do we convert this to something about scattering of "particles" or into something about "cross section" ? Well, imagine our usual "ring" around the incoming axis and call the area of this annular ring $d\sigma$ . Imagine there is a "probability flux" coming in from the $p_i$ direction hitting this ring in a constant flow. The dimensions of this flux is probability/area/sec. It is like a current density J but without the charge. I am not sure what this would be in E&M (but I could probably figure it out), but in the Dirac theory we know exactly what this probability flux is. It is $\mathbf{j} = \bar{\psi}\,\boldsymbol{\gamma}\,\psi$ and in particular $j_{incident} = \mathbf{j} \bullet \hat{\mathbf{p}}_i = \bar{\psi}\,\boldsymbol{\gamma}\,\psi \bullet \hat{\mathbf{p}}_i$. So I am then happy saying this:

$j_{incident}$ (probability/area/sec) *$d\sigma$ (area) = rate (probability/sec).

By the way, notice that our scattering potential is at the center of our "box", since 1/r potential.

**Side Exercise with $j^\mu$.** Consider $j^\mu = c\,\bar{\psi}\,\gamma^\mu\,\psi$ = which we know is a 4-vector. Let's compute this for the $w^1$ spinor in the "rest frame". The phases whatever they are cancel, and we have $j^\mu = c\,(\gamma^\mu)_{00}$ according to my scratch calculation. But only $\gamma^0$ is non-vanishing in that corner, so we have $j^\mu = c\,\delta_{\mu,0}$ which says that $j^\mu = c(1,0,0,0)$. This can be *compared* to a rest particle which has $p^\mu = (mc^2,0,0,0)$.

Now suppose we boost $p^\mu$ out to where it has $\mathbf{p}$. We then have $p^\mu = (E\ ,\ \mathbf{p})$ where $E = \gamma mc^2$ and where $\mathbf{p} = \gamma\boldsymbol{\beta}mc = (\gamma\boldsymbol{\beta}/c)\,mc^2$. Therefore, if we apply this same boost to $j^\mu$ we should get $(j^0, \mathbf{j})$ where $j^0 = \gamma c$ and where $\mathbf{j} = (\gamma\boldsymbol{\beta}/c)c = \gamma\boldsymbol{\beta}$, where we just replace $mc^2$ by c**.** Thus, we have $j^\mu = (\gamma c, \gamma\boldsymbol{\beta})$. So in the boosted frame where an electron has $p^\mu$, we just have this for $j^\mu$. This says $\mathbf{j} = \gamma\boldsymbol{\beta} = c\gamma\mathbf{v}$ .

But BD are claiming that $\mathbf{j} = \mathbf{v}/V$, so we have a problem Houston. OK, in my computation of $j^\mu$ above I assumed that $w^1$ had no overall normalization factor. Just spinor times expo, and that is the basis on which I computed $j^\mu = c\,\delta_{\mu,0}$ in the rest frame. See 3.1 page 28 for example. But in this chapter we have a norm factor as in 7.2 which puts an extra m/EV. So we then get

$j^\mu = (j^0, \mathbf{j})$ where $j^0 = \gamma c(m/EV)$ and $\mathbf{j} = c\gamma\mathbf{v}\,(m/EV)$ where $E = \gamma mc^2$.

This says

$j^0 = 1/(cV)$ $\qquad$ $\mathbf{j} = \boldsymbol{\beta}/V = \mathbf{v}/(cV)$ $\qquad$ and BD then say $\mathbf{j} = \mathbf{v}/V$

So, we now know what $j^\mu$ is for a plane wave normalized to a box in this way! The direct calculation would have given the same result.

So back to our result above

$d\sigma$ = rate/$j_{inc}$ = rate * V/v

The volumes cancel, we get a v in the denominator, and we write out $d^3p$ as usual to get 7.10, so I am now happy with 7.10 (that did take a while). Notice that once we have pulled out $d\Omega$, we then want to integrate over the $p_f$ variable because we want all momenta that go into this cross section angle.

Now comes the final trick which is this:

$p^2dp\ \delta(E-E_i)/E$ : $E^2 = p^2 + m^2$ so $EdE = pdp$ so $p^2dp/E = p(EdE)/E = pdE$

so that

$p^2dp/E * \delta(E-E_i) = pdE\ \delta(E-E_i) = \gamma mv\ \delta(E-E_i)$

So doing the $p_f$ integral just gives us a factor $\gamma mv$. Now we use a fact not mentioned by BD which is that this is elastic scattering. Then we know that $E_i = \gamma mc^2$ and then $\gamma mv/E_i v = \gamma mv/\gamma mc^2 v = 1$.

Thus we have arrived at the very compact result 7.11 for our differential cross section $d\sigma/d\Omega$ and I have verified every single step along the way. Of course V does not appear so we now take it to $\infty$.

Now here are some comments on the NR limit (again using elastic)

$\mathbf{q} = \mathbf{p}_i - \mathbf{p}_f$ so $q^2 = 2p^2 - 2p^2\cos\theta = = 2p^2(1 - \cos\theta) = 4p^2\sin^2(\theta/2) = q^2$

Therefore

$1/q^4 = 1/[\ 16\ p^4\sin^4(\theta/2)\ ]$

In the NR limit, we have $p^2 = 2mE$ so we get

$1/q^4 = 1/[\ 64\ m^2\ E^2\ \sin^4(\theta/2)\ ]$

The spinor thing gives 1, so we have

$d\sigma/d\Omega = 4Z^2\alpha^2m^2/\ /[\ 64\ m^2\ E^2\ \sin^4(\theta/2)\ ] = Z^2\alpha^2/[\ 16\ E^2\ \sin^4(\theta/2)\ ]$

which agrees with our Goldstein result quoted above

$\sigma(\Theta) = 1/4\ (ZZ'e^2/2E)^2\ \csc^4(\Theta/2)$

A few extra comments. The formula 7.11 is for a particular $s_i$ to a particular $s_f$. In the NR limit, if $s_i$ = up, then only $s_f$ = up has a non-vanishing spinor matrix element. Thus, "summing over final polarizations' does not give an extra factor of 2. Nor of course does "averaging over initial polarizations". The Goldstein result was done assuming electron spin did not exist. But here we see that in the Dirac theory, spin does exist and it affects the cross section if you are doing a spin-specific experiment.

And what about the "elastic issue" ? We have a momentum transfer $\mathbf{q} \neq 0$, so this somehow must be absorbed by the potential. But then it must absorb some energy as well, so then why am I claiming this is elastic? Well, the potential here is "infinitely massive", so $q^2/2M = 0$ and it absorbs no energy. Fine.

Spinor summation. This is the next world of micro-detail. I agree with bottom of page E and I agree that it has the form shown in 7.13 where in our case $\Gamma = \gamma^\circ$. And I have scratch verified all the special cases shown below it as equations A, so next stop is B,C and D.

Equation B follows at once from my notes above concerning equation 3.9c. The spin sum shown is the same as summing over the r = 1,2 spinors and this gives $1^+$ which is the positive energy projector. Their method of deriving this I do not follow. In the first line, they already assume the result. But fine, I know how to get the result.

Now stare at the bottom line on page 102. We see in there the two groupings:

$$\Sigma_i u_\beta(i)\, \bar{u}_\delta(i) = \Lambda^+_{\beta\delta}(i)$$
$$\Sigma_f u_\sigma(f)\, \bar{u}_\alpha(f) = \Lambda^+_{\sigma\alpha}(f)$$

We can be more general going to the form shown in 7.13. Write as

$$= \Sigma_{i,f} \quad \bar{u}_\alpha(f)\Gamma_{\alpha\beta}\, u_\beta(i) \quad \bar{u}_\delta(i)\, \bar{\Gamma}_{\delta\sigma}\, u_\sigma(f)$$

which is a sum of products of two Dirac scalars. We can convert each spin sum into a Dirac index sum as follows:

$$= \Sigma_i\, u_\beta(i)\, \bar{u}_\delta(i)\ \bar{\Gamma}_{\delta\sigma}\ \ \Sigma_f\, u_\sigma(f)\, \bar{u}_\alpha(f)\, \Gamma_{\alpha\beta} = \Lambda^+_{\beta\delta}(i)\, \bar{\Gamma}_{\delta\sigma}\, \Lambda^+_{\sigma\alpha}(f)\, \Gamma_{\alpha\beta} = \mathrm{trace}(\Lambda^+(i)\, \bar{\Gamma}\, \Lambda^+(f)\, \Gamma) \quad \Rightarrow$$

$$|\bar{u}(f)\, \Gamma\, u(i)|^2 = \mathrm{tr}[\, \Gamma\, \Lambda^+(i)\, \bar{\Gamma}\, \Lambda^+(f)\, ]$$

Notice how the first index β links to the last index β causing the thing to be a trace. It seems pretty obvious that you can write the matrices in the trace in any cyclic order you want, it is the same sum. So this is the very famous result I remember now, that you replace spin sums with traces when doing unpolarized cross sections. And we get a factor of ½ from averaging over initials, already in our formula.

And so finally we arrive at 7.14 and we are done with this complicated section! All that remains is to compute the trace.

**7.2 Trace Theorems and Final Evaluation of the Mott cross section (103)** .

Theorem 1 is based on $\gamma_5^2 = 1$ and fact that $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$. You slide $\gamma_5$ through all our $\not{a}$ things and get overall - sign if odd.

Theorem 2 is pretty simple. says $\mathrm{tr}(\not{a}\not{b}) = 4a.b$ which we will use below at once.

Theorem 3 is a little heavier duty. Here is an example in compact notation where 1 means $\gamma^1$ and (12) means a1.a2. We just repeatedly apply the rule $12 = -21 + 2g^{12}$.

$\mathrm{tr}(123456) = 2(12)\, \mathrm{tr}(3456) - \mathrm{tr}(213456)$
$\mathrm{tr}(213456) = 2(13)\mathrm{tr}(2456) - \mathrm{tr}(231456)$
$\mathrm{tr}(231456) = 2(14)\mathrm{tr}(1235) - \mathrm{tr}(234156)$
$\mathrm{tr}(234156) = 2(15)\mathrm{tr}(2346) - \mathrm{tr}(234516)$
$\mathrm{tr}(234516) = 2(16)\mathrm{tr}(2345) - \mathrm{tr}(234561)$

Put all the pieces together to get

$$tr(123456) = 2(12)tr(3456) - 2(13)tr(2456) + 2(14)tr(1235) - 2(15)tr(2346) + 2(16)tr(2345) - tr(123456)$$

but move last term to the LHS and cancel all 2's to get

$$tr(123456) = 2(12)tr(3456) - 2(13)tr(2456) + 2(14)tr(1235) - 2(15)tr(2346) + 2(16)tr(2345)$$

or

$$tr(\acute{a}_1\acute{a}_2\acute{a}_3\acute{a}_4\acute{a}_5\acute{a}_6) =$$
$$a_1.a_2\ tr(\acute{a}_3\acute{a}_4\acute{a}_5\acute{a}_6) - a_1.a_3 tr(\acute{a}_2\acute{a}_4\acute{a}_5\acute{a}_6) + a_1.a_4\ tr(\acute{a}_1\acute{a}_2\acute{a}_3\acute{a}_5) - a_1.a_5\ tr(\acute{a}_2\acute{a}_3\acute{a}_4\acute{a}_6) + a_1.a_6\ tr(\acute{a}_2\acute{a}_3\acute{a}_4\acute{a}_5)$$

The general idea is that you can knock the list down by 2 $\acute{a}$ factors with each application of this idea. We would insert the 4-$\acute{a}$ result four times into the above to get all dot products! Notice of course that this result must be true for any cyclic permutation of the indices!

The result with only 4 is then this:

$$tr(\acute{a}_1\acute{a}_2\acute{a}_3\acute{a}_4) = a_1.a_2\ tr(\acute{a}_3\acute{a}_4) - a_1.a_3 tr(\acute{a}_2\acute{a}_4) + a_1.a_4\ tr(\acute{a}_2\acute{a}_3)$$
$$= 4\ \{\ a_1.a_2\ a_3.a_4\ \ - a_1.a_3\ a_2.a_4 + a_1.a_4\ a_2.a_3\}$$

Theorem 4. The first item is trivial. The second item I commented long ago I was unable to show, and today it took lots of effort, but B&D don't acknowledge it. Here is my method:

Consider $tr(5\mu\nu)$ : If $\mu=\nu$ then this is $g^{\mu\mu}\ tr(5) = 0$, so we have only to consider $\mu \neq \nu$. But if different, then we know that $tr(5\mu\nu) = -tr(5\nu\mu)$ because $\mu\nu = -\nu\mu$ when $\mu\neq\nu$.

Now we go off on this strange tangent: we know these two facts:

$$0\mu^{\dagger}0 = \mu \ \ \Rightarrow\ 0^{*}\mu^{T}0^{*} = \mu^{*} \ \Rightarrow\ 0\mu^{T}0 = \mu^{*} \ \Rightarrow 0\ \mu^{*}0 = \mu^{T}$$
$$2\mu^{*}2 = \mu \ \ \Rightarrow\ \ 2^{*}\mu\ 2^{*} = \mu^{*} \ \Rightarrow (-2)\ \mu(-2) = \mu^{*} \ \ \Rightarrow 2\mu2 = \mu^{*}$$

Thus we know that

$$02\mu20 = 0\mu^{*}0 = \mu^{T}$$

So, consider again our starting point and use the fact that $tr(A) = tr(A^{T})$:

$$tr(5\mu\nu) = tr(\nu^{T}\mu^{T}5^{T}) = tr(02\nu2002\mu205) = - tr(02\nu\mu205) = - tr(20502\nu\mu) = - tr(20025\nu\mu) = +tr(5\nu\mu)$$

Thus we have shown that for all $\mu,\nu$ we have $tr(5\mu\nu) = +tr(5\nu\mu)$. In particular, this must be true when $\mu \neq \nu$. But earlier was showed that in this case $tr(5\mu\nu) = -tr(5\nu\mu)$. Thus, we have shown that when $\mu \neq \nu$, we must have $tr(5\mu\nu) = 0$. But we already showed up front that this is true when $\mu=\nu$. Therefore we have proven after much pain that:

$$tr(5\mu\nu) = 0$$

The next item in theorem 4 asks us to compute

$\mathrm{tr}(5\mu\nu\alpha\beta)$

If any pair of the four indices are equal, we can slide them together perhaps picking up a minus sign. But then we will be left with $\mathrm{tr}(5\rho\sigma)$ after the two annihilate, and this is 0. Therefore, all four indices must be different. Thus we know that

$\mathrm{tr}(5\mu\nu\alpha\beta) = k\,\varepsilon^{\mu\nu\alpha\beta}$ proper tensor matching!

and we now have to find k. So consider the case

$\mathrm{tr}(50123) = k\,\varepsilon^{0123} = -k$

Notice this crucial fact. The way BD define ε is such that $\varepsilon_{0123} = +1$. If we raise all four indices, we get a minus sign, which explains the -k above. Now recall that $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$ where c is a scalar. Thus

$\mathrm{tr}(50123) = \mathrm{tr}(5\ 5/i) = (1/i)\,\mathrm{tr}(5^2) = (1/i)\,\mathrm{tr}(1) = -4i = -k.$

Thus we find that

$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = +4i\,\varepsilon^{\mu\nu\alpha\beta}$ and $\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) = +4i\,\varepsilon_{\mu\nu\alpha\beta}$

where the second form follows from normal tensor ways of raising and lower indices using $g_{\mu\mu}$ etc etc.

And then we will get

$\mathrm{tr}(\gamma_5 \not{a}\not{b}\not{c}\not{d}) = a_\mu\,b_\nu\,c_\alpha\,d_\beta\,\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = +4i\,\varepsilon^{\mu\nu\alpha\beta}\,a_\mu\,b_\nu\,c_\alpha\,d_\beta = +4i\,\varepsilon_{\mu\nu\alpha\beta}\,a^\mu\,b^\nu\,c^\alpha\,d^\beta$

giving the result that BD claim.

<u>Theorem 5</u>: First item is $\gamma_\mu\gamma^\mu = g_{\mu\mu}(\gamma^\mu)^2 = 1$ for each value of μ. Thus, the summed $\gamma_\mu\gamma^\mu = 4$.

Second item: $\gamma_\mu\gamma^a\gamma^\mu = \gamma_\mu(-\ \gamma^\mu\gamma^a + 2g^{\mu a}\,1) = -\ 4\gamma^a + 2\,\gamma^a = -2\gamma^a$

Third item: $\gamma_\mu\gamma^b\gamma^a\gamma^\mu = \gamma_\mu\gamma^b(-\ \gamma^\mu\gamma^a + 2g^{\mu a}\,1) = (-\ \gamma^b\gamma_\mu + 2g^b{}_\mu\,1)\,(-\ \gamma^\mu\gamma^a + 2g^{\mu a}\,1)$

$= 4\,\gamma^b\gamma^a - 2\,\gamma^b\gamma^a - 2\,\gamma^b\gamma^a + 4g^{ba} = 4\,g^{ba}$ QED.

Fourth item

$\gamma_\mu\gamma^b\gamma^c\gamma^a\gamma^\mu = (-\ \gamma^b\gamma_\mu + 2g^b{}_\mu\,1)\,\gamma^c\,(-\ \gamma^\mu\gamma^a + 2g^{\mu a}\,1)$

$$= + \gamma^b\gamma_\mu\,\gamma^c\,\gamma^\mu\,\gamma^a - 2\gamma^b\gamma^a\gamma^c - 2\gamma^c\gamma^b\gamma^a + 4g^{ba}\gamma^c$$

$$= \gamma^b(-2\gamma^c)\,\gamma^a - 2\gamma^b\gamma^a\gamma^c - 2\gamma^c\gamma^b\gamma^a + 4g^{ba}\gamma^c$$

$$= -2\,\gamma^b\gamma^c\gamma^a - 2\gamma^b\gamma^a\gamma^c - 2\gamma^c\gamma^b\gamma^a + 4g^{ba}\gamma^c$$

But look at the second and fourth terms:

$$2(-\gamma^b\gamma^a + 2g^{ba})\,\gamma^c = 2\,\gamma^a\gamma^b\gamma^c$$

so result is then

$$= -2\,\gamma^b\gamma^c\gamma^a + 2\,\gamma^a\gamma^b\gamma^c - 2\gamma^c\gamma^b\gamma^a$$

Now if we swap the b and c in both the last terms, the g contribution will cancel between the two cases, and we get each term changing sign so get

$$= -2\,\gamma^b\gamma^c\gamma^a - 2\,\gamma^a\gamma^c\gamma^b + 2\gamma^b\gamma^c\gamma^a$$

But now the first and third terms cancel and we get this result

$$\gamma_\mu\gamma^b\gamma^c\gamma^a\gamma^\mu = -\,2\,\gamma^a\gamma^c\gamma^b$$

where we have a reversal of order of the gammas. This then gives the result shown. I think there was an easier way! Let's apply that easier way on the next one:

$$\gamma_\mu\gamma^a\gamma^b\gamma^c\gamma^d\gamma^\mu = \gamma_\mu\gamma^a\gamma^b\gamma^c(-\gamma^\mu\gamma^d + 2g^{\mu d}) = -(\gamma_\mu\gamma^a\gamma^b\gamma^c\,\gamma^\mu)\gamma^d + 2\,\gamma^d\gamma^a\gamma^b\gamma^c$$

$$= +2\,\gamma^c\gamma^b\gamma^a\,\gamma^d + 2\,\gamma^d\gamma^a\gamma^b\gamma^c = 2\,[\gamma^c\gamma^b\gamma^a\,\gamma^d + \gamma^d\gamma^a\gamma^b\gamma^c]$$

and this agrees with their ordering.

Theorem 6: Recall our result above that $02\mu20 = 0\mu^*0 = \mu^T$

So consider

$$\mathrm{tr}(123...N) = \mathrm{tr}(N^T....3^T2^T1^T) = \mathrm{tr}(02N20...\ 02320\ \ 02220\ \ 02120)$$

There are N occurrences of 2002 = -1 and N is even, so just remove all of these to get

$$= \mathrm{tr}(N...\ 3\ \ 2\ \ 1) = \text{desired result, QED.}$$

Now finally go back to 7.14 and we want now to evaluate the traces. There are four of them: The first of the four is this one:

$(p_i)_\alpha (p_f)_\beta \, tr(\gamma^0 \gamma^\alpha \gamma^0 \gamma^\beta) = (p_i)_\alpha (p_f)_\beta \, 4[\, g^{0\alpha} g^{0\beta} - g^{00} g^{\alpha\beta} + g^{0\beta} g^{\alpha 0}] = (p_i)_\alpha (p_f)_\beta \, 4[\, 2 g^{0\beta} g^{\alpha 0} - g^{\alpha\beta}]$

$= 8(p_i)^0 (p_f)^0 - 4\, p_i . p_f$

The cross terms vanish because they each have 3 gammas. The fourth term is this

$tr(\gamma_0^2) = 4$

So including the m factors we get

$Tr = (1/2m)^2 [8(p_i)^0 (p_f)^0 - 4p_i . p_f] + ¼*4 = (1/2m)^2 [8(p_i)^0 (p_f)^0 - 4p_i . p_f] + 1$

$= (1/2m)^2 [8E_i E_f - 4p_i . p_f] + 1 = (1/2m)^2 [8E_i E_f - 4p_i . p_f + 4m^2]$

so we have

$4m^2 Tr = [8E_i E_f - 4p_i . p_f + 4m^2]$

So this removes the $4m^2$ from the numerator of 7.14 and we get 7.21.

Now comes the final step. Recall from above that

$4p^2 \sin^2(\theta/2) = q^2$

and put in the dot product as shown to get

$[8E_i E_f - 4p_i . p_f + 4m^2] = 8E^2 - 4 (m^2 + 2p^2 \sin^2(\theta/2)) + 4m^2 = 8E^2 - 8p^2 \sin^2(\theta/2)$

$= 8E^2 - 8\beta^2 E^2 \sin^2(\theta/2) = 8E^2 ( 1 - \beta^2 \sin^2(\theta/2) )$

So our result is then

$Z^2\alpha^2 / \{ 2 * 16\, p^4 \sin^4(\theta/2)\} * 8E^2 ( 1 - \beta^2 \sin^2(\theta/2) )$

$= Z^2\alpha^2 / \{ 4\, p^4 \sin^4(\theta/2)\} * p^2/\beta^2 ( 1 - \beta^2 \sin^2(\theta/2) )$

$= Z^2\alpha^2 / \{ 4\, p^2 \beta^2 \sin^4(\theta/2)\} * ( 1 - \beta^2 \sin^2(\theta/2) )$

and finally we have the Mott result 7.22 for the relativistically correct unpolarized cross section to lowest order in $\alpha^2$ . Corrections to this would involve graphs like those on page 124 which have three vertices and I expect these terms to be down by $\alpha^2$ because we add another $e^2$ to the amplitude, and thus another $e^4$ to the rate, and 137*137 = 18769 so correction might be 1 part in 20,000 which is pretty small and hard to imagine they could measure this that accurately anyway. Of course there are also recoil corrections which might be larger than this.

But overall, this is a great victory for me today. Getting the Mott formula took a lot of platform material. Maybe next calculations will be easier??

**7.3 Coulomb Scattering of Positrons (106)** .

*I paused here on 9/24/08 and did several digressionary things until 10/8/08. I started looking at the Moore notes on the Lorentz Group, then I wrote up lots of stuff about $RVR^{-1} = R^{-1}V$ and learned how various objects transform, such as classical field amplitudes and quantum field operators, etc etc. Then I did a 50 page writeup on the Lorentz Group motivated by Moore's notes. I was trying to understand what exactly the "Dirac representation" was, and now I know. I then computed an explicit form for the spinors u(p,s) for arbitrary spin direction. Then I reviewed all the BD meta notes up to this point.*

Here are my comments on this short book section:

(1) When you follow the "Feynman Rules" as I shall loosely call them, and when you think in the model where you have a negative-energy electron scattering backwards in time as the page 107 picture shows, you install the plane waves shown in 7.24 and 7.25. We know that a forward plane wave has $e^{-ip\cdot x}$ and a backward one has $e^{+ip\cdot x}$ so we can regard the entire p 4-vector as changing sign, not just the energy component. And recall that $v(p,s) = w^4$ has "spin-down" in terms of the negative energy state, but is spin up for the postive energy positron state.

(2) We also follow the rules that the "matrix element" has an **Initial** state and a **Final** state, where I am now using caps so that Initial means the right or starting side of the matrix element, and Final means the left or ending side of the matrix elements. If you were to draw an electron flow arrow, the arrow would start at Initial and end at Final.

But the words **initial** and **final** have (no caps) have different meaning. These refer to initial and final states of physical particles in a real scattering experiment. So, in this problem, our initial state is a physical positron with momentum $p_i$ , but it is the Final state in the matrix element. So we end with $\overline{v}(p_i,s_i)$ as in equation A. And similarly for the other spinor.

How should we now interpret (7.24)? I would write it as

$$\psi_I(x) \approx v(p_f)e^{+ipf\cdot x} = w^4\, e^{+ipf\cdot x}$$

We use "I" because this is the start or right side of the matrix element, the start of the electron arrow. The $v(p_f) \approx w^4$ is a negative energy spinor, and $e^{+ipf\cdot x}$ is a negative energy phase.

As a reminder, back in Section 5.2 we defined a "charge conjugated" wavefunction $\psi_{Ic}$ where

$$\psi_{Ic} \equiv C\gamma^0\psi_I^* = (i\gamma^2)\,\psi_I^* = w^1\, e^{-ipf\cdot x} = u(p_f)\, e^{-ipf\cdot x}$$

(3) Consider the following algebra in the abstract, which appears as B,C,D on page 107. First,

$$\psi_F \rightarrow \psi_{Fc} = C\gamma^0\psi_F^*$$
$$\psi_I \rightarrow \psi_{Ic} = C\gamma^0\psi_I^* \quad \Rightarrow \quad \psi_I^T \rightarrow \psi_{Ic}^T = \psi_I^\dagger\gamma^0 C^T \text{ so } \psi_{Ic}^\dagger = \psi_I^T\gamma^0 C^\dagger = \psi_I^T\gamma^0 C^{-1}$$

Then we get

$\bar{\psi}_{Ic} \not{A} \psi_{Fc}$

$= \psi_{Ic}^{\dagger}\gamma^0 \not{A} \psi_{Fc}$ expose the left side $\gamma^0$

$= \psi_I^T\gamma^\circ C^{-1} \gamma^\circ \not{A} C\gamma^\circ\psi_F^*$ insert above expressions for $\psi_{Fc}$ and for $\psi_{Ic}^{\dagger}$

$= -\psi_I^T C^{-1} \not{A} C\gamma^\circ\psi_F^*$ because $\gamma^2$ and $\gamma^0$ anticommute and $\gamma^{\circ\,2} = 1$

$= -A_\mu \psi_I^T C^{-1} \gamma^\mu C\gamma^\circ\psi_F^*$ writing out the slash

$= +A_\mu\psi_I^T (\gamma^\mu)^T \gamma^\circ\psi_F^*$ using the rule above for C sandwiched around $\gamma^\mu$

$= +A_\mu\ \psi_F^{\dagger}\gamma^\circ \gamma^\mu\psi_I$ write $a^TXb = b^TX^Ta$ , just rearrange the indices on things

$= \bar{\psi}_F \not{A} \psi_I$

which says

$$2 = 3$$

$$\bar{\psi}_{Ic} \not{A} \psi_{Fc} = \bar{\psi}_F \not{A} \psi_I$$

Item 3 here is basically equation A, our Feynman-rules matrix element, where $\psi_I(x) \approx v(p_f)e^{+ipf.x}$ is our negative energy electron of momentum $-p_f$ (defining *momentum* as the p in $e^{-ip.x}$ ). Item 2 has a Final state being $\psi_{Ic} \approx u(p_f)\ e^{-ipf.x}$ which is our positive energy positron with momentum $p_f$. But this is what we used as a final state in our *electron*-Coulomb scattering. Thus, we expect our first-order positron result to be *identical* to the first-order electron result, and this is in fact true.

It turns out when you use the 3-form above to do the calculation, you use v type spinors, and your calculation results in the Dirac index sum shown in E which is $-\Lambda^-(p)$ and this brings in $(\not{p} - m)$ into the trace instead of $(\not{p} + m)$. But the linear m terms in the trace involve the trace of three gammas which we know is zero, so this sign makes no difference in the result. And of course once we get to F, we see that the final cross section result is for positron-Coulomb scattering will be the same Mott formula as for electron-Coulomb scattering.

(4) What sign charge did we select for our Coulomb scattering center? In our problem set-up on page 100, the Coulomb center has a positive charge all the time. Thus, for electron scattering we have an attractive center, and for positron scattering a repulsive center. We know from Goldstein (which I just revisited on this very issue) that the non-rel cross section is the *same* in both cases, attractive or repulsive, so we are not surprised to find that the Mott formula applies to both these cases. BD remark, however, that this is not true when higher order graphs are included!

In our first order calculation, the sign of $A^\mu$ (in the form of $A^0 = -Ze/(4\pi r)$ ) does not affect the answer because we have $S_{fi}^2$ for our rate. If we write out $S_{fi}$ to second order in perturbation theory using our standard formula quoted above

$$S_{fi} = \delta_{fi} - \varepsilon_f\, i\, \bar{\psi}_f^s\, e\not{A}\, \psi_i - \varepsilon_f\, i\, \bar{\psi}_f^s\, e\not{A}\, S_F\, e\not{A}\, \psi_i + \ldots \quad \text{form 7} \qquad \mathbf{6.57}\ \ \varepsilon_f = 1$$

then when we square $S_{fi}$ we will get cross terms which are order$(A^\mu)^3$ and then we can see that the sign *might* affect the result, and in fact it does.

Recall from Goldstein that orbits are affected by special relativity, for example elliptic orbits precess. I would expect that if you solved the classical Rutherford problem relativistically, you might find that the attractive and repulsive Rutherford scattering formulas are different. Goldstein does not attempt such a

problem. I found www.gsi.de/~wolle/BUCH/HTML/REL_**RUTHERFORD**/rel_**rutherford**.ps which claims to address this classical problem by modifying mass and velocity by γ and β factors, but it is not clear to me that the starting formula they work with is still correct in high γ situation. They only apply their work to small angle scattering of heavy nuclei off each other. I think the Dirac theory Mott formula is more trustworthy of course.

**7.4 Electron scattering from a Dirac proton (108) (Moller potential idea)**

The plan is to consider the proton as somehow creating a classical EM field $A^\mu$ which we then put into our first order formula for $S_{fi}$.

Digression to derive (7.27) First, recall our E&M in the H-L gauge, taken from above

| | | |
|---|---|---|
| div**E** = ρ (1) | curl **E** = - ∂**B**/∂t (2) | **F** = q (**E** + **v**x **B** ) |
| div **B** = 0 (3) | curl **B** = **J** + ∂**E**/∂t (4) | 1/137 = α = $e^2$/(4π) |
| **E** = - ∂**A**/∂t - ∇φ | **B** = ∇ x **A** | |

Jackson on page 178-179 deals with the required subject here. First, he states the Maxwells bottom of page 187, and we have to set c = 1 and 4π = 1 in his equations to get our H-L results above. We can say **B** = ∇ x **A** because such a **B** will satisfy div **B** = 0. We can then say **E** + $\partial_t$**A** = -∇φ because such an **E** satisfies curl **E** = - ∂**B**/∂t. So the point is that if we define $A^\mu$ as shown on the bottom two lines above, we satisfy Maxell 2 and 3 above. If we then stuff the so-defined $A^\mu$ into Maxell 1 and 4 we get Jackson 6.32,33 page 180 which are a bit messy since **A** and φ are cross-coupled (and again, we need to set c=1 and 4π = 1 to get our H-L versions of these equations).

Jackson next points out that if you do $A^\mu \rightarrow A^\mu + \partial^\mu\Lambda$, then neither **E** nor **B** changes. We can write this as $\partial_\mu A^\mu \rightarrow \partial_\mu A^\mu + \partial_\mu \partial^\mu\Lambda$. So the trick is to select function Λ such that $\partial_\mu A^\mu = 0$. If you do this, then you have selected "the Lorentz gauge", although certainly other "gauges" are possible, such as "the transverse gauge" I recall from the distant past which I think might be ∇•**A** = 0, but no matter. So,

$\partial_\mu A^\mu = 0$ the Lorentz gauge = the Lorentz condition

If you make this gauge choice, the two coupled equations 6.32,3 magically become decoupled and can be combined into a single equation which says

$(\partial_a \partial^a)A^\mu = + J^\mu$ using the BD metric $g^{\mu\nu}$ $(\partial_a \partial^a) = \Box$

and this is (7.27) of BD page 109!

This is the Klein-Gordon equation for a massless particle driven by a current, just to give it a name.

Digression to derive various photon propagator results . So now we have a new differential equation to mess with

$\Box A^\mu = + J^\mu$ // Lu(x) = f(x) **(7.27)**

and as usual, we define a (relativistic) Green's function as follows

$$\Box_x D_F(x-y) = \delta^4(x-y) \qquad // \quad Lg(x,x') = \delta(x-x')$$

If we can find $D_F$, we can then solve our ODE as follows

$$A^\mu(x) = \int d^4y\, D_F(x-y)\, J^\mu(y) \qquad // \; u(x) = \int dx'\, g(x',x)\, f(x') \qquad \textbf{(7.31)}$$

So solve for $D_F$ , we "elevate" $D_F$ into a Fourier momentum space representation as in equation A

$$D_F(x) = \int d^4q/(2\pi)^4 * D_F(q)\, e^{-iq.x}$$

We then apply $\Box_x$ to both sides. Note that $\Box_x\, e^{-iq.x} = -\, q^2\, e^{-iq.x}$ so we then have

$$\delta^4(x-y) = \int d^4q/(2\pi)^4 * D_F(q)\, e^{-iq.x}\,(-q.q) \qquad => \qquad D_F(q) = -\,1/q.q$$

Fine, but this has a poles at $(q^0)^2 = (\mathbf{q})^2$ or $q^0 = \pm\,|\mathbf{q}|$ . Our boys claim that if you move the poles in $q^0$ according to this

$$D_F(q^2) = -\,1/(q^2+i\varepsilon) \qquad // \text{ quietly changing the functional form of } D_F$$

then we get a situation similar to 6.44 page 94 for the Dirac case, where we have positive energy plane wave components ONLY in the integration when $\Delta t > 0$, ie, when we go into the future. And similarly, we have only negative energy plane waves going into the past. The explicit form of $D_F(x-y)$ is not computed here, but it is in the BD2 appendices and involves $J_r$ Bessel functions and such things.

So we are now "on plan". We have found the $A^\mu$ generated by $J^\mu$ , namely

$$A^\mu(x) = \int d^4y\, D_F(x-y)\, J^\mu(y) = D_F\, J^\mu \qquad \text{in matrix notation}$$

and we just wedge this into our standard first order scatteringmatrix element

$$S_{fi} = \delta_{fi} - \varepsilon_f\, i\, \bar{\psi}_f^s\, e\not{A}\, \psi_i$$

and for electron scattering $\varepsilon_f = 1$ and we then have

$$S_{fi} \approx -i\, \bar{\psi}_f^s\, e\not{A}\, \psi_i = -i\, \bar{\psi}_f\, e\, D_F\, \not{J}\, \psi_i = -i\, [e_e\, \bar{\psi}_f\, \gamma_\mu\, \psi_i]\, D_F\, J^\mu \qquad // \text{ integrals implied}$$

But I recognize the bracket as the Dirac current (with charge attached, so an electric current, not just a probability current), so I might call this

$(J_e)_\mu = e_e\, \bar{\psi}^e{}_f\, \gamma_\mu\, \psi^e{}_i$ $\qquad\qquad$ $(J_p)_\mu = e_p\, \bar{\psi}^p{}_f\, \gamma_\mu\, \psi^p{}_i$

But then the proton, being also a Dirac particle, has the same current form as shown above right, we we then have this for our $S_{fi}$

$S_{fi} \approx -i\, (J_e)_\mu\, D_F\, (J_p)^\mu$

So now we are happy with (7.32) and (7.34) and with the Feyman diagram page 111. I notice that BD have carefully avoided using the word "photon" anywhere in the text so far.

This idea of saying the following

$A^\mu(x) = D_F\, (J_p)^\mu$ $\qquad\qquad$ = "the Møller potential of a Dirac proton"

The notation is going to be to use P as momentum for the proton, and p for the electron, same idea for spin.

**Comment**: Christian Moller was a Danish physicist who I think was the first to use the above idea where you model the *classical* potential $A^\mu$ due to the proton as shown above. Of course this same idea would apply in electron-electron scattering. The photon as a *particle* gets no mention in this approach, it is just a model for the classical $A^\mu$. Moller did this in the early 1930's and of course knew the 1928 Dirac theory. The term **Moller scattering** today refers to electron-electron scattering. I think it was only later that people realized this was just a canonical application of the Feynman rules of QED. I suspect there are some Pauli principle statistics that get into the electron-electron scattering problem not present in electron-proton. And of course our electron-proton treatment ignores the proton's internal structure which makes it not be an ideal Dirac particle, it has some kind of "form factor" to deal with. Moller wrote a book on relativity in 1952, you can buy it on the web .

Verification of 7.35: We start with 7.32 with -i in front and dx and dy integrations. Each plane wave has a by-now standard normalization as shown in 7.2 with $\sqrt{m/EV}$ and we see these four factors appearing correctly, he uses a bolded E for the proton energy and M for its mass, the V's are pulled out. All spatial x and y dependence is just plane waves except for $D_F$, but we elevate it as in 7.30 and then ALL spatial is expo and we add a dq integration. Notice that half of the $e^{-iq\cdot(x-y)}$ goes to the y integral and half to the x. Each integral does a delta pin. So let's do the $d^4x$ . The expo for it shows -iqx, $-ip_i x$, $+ip_f x$ so get $(2\pi)^4\delta^4(q - [p_f - p_i])$ . The $d^4y$ integral gives a similar result $(2\pi)^4\delta^4(q - [P_i - P_f])$. Then do the $d^4q$ to use up one of these deltas. One of the $(2\pi)^4$cancels that in the elevation of $D_F$ in 7.30, the other remains so $(2\pi)^4\, \delta^4([p_f - p_i] - [P_i - P_f])$ where the two f ones are + and the other two -. The constant factor is -i start, then e from the elctron current and -e from the proton current, so $-ie^2$. Finally, in $D(q^2)$ we select one or the other, so he selects $q = [p_f - p_i]$ and there is our $+i\varepsilon$. The residual spinor factors are "identical". So, I am completely happy with 7.35.

Page 111: the word "photon" is snuck in for the first time. The wavy line for $D_F$ is called a "virtual photon", not a photon propagator, but that is what it is. Equation 7.36 is correct as stated, using VT = $(2\pi)^4\delta^4(0)$ in our usual gimmick (that I am also happy with). The "spinor thing" $M_{fi}$ is noted to be a world-scalar, and is therefore called "the invariant amplitude".

Page 112: Now we have to face the question of cross section, flux, etc. For Coulomb scattering we said

$j_{incident}$ (probability/area/sec) *dσ (area) = rate (probability/sec).

$\mathbf{j}_{inc} = \boldsymbol{\beta}/V = \mathbf{v}/(cV)$

and we justified both these claims. But now we have two incident fluxes inside our box, but I guess we are just supposed to think of the electrons as incident. This sounds like an approximation to me. We now have a replacement for the first item above

$j_{incident}$ (probability/area/sec) *dσ (area) = rate (probability/sec/volume) * V

=> dσ (area) = $w_{fi}$ * V/ $J_{inc}$

The phase space is just two factors similar to what we had before on page 101 as V $d^3p_f/(2\pi)^3$.

Factors of V: $1/V^4$ (in $w_{fi}$), $V^2$ (phase space), V( from dσ formula above) = 1/V. So I agree with the first equality in **7.39.** And we just install $w_{fi}$ and group factors in an odd way, it is perfectly correct, so on goes our "red check" (red for complicated results, pencil for simple).

Now we are going to group the energy factors in 7.39 in a very special way:

(1) for final particles, we have $d^3p_f/2E_f$ = the famous formula shown in 7.40, which just shows that this factor is a world-scalar. I am fine with that (the 2 is then removed and put elsewhere).

(2) Now we digress and write the flux as in A on page 113. I would agree with this as a non-relativistic result. In the rest frame of the proton, you see electrons coming at v+V. So assume non-rel for the moment and we then accept A since it is the same as our previous $j_{incident}$ expression back in the Coulomg section and which I just quoted above.

Now, we go off on scratch paper (a whole page is needed) and verify that both equalities in **7.41** are correct, DONE, just brute force. So in the non-rel limit they agree. If we then use the world-scalar quantity on the right, then that "continues" our result, so to speak, to allow arbitrary incoming velocities. But then it must be that the velocity sum is the right thing to put for $J_{inc}$ even if you get c + c = 2c! This is a tricky argument but I think it is correct. In other words, you don't use relativistic velocity addition, and the reason must be explainable in some kind of Lorentz contraction argument that BD don't get into. They are just "silent", I can just hear the students asking the obvious question.

So we then arrive at a famous result **(7.42)** which I *claim* in pencil applies to any 2-in 2-out scattering process even of non-Dirac particles and whether elastic or inelastic. In the general case, the various factors will be as shown so as to have a world-scalar result. And $|M|^2$ will also be some scalar, but in general it won't be something simple from the Dirac+Dirac model we have just used here. But 4-momentum will be conserved only if elastic, so maybe my claim is wrong! This must be "the elastic 2→2 scattering formula" only! By definition: "is a collision in which the total kinetic energy of the colliding bodies after collision is equal to their total kinetic energy before collision." In particle collisions, there is not potential energy difference in distant initial and final states, so elastic means energy is conserved.

Verification of page 113 equation B. Now everything so far has assumed collinear beams where the cross section is perp to beams and so is a Lorentz scalar. If beams meet at an angle, you have more work to do. We know that $w_{fi}$ in (7.36) is the reaction rate (probability per second per volume) where we have put exactly one of each incident particle in the box of volume V. This last fact follows because integral $\bar{\psi}\psi$ for each incident particle is 1, that is how we normalized things. And, for each incident particle, our density was uniform throughout the box (we used plane waves).

Consider a volume $d^3x$ within the box and assume $\rho 1$ and $\rho 1$ are the particle count densities of the incident beams in that volume. Certainly the rate from that volume will be $K * \rho 1\ \rho 2\ w_{fi}\ d^3x$ based on the same argument used for chemical reaction rates, so the only problem is to find the constant K.

total rate from volume $d^3x$ = $K * \rho 1\ \rho 2\ w_{fi}\ d^3x$ * phase space

total rate from box of volume V = $K\ w_{fi} * \int_V d^3x\ \rho 1\ \rho 2$ * phase space

If $\rho 1$ and $\rho 2$ were some fixed uniform densities like $\rho 1 = 27$ electrons/$cm^3$ , then I would expect the total rate from the box to be proportional to the volume of the box. In this case, $\int_V d^3x = V$ and then the V factors on the RHS of the above are (not including K): $1/V^4\ (w_{fi}) * V(\int_V d^3) * V^2$(phase space) = 1/V. Thus, I would set $K = V^2$ to make that total rate be proportional to V. So my result is then this:

$dN/dt$ = total rate from box of volume V = $V^2\ w_{fi} * \int_V d^3x\ \rho 1\ \rho 2$ * phase space

Now the count of the visible V factors on the right is $V^{2*}\ 1/V^4 * V^2 = V^0$ as desired. So we have therefore derived equation B on page 113

Now what about our $|M|^2$ ? I agree with A and B on page 114 top. The ¼ is because we average the spin of *each* incoming particle. The sums are all with u's, not v's, so no special minus signs enter. The trace form in B is just as we had back in 7.14 where we only had $\gamma^0$ . And we can use our now-known trace theorems to compute the traces. I did this quickly on scratch and C is correct.

Digression to verify page 114 D, based on C.

I need a computer notation to do this. Let's try this idea where i means $p_i$ and I means $P_i$ etc.

$m^2\ Tr(p)^{\mu\nu} = [\ f^\mu i^\nu + i^\mu f^\nu\ - g^{\mu\nu}\ (fi - m^2)\ ]$
$M^2\ Tr(P)_{\mu\nu} = [\ F_\mu I_\nu + I_\mu F_\nu\ - g_{\mu\nu}\ (FI - M^2)\ ]$

Then

$m^2\ Tr(p)^{\mu\nu}\ M^2\ Tr(P)_{\mu\nu} = [\ f^\mu i^\nu + i^\mu f^\nu\ - g^{\mu\nu}\ (fi - m^2)\ ]\ [\ F_\mu I_\nu + I_\mu F_\nu\ - g_{\mu\nu}\ (FI - M^2)\ ]$

$= \{\ fF\ iI + fI\ iF - fi\ (FI - M^2) + iF\ fI + iI\ fF - if\ (FI - M^2)\ - FI\ (fi - m^2) - FI\ (fi - m^2)\ + 4\ (fi - m^2)\ (FI - M^2)\}$

$= \{ 2\, fF\, iI + 2fI\, iF - 2\, fi\, (FI - M^2) - 2\, FI\, (fi - m^2) + 4\, (fi - m^2)\, (FI - M^2)\}$

=>

$\frac{1}{2}\, m^2\, Tr(p)^{\mu\nu}\, M^2\, Tr(P)_{\mu\nu} = \{ fF\, iI + fI\, iF - fi\, (FI - M^2) - FI\, (fi - m^2) + 2(fi - m^2)\, (FI - M^2)\}$

$= x = \{ fF\, iI + fI\, iF - fi\, FI + fi\, M^2 - FI\, fi + FI\, m^2 + 2(fi - m^2)\, (FI - M^2)\}$

But now

$(fi - m^2)\, (FI - M^2) = fi\, FI - fi\, M^2 - FI\, m^2 + m^2 M^2$ so insert into the above

$x = \{ fF\, iI + fI\, iF - fi\, FI + fi\, M^2 - FI\, fi + FI\, m^2 + 2\, fi\, FI - 2\, fi\, M^2 - 2\, FI\, m^2 + 2\, m^2 M^2\}$

$= \{ fF\, iI + fI\, iF + fi\, FI - fi\, M^2 - FI\, fi - FI\, m^2 + 2\, m^2 M^2\}$

$= \{ fF\, iI + fI\, iF - fi\, M^2 - FI\, m^2 + 2\, m^2 M^2\}$

$= \{ Ff\, Ii + Fi\, If - m^2\, FI - M^2\, fi + 2\, m^2 M^2\}$

and this { } agrees with the square bracket quantity in (7.43) which I am calling x. So we then have

$|M|^2 = \frac{1}{4}\, Tr(p)^{\mu\nu}\, Tr(P)_{\mu\nu}\, (e^4/q^4) = \frac{1}{2}\, (1/(m^2 M^2))\, (e^4/q^4)\, [\frac{1}{2}\, m^2\, Tr(p)^{\mu\nu}\, M^2\, Tr(P)_{\mu\nu}]$

$= \frac{1}{2}\, (1/(m^2 M^2))\, (e^4/q^4)\, x$

and (7.43) has been verified.

The rest of page 114 computes the result in the "lab frame" where protons are at rest (and will of course recoil). It is easy to imagine this experiment, firing an electron beam into liquid $H_2$ perhaps to supply lots of isolated protons. The authors were big SLAC folks and they did lots of electron beams there, so that may explain their interest in expressing the result here in many ways.

So let's carry through all this algebra here. They are using the following notation now:

| For the electron: | | For the proton: | |
|---|---|---|---|
| $p_i = p$ | $p_f = p'$ | $P_i = 0$ | $P_f$ = itself |
| $E_i = E$ | $E_f = E'$ | $E_i = M$ | $E_f$ = itself |

We then use 7.42 but the first factor is

mM/sqrt = mM/(pM) = m/p

and we write it all out as

$d\sigma = (m/p) \int |M|^2\, (2\pi)^{-2} \delta^4\, (P_f - P_i + p' - p)\, m\, (p'E'dE'\, d\Omega'/E')\, M\, (d^3 P_f / 2E_f) * 2$

$$\Rightarrow \; d\sigma/d\Omega' = 2\, m^2 M\, (1/p)\, (2\pi)^{-2} \int |M|^2 \delta^4 (P_f - P_i + p' - p)\, (p'dE')\, (d^3P_f / 2E_f)$$

$$= 2\, m^2 M\, (1/p)\, (2\pi)^{-2} \int |M|^2 \delta^4 (P_f - P_i + p' - p)\, (p'dE') \int d^4P_f\, \delta(P_f^2 - M^2)\theta(E_f)$$

$$= 2\, m^2 M\, (1/p)\, (2\pi)^{-2} \int p'dE'\, |M|^2\, \delta([P_i - p' + p]^2 - M^2)\theta(M - E' + E)$$

which is equation F. Now the θ function says E' < E+M and of course we know E' > M, so we have this banding: M < E' < M+E which explains the integration limits in equation G. Now we write out

$$[P_i - p' + p]^2 - M^2 = (M - E' + E)^2 - (\mathbf{p} - \mathbf{p}')^2 - M^2 = x$$

but now

$$(M - E' + E)^2 = M^2 + E'^2 + E^2 + 2ME - 2ME' - 2EE'$$
$$(\mathbf{p} - \mathbf{p}')^2 = p^2 + p'^2 - 2\mathbf{p} \bullet \mathbf{p}'$$

So we get

$$x = M^2 + E'^2 + E^2 + 2ME - 2ME' - 2EE' - p^2 - p'^2 + 2\mathbf{p} \bullet \mathbf{p}' - M^2$$

$$= E'^2 + E^2 + 2ME - 2ME' - 2EE' - p^2 - p'^2 + 2\mathbf{p} \bullet \mathbf{p}'$$

$$= m^2 + m^2 + 2ME - 2ME' - 2EE' + 2\mathbf{p} \bullet \mathbf{p}'$$

The delta pins x = 0, so we then get

$$0 = m^2 + ME - ME' - EE' + \mathbf{p} \bullet \mathbf{p}'$$

or

$$m^2 + ME = ME' + EE' - \mathbf{p} \bullet \mathbf{p}' = E'(M+E) - pp'\cos\theta = EM + m^2 \qquad \text{which is } \mathbf{(7.45)}$$

Meanwhile, our delta function was this

$$\delta(x) = \delta(2\, m^2 + 2ME - 2ME' - 2EE' + 2\, pp'\cos\theta)$$

so we now have derived equation G in full, on goes its check. We want now to rewrite this δ as stuff*δ(E'-stuff) so we can then do the dE' integration. We have

$$\delta(2\, m^2 + 2ME - 2ME' - 2EE' + 2\, pp'\cos\theta) = \delta(f(E'))$$

where dependence also includes the p' factor. We compute

$df/dE' = -2(M+E) + 2p\cos\theta\, dp'/dE' = -2(M+E) + 2p\cos\theta(E'/p')$

so we then know that

$\delta(x) = 1/[+2(M+E) - 2p\cos\theta(E'/p')]\ \delta(E' - \text{hit value})$

where I assume the 2M term wins so when we do |f(E')| it is as I have shown. Then assuming (we could check) that we hit within the integration range, we get

$d\sigma/d\Omega' = 2\, m^2 M\, (1/p)\, (2\pi)^{-2} p'\, |M|^2\ \ 1/[+2(M+E) - 2p\cos\theta(E'/p')]$

$= m^2 M/(4\pi^2)\, (p'/p)\, |M|^2/ [M+E - (pE'/p')\cos\theta]$ which is **(7.44)**

where as noted we have the relation shown in 7.45 which connects E',p', E,p so now all are free. Really since E and p are given initial values, this connects E' and p' in addition to their normal mass connection.

And so with sickle and axe, we have battled our way through page 114.

Now, remember that $M_p$ = 938 MeV and $m_e$ = .51 MeV so we have a 2000 ratio here. So if you shoot in electrons that are doing perhaps 5 MeV, they are extremely relativistic, but still we have E << M, and that is the limit being probed in the text here.

Page 115: Result A comes from 7.44 for $d\sigma/d\Omega'$ were we take E<<M limit so denom = M and p = p' and E = E' from 7.45. The result is now elastic. Then Result B comes from 7.43 where we set things like $P_f.p_i = ME_i = ME$ because $\mathbf{P_f}$ is so small and we only maintain the $p_i.p_f$ term "as is". I have written in the 7.43 factor limit. Also in this limit we have $q^\mu = (E'-E, \mathbf{p'} - \mathbf{p}) \approx (0, \mathbf{p'} - \mathbf{p}) = (0,\mathbf{q})$ so q.q = $-\mathbf{q}^2$ and $q^4 = \mathbf{q}^4$. So, when you combine A with B you do indeed get 7.21 which is the Mott cross section. So this limit applies when M of the proton remains huge compared to E of the electrons, even though the electrons may still be relativistic. So don't call this the non-rel limit please!

Question: If we are not in this E<<M limit, would you say that e-p scattering was "elastic"? Well, when we are not in this limit, we do have E' ≠ E for sure, and that means p ≠ p', and momentum was transferred from electron to proton. But of course $E_{tot}$(initial) = $E_{tot}$(final), and this is what elastic means.

Now mid page 115, BD want to take a *different limit* of our general formula. Suppose E >> m, so incoming electron is extreme relativistic. Then in 7.44 we have p'/p ~ E'/E and

$M+E - \ldots = M + E - p\sin\theta = M + E(1-\cos\theta) = M+2E\sin^2(\theta/2)$

and then we have equation C. I am now going to stop doing algebra, but I see the point. We then take 7.43 and take this same limit and we end up (they claim) with F. We then plug this into C to get (7.46) which is the cross section in this limit. This formula has $q^2$ and E as variables as shown. At least here we get the cross section as a *single expression* and it is simpler than the general result 7.44 + 7.43 with its 4-vector dot products.

**7.5 Higher order corrections to Electron-Proton Scattering (116).**

We open here by stating the second-order perturbation term -- an electron hits the EM potential $A^\mu$ twice. Recall from above the series expansion

$$S_{fi} = \delta_{fi} - \varepsilon_f\, i\, \bar\psi_f\, e\not{A}\, \psi_i - \varepsilon_f\, i\, \bar\psi_f\, e\not{A}\, S_F\, e\not{A}\, \psi_i + \ldots \qquad \text{form 7}$$

and $\varepsilon_f = +1$ for the final state being a positive-energy electron, which is the case here, hence 7.47. This term can be written as

$$S_{fi} \approx -ie^2\, \bar\psi_f\, \not{A}\, S_F\, \not{A}\, \psi_i = -ie^2\ \bar\psi_f\, \gamma_\mu\, S_F\, \gamma_\nu\, \psi_i\, A^\mu A^\nu = +\,[\bar\psi_f\, e\gamma_\mu\, iS_F\, e\gamma_\nu\, \psi_i]\,(-iA^\mu)\,(-iA^\nu)$$

$$= \int d^4x\, d^4y\ [\bar\psi_f(x)\, e\gamma_\mu\, iS_F(x-y)\, e\gamma_\nu\, \psi_i(y)]\,(-iA^\mu(x))\,(-iA^\nu(y))$$

Remember that the old potential V(x) is here $A^0 = e_p/r$ for example (modulo k1). Recall my notes on how iG and -iV are the correct places to put factors of i and then all is positive. Now, BD refer to the square bracket object as a "second order electron current". It is in fact some kind of tensor $J_{\mu\nu}(x,y)$ object, and we are then saying $S_{fi} = -\int d^4x\, d^4y\, J_{\mu\nu}(x,y)A^\mu(x)A^\nu(y)$.

The form of the $S_F$ propagator follows from my notes above concerning equation 6.26 on page 86, which gets doubled for $S_F$ compared to non-rel G. The "n" in the sums refers to the eigenvalues of the basis functions you choose for the eigenfunctions of the defining differential operator L which for $S_F$ is the Dirac equation operator $(\hat{\not{p}}-m)$ as shown top of page 93. Usually these eigenvalues are momentum p and are our plane waves with spinors. So think of $\Sigma_n = \int d^3p$ roughly.

Several things are now unclear. (1) why do BD bother to write the two terms in $S_F$ as shown instead of just writing $S_F$? (2) what do they mean in the sentence here referring to "transition currents"?

I think we should regard 7.48 as just a hypothesis for what the $A^\mu A^\nu$ "second order potential" might look like (I am looking at the second equality in 7.48). In this hypothesis, we get the desired symmetry that the second order *proton* current appears in a fashion identical to the way the second order *electron* current $J_{\mu\nu}$ appears in $S_{fi} = -J_{\mu\nu}A^\mu A^\nu$. So we get roughly $S_{fi} = -J^e{}_{\mu\nu}\, D_F D_F J_p{}^{\mu\nu}$. When we related $A^\mu$ to $J^\mu$ earlier, we got the single integral as in 7.31, so seems reasonable that we get a double integral with two $D_F$ here.

Well, what we are really doing is making up words to support the Feynman diagram on page 116. You match this to the factors in 7.48. The left electron line is the second order electron current which we wrote above as $[\bar\psi_f(x)\, e\gamma_\mu\, iS_F(x-y)\, e\gamma_\nu\, \psi_i(y)]$. You see how we put $e\gamma_\mu$ at each vertex, and $iS_F$ as the linkage, and how vertices are labelled with spacetime coordinates. The proton line plus the two photon propagators combined is the object $A^\mu A^\nu$ which I can call the second-order potential, again all classical. You can see how the picture on page 116 exactly matches the expression in 7.48. Later we will have a better basis from QFT, but for now the hypothesis is reasonable.

We now include the diagram on page 117 and its expression with the words " electron does now know which photon hits first since photons are identical". Again, just blather, this is just another term in the QFT expansion we will later learn.

So we end up with 7.49 for our amplitude $S_{fi}$ and we are now integrating all four coordinates over all spacetime to get the total amplitude.

BD then do my discussion above about where the factors of i appear, they agree with me. Each vertex is a V(x) thing and gets a -i. Each propagator gets a +i. Then all signs are plus.

Now 7.50 is very good. It shows many good things:

(1) It shows the four plane waves in their correct positions including phases, but the plane wave normalizing factors are collected together with the V's are put way in the front.
(2) It elevates each propagator into a corresponding (conjugate) momentum space variable.
(3) The same notation for electron and proton momenta are used as done earlier, caps for proton.

Notes:

- As in 7.29, the photon propagator has a -1 in the numerator, but there are two of them so these minuses cancel and we see no such signs.
- The $S_F$ has a +1 in the numerator, as shown
- the factors of i are not shown. They would be $(+i)^4$ [ propagators] * $(-i)^4$ [ vertices] = +1

Now, each spatial integral acting on "it's" exponential phase factors is going to create a momentum conserving $(2\pi)^4\delta^4$ for each momentum-space vertex. So this is four delta functions. Three of these are killed off by three of the momentum-space elevation integrals, and you are left with one delta as the overall momentum conservor as shown top page 119. The result is 7.51 and it goes with a momentum-space Feynman diagram at the bottom of page 119 and I like it.

The other Feynman momentum-space diagram is 120 and, as they say, the corresponding expression to 7.51 differs only in the argument of the proton propagator.

We are left then with two terms of the form of 7.51 and we are facing a $d^4q_1$ integration. of a complicated scalar function of the four external momentum variables and the spins and masses m and M. BD suggest that nobody has been able to compute this thing except in a large M limit ( and that was Dalitz in 1951). Of cours this same situation is going to arise in electron-electron scattering later in this chapter.

Note on Richard Dalitz. He was Australian, died in 2006, and the web has a little 2-page PDF put up regarding his life work. This is probably a trend and such memorials may last forever. His calculation here is not mentioned. Sometimes the second order diagrams are called the "higher Born approximations", but the word Born has not appeared in BD.

**Question**: Why are there two Feynman diagrams? Is this fact included in our perturbation expansion formula $S_{fi} \approx -\varepsilon_f\, i\, \bar{\psi}_f\, e\not{A}\, S_F\, e\not{A}\, \psi_i + \ldots$ ? This formula has two $\not{A}$ interactions which you could put behind a black curtain in the graph on page 116 and 117. So our $S_{fi}$ formula does not tell us how many terms there are having this second-order form. This is really a QFT issue and we are sort of faking things here a bit. If the virtual photons are indistinguishable Bose particles, then the electron cannot *in principle* distinguish which one it captured first, so we have to add the second graph with a + sign so the resulting amplitude is + under exchange of the two photon sources (think source on the proton).

**7.6 Bremsstrahlung (120) ( Bethe-Heitler 1934)**

The term means "braking radiation" and that is where it was first observed, electron beam decelerating as it hits a target. The term means any radiation by a free charged particle as it accelerates, such as a beam going around a ring. Sometimes called free-free radiation.

Preliminaries about the photon field.

(1) It is a "vector" field, meaning vector with respect to the Lorentz group. The 4-spinor $\psi$ of the Dirac representation is now the $\varepsilon^\mu$ polarization vector of the vector representation. We know we can work in a Coulomb gauge where $\nabla\bullet\mathbf{A} = 0$ without losing any of our observable theory, and for plane waves this says $\mathbf{k}\bullet\boldsymbol{\varepsilon} = 0$ which means there are only two "degrees of freedom" for $\boldsymbol{\varepsilon}$ and we take $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ as perp and both transverse to the motion direction of the photon. In theory, $\varepsilon^\mu$ has four basis functions, but the claim is that only two matter.

Suppose we have a Lorentz frame in which $\varepsilon = (0,\boldsymbol{\varepsilon})$ with $\varepsilon.\varepsilon = -\boldsymbol{\varepsilon}^2 = -1$. This would be a spacelike 4-vector and for such a vector, we know we can boost away the time component. For the timelike 4-vector $p^\mu$ we are comfortable talking about "the rest frame" (momentumless, and one of many) in which it has the form $(M,\mathbf{0})$. Similarly, we should be comfortable talking about the "timeless frame" (one of many) in which a spacelike 4-vector like $\varepsilon^\mu$ has the form $\varepsilon = (0,\boldsymbol{\varepsilon})$. In either case, rotations don't change the rest or timeless nature of the frame.

So OK, I think I can be happy with the normalization $\varepsilon_\mu\varepsilon^\mu = -1$ as a world scalar for $\varepsilon^\mu$. In that special Lorentz frame where $\varepsilon^\mu = (0,\boldsymbol{\varepsilon})$ we have $\varepsilon^\mu k_\mu = -\boldsymbol{\varepsilon}\bullet\mathbf{k}$ and this is 0 in the Coulomb gauge (also called the transverse gauge or radiation gauge), so reasonable to say that $\varepsilon^\mu k_\mu = 0$. Let's compare to Dirac spin:

| | |
|---|---|
| $s^\mu = (0,\mathbf{s})$ in rest frame, spacelike | $\varepsilon^\mu = (0,\boldsymbol{\varepsilon})$ in some frame, spacelike |
| $s^\mu s_\mu = -\mathbf{s}\bullet\mathbf{s} = -1$ | $\varepsilon^\mu\varepsilon_\mu = -\boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon} = -1$ Coulomb gauge: ↓ |
| $p^\mu = (m,0)$ in rest frame, timelike | $k^\mu = (k,\mathbf{k})$ in some frame, lightlike $\mathbf{k}\bullet\boldsymbol{\varepsilon} = 0$ |
| $p^\mu p_\mu = m^2$ | $k^\mu k_\mu = 0$ since massless, $m^2 = 0$ |
| $s^\mu p_\mu = 0$ in rest frame and thus all frames | $\varepsilon^\mu k_\mu = 0$ in rest frame and thus all frames |

So things are quite similar. Web article notes that a massive vector particle will have 3, not 4, physical polarizations.

(2) BD show both positive and negative energy parts, which is different from our Dirac plane wave where we assigned either one or the other to an electron state. But again, we are classical here on $A^\mu$ so it must be real, hence the two parts.

(3) What is $\mathbf{B} = \nabla\text{x}\mathbf{A}$ ? On the first phase $e^{-ik.x}$ we see that $\nabla = +i\mathbf{k}$ etc, so we get

$$\nabla\text{x}\mathbf{A} = N\{ (+i\mathbf{k})\text{ x }\boldsymbol{\varepsilon}\ e^{-ik.x} + (-i\mathbf{k})\text{ x }\boldsymbol{\varepsilon}\ e^{+ik.x}\} = ikN\{\hat{\mathbf{k}}\text{ x }\boldsymbol{\varepsilon}\ (e^{-ik.x} - e^{+ik.x})\} = -$$

$$= ikN(-2i)\,\hat{\mathbf{k}}\text{ x }\boldsymbol{\varepsilon}\ \sin(k.x) = +2kN\ \hat{\mathbf{k}}\text{ x }\boldsymbol{\varepsilon}\ \sin(k.x) = \mathbf{B}$$

We know that E = B in magnitude for a plane wave, so energy U is then 1 * $B^2$ space integral. So what is $B^2$?

$B^2 = (2kN)^2 \sin^2(k.x)$ $\qquad$ $U = \int d^3x\, B^2 = (2kN)^2 \int d^3x \sin^2(\omega t - \mathbf{k}\bullet\mathbf{x})$

So what is this integral? Assume a very large box. The average value of $\sin^2(\varphi)$ is ½, so the integral ought therefore to be V/2. So we get U = $(2kN)^2$* V/2. We set this to the energy ω of 1 photon and we then get

$\omega = (2kN)^2 * V/2 \Rightarrow$ $\qquad$ $2\omega/(V4k^2) = N^2$ $\qquad$ but k = ω so $\qquad$ $N^2 = 1/(2kV)$

and this is how we justify the normalization factor in 7.53 ! Very good.

So, what is 7.53 really saying? It is a photon particle plane wave! On page 121 they say "photon", but still this represents the potential $A^\mu$ of a classical EM field in the box, albeit normalized to have the energy of exactly one photon.

Now, back to Bremsstrahlung. Why can't we have a single vertex Feynman diagram? Need

$p_i^\mu = p_f^\mu + k^\mu \quad \Rightarrow \quad k^\mu = p_i^\mu - p_f^\mu \quad \Rightarrow \quad 0 = 2m^2 - 2\, p_i.p_f$

But the most the RHS can be occurs when $p_i.p_f$ is the least it can be which is $E_iE_f - p_ip_f$ . You have to do the algebra to see that RHS is always < 0, I did this on scratch (and here below) and it is not at all obvious to me, but it is true. That is, I have shown that

$E_iE_f - p_ip_f > m^2$ $\qquad$ which is the same as $\sqrt{m^2 + p_i^2}\,\sqrt{m^2 + p_f^2} > (m^2 + p_ip_f)$

or

$(m^2 + p_i^2)(m^2 + p_f^2) > (m^2 + p_ip_f)^2$

or

$m^2 (p_i^2 + p_f^2) > 2\, m^2\, p_ip_f$

or $\quad (p_i - p_f)^2 > 0$

So let's start with the usual Coulomb potential and draw the two Feynman diagrams top page 122. The $S_{fi}$ in coordinate space is shown in 7.56 which we can read right off the diagrams. Again, i with propagator, -i with vertex, plus sign overall. We have dx and dy integrals. In each of the two graph terms, after we elevate the propagator, we can do one of the $d^4x$ space integrals, but the other has that 1/r from the Coulomb potential to deal with. So the second space integral does this: the $d^3x$ on the phase times 1/r gives the $1/q^2$ as we saw on page 100 equation B. Then the remaining $dx^0$ gives just the energy delta function. So point to remember: in Coulomb you only conserve energy, you do not conserve 3-momentum because the Coulomb can absorb momentum. This is pretty obvious in the simpler Coulomb scattering situation with the 1 vertex graph.

The next point is that an external photon has $\varepsilon^\mu$ from its $A^\mu$ so that turns into $\not{\varepsilon}$ and we get a new Feynman rule for external photon: (1) put $-i\not{\varepsilon}$ at its vertex; (2) put the right normalization factor as shown (which is completely different from the electron leg norm factor).

Next point is that we ignore the "other phase" term in 7.53 because that would be an incoming photon and that is now what we are doing. Sort of like selecting a particular electron spinor for the process you are looking at.

Now that we have $S_{fi}$ , what might we try to calculate? One particle in, two out with Coulomb absorbing momentum. I think we will need a separate $d\Omega$ for the photon and for the final electron for our cross section definition. Ie, rate into those two chunks of solid angle, each coming from its usual phase space factor.

Comment: Sakurai gives a similar setup for the Bremsstrahlung $S_{fi}$ on page 230, where he uses $\gamma^4$ to mean $\gamma^0$, and he presents $\not{\varepsilon}$ as $\varepsilon.\gamma$ , so various notational differences (and mixed metric, etc etc). BUT, he does nothing with it, so ths book is not going to give you any final results. The full result is called Bethe-Heitler, but it is beyond all my owned texts.

Comment: The term Born series is in fact used for what we are doing. Here is a quote from the web

**The Born approximation**

$$\phi_k(\mathbf{r}) = e^{ik_i\cdot\mathbf{r}} + \int d^3r' G^+(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')e^{ik_i\cdot\mathbf{r}'} + \int d^3r' \int d^3r'' G^+(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')G^+(\mathbf{r}-\mathbf{r}'')U(\mathbf{r}'')\phi_k(\mathbf{r}'') \cdots$$

This procedure can be repeated and yields the **Born expansion**. The **Born approximation** is the first term in the Born expansion.

and here is a figure using this language,

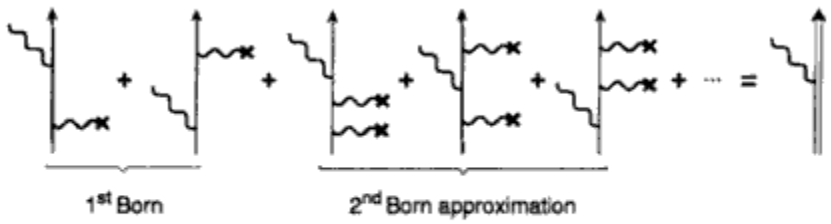



Fig. 3.1 Sum of Feynman-Dyson graphs for bremsstrahlung and the equivalent graph of the Furry picture.

This is from a Google reviewed book *The Elementary Process of Bremsstrahlung* . This book purports to present the **Bethe-Heitler formula** in all its glory, but things are very messy. Well, here is the result from this book. First, here is the physics package with all spins summed and averaged

$$\frac{1}{2}\sum_{\mathbf{e}} \mathbf{S}^{p_1}\mathbf{S}^{p_2}|M|^2 = \frac{16\pi^2 a^2}{\epsilon_1\epsilon_2 q^4}\left\{\frac{4\epsilon_2^2-q^2}{D_1^2}(\mathbf{p}_1\times\hat{\mathbf{k}})^2 + \frac{4\epsilon_1^2-q^2}{D_2^2}(\mathbf{p}_2\times\hat{\mathbf{k}})^2 - 2\frac{4\epsilon_1\epsilon_2-q^2}{D_1D_2}(\mathbf{p}_1\times\hat{\mathbf{k}})\cdot(\mathbf{p}_2\times\hat{\mathbf{k}}) + 2\frac{(\mathbf{q}\times\mathbf{k})^2}{D_1D_2}\right\}. \qquad (3.80)$$

where $\varepsilon_1$= E and $\varepsilon_2$ = E' (energies), and

$$D_1 = (\mathbf{p}_1 - \mathbf{k})^2 - p_2^2 = 2(\epsilon_1 k - \mathbf{p}_1 \cdot \mathbf{k}) , \qquad (3.55)$$
$$D_2 = p_1^2 - (\mathbf{p}_2 + \mathbf{k})^2 = 2(\epsilon_2 k - \mathbf{p}_2 \cdot \mathbf{k}) . \qquad (3.56)$$

Then here is the corresponding cross section,

From (3.24) the differential cross section for the elementary process of bremsstrahlung in Born approximation (Bethe-Heitler formula) has the form [11]

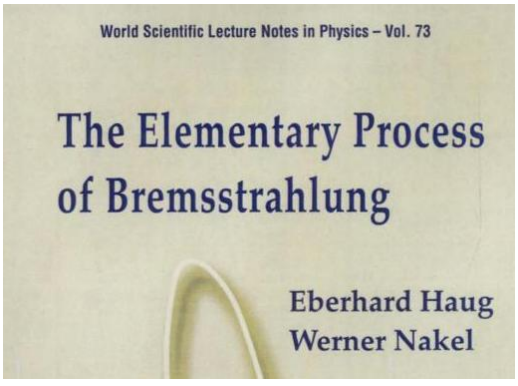
$$\frac{d^3\sigma_B}{dk\, d\Omega_k\, d\Omega_{p_2}} = \frac{\alpha Z^2 r_0^2}{\pi^2} \frac{p_2}{k p_1 q^4} \left\{ \frac{4\epsilon_2^2 - q^2}{D_1^2} (\mathbf{p}_1 \times \mathbf{k})^2 + \frac{4\epsilon_1^2 - q^2}{D_2^2} (\mathbf{p}_2 \times \mathbf{k})^2 \right.$$
$$\left. - 2\, \frac{4\epsilon_1\epsilon_2 - q^2}{D_1 D_2} (\mathbf{p}_1 \times \mathbf{k}) \cdot (\mathbf{p}_2 \times \mathbf{k}) + \frac{2k^2}{D_1 D_2} (\mathbf{q} \times \mathbf{k})^2 \right\} , \qquad (3.81)$$

where $r_0 = (\hbar/mc)\alpha \approx 2.818 \cdot 10^{-15}$ m is the classical electron radius. The corresponding cross section including photon polarization is obtained by substituting the curly brackets of Eq. (3.76) for that of (3.81).

and if you don't want to sum over photon polarizations, you replace the curly bracket above with this:

$$\tfrac{1}{2}\mathbf{S}^{p_1}\mathbf{S}^{p_2}|M|^2 = \frac{16\pi^2\alpha^2}{\epsilon_1\epsilon_2 q^4} \left\{ \frac{4\epsilon_2^2 - q^2}{D_1^2} |\mathbf{e} \cdot \mathbf{p}_1|^2 + \frac{4\epsilon_1^2 - q^2}{D_2^2} |\mathbf{e} \cdot \mathbf{p}_2|^2 \right.$$
$$\left. - 2\frac{4\epsilon_1\epsilon_2 - q^2}{D_1 D_2} \mathrm{Re}(\mathbf{e} \cdot \mathbf{p}_1)(\mathbf{e}^* \cdot \mathbf{p}_2) + \frac{(\mathbf{q} \times \mathbf{k})^2}{D_1 D_2} \right\}. \qquad (3.76)$$

where **e** is the polarization vector of the outgoing photon. So I guess the final result is not so bad after all, but there is a lot of algebra in this 2004 book leading up to this result




Comment: When you bend a particle beam with magnets, they are providing the "Coulomb potential" which causes Bremsstrahlung with the standard graphs. You might install **A** = ½ **B** x **r** as on BD1 page 13. This would be a similar "application". I could not find a simpatico presentation on the web but suspect it would work. This is synchrotron radiation.

Derivation of 7.58: First line is just a copy of the square bracket in 7.57. In the second line, we drop k in each numerator, and drop $k^2$ in each denominator. In the first term, slide $\not\epsilon$ to the right, and in the second term to the left, using the slash-slider rule. The purpose is to expose the factor ($\not p$ - m) of the correct type to act on a spinor to give 0. For the first term, it acts to the left, recall 3.9a on page 30. In the second term it acts to the right. This kills off 4 of the 6 terms, leaving you with the third line as claimed.

If we compare the combination of 7.57 + 7.58, I showed that it is just $(ie)/\sqrt{2kV}$ times the Coulomb scattering result 7.5. Thus, when you square this $S_{fi}$, you get factor $e^2/(2kV)$. But then you now have an extra phase space factor for the photon, and that makes a V which cancels our V here, leaving us with just $e^2/(2k)$ which you see out front in equation D page 123 (taking to $e^6$). You can also compare this D to the cross section in 7.10 page 102. Lets compare the factors in these two cases:

(7.10) $\quad 4\,\alpha^2 * (e^2/2k) = 4e^6/(16\pi^2) * (1/2k) = e^6/(2\pi)^2 * (1/2k)$

D $\quad e^6 * (1/2k) * 2\pi * (1/2\pi)^3 = e^6 * (1/2k) * (1/2\pi)^2$

and so when we multiply 7.10 by $(e^2/2k)$ we do in fact get our D, apart from the extra phase space and the adjustment factor. So I am completely happy now with equation D.

We are let at once to 7.59 -- all factors are fine -- and they have added a Heaviside to say $E_i > m+k$ which just says you need enough incoming zip to make the photon and an at-rest final electron. Recall the braking notion (my friend Sommerfeld coined the Bremsstrahlung word).

This Bremsstrahlung think is losing energy to the photon, so it is "inelastic". But our Coulomb scattering conserved energy so was "elastic". So page 123 is now happy.

As $k \to 0$ we have $1/k * k^2 * 1/k^2 = 1/k$ and the cross section blows up! This is a dk/k log problem at k = 0. It has the famous name **"infrared catastrophe"** and is an example of a QED divergence. A good explanation is given (without proof) which basically says you need to include all double-photon graphs for *elastic* scattering in your analysis, because if you measure $k \approx 0$ photons, you also get elastic scattering electrons in your final electron detector. So you have to include the box graphs we talked about in the previous section along with those of the form shown on page 124, the vertex correction and propagator correction. All fascinating. I guess one could just look at the small k limit of these graphs.

What about the double Coulomb graphs shown above? Well, these are really the box graphs we just talked about above in the heavy M limit.

Claim: Replace $\not\epsilon$ by $\not k$ and the two terms in 7.57 cancel. Yes, it is true on scratch. If we ignore all i factors, we have

$T1 = \not k\,(\not p_f + \not k + m)\,\gamma^0 / D1$
$T2 = \gamma^0\,(\not p_i - \not k + m)\,\not k / D2$

where
$D1 = (p_f + k)^2 - m^2 = 2p_f.k$
$D2 = (p_i - k)^2 - m^2 = -2p_i.k$

Meanwhile, we know that

$\not{a}\not{b} + \not{b}\not{a} = 2a.b$ => $2\ \not{k}\not{k} = 2k.k = 0$ so conclude that $\not{k}\not{k} = 0$ so we have

$T1 = \not{k}\,(\not{p}_f + m)\,\gamma^0 / D1$
$T2 = \gamma^0\,(\not{p}_i + m)\,\not{k} / D2$

Now move $\not{k}$ to the right in T1 and to the left in T2 to get

$T1 = (-\not{p}_f + m)\,\not{k}\,\gamma^0 / D1 \quad + 2p_f.k\,\gamma^0 / D1 = 2p_f.k / D1 \quad \gamma^0 = \gamma^0$
$T2 = \gamma^0\not{k}\,(-\not{p}_i + m) / D2 \quad + \gamma^0\,2\,p_i.k / D2 = 2p_i.k / D2 \quad \gamma^0 = -\gamma^0$

where we use the slider rule and the fact that $(\not{p}-m)u = 0$ left for T1 and right for T2. So T1+T2 = 0 as claimed.

Derive 7.61 : This is quite fancy, a Feynman "trick" they observe.

(1) First write the RHS of 7.60 like this

$RHS = \varepsilon_\mu\,(f/k.f - i/k.i)^\mu\ \varepsilon_\nu\,(f/k.f - i/k.i)^\nu = \varepsilon_\mu\,\varepsilon_\nu\,J^{\mu\nu}$ where f means $p_f$

$J^{\mu\nu} = (f/k.f - i/k.i)^\mu (f/k.f - i/k.i)^\nu$ = symmetric

so at least we know what $J^{\mu\nu}$ is. So we have (7.60) as a scalar equality so can evaluate in any frame. Go to a frame where $k^\mu = (k,k,0,0)$ [ going in x direction] and set the two polarizations to be y and z as shown. In this special frame, the $\Sigma_{pol}\ \varepsilon_\mu\,\varepsilon_\nu\,J^{\mu\nu} = J^{22} + J^{33}$ where $\Sigma_{pol}$ means sum over the allowed photon polarizations and only the two shown are allowed.

(2) Write $J^\mu{}_\mu = g^{\mu\nu}J^{\mu\nu} = J^{00} - J^{11} - J^{22} - J^{33}$ and we get second equality in B.

(3) Note that $k_\mu J^{\mu\nu} = 0$ because $k_\mu\ (f/k.f - i/k.i)^\mu = 1-1 = 0$, similarly on other index. In our special frame, evaluate and end up as it says with $J^{11} = J^{00}$ so end up with equation C! This is telling us then how to do a polarization sum!

(4) We can generalize to the case that $J^{\mu\nu} = a^\mu b^\nu$ where k.a = k.b = 0 (both are conserved currents). Then we get

$\Sigma_{pol}\ \varepsilon_\mu\,\varepsilon_\nu\,J^{\mu\nu} = \Sigma_{pol}\ \varepsilon_\mu\,a^\mu\,\varepsilon_\nu\,b^\nu = -\,a^\mu b_\mu = -\,a.b$

and this is 7.61. Pretty fancy !

(5) So let's apply this polarization summation trick to 7.60 and we get

$\Sigma_{pol}\ (\varepsilon.f/k.f - \varepsilon.\ i/k.i)^2 = -\,(f/k.f - i/k.i)^2 = -\,\{\ f.f/(k.f)^2 + i.i/(k.i)^2 - 2i.f\ /\ [(k.f)(k.i)]\ \}$

$= -\,\{\ m^2/(k.f)^2 + m^2/(k.i)^2 - 2i.f\ /\ [(k.f)(k.i)]\ \} = \{\ +2i.f\ /\ [(k.f)(k.i)] - m^2/(k.f)^2 - m^2/(k.i)^2\ \}$

= the square bracket in 7.62.

The rest of 7.62 is just checking each factor one at a time, it is correct on scratch so red check on 7.62. This is the Bremsstrahlung cross section summed over the two possible final polarization states of the final physical photon. Notice that ε's no longer appear of course.

So in 7.62 there are three angular integrals to do. The last two integrals are each exactly -1, so together they provide -2 . The first integral cannot I guess be done all the way, we are left with a single definite integral showing there. But then we do the NR and ER limits and get the two results B in these limits. Then NR limit just means incoming electron has $\beta << 1$ and you can see the result is proportional to $\beta^2$ so you lose your Bremsstrahlung juice when the incoming slows down. The ER limit applies as long as the $q^2$ shown bottom page 126 is $>> m^2$, so with a hot beam, this will be true except at very small electron deflection angles. In this case, the Bremsstrahlung grows log with $q^2$.

Comments:

(1) We imagine we only capture photons in the range $k_{min}$ to $k_{max}$. But we have a $4\pi$ steradian detector so that is why we do the $d\Omega_k$ integral in 7.62 which took all of page 126 to do. Then imagine we capture electrons only in some small cone at $\Omega_f$ (well, Mott unpolarized electron cross section has no $\varphi$ dependence, so imagine $\Omega_f$ is cone shaped thing going all around the beam direction). Then the rate of photon detection is predicted by 7.64. It is the Mott Coulomb electron cross section number times the new factors shown. I wonder if this has been subject to an experiment? I could look in the data base, but not right this minute.

(2) This is just a "soft photon" result which means $k_{min}$ and $k_{max}$ are both small. But they can't be *too* small, or you need to add the corrections from those other graphs we talked about, and these affect the Mott cross section in 7.64. So I guess we are supposed to understand the "Mott" thing there (my label) as including these corrections, they you can go as low as you want with k.

(3) Angle $\theta$ appearing in 7.64 NR case is the electron deflection angle. $q^2$ is the "momentum transfer" to the Coulomb sink but for small k this is also $(p_f - p_i)^2$. We always assume that $q^2 >> k^2$ I think.

Comment: Recall our various integral equations for scattering. Instead of computing higher order terms using plane waves, you can compute these in effect by using "modified wave functions" in the lower order calculations. For example, you could do the first Born term we did, but put in "Coulomb wave functions" in place of our plane waves and then you would in effect be doing some of the higher order graphs. This is the language I think many writers use.

**7.7 Compton Scattering (127) (Klein-Nishina 1929)**

This is $e+\gamma \rightarrow e+\gamma$, so like the Bremsstrahlung case with these differences;

Naming. The outgoing photon is now given primes ε' and k'. The non-primed ε and k apply to the new incoming photon.

Physics Package: Compare the new 7.67 with the Bremsstrahlung 7.57. If we look at the $M_{fi}$ (physics package) for the Bremsstrahlung case, we should replace the previous $\not{\epsilon}$ with $\not{\epsilon}'$ and replace the previous $\gamma^\circ$ with $\not{\epsilon}$ . Also, the internal electron propagator of the first graph that was $p_f + k$ becomes $p_f + k'$ which is same as $p_i + k$. The second graph contained $p_i - k$ which becomes $p_i - k'$.

$S_{fi}$ . Compare the new 7.67 with the e-p scattering 7.35. We replace $M/(E_i V) \rightarrow 1/(2kV)$ twice, once for each proton converted to a photon leg, so this adjusts the normalization factors. The V powers are the same. As for the i-factor :

What about the leading power of i in $S_{fi}$ ? Let's backtrack to 7.32 for e-p scattering and ponder the leading -i there. It is the same as the -i in the $S_{fi} = \delta_{fi} - \varepsilon_f\, i\, \bar{\psi}_f^s\, e\not{A}\, \psi_i$ Born term of S, and I think of it as coming from $(-i\not{A})$. The connection 7.31 [ p 109] adds no factors. We now absorb the i with $D_F$ and the - sign as shown in 7.29, so then we interpret 7.35 as all + except imagine $-i/q^2$ as the photon propagator.

Now shift forward to 7.67. We have no photon propagators, so the -i you see leading in 7.35 is not present and we see just the "true +" as the leading factor in 7.67. I think BD should have put the -i over the $q^2$ in 7.35 and this would have helped, but all OK now.

Attempt modification of the former 7.39 for dσ. In the second line, the only changes here are the ones already noted above having to do with normalization, $M/(E_i V) \rightarrow 1/(2kV)$ twice. And $d^3k'$ would replace $d^3P_f$ for the phase space of one outgoer. So this second line would read:

$$d\sigma = (2\pi)^{-2}\, d^3p_f\, d^3k_f * m/E_i * m/E_f * 1/(2k) * 1/(2k') * \delta^4\, |M|^2\, /(JV)$$

The incident photon flux J. Before we did this for two massive particles and got page 113 A which led to 7.41. Back in my notes for simple Coulomb scattering (Section 7.1), I derived the fact that $\mathbf{j} = \boldsymbol{\beta}/V = \mathbf{v}/(cV)$ which was the electron probability 3-current (charge elided).

You can go to the rest frame of the incident electrons, but you cannot go to the incident photon rest frame, so it seemed more reasonable to me to talk about an incoming beam of photons, and when talking cross section, you would want to talk about $J_{inc}$ of this incoming species.

What is the *photon* probability current? Our incoming photon wave function is $A^\mu = \varepsilon^\mu / \sqrt{2kV}\, e^{-ik.x}$ We normalized to have one photon in the box V, so probability of it being in volume dA Δz must be dA Δz/V. If we want this entire volume to flow through area dA in time Δt, then need $\Delta z = c\, \Delta t$ so prob in volume is dA c Δt/V and the prob flux J is just c/V which is just 1/V in our units! This is then compatible with our result $\mathbf{j} = \boldsymbol{\beta}/V$ with $\beta = 1$. So JV = 1 and we can rewrite the above as

$$d\sigma = (2\pi)^{-2}\, d^3p_f\, d^3k_f * m/E_i * m/E_f * 1/(2k) * 1/(2k') * \delta^4\, |M|^2$$

$$= (2\pi)^{-2}\, (1/E_i)(m/2k)\, (d^3k'/2k')\, (m\, d^3p_f/E_f)\, \delta^4\, |M|^2 \qquad \text{7.68 with "v" = 1}$$

but alas, this agrees with 7.68 but they have an extra 1/v flux factor because they used the incoming electron flux which we know exactly adds this factor of v.

But how do we justify either of these claims for "flux"? I am now stumped, time to call for help. Sakurai on page 216 says the flux is $v_{rel}/V$ valid in a collinear arrangement and he explicitly says this depends on frame, and that $v_{rel}$ can be greater than c.

Now, suppose in 7.68 we interpret "v" as meaning this "$v_{rel}$". In the "lab" frame with electron at rest, it seems to me we would have $v_{rel} = c = 1$. On page 129 after the equation BD say $v = m/E_i$ which in fact is 1 since $E_i = m$ in the lab frame.

But how do you write this in a scalar way? Equation 7.41 does say this:

$$E_1E_2\, v_{rel} = \sqrt{(P_1.P_2)^2 - m_1^2m_2^2} \quad = \text{world scalar}$$

and this appears in Sakurai as well top page 217 where 1 and 2 are called + and – and both are electrons or mass m. Now let's try to set 2 = photon in the above, so $m_2 = 0$ and we then get

$$E_1E_2\, v_{rel} = (P_1.P_2) \qquad \Rightarrow \qquad kE_i\, v_{rel} = p_i.k \qquad \text{in our Compton notation}$$

So let's go back to our earlier form

$$d\sigma = (2\pi)^{-2}\, d^3p_f\, d^3k_f * m/E_i * m/E_f * 1/(2k) * 1/(2k') * \delta^4\, |M|^2\, /(JV)$$

and write $J = v_{rel}/V$ to get

$$d\sigma = (2\pi)^{-2}\, d^3p_f\, d^3k_f * m/E_i * m/E_f * 1/(2k) * 1/(2k') * \delta^4\, |M|^2\, / v_{rel}$$

$$\Rightarrow$$

$$d\sigma = (2\pi)^{-2}\, d^3p_f\, d^3k_f * m/E_i * m/E_f * 1/(2k) * 1/(2k') * \delta^4\, |M|^2\, * (kE_i/\, p_i.k)$$

$$\Rightarrow$$

$$d\sigma = (2\pi)^{-2}\, (m/2)\, (d^3k'/2k')\, (m\, d^3p_f/E_f)\, \delta^4\, |M|^2 * (p_i.k\,)$$

and we are happy now to see $d\sigma$ as a Lorentz scalar quantity!

So I am now happy with 7.68 with the understanding that v means

$$v_{rel} = (p_i.k)/\, kE_i.$$

I am now happy to go to the "lab frame" and set v = 1.

Comment on Leg Norms. A Dirac particle leg gets the $\sqrt{m/EV}$ factor because

$$\psi^\dagger\psi \;\; = (m/EV)\, u^\dagger\, u \;\; = (m/EV)\, w^{\dagger 1}\, w^1 = (m/EV)(E/m) = 1/V$$

In other words, the strange m/E has to do with the way a spinor makes probability with Lorentz contraction. This causes there to be 1 Dirac particle in the box. The photon norm is different, you set it to make $\hbar\omega$ be the energy in the box, because we know that will put one photon in the box. So you cannot just take the $m \to 0$ limit of the Dirac norm and get the photon norm, it is a different kind of particle! This is why 7.42 does not apply to Compton scattering in this limit, for example.

P 128: On bottom of page 127 BD talk about **"three additional terms"** and they just mean the other combinations that arise from the two exponentials in 7.65 and 7.53. The "crossing symmetry seems pretty reasonable, the sum of the graphs ought to be the same if you swap momenta with a sign change. The ε happens to now have a sign change, similar to spin in u(p,s). Perhaps CPT on a leg, not a lot of detail here, but of course in hadronics I remember this idea well.

P 129. I am happy with 7.68 with v = 1, see details above. So we will work in the **incident-electron-at-rest lab frame.** The algebra on the rest of the page is straightforward, the usual df/dk' delta function rule, the δ is just energy conservation called **"the Compton condition"** where θ is the photon scattering angle (they forgot to mention that). So we quickly obtain 7.71 which is amazingly simple for the kinematics package. I guess for each $A^\mu$ you can separately do an arbitrary gauge shift to force transverse polarization as shown page bottom.

P 130: The gauge choice has a huge benefit as we shall soon see again and again. Equation A at page top is correct, you do the anticommutes as I show with arrows. The dot product pick-up terms vanish due to the "the benefit", and then Dirac equation kills most terms leaving you exactly with the RHS shown, where both k's in the numerator are slashed (I have trouble seeing the slashes! ). When this thing is squared, the overall minus goes away, you have two positive energy projectors as usual, and the ½ is from initial spin averaging on the electron. So I am happy with 7.72 and now the traces begin.

**Trace $T_1$** : Here is why the two m and the one $m^2$ terms vanish: Each linear m term has an 7 number of slashers so zero. In the $m^2$ term, you get $\not{k}\not{k} = 0$ so it is gone. So the first step is clear. In the second step we say $\not{k}\not{p}_i\not{k} = 2p_i.k\,\not{k}$. In the third step recall ε.k = 0. As we slide $\not{\varepsilon}$ to the left, we get a minus from anti-commute, then another minus since ε.ε = -1. So we are then down to four slashers in the trace. We cannot move $\not{\varepsilon}'$ to the left because we don't have ε'.k = 0, so we use the 4-slasher formula and ε'.ε' = -1 and two of the three terms are the same and we get our next step. We then digress to derive the left expression in 7.73 which arises from $(k-p_f)^2 = (k' - p_i)^2$ , so we can then replace $k.p_f$ by $k'.p_i$ to get E. So this one trace required a lot of pondering. You can see that Compton in this gauge has a lot of zero dot products and things get simplified on that account.

**Trace $T_2$**; This is the one you get by taking the second factor in both parens. We can see that making the changes shown maps $T_2$ back to $T_1$ . The reason we don't do k ↔ +k' is that this would cause our energy conservation condition to be violated. That is, either sign in k ↔ ± k' would map $T_2$ into $T_1$, but since we used EC in the evaluation, we must use the minus sign. This reminds us of "crossing" but the connection is not quite clear to me.

**Trace $T_3$.** This one involves the first term in the first paren and the second term in the second paren in 7.72. In the first line, we replace $p_f$ on the left, and this creates the (k'k') slash term. Then things become very ugly, so here are some algebra traces.

$T_3$ line 2 first term processing:

| | |
|---|---|
| $= (\not{p}_i + m)\,\not{\varepsilon}'\,\not{\varepsilon}\,\not{k}\,(\not{p}_i + m)\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}$ | starting position, start moving the left $\not{\varepsilon}$ to the right |
| $= -(\not{p}_i + m)\,\not{\varepsilon}'\,\not{k}\,\not{\varepsilon}\,(\not{p}_i + m)\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}$ | since k.ε = 0 |

$= -(\not{p}_i + m)\,\not{\varepsilon}'\ \not{k}\ (-\not{p}_i + m)\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\not{\varepsilon}$ since $\varepsilon.p_i = 0$; now move left $\not{\varepsilon}'$ to the right

$= +(\not{p}_i + m)\,\not{k}\,\not{\varepsilon}'\ (-\not{p}_i + m)\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}\ -(\not{p}_i + m)\,2k.\varepsilon'\,(-\not{p}_i + m)\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\ \not{\varepsilon}$ // pick-up term

$= +(\not{p}_i + m)\,\not{k}\ (+\not{p}_i + m)\,\not{\varepsilon}'\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}\ -(\not{p}_i + m)\,2k.\varepsilon'\,(-\not{p}_i + m)\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\not{\varepsilon}$

$= +(\not{p}_i + m)\,\not{k}\ (\not{p}_i + m)\,\not{\varepsilon}'\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}$

where the pick-up term vanishes because $(\not{p}_i + m)(-\not{p}_i + m) = 0$ ! Now move the $\not{k}'$ to the left. The pick-up term here will be $(\not{p}_i + m)\,\not{k}\ (\not{p}_i + m)\,\not{\varepsilon}'\,2k'.\varepsilon\,\not{\varepsilon}'\,\not{\varepsilon}$, and at this point we then have

$= -(\not{p}_i + m)\,\not{k}\ (+\not{p}_i + m)\,\not{\varepsilon}'\,\not{k}'\,\not{\varepsilon}\ \not{\varepsilon}'\,\not{\varepsilon}\ + (\not{p}_i + m)\,\not{k}\ (\not{p}_i + m)\,\not{\varepsilon}'\,2k'.\varepsilon\,\not{\varepsilon}'\,\not{\varepsilon}$

But this pick-up term also vanishes as we now shall see. It is

$\text{pickup} = (\not{p}_i + m)\,\not{k}\ (\not{p}_i + m)\,\not{\varepsilon}'\,2k'.\varepsilon\,\not{\varepsilon}'\,\not{\varepsilon} = -(\not{p}_i + m)\,\not{k}\ (\not{p}_i + m)\,2k'.\varepsilon\ \not{\varepsilon}$

$= -\not{\varepsilon}\,(\not{p}_i + m)\,\not{k}\ (\not{p}_i + m)\,2k'.\varepsilon = -(-\not{p}_i + m)\,\not{\varepsilon}\not{k}\ (\not{p}_i + m)\,2k'.\varepsilon$

$= -2k'.\varepsilon\,(\not{p}_i + m)(-\not{p}_i + m)\,\not{\varepsilon}\not{k} = 0$ again since $(\not{p}_i + m)(-\not{p}_i + m) = 0$

Now in our one surviving term move $\not{k}'$ left one more position to get

$= +(\not{p}_i + m)\,\not{k}\ (+\not{p}_i + m)\,\not{k}'\,\not{\varepsilon}'\ \not{\varepsilon}\ \not{\varepsilon}'\,\not{\varepsilon}$

which agrees with their third line first term.

$T_3$ line 2 second term processing:

$= (\not{k} - \not{k}')\,\not{\varepsilon}'\,\not{\varepsilon}\,\not{k}\,(\not{p}_i + m)\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}$

$= \not{\varepsilon}'\,\not{\varepsilon}\,(\not{k} - \not{k}')\,\not{\varepsilon}'\,\not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}')$ because trace of 7 gammas is 0; swing two to the left

$= \not{\varepsilon}'\,\not{\varepsilon}\ \not{k}\ \not{\varepsilon}'\,\not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}') - \not{\varepsilon}'\,\not{\varepsilon}\,\not{k}'\,\not{\varepsilon}'\,\not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}')$ // just write as two terms

In the first term, move left $\not{k}$ to the right to kill against the other $\not{k}$ . But along the way pick up

$\not{\varepsilon}'\,\not{\varepsilon}\,2k.\varepsilon'\ \not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}') = -2k.\varepsilon'\,(\not{k}\,\not{p}_i\,\not{k}') = -2k.\varepsilon'\,(\not{k}\,\not{p}_i\,\not{k}')\,\not{\varepsilon}'$

In the second term, move left $\not{k}'$ to the left to kill against the other $\not{k}'$. But along the way pick up

$-\not{\varepsilon}'\,2\varepsilon.k'\,\not{\varepsilon}'\,\not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}') = +\not{\varepsilon}\,2\varepsilon.k'\ \not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}') = +2\varepsilon.k'\ \not{\varepsilon}\,(\not{k}\,\not{p}_i\,\not{k}')$

So we then get the rightmost two terms in their third line, so entire third line is now checked.

$T_3$ line 3 first term processing:

$= (\not{p}_i + m)\,\not{k}\ (+\not{p}_i + m)\,\not{k}'\,\not{\varepsilon}'\ \not{\varepsilon}\ \not{\varepsilon}'\,\not{\varepsilon}$ // starting position

We will move $\not{k}$ to the right, it will make $(-\not{p}_i + m)$ which will kill as usual, pick-up term will be

$= (\not{p}_i + m) 2k.p_i \not{k}' \not{\varepsilon}' \not{\varepsilon} \not{\varepsilon}' \not{\varepsilon}$
$= \not{p}_i 2k.p_i \not{k}' \not{\varepsilon}' \not{\varepsilon} \not{\varepsilon}' \not{\varepsilon}$ // linear m is odd gamma
$2k.p_i \not{p}_i \not{k}' \not{\varepsilon}' \not{\varepsilon} \not{\varepsilon}' \not{\varepsilon}$ // which is first term on line 4.

The other terms are these

$- 2k.\varepsilon' \text{ tr} (\not{k} \not{p}_i \not{k}') \not{\varepsilon}' + 2\varepsilon.k' \text{ tr} \not{\varepsilon} (\not{k} \not{p}_i \not{k}')$

But we have

$\text{tr} (\not{k} \not{p}_i \not{k}') \not{\varepsilon}' = 4 \{ k.p_i \, 0 - k.k' \, 0 + k.\varepsilon' \, p_i.k' \} = 4 \, k.\varepsilon' \, p_i.k'$

$\text{tr} (\not{k} \not{p}_i \not{k}') \not{\varepsilon} = 4 \{ k.p_i \, k'.\varepsilon - 0 + 0 \} = 4 \, k.p_i \, k'.\varepsilon$

so these two terms become

$- 2k.\varepsilon' \text{ tr} (\not{k} \not{p}_i \not{k}') \not{\varepsilon}' + 2\varepsilon.k' \text{ tr} \not{\varepsilon} (\not{k} \not{p}_i \not{k}') = - 8 \, k.\varepsilon' \, k.\varepsilon' \, p_i.k' + 8 \, \varepsilon.k' \, k.p_i \, k'.\varepsilon$

$= - 8 (k.\varepsilon')^2 \, p_i.k' + 8 (k'.\varepsilon)^2 \, p_i.k$ // which are the remaining terms on line 4

So we have now verified all of line 4. I now have only to verify line 5 and fiddle a bit to get to the K-N, but it is 10AM and I think I am out of time, heading to Torrey soon. // Back, it's Monday 10.20.08.

Verifying Line 5. Start with the long first trace of 6 items and swap the inner two ε's, then do the 2-trace and the 4-trace (which has only one term) to get

$\text{tr}(\not{p}_i \not{k}' \not{\varepsilon}' \not{\varepsilon} \not{\varepsilon} \not{\varepsilon}) = - 4 \, p_i.k' + 8 (\varepsilon.\varepsilon')^2 \, p_i.k'$

Then just assemble this with the same pieces from the end of line 4 and we have line 5.

Now ready to assemble all the pieces to verity the K-N 7.74:

Start with 7.72. Let's write the various terms including all factors:

$\alpha^2/2 * (k'/k)^2 (1/2m)^2 * (1/2k.p_i)^2 * 8 \, k.p_i \, [ k'.p_i + 2(k.\varepsilon')^2]$ // $T_1$ term
$- \alpha^2/2 * (k'/k)^2 (1/2m)^2 * (1/2k'.p_i)^2 * 8 \, k'.p_i \, [ -k.p_i + 2(k'.\varepsilon)^2]$ // $T_2$ term
$2*\alpha^2/2 * (k'/k)^2 (1/2m)^2 * (1/2k.p_i) (1/2k'.p_i) *$ { line 5} // $T_3 + T_4$ term

If we factor out the common factor $\alpha^2/4m^2 (k'/k)^2$ we are left with

$1/2 * (1/2k.p_i)^2 * 8 \, k.p_i \, [ k'.p_i + 2(k.\varepsilon')^2]$ // $T_1$ term
$- 1/2 * (1/2k'.p_i)^2 * 8 \, k'.p_i \, [ -k.p_i + 2(k'.\varepsilon)^2]$ // $T_2$ term

$$* (1/2k.p_i)\,(1/2k'.p_i) * \quad \{\text{ line 5}\} \qquad // \; T_3 + T_4 \text{ term}$$

=

$$(1/k.p_i) * [\, k'.p_i + 2(k.\varepsilon')^2] \qquad // \; T_1 \text{ term}$$

$$-(1/k'.p_i) * [\, -k.p_i + 2(k'.\varepsilon)^2] \qquad // \; T_2 \text{ term}$$

$$(1/4)* (1/k.p_i)\,(1/k'.p_i) * \{\text{ line 5}\} \qquad // \; T_3 + T_4 \text{ term}$$

The last line above is this:

$$(1/4)* (1/k.p_i)\,(1/k'.p_i) * 8\ \{k.p_i\ k'.p_i\ [\, 2\,(\varepsilon.\varepsilon')^2 - 1] \; - (k.\varepsilon')^2\ k'.p_i + (k'.\varepsilon)^2\ k.p_i\ \}$$

$$= 2* (1/k.p_i)\,(1/k'.p_i) * \quad \{k.p_i\ k'.p_i\ [\, 2\,(\varepsilon.\varepsilon')^2 - 1] \; - (k.\varepsilon')^2\ k'.p_i + (k'.\varepsilon)^2\ k.p_i\ \}$$

$$= 2\ [\, 2\,(\varepsilon.\varepsilon')^2 - 1] - 2\,(k.\varepsilon')^2\ k'.p_i\ (1/k.p_i)\,(1/k'.p_i) + 2\,(k'.\varepsilon)^2\ k.p_i\ (1/k.p_i)\,(1/k'.p_i)$$

$$= 2\ [\, 2\,(\varepsilon.\varepsilon')^2 - 1] - 2\,(k.\varepsilon')^2\ (1/k.p_i) + 2\,(k'.\varepsilon)^2\ \ (1/k'.p_i)$$

So our total result (apart from our common factor taken out above) is this:

$$\text{all terms} = (1/k.p_i) * [\, k'.p_i + 2(k.\varepsilon')^2] - (1/k'.p_i) * [\, -k.p_i + 2(k'.\varepsilon)^2]$$
$$+ 2\ [\, 2\,(\varepsilon.\varepsilon')^2 - 1] - 2\,(k.\varepsilon')^2\ (1/k.p_i) + 2\,(k'.\varepsilon)^2\ \ (1/k'.p_i)$$

Now regroup by terms having the same denominator:

$$\text{all terms} = (1/k.p_i) * [\, k'.p_i + 2(k.\varepsilon')^2] - 2\,(k.\varepsilon')^2\ (1/k.p_i)$$

$$- (1/k'.p_i) * [\, -k.p_i + 2(k'.\varepsilon)^2] + 2\,(k'.\varepsilon)^2\ \ (1/k'.p_i)$$

$$+\ [\, 4\,(\varepsilon.\varepsilon')^2 - 2]$$

In each of the first two lines, the last two terms cancel, so we have

$$\text{all terms} = (1/k.p_i) * [\, k'.p_i] \; - (1/k'.p_i) * [\, -k.p_i] \; + \; [\, 4\,(\varepsilon.\varepsilon')^2 - 2]$$

$$= (1/k.p_i) * (k'.p_i) \; + (1/k'.p_i) * (k.p_i) \; + \; [\, 4\,(\varepsilon.\varepsilon')^2 - 2]$$

But we are in the lab frame all this time, so $k.p_i = mk$ and $k'.p_i = mk'$ so simplify to

$$\text{all terms} = (k'/k) + (k/k') \; + \; [\, 4\,(\varepsilon.\varepsilon')^2 - 2]$$

Since this result is needed later, let's be very clear about what we just found. The result 7.72 is given by the above "all terms" times the factor we extracted, so we get

$$7.72 \;=\; \alpha^2/4m^2\ (k'/k)^2 \; * \text{"all terms"} \;=\; \alpha^2/2\ (k'/k)^2 \; * \text{Tr}(....)$$

$$\Rightarrow \qquad 1/4m^2 \ * \text{"all terms"} \ = \ 1/2 \ * \text{Tr}(....)$$

$$\Rightarrow \qquad \text{Tr}(....) = \text{"all terms"}/(2m^2) \ = \{(k'/k) + \ (k/k') \ + \ [\ 4\ (\varepsilon.\varepsilon')^2 - 2]\ \}/(2m^2)$$

and this result will be used later in pair production.

This then concludes our complete derivation of the **Klein-Nishina formula 7.74** (1929)! In the low k energy limit, overall energy conservation tells us k + m = k' + m, so we have photons scattering elastically and k = k'. in this case, three of the terms in the 7.74 bracket cancel, and the power is 1 out front, the 4's cancel, and we get the limit shown.

Now why is this the result of "classical Thomson scattering" ? I will have to sidetrack just a bit on that. Where do I have this?

**Thomson Scattering and some History.** I have derived all the relevant **Thomson** cross section stuff in the last few days in a document "thomson scattering.doc" (JJ Thomson 1906). There I show that BD1 page 131 equation D is in fact what the Thomson theory predicts, and $\boldsymbol{\varepsilon} \bullet \boldsymbol{\varepsilon}' = \sin\psi$. The key fact about Thomson scattering is that you jiggle an electron and it radiates at the same frequency it is zapped with, k' = k we would say here. But as you increase $\hbar\omega$ so it gets closer to $mc^2$, the collision nature of the process appears, and the final photon has k' < k since the recoiling electron marches off with some energy. This lessening of the wavelength of the scattered beam is what **Compton** discovered in 1923. The formula relating k' to k is given by our BD1 page 129 7.70, and I presume Compton came up with this formula based on simple scattering theory. So this downshift of wavelength is called the Compton Effect, and it does not exist in the realm of Thomson scattering.

Yoshio **Nishina** died in 1950 or so, and was head of one of Japan's nuke programs during WWII. Oskar **Klein** died in 1977 and was a Swedish theoretical guy. They both worked with Bohr in Copenhagen. They came up with their K-N formula in 1929, but I imagine their path to this formula is a little different than ours. Again, remember that Dirac theory was 1928 which launched all this stuff.

Original paper:

http://books.google.com/books?id=yqp7tcnXEEwC&pg=PA113&lpg=PA113&dq=klein+nishina++%22on+the+scattering+of+radiation%22&source=bl&ots=oaYANVGb9R&sig=pSskB4XkoQJQ6_D2E4C6Z4vYkmA&hl=en&sa=X&oi=book_result&resnum=4&ct=result#PPA113,M1

It was not the routine calculation it is now! Nishina was assigned the problem as a kid, and then Klein helped him out, lots of history in the above book.

Doing the polarization sums and averaging to get equation F.

I did this in my Thomson document and copy it here now:

Assuming incoming along $\hat{\mathbf{z}}$ so we have these incoming polarizations

$$\hat{\boldsymbol{\varepsilon}}^{(1)} = \hat{\mathbf{x}}$$
$$\hat{\boldsymbol{\varepsilon}}^{(2)} = \hat{\mathbf{y}}$$

The outgoing photon has these possible states:

$\hat{\varepsilon}'^{(1)} = \hat{\varphi}$

$\hat{\varepsilon}'^{(2)} = \hat{\theta}$

Then I can put ½ *in front* of KN, then just *manually* sum on these polarizations:

$$\Sigma_{pol}\,(\varepsilon \bullet \varepsilon')^2 = (\hat{\varepsilon}^{(1)} \bullet \hat{\varepsilon}'^{(1)})^2 + (\hat{\varepsilon}^{(1)} \bullet \hat{\varepsilon}'^{(2)})^2 + (\hat{\varepsilon}^{(2)} \bullet \hat{\varepsilon}'^{(1)})^2 + (\hat{\varepsilon}^{(2)} \bullet \hat{\varepsilon}'^{(2)})^2$$

$$= (\hat{x} \bullet \hat{\varphi})^2 + (\hat{x} \bullet \hat{\theta})^2 + (\hat{y} \bullet \hat{\varphi})^2 + (\hat{y} \bullet \hat{\theta})^2$$

$$= \sin^2\varphi + \cos^2\theta\cos^2\varphi + \cos^2\varphi + \cos^2\theta \sin^2\varphi =$$

$$= 1 + \cos^2\theta$$

Of course constant terms in the NK formula will do this under the same summation

$\Sigma_{pol}\, a = 4a$

So, if the K-N formula were this:

$KN = K * (a + 4(\varepsilon \bullet \varepsilon')^2 - 2)$

Then our result would be

$KN = K/2 * (4a + 4 + 4\cos^2\theta - 8) = 2K * (a + 1 + \cos^2\theta - 2)$

$= 2K * (a + \cos^2\theta - 1) = 2K ( a - \sin^2\theta).$

This then explains how we get from BD 7.74 to BD F. We have a = k'/k + k/k' and we pick up the factor of 2 as just shown. This arises because we do ½ for averaging, but pick up 4 from dual summing the constant terms.

Doing the integrals in 7.75: Define the integral T to be

$\bar{\sigma} = \pi\alpha^2/m^2 * T$

and define a = m/k . Then Maple says:

$$T := a^3 \int_{-1}^{1} \frac{1}{(a+1-z)^3}\,dz + a \int_{-1}^{1} \frac{1}{a+1-z}\,dz - a^2 \int_{-1}^{1} \frac{1-z^2}{(a+1-z)^2}\,dz$$

$$T := 2\frac{a\,(a+1)}{(a+2)^2} + a\,(-\ln(a) + \ln(a+2)) - a^2\,(-4 - 2\ln(a)\,a - 2\ln(a) + 2\ln(a+2)\,a + 2\ln(a+2))$$

which I can write as

T = 2a (a+1)/ (a+2)² + a ln (1+2/a) - a²[ -4 - 2(a+1) ln(a) + 2(a+1)ln(a+2) ]

= 2a (a+1)/ (a+2)² + a ln (1+2/a) - a²[ -4 + 2(a+1) ln (1+2/a) ]

= 2a (a+1)/ (a+2)² + 4a² + [a - 2a²(a+1)] ln (1+2/a) // exact result

Now look at the **large-a limit.** This is a little tricky because leading terms cancel. Rewrite as

= 2 + order(1/a) + 4a² + [a - 2a²(a+1)] ln (1+2/a)

Expand the log out to several terms to get

= 2 + 4a² + [ -2a³ - 2a² + a ] ( 2/a - (2/a)²/2 + (2/a)³/3 + ...]

= 2 + 4a² + [ -2a³ - 2a² + a ] ( 2/a - 2/a² + 8/3 * 1/a³ + ...]

= 2 + 4a² + 2 [ -2a³ - 2a² + a ] ( 1/a - 1/a² + 4/3 * 1/a³ + ...]

= 2 + 4a² + 2 [ -2a² + 2a - 8/3 -2a + 2 + 1 ]

= 2 + 2 [ - 8/3 + 2 + 1 ] = 2 + 2 [ 1/3] = 2 + 2/3 = 8/3

So in this limit we get

$\bar{\sigma}$ = πα²/m² * 8/3 = 8π/3 α²/m²

Now look at the **small-a** limit: // a = m/k

T = 2a (a+1)/ (a+2)² + 4a² + [a - 2a²(a+1)] ln (1+2/a)

= a/2 + order(a²) + 4a² + a ln (1+2/a) = a/2 + order(a²) + 4a² + a ln[ (2/a) (1+a/2)]

$= a/2 + \text{order}(a^2) + 4a^2 + a \ln(2/a) + a \ln(1+a/2)]$

$\approx a/2 + \text{order}(a^2) + 4a^2 + a \ln(2/a) + a * a/2 + \text{order}(a^3)$

$= a[ 1/2 + \text{order}(a^2)/a + 4a + \ln(2/a) + a/2 + \text{order}(a^2) ]$

Now we need to expand:

$2a(a+1)/(a+2)^2 = ¼\ 2a\ (1+a)(1+a/2)^{-2} \approx ¼\ 2a\ (1+a)\ (1-2(a/2)) = ¼\ 2a\ (1+a)\ (1-a)$ so no $\text{order}(a^2)$

so now our result is

$T = a[ 1/2 + \ln(2/a) + \text{order}(a) + \text{order}(a^2) ]$

$T = a\ [1/2 + \ln(2/a) + \text{order}(a) ]$

which we can compare with their result

$T = a\ [ \ln(2/a) + ½ + \text{order}[a \ln(1/a) ]$ $\quad a = m/k$

So I agree with the first two terms, but I question the third order term they show. I think the third term should be order a without any logs.

**7.8 Pair annihilation into gamma rays (132) (Dirac 1930)**

The form given in 7.76 is correct by all the rules I understand so far. Our first big question here is this: "This $S_{fi}$ sure looks a lot like the Compton scattering graphs $S_{fi}$, exactly what is the relation?" The answer is very clear and I will detail it out.

Start with 7.10 graphs and, from them, come up with the 7.11 graphs:

(1) identify the outgoing k',ε' with $k_2,\varepsilon_2$. This will make the left side of 7.10 become the left side of 7.11 when we are done. This is the photon connected to the Final electron leg -- the one at the end of the electron flow arrow -- in the left figures.

(2) Similarly, identify $p_i,s_i$ with $p_-,s_-$ to make the above work.

(2) swing the incoming photon in 7.10 so it is outgoing, so then $k_1 = -k$ and $\varepsilon_1 = \varepsilon$.

(3) swing the outgoing electron in 7.10 around so it becomes incoming positron. This means $p_+ = -p_f$ but the spin stays the same so $s_+ = s_f$. So the $u(p_f, s_f)$ becomes $u(-p_+, s_+) = v(p_+, s_+)$. Recall the careful way that v(p,s) was defined with respect to the spin projection operator [ see "(2) How is the spin operator defined and why? " in my section 3.2 notes above. ]

I am quite happy with all this as expressed in 7.77, as I am with their more general notion in terms of the two equations A and B which show processes related by doing leg swing-arounds to get anti-particles. Since photon is its own antiparticle, we don't see a u→v thing going on with the photon. The ε just stays the same. They keep calling this notion a "substitution rule".

Derive cross section 7.78: I have pencil-checked off the parts that are obviously correct when we compare this to 7.68 + 7.72 for Compton. The 2π's are obvious, the $e^4$ obvious, the $\delta^4$ and phase space obvious (substitutions as noted above inside $\delta^4$). The Dirac equation parts of the Trace are again obvious from the sub rules. But now we have to ponder the other factors a bit.

The electron and positron leg factors are $\sqrt{m/EV}$ . We are in the electron lab frame, so we have

$$\sqrt{m/E_-V}\,\sqrt{m/E_+V} = \sqrt{1/V}\,\sqrt{m/E_+V} \quad \text{squared} \Rightarrow \quad V^{-2}\, m/E_+$$

The photon leg factors give

$$\sqrt{1/2k_+V}\,\sqrt{1/2k_-V} \quad \text{squared} \Rightarrow \quad V^{-2}\ (1/2k_+)\ (1/2k_-)$$

Each final particle phase space is $V\, d^3k/(2\pi)^3$ and we have already commented on the 2π factors. So these two phase space factors cancel the V's and we can move the $(1/2k_+)$ under respective $d^3k$ and then they are taken care of. The squared $S_{fi}$ picks up a V which cancels that from the flux, and all that is left is to divide by $v_{rel}$ which is $\beta_+$. Thus, we get the $m/(E_+\beta_+)$ leading factor shown and the invariant phase spaces as shown.

But we are not done yet! The ¼ comes from doing two initial Dirac particle spin averages. The (-1) comes from the fact shown page 107 E that $\Sigma v\bar{v} = -\Lambda^-$. The sign is called "potentially treacherous" because if you forget it, you will get a negative cross section!

The only things that remain are the paren quantities. Here is a sample:

$\not\varepsilon'\not\varepsilon\not k \rightarrow \not\varepsilon_2\not\varepsilon_1(-\not k_1) = \not\varepsilon_2\not k_1\not\varepsilon_1$ but $k.p_i \rightarrow -k_1.p_-$

$\not\varepsilon\not\varepsilon'\not k' \rightarrow \not\varepsilon_1\not\varepsilon_2(+\not k_2) = -\not\varepsilon_1\not k_2\not\varepsilon_2$ but $k',p_i \rightarrow k_2.p_-$

Thus, the first "paren factor" in 7.72 maps into minus the first "paren factor" in 7.78. Now consider the second paren factor:

$\not k\not\varepsilon\not\varepsilon' \rightarrow (-\not k_1)\,\not\varepsilon_1\,\not\varepsilon_2 = +\,\not\varepsilon_1\not k_1\,\not\varepsilon_2$ but $k.p_i \rightarrow -k_1.p_-$

$\not k'\not\varepsilon'\not\varepsilon \rightarrow \not k_2\not\varepsilon_2\not\varepsilon_1 = -\not\varepsilon_2\not k_2\not\varepsilon_1$ but $k',p_i \rightarrow k_2.p_-$

and so the second paren factor maps into minus the second paren factor, and the two minus signs cancel. The authors have chosen to put the $\not k$ factor in the middle of the $\not\varepsilon$ sandwiches so the two terms in the paren factors have a relative + sign.

So I am now happy with every single symbol in the very complicated 7.78.

**Traces**: We are not really going to use the explicit physics package as written in 7.78. We will instead take the sum of the four traces we got in the Compton problem and just make our 7.77 substitutions in the

result! Remember that the physics package for pair annihilation is the same as that for Compton scattering with these subs, and this fact goes through into the traces.

Recall from above that the sum of our traces was this:

$$\text{Tr}(....) = \text{"all terms"}/(2m^2) = \{(k'/k) + (k/k') + [\,4(\varepsilon.\varepsilon')^2 - 2]\,\}/(2m^2)$$

so we can translate this into

$$\text{Tr}(....) = \text{"all terms"}/(2m^2) = \{-(k_2/k_1) - (k_1/k_2) + [\,4(\varepsilon_1.\varepsilon_2)^2 - 2]\,\}/(2m^2)$$

and we see the {...} factor appearing in 7.80.

The next step is processing the phase space. We will use up the $d^3k_2$ against the delta function and use our usual 7.40 idea p 112. The single delta needs analysis:

$$\delta(k_2.k_2 - m_2^2) = \delta(k_2.k_2) = \delta([\,p_+ + p_- - k_1]^2) = \delta((p_+ + p_-)^2 - 2\,k_1.(p_+ + p_-))$$

and similarly for the $\theta(k_2^0)$. Recall that in our lab frame, $E_- = m$. Now some algebra

$$(p_+ + p_-)^2 - 2\,k_1.(p_+ + p_-) = 2m^2 + 2(E_+m) - 2k_1(E_+ + m) + 2\mathbf{k_1}\bullet\mathbf{p_+}$$

$$= \text{stuff} + 2k_1p_+\cos\theta \qquad \theta = \text{positron angle to photon1 angle.}$$

To extract $\delta(k_1 - \text{stuff})$ we need the derivative $d(\text{guts})/dk_1 = 2(m + E_+ - p_+\cos\theta)$ so this derivative then ends up in the bottom, the $\delta$ kills off the integral and $k_1$ gets pinned to the following:

$$2m^2 + 2(E_+m) = 2k_1[E_+ + m - p_+\cos\theta\,]$$

$$k_1 = \{2m^2 + 2(E_+m)\}/\{\,2[E_+ + m - p_+\cos\theta\,]\} = m(m+E_+)/(m + E_+ - p_+\cos\theta)$$

so we then get

$$\{2m^2 + 2(E_+m)\}/\{\,2[E_+ + m - p_+\cos\theta\,]\}^2 = \tfrac{1}{2}(m^2 + E_+m)/[E_+ + m - p_+\cos\theta\,]^2$$

This gives a ½ which combines with the original ½ to make ¼. And so we have the final form shown in 7.79.

Now a final assembly of factors going back to 7.78:

$$e^4/(2\pi)^2\, m/(E_+\beta_+)(-1/4)\,(1/4)(m^2 + E_+m)/[E_+ + m - p_+\cos\theta\,]^2$$
$$\{-(k_2/k_1) - (k_1/k_2) + [\,4(\varepsilon_1.\varepsilon_2)^2 - 2]\}/(2m^2)$$

$$e^4/(4\pi)^2\, m^2/(E_+\beta_+)(-1/4)\,(m + E_+)/[E_+ + m - p_+\cos\theta\,]^2\,\{-(k_2/k_1) - (k_1/k_2) + [\,4(\varepsilon_1.\varepsilon_2)^2 - 2]\}/(2m^2)$$

$$-\alpha^2(1/8\,p_+)(m + E_+)/[E_+ + m - p_+\cos\theta\,]^2\,\{-(k_2/k_1) - (k_1/k_2) + [\,4(\varepsilon_1.\varepsilon_2)^2 - 2]\}$$

and this agrees with the first line of 7.80. The second line is then trivial once you compute $k_2$ as on top page 135, which is trivial algebra. The expression for $k_1$ we have already stated above.

Now let's get this ½ identical particles thing done. But suppose we had a process of 3 outgoing identical photons instead. We treat them as distinct and we get $d\sigma/d\Omega_1\, d\Omega_2\, d\Omega_3$ We integrate over 2 and 3 and we get some resulting $d\sigma/d\Omega_1$ that we predict our detector will measure. If the photons were really "marked", we could make our detector ignore its capture of 2 and 3 marked photons and the formula would be correct as stated. But since unmarked, our detector picks up the 1 and 2 and 3 photons and our reading is 3x too large, so we would correct with a 1/3 factor. My point is that this is not a 1/3! factor ! So I think I would argue that 7.80 -- the differential cross section $d\sigma/d\Omega_1$ -- should also have the factor ½ that they put in 7.81 for the total cross section.

What about the polarization sum? I computed this by hand for the Compton case with one ingoing and one outgoing and I got $\Sigma = 1 + \cos^2\theta$ where $\theta$ was the angle between the photon directions, one incoming and one outgoing. Here we have two outgoing and for very small $p_+$, they will be travelling in opposite directions in the lab frame, so $\theta = \pi$ and then $\Sigma = 2$. Since two outgoing, the average is then going to be 2/4 = ½.

Now let's take 7.80 to the non-rel limit. $E_+ \rightarrow m$ etc and we get:

$$d\sigma/d\Omega = -\alpha^2/(16m^2\beta+) [-4 + 4(\varepsilon_1.\varepsilon_2)^2] = \alpha^2/(4m^2\beta_+) [1 - (\varepsilon_1.\varepsilon_2)^2]$$

since $k_1 = k_2 = m$ in this limit. This is the cross section into one of 4 possible final polarization combinations. Suppose the average value of $(\varepsilon_1.\varepsilon_2)^2$ over these 4 states were ½. Then

$$¼ \Sigma_{4\ states} (\varepsilon_1.\varepsilon_2)^2 = ½ \Rightarrow \quad \Sigma_{4\ states} (\varepsilon_1.\varepsilon_2)^2 = 2.$$

and

$$\Sigma_{4\ states} (1) = 4$$

In this case, when we sum over the four states, we should get

$$\Sigma_4\, d\sigma/d\Omega = \alpha^2/(4m^2\beta_+)* [4 - 2] = \alpha^2/(2m^2\beta_+)$$

Now add the Bose ½ shown in 7.81,and add $4\pi$ for integrating over solid angle, and we get

$$\sigma = \alpha^2/(2m^2\beta_+)* ½ * 4\pi = \alpha^2\pi/(m^2\beta_+)$$ agrees with 7.81

What about that correction term they show $O(\beta^2)$? Looking at 7.80, if we ignore the $1/\beta_+$ overall factor, things are a function of $E_+$ (which is a function of $p^2$), and a function of $p\cos\theta = pz$. Thus, each of the three integrals in 7.80 has the form

$$\int_{-1}^{1} dz\, f(pz; p^2) = (1/p) \int_{-p}^{p} dx\, f(x; p^2) = (1/p)\ F(x; p^2)|^{p}_{-p} = (1/p) \{ F(p; p^2) - F(-p; p^2) \} = g(p^2)$$

which the product of two functions odd in p, and so is a function even in p which we write as $g(p^2)$. The low frequency limit of our exact result will then look like this

$$d\sigma/d\Omega = 1/\beta_+ * g(p^2) \approx 1/\beta_+ [\, g(0) + p^2 g'(p^2) \,]$$

and this is why the correction term is of order $\beta_+^2$ and not $\beta_+$

The footnote says this is a bad approximation for the NR limit and I think the reason is that the two particles form S ground state positronium and then annihilate so our plane wave approximation is particularly bad. Better to put them both into an S orbital wavefunction somehow, and I guess that is where the "Coulomb wave functions" come into play. Sakurai discusses positronium decaying from the S state, but does not mention these wavefunctions.

**Coulomb Wave Functions.** Look at the central potential radial equation Saxon page 276 where things are converted to R = u/r. Change sign and it says:

$$(\hbar^2/2m)\, u_{rr} + [\, E - V(r) - (\hbar^2/2m)\ell(\ell+1)/r^2 \,]\, u = 0$$

Mult through to get

$$u_{rr} + [\, (2m/\hbar^2)(E-V) - \ell(\ell+1)/r^2 \,]\; u = 0$$

or

$$u_{rr} + [\, c(E-V) - \ell(\ell+1)/r^2 \,]\; u = 0 \qquad\qquad c = 2m/\hbar^2$$

Now define w(br) = u(r). Let ρ = br. Then

$$w_\rho(\rho) = \partial w/\partial\rho = (1/b)\partial w/\partial r = (1/b)\partial/\partial r\, [\, u(r)] = (1/b)u_r$$

$$w_{\rho\rho} = (1/b)\partial/\partial r\, \{\; (1/b)u_r \;\} = (1/b^2)\, u_{rr}$$

Then the above becomes

$$b^2\, w_{\rho\rho} + [\, c(E-V) - \ell(\ell+1)/r^2 \,]\, w = 0$$

or

$$w_{\rho\rho} + [\, c(E-V)/b^2 - \ell(\ell+1)/b^2 r^2 \,]\; w = 0$$

or

$$w_{\rho\rho} + [\, c(E-V)/b^2 - \ell(\ell+1)/\rho^2 \,]\; w = 0$$

Select b so that $cE/b^2 = 1$, so this means

$$b^2 = cE = 2mE/\hbar^2 \qquad = M\, E/L^2 = M\; (P^2/2M) / (RP)^2 = 1/R^2 \;\text{ so b = 1/R in dimension}$$

so ρ = br = dimensionless. Then we have

$$w_{\rho\rho} + [\; 1 - (V)/b^2 - \ell(\ell+1)/\rho^2 \,]\; w = 0$$

Now set V = K/r for a Coulomb potential, and define $2\eta = K/b$ so that $-V/b^2 = -(K/b)/(rb) = -2\eta/\rho$

$$w_{\rho\rho} + [\; 1 - 2\eta/\rho - \ell(\ell+1)/\rho^2 \,]\; w = 0$$

Finally we have the differential equation shown on page 538 A&S whose solutions are the Coulomb Wave Functions. So the SE solution is then

$$\psi_{\ell E} = R_{E\ell}(r)\, Y_{\ell m}(\theta,\varphi) = u_{E\ell}(r)/r * Y_{\ell m}(\theta,\varphi) = w_{\eta(E)\ell}(b_E r)/r * Y_{\ell m}(\theta,\varphi)$$

where $b_E = \sqrt{2mE}/\hbar$ and $\eta = K/(2b) = K\hbar/(2\sqrt{2mE})$.

So, when people speak of Coulomb Wave Functions, they are speaking of the dimensionless form of the hydrogen-like atom radial equation solution. Probably Sakurai will comment on spin selection rules for this to happen from positronium.

I shall not bother to extract the ER limit of 7.80 since I know how to do it and the result is of no interest to me right now. On the day it becomes of interest, I can start with 7.80 and verify it.

**7.9 Electron-electron and Electron-positron Scattering.**

**A. Electron-electron scattering (Møller 1932)**

We now see what e-p scattering was done first. Here we have the complication of the second diagram in order to generate an antisymmetric amplitude for the final electrons (and also for the initial ones). The first term has -i on the propagator and that explains all signs, i's, and other factors. Then the second term just be added with a Fermi minus sign. So 7.82 is fine by me. You can also look back at the e-p case and see it is the same as 7.35 on page 110.

So page 136 is all OK except I don't buy the usefulness of the last comment. Seems to me that both terms contribute to the forward direction since you don't know whether 1 or 2 is the forward particle.

Page 137: Now of course we have the issue of cross terms when we square $S_{fi}$, and we have seen this before in our other 2-graph applications. So time to invent my own compact notation. First, replace 1' = 3 and 2' = 4. Then we can write the physics package like so [ and recall $\delta(3+4-1-2)$ means 1+2 = 3+4 ]

$$S_{fi}(\text{physics package}) = \bar{3}\mu 1\, \bar{4}\mu 2/(1-3)^2 - \bar{3}\mu 2\, \bar{4}\mu 1/(1-4)^2 = f(1,2,3,4)$$

If we swap 3 and 4, the minus sign is very obvious. If we swap 1 and 2, then $(1-3)^2 \rightarrow (2-3)^2 = (1-4)^2$ and again the minus sign is obvious. Now make up names for the photon propagators like $P_{13}$ and we have

$$S_{fi}(\text{physics package}) = P_{13}\, \bar{3}\mu 1\, \bar{4}\mu 2 - P_{14}\, \bar{3}\mu 2\, \bar{4}\mu 1$$

Now we are ready to square this thing. We know from way back on page 103 that this is true

$$|\bar{3}\mu 1|^2 = \bar{3}\mu 1\, \bar{1}\mu 3 = a^2\, \mathrm{Tr}[(3+m)\mu(1+m)\mu] \qquad \text{where } a = 1/(2m)$$

where I have installed the projectors in long hand. So, here we go:

$|S_{fi}(\text{physics package})|^2 = |P_{13}\,\bar{3}\mu1\,\bar{4}\mu2 - P_{14}\,\bar{3}\mu2\,\bar{4}\mu1|^2$

$= (P_{13}\,\bar{3}\mu1\,\bar{4}\mu2 - P_{14}\,\bar{3}\mu2\,\bar{4}\mu1)(\,P_{13}\,\bar{3}\nu1\,\bar{4}\nu2 - P_{14}\,\bar{3}\nu2\,\bar{4}\nu1)^*$

$= (P_{13}\,\bar{3}\mu1\,\bar{4}\mu2 - P_{14}\,\bar{3}\mu2\,\bar{4}\mu1)(\,P_{13}\,\bar{1}\nu3\,\bar{2}\nu4 - P_{14}\,\bar{2}\nu3\,\bar{1}\nu4)$ = four terms

$T_1 = P_{13}^2\;\bar{3}\mu1\,\bar{4}\mu2\;\bar{1}\nu3\,\bar{2}\nu4 = P_{13}^2\;\bar{3}\mu1\,\bar{1}\nu3\;\bar{4}\mu2\,\bar{2}\nu4$

$= P_{13}^2\,a^4\;\text{Tr}[\,(3+m)\mu(1+m)\nu]\;\;\text{Tr}[\,(4+m)\mu(2+m)\nu]$ // agrees with 7.83 {...}

$\equiv T_1(1,2,3,4)$

$T_4 = P_{14}^2\;\bar{3}\mu2\,\bar{4}\mu1\;\bar{2}\nu3\,\bar{1}\nu4 = P_{14}^2\;\bar{3}\mu2\,\bar{2}\nu3\;\bar{4}\mu1\,\bar{1}\nu4$

$= P_{14}^2\,a^4\;\text{Tr}[\,(3+m)\mu(2+m)\nu]\;\;\text{Tr}[\,(4+m)\mu(1+m)\nu]$

$= T_1(2,1,3,4)$ // swap 1 and 2 in $T_1$

$-T_3 = P_{13}\,\bar{3}\mu1\,\bar{4}\mu2\,P_{14}\;\bar{2}\nu3\,\bar{1}\nu4 = P_{13}\,P_{14}\,\bar{3}\mu1\,\bar{1}\nu4\,\bar{4}\mu2\,\bar{2}\nu3$

$= P_{13}\,P_{14}\,a^4\;\text{Tr}[(3+m)\mu(1+m)\nu(4+m)\mu(2+m)\nu\,]$ // agrees with 7.83 {...}

$\equiv -\,T_3(1,2,3,4)$

$-T_4 = P_{14}\,\bar{3}\mu2\,\bar{4}\mu1\,P_{13}\;\bar{1}\nu3\,\bar{2}\nu4 = P_{13}\,P_{14}\;\bar{3}\mu2\;\bar{2}\nu4\;\bar{4}\mu1\,\bar{1}\nu3$

$= P_{13}\,P_{14}\,a^4\;\text{Tr}[(3+m)\mu(2+m)\nu(4+m)\mu(1+m)\nu\,]$

$\equiv -\,T_3(2,1,3,4)$

So we can summarize as follows:

$|S_{fi}(\text{physics package})|^2 \equiv |pp|^2 = \{\,T_1(1,2,3,4) + T_3(1,2,3,4)\} + (1\leftrightarrow2)$

where

$T_1 = P_{13}^2\,a^4\;\text{Tr}[\,(3+m)\mu(1+m)\nu]\;\;\text{Tr}[\,(4+m)\mu(2+m)\nu]$ $\qquad P_{nk} = 1/(n-k)^2$

$T_3 = -\,P_{13}\,P_{14}\,a^4\;\text{Tr}[(3+m)\mu(1+m)\nu(4+m)\mu(2+m)\nu\,]$ $\qquad a = 1/(2m)$

The graphical aids at the bottom of page 137 when drawn for my ordering should have all the arrows reversed and 3 and 4 installed. You are supposed to add momenta at vertices to get the right photon propagators, and each electron line just gets an energy projector, and yes, this can be thought of as a full propagator without the squared denominator, and that is what the circles are for. I think my notation shown above is better.

Now let's check the kinematics package in 7.83 and make sure all factors are correct. I will just assemble my own factors. In 7.82 we can factor out an overall $(-i)^3$ and then just ignore it since we do abs squared on $S_{fi}$.

$\text{prob} = \tfrac{1}{4}\,e^4\,m^4\,V^{-4}\,1/(E_1E_2E_3E_4)\;(2\pi)^4\,VT\,\delta^4(3+4-1-2)\,Vd^3p_3/(2\pi)^3\,Vd^3p_4/(2\pi)^3\,V/v_{rel}\,|pp|^2$

where pp means the physics package shown above. The ¼ is from spin averaging on both inbounds. The π's work as usual, so one more time:

$$d\sigma = \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(E_1E_2E_3E_4)\,\delta^4(3+4-1-2)\,d^3p_3\,d^3p_4\,/v_{rel}\,|pp|^2$$

Now we go the CM frame and I first wrote all this out:

| | |
|---|---|
| $p_1 = (E_1, \mathbf{p_1})$ | $1.2 = E_1^2 + \mathbf{p_1}^2$ |
| $p_2 = (E_1, -\mathbf{p_1})$ | $1.3 = E_1E_3 - \mathbf{p_1}\bullet\mathbf{p_3}$ |
| $p_3 = (E_3, \mathbf{p_3})$ | $1.4 = E_1E_3 + \mathbf{p_1}\bullet\mathbf{p_3}$ |
| $p_4 = (E_3, -\mathbf{p_3})$ | $2.3 = E_1E_3 + \mathbf{p_1}\bullet\mathbf{p_3} = 1.4$ |
| | $2.4 = E_1E_3 - \mathbf{p_1}\bullet\mathbf{p_3} = 1.3$ |
| | $3.4 = E_3^2 + \mathbf{p_3}^2$ |

but energy conservation from the $\delta^4$ tells us that $|\mathbf{p_1}| = |\mathbf{p_3}|$ (3-vectors) and $E_1 = E_3$. So have

| | | |
|---|---|---|
| $p_1 = (E, \mathbf{p_1})$ | $1.2 = E^2 + p^2$ | |
| $p_2 = (E, -\mathbf{p_1})$ | $1.3 = E^2 - p^2\cos\theta$ | |
| $p_3 = (E, \mathbf{p_3})$ | $1.4 = E^2 + p^2\cos\theta$ | |
| $p_4 = (E, -\mathbf{p_3})$ | $2.3 = E^2 + p^2\cos\theta$ | $= 1.4$ |
| | $2.4 = E^2 - p^2\cos\theta$ | $= 1.3$ |
| | $3.4 = E^2 + p^2$ | $= 1.2$ |

which says all dot products are equal to "the other pair", so we could take 1.2, 1.3, 2.3 as our basic ones

We know that $v_{rel} = 2\beta$ and here is what the above looks like in the CMS frame:

$$d\sigma = \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,\delta^4(3+4-1-2)\,d^3p_3\,d^3p_4\;|pp|^2$$

and this **agrees now with 7.83**, and we have

$$|pp|^2 = \{\,T_1(1,2,3,4) + T_3(1,2,3,4)\} + (1\leftrightarrow 2)$$

$$T_1 = P_{13}^2\,a^4\,Tr[\,(3+m)\mu(1+m)\nu]\;\;Tr[\,(4+m)\mu(2+m)\nu] \qquad P_{nk} = 1/(n-k)^2$$

$$T_3 = -P_{13}\,P_{14}\,a^4\,Tr[(3+m)\mu(1+m)\nu(4+m)\mu(2+m)\nu\,] \qquad a = 1/(2m)$$

$$d\sigma = \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,\delta^4(3+4-1-2)\,d^3p_3\,d^3p_4\;|pp|^2$$

If you compute these trances and install them, you get the **Møller Cross Section**!
(the Fermi ½ is not added yet in the above) . I have evaluated the smaller traces
below in full, but did not evaluate the large trace (but I could if I wanted to).

How hard are these traces to do I wonder:

$$\mathrm{Tr}[\,(3+m)\mu(1+m)\nu\,] = \mathrm{Tr}[\,3\mu 1\nu\,] + m^2\,\mathrm{Tr}[\mu\nu] = 3_\alpha 1_\beta\,\mathrm{Tr}[\,\alpha\mu\beta\nu\,] + m^2\,\mathrm{Tr}[\mu\nu]$$

$$= 3_\alpha 1_\beta\, 4\,\{\, g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\,\} + m^2\, 4\{g^{\mu\nu}\}$$

$$= 4\,\{\, 3^\mu 1^\nu - 1.3\, g^{\mu\nu} + 3^\nu 1^\mu + m^2\, g^{\mu\nu}\,\} = 4\,\{\, 3^\mu 1^\nu + 3^\nu 1^\mu + g^{\mu\nu}(m^2 - 1.3)\,\}$$

Then just change 3→4 and 1 → 2 to get the other trace of this size:

$$\mathrm{Tr}[\,(4+m)\mu(2+m)\nu] = 4\,\{\, 4^\mu 2^\nu + 4^\nu 2^\mu + g^{\mu\nu}(m^2 - 2.4)\,\}$$

Then the product of these two traces is

$$\mathrm{Tr}\,\mathrm{Tr} = 16\,\{\, 3^\mu 1^\nu + 3^\nu 1^\mu + g^{\mu\nu}(m^2 - 1.3)\,\}\,\{\, 4^\mu 2^\nu + 4^\nu 2^\mu + g^{\mu\nu}(m^2 - 2.4)\,\}$$

$$= 16\,\{\, 3.4\ 1.2 + 3.2\ 1.4 + 3.1\ (m^2 - 2.4) + 3.2\ 1.4 + 3.4\ 1.2 + 3.1\ (m^2 - 2.4)$$
$$2^*\ 4.2\ (m^2 - 1.3) + 4(m^2 - 1.3)\ (m^2 - 2.4)\,\}$$

$$= 16\,\{1.2\ 1.2 + 3.2\ 2.3 + 3.1\ (m^2 - 1.3) + 3.2\ 2.3 + 1.2\ 1.2 + 3.1\ (m^2 - 1.3)$$
$$2^*\ 1.3\ (m^2 - 1.3) + 4(m^2 - 1.3)\ (m^2 - 1.3)\,\}$$

where only in the last line have we assumed CMS. So we then have

$$= 16\,\{2\ (1.2)^2 + 2\ (2.3)^2 + 4(3.1)\ (m^2 - 1.3) + 4(m^2 - 1.3)\ (m^2 - 1.3)\,\}$$

$$= 32\ \{(1.2)^2 + (2.3)^2 + 2\ (3.1)\ (m^2 - 1.3) + 2(m^2 - 1.3)\ (m^2 - 1.3)\,\}$$

which is not TOO bad. So this means we just got

$$T_1 = P_{13}^2\, a^4\, \mathrm{Tr}[\,(3+m)\mu(1+m)\nu]\ \ \mathrm{Tr}[\,(4+m)\mu(2+m)\nu]$$

$$= 32\, P_{13}^2\, a^4\,\{(1.2)^2 + (2.3)^2 + 2\ (3.1)\ (m^2 - 1.3) + 2(m^2 - 1.3)\ (m^2 - 1.3)\,\}$$

So I have a complete evaluation of the $T_1$ term above.

Now dare I step off the curb and attempt the trace for the $T_3$ term?

$$\mathrm{Tr}[(3+m)\mu(1+m)\nu(4+m)\mu(2+m)\nu\,] = \mathrm{Tr}[3\mu 1\nu 4\mu 2\nu]$$

$$+ m^2\{\,\mathrm{Tr}[\mu\nu 4\mu 2\nu] + 5 \text{ similar } m^2 \text{ terms}$$

$$+ m^4\,\mathrm{Tr}[\,\mu\nu\mu\nu\,]$$

I could do it all, but don't see the point of doing this all today. It is just turning the crank. And I don't have the final answer so I can't verify that I did it right.

What about the **"high energy limit"** ? I guess we could set m=0 and get:

$T_1 = P_{13}^2 a^4 \, Tr[(3+m)\mu(1+m)\nu] \; Tr[(4+m)\mu(2+m)\nu]$

$= 32 P_{13}^2 a^4 \{(1.2)^2 + (2.3)^2 + 2(3.1)(m^2 - 1.3) + 2(m^2 - 1.3)(m^2 - 1.3)\}$

$= 32 P_{13}^2 a^4 \{(1.2)^2 + (2.3)^2 + 2(3.1)(-1.3) + 2(-1.3)(-1.3)\}$

$= 32 P_{13}^2 a^4 \{(1.2)^2 + (2.3)^2\}$

Then in the large trace we keep only the leading term so we have:

$T_3 = -P_{13} P_{14} a^4 \, Tr[(3+m)\mu(1+m)\nu(4+m)\mu(2+m)\nu] = -P_{13} P_{14} a^4 \, Tr[3\mu 1\nu 4\mu 2\nu]$

BD show a limit of this messy trace, how are they doing it? Consider this guts section

$\nu 4\mu 2\nu = \gamma_\nu 4\not{a}2\gamma^\nu$ where $a_\alpha = \delta_\alpha^\mu$

Page 105 Theorem 5 says this

$\nu 4\mu 2\nu = \gamma_\nu 4\not{a}2\gamma^\nu = 2\not{a}4 = -2 * 2\mu 4$

So we then have

$Tr[3\mu 1\nu 4\mu 2\nu] = -2 \, Tr[3\mu 12\mu 4]$ // agrees page 138 top

Now we can take some guts and do this again:

$\mu 12\mu = 4 * 1.2$

so we then have

$Tr[3\mu 1\nu 4\mu 2\nu] = -2 \, Tr[3\mu 12\mu 4] = -8 \, 1.2 \, Tr[34] = -32 \, 1.2 \, 3.4$ // all this agrees!

So we have found that

$T_3 = -P_{13} P_{14} a^4 \, Tr[3\mu 1\nu 4\mu 2\nu] = +32 P_{13} P_{14} a^4 \, 1.2 \, 3.4 = +32 P_{13} P_{14} a^4 (1.2)^2$

$T_1 = 32 P_{13}^2 a^4 \{(1.2)^2 + (2.3)^2\} = 32 P_{13}^2 a^4 \{(1.2)^2 + (2.3)^2\}$

But now recall that

$|S_{fi}(\text{physics package})|^2 = \{T_1(1,2,3,4) + T_3(1,2,3,4)\} + (1 \leftrightarrow 2)$ $\quad 1 + 2 = 3 + 4$

so let's now add in those $(1 \leftrightarrow 2)$ terms. Just looking at $T_3$ shows that $(1 \leftrightarrow 2)$ gives the same term.

$T_3 + (1\leftrightarrow 2) = 2T_3 = +64\, P_{13}\, P_{14}\, a^4\, (1.2)^2 = 64\, P_{13}\, P_{23}\, a^4\, (1.2)^2$

But not also for $T_1$ and in fact we have

$T_1 + (1\leftrightarrow 2) = 32\, P_{13}^2\, a^4\, \{(1.2)^2 + (2.3)^2\} + 32\, P_{23}^2\, a^4\, \{(1.2)^2 + (1.3)^2\}$

The propagators simplify as follows:

$P_{13} = 1/(1-3)^2$ but $(1-3)^2 = m^2 + m^2 - 2*1.3 = -2*1.3$ etc

$\Rightarrow P_{13} = -1/[2(1.3)]$ $P_{23} = -1/[2(2.3)]$

So we then have

$T_3 + (1\leftrightarrow 2) = 64\, P_{13}\, P_{23}\, a^4\, (1.2)^2 = 64\, a^4\, ¼\, (1.2)^2/[\,1.3\; 2.3\,] = 16\, a^4 (1.2)^2/[\,1.3\; 2.3\,]$

$T_1 + (1\leftrightarrow 2) = 32\, P_{13}^2\, a^4\, \{(1.2)^2 + (2.3)^2\} + 32\, P_{23}^2\, a^4\, \{(1.2)^2 + (1.3)^2\}$

$= 32\, ¼\, a^4\, \{(1.2)^2 + (2.3)^2\}/(1.3)^2 + 32\, ¼\, a^4\, \{(1.2)^2 + (1.3)^2\}/(2.3)^2$

$= 8\, a^4\, [\{(1.2)^2 + (2.3)^2\}/(1.3)^2 + \{(1.2)^2 + (1.3)^2\}/(2.3)^2\,]$

Now looking above in this limit that E = p, we have (agreeing with page 138)

$1.2 = 2E^2$
$1.3 = E^2(1-\cos\theta) = 2E^2\sin^2(\theta/2)$
$2.3 = E^2(1+\cos\theta) = 2E^2\cos^2(\theta/2)$

$T_3 + (1\leftrightarrow 2) = 16\, a^4 (1.2)^2/[\,1.3\; 2.3\,]$
$T_1 + (1\leftrightarrow 2) = 8\, a^4\, [\,\{(1.2)^2 + (2.3)^2\}/(1.3)^2 + \{(1.2)^2 + (1.3)^2\}/(2.3)^2\,]$

$|S_{fi}(\text{physics package})|^2 = T_3 + (1\leftrightarrow 2) + T_1 + (1\leftrightarrow 2)$

This looks pretty ugly to me.

$\{(1.2)^2 + (2.3)^2\}/(1.3)^2 = \{(2E^2)^2 + (2E^2\cos^2(\theta/2))^2\}/(2E^2\sin^2(\theta/2))^2$

$= \{\,1 + \cos^4(\theta/2)\,\}/\sin^4(\theta/2)$

$\{(1.2)^2 + (1.3)^2\}/(2.3)^2 = \{(2E^2)^2 + (2E^2\sin^2(\theta/2))^2\}/(2E^2\cos^2(\theta/2))^2$

$= \{\,1 + \sin^4(\theta/2)\,\}/\cos^4(\theta/2)$

$(1.2)^2/ [\ 1.3\ 2.3\ ] \quad = (2E^2)^2/ [2E^2\sin^2(\theta/2)\ 2E^2\cos^2(\theta/2)\ ] \ = 1/ [\sin^2(\theta/2)\ \cos^2(\theta/2)\ ]$

So we now have:

$T_3 + (1\leftrightarrow 2) \ = 16\ a^4(1.2)^2/ [\ 1.3\ 2.3\ ] \ = 16\ a^4 / [\sin^2(\theta/2)\ \cos^2(\theta/2)\ ]$

$T_1 + (1\leftrightarrow 2) \ = 8\ a^4\ [\ \{\ 1 + \cos^4(\theta/2)\ \} / \sin^4(\theta/2) \ + \{\ 1 + \sin^4(\theta/2)\ \} / \cos^4(\theta/2)\ ]$

So I will now assemble all these physics package pieces

$|S_{fi}(\text{physics package})|^2 \ = T_3 + (1\leftrightarrow 2) + T_1 + (1\leftrightarrow 2)$

$= \ 16\ a^4 \ \ 1/ [\sin^2(\theta/2)\ \cos^2(\theta/2)\ ]$
$\quad + 8a^4\ [\ \{\ 1 + \cos^4(\theta/2)\ \} / \sin^4(\theta/2) \ + \{\ 1 + \sin^4(\theta/2)\ \} / \cos^4(\theta/2)\ ]$

$= \ 8a^4 \ \ 2/ [\sin^2(\theta/2)\ \cos^2(\theta/2)\ ]$
$\quad + 8a^4\ [\ \{\ 1 + \cos^4(\theta/2)\ \} / \sin^4(\theta/2) \ + \{\ 1 + \sin^4(\theta/2)\ \} / \cos^4(\theta/2)\ ]$

$= 8a^4 \ * f(\theta/2)$

where

$f(\theta/2) = 2/ [S^2\ C^2] \ + [\ \{\ 1 + C^4\ \} / S^4 \ + \{\ 1 + S^4\ \} / C^4\ ]$

and this agrees exactly with the square bracket in 7.84. Then we have

$d\sigma = \ ¼\ (2\pi)^{-2}e^4m^4/(2\beta E^4)\ \delta^4(3+4-1-2)\ d^3p_3\ d^3p_4 \ |pp|^2$

$= \ ¼\ (2\pi)^{-2}e^4m^4/(2\beta E^4)\ \delta^4(3+4-1-2)\ d^3p_3\ d^3p_4\ f(\theta/2)\ 8a^4$

$= ¼\ (2\pi)^{-2}e^4m^4/(2\beta E^4)\ \delta^1(3+4-1-2)\ f(\theta/2)\ p_3^2\ dp_3\ d\Omega_3 \ \ 8a^4$

$= ¼\ (2\pi)^{-2}e^4m^4/(2\beta E^4)\ \delta^1(3+4-1-2)\ f(\theta/2)\ E_3^2\ dE_3\ d\Omega_3\ 8a^4$

$= ¼\ (2\pi)^{-2}e^4m^4/(2\beta E^4)\ f(\theta/2)\ E^2 d\Omega_3 \ \ 8a^4$

$= ¼\ (2\pi)^{-2}e^4m^4/(2\beta E^2)\ f(\theta/2) d\Omega_3\ 8a^4 \qquad\qquad e^4/4\pi^2 \ = 4\ (e^2/4\pi)^2 \ = 4\alpha^2$

$= ¼\ 4\alpha^2\ m^4/(2\beta E^2)\ f(\theta/2) d\Omega_3 \ \ 8a^4$

$= \alpha^2\ m^4/(2\beta E^2)\ f(\theta/2) d\Omega_3 \ \ 8(1/16m^4)$

$= ¼\ \alpha^2\ /(\beta E^2)\ f(\theta/2) d\Omega_3 \qquad\qquad$ but $\beta = 1$

so if we *do* add the extra ½ for identical particles, we get

$$d\sigma/d\Omega_3 = \tfrac{1}{2}\,\tfrac{1}{4}\,\alpha^2/E^2\, f(\theta/2) = \alpha^2/(8E^2)\, f(\theta/2)$$
$$= \alpha^2/(8E^2)[\ 2/(S^2 C^2) + \{1 + C^4\}/S^4 + \{1 + S^4\}/C^4\ ]$$

and this agrees with 7.84, so I have derived this high-energy limit exactly. As they say, the $2/(S^2 C^2)$ term comes from the $T_3/T_4$ interference terms, while the other terms come from $T_1$ and $T_2$

**Low-energy limit?** Let's take a quick shot at this. Keep the largest m terms now in the traces:

$$|pp|^2 = \{ T_1(1,2,3,4) + T_3(1,2,3,4)\} + (1\leftrightarrow 2)$$

$$T_1 = P_{13}^2\, a^4\, Tr[(3+m)\mu(1+m)\nu]\ \ Tr[(4+m)\mu(2+m)\nu] \qquad P_{nk} = 1/(n-k)^2$$
$$T_3 = -P_{13}\, P_{14}\, a^4\, Tr[(3+m)\mu(1+m)\nu(4+m)\mu(2+m)\nu\ ] \qquad a = 1/(2m)$$

$$d\sigma = \tfrac{1}{4}(2\pi)^{-2} e^4 m^4/(2\beta E^4)\, \delta^4(3+4-1-2)\, d^3p_3\, d^3p_4\ |pp|^2$$

$$T_1 \approx P_{13}^2\, a^4\, m^4\ Tr[\mu\nu]\ \ Tr[\mu\nu] = P_{13}^2\, a^4\, m^4\ 4g_{\mu\nu}\ 4g^{\mu\nu} = P_{13}^2\, a^4\ (64m^4)$$
$$T_3 \approx -P_{13}\, P_{14}\, a^4\, m^4 Tr[\mu\nu\mu\nu\ ] = -P_{13}\, P_{14}\, a^4\, m^4\, (-32) = -P_{13}\, P_{14}\, a^4\ (-32m^4)$$

So adding second terms we get

$$T_1 + (1\leftrightarrow 2) = (P_{13}^2 + P_{23}^2)\ a^4\ (64m^4)$$
$$T_3 + (1\leftrightarrow 2) = 2T_3 = P_{13}\, P_{14}\, a^4\ (64m^4)$$

What about these propagators? In the CMS frame we have $E_1 = E_3$ etc and we get

$$(1\text{-}3)^2 = m^2 + m^2 - 2\ 1.3 = 2m^2 - 2(E_1E_3 - \mathbf{p}_1\bullet\mathbf{p}_3) = 2m^2 - 2(E^2 - p^2\cos\theta) =$$
$$= 2(m^2\text{-}E^2) + 2p^2\cos\theta = -2p^2 + 2p^2\cos\theta = -2p^2(1 - \cos\theta) = -4p^2\sin^2(\theta/2)$$

$$(1\text{-}4)^2 = m^2 + m^2 - 2\ 1.4 = 2m^2 - 2(E_1E_4 + \mathbf{p}_1\bullet\mathbf{p}_3) = 2m^2 - 2(E^2 + p^2\cos\theta) =$$
$$= 2(m^2\text{-}E^2) - 2p^2\cos\theta = -2p^2 - 2p^2\cos\theta = -2p^2(1 + \cos\theta) = -4p^2\cos^2(\theta/2)$$

which says that

$$P_{13} = -1/(4p^2S^2) \qquad P_{14} = P_{23} = -1/(4p^2C^2)$$

So we then have

$$|pp|^2 = 64m^4a^4\,(1/16)(1/p^4)\,\{1/S^4 + 1/C^4 + 1/S^2C^2\}$$
$$= 8\,a^4\,[\ \tfrac{1}{2}\,m^4/p^4\,\{1/S^4 + 1/C^4 + 1/S^2C^2\}\ ]$$

$$= 8a^4\,F(\theta/2) \qquad F(\theta/2) = \tfrac{1}{2}\,m^4/p^4\,\{1/S^4 + 1/C^4 + 1/S^2C^2\}$$

where I am making this look like the ER result format. Now we need to re-examine our processing of dσ as shown above and not make the ER approx. So we have

$$d\sigma = \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,\delta^4(3+4-1-2)\,d^3p_3\,d^3p_4\,|pp|^2$$

$$= \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,\delta^4(3+4-1-2)\,d^3p_3\,d^3p_4\,F(\theta/2)\,8a^4$$

$$= \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,\delta^1(3+4-1-2)\,F(\theta/2)\,p_3^2\,dp_3\,d\Omega_3\;8a^4$$

$$= \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,\delta^1(3+4-1-2)\,F(\theta/2)\,p_3\;E_3\;dE_3\,d\Omega_3\,8a^4$$

$$= \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2\beta E^4)\,F(\theta/2)\,p\,E\,d\Omega_3\;8a^4$$

$$= \tfrac{1}{4}(2\pi)^{-2}e^4m^4/(2m^2)\,F\,(\theta/2)d\Omega_3\,8a^4 \qquad e^4/4\pi^2 = 4\,(e^2/4\pi)^2 = 4\alpha^2$$

$$= \tfrac{1}{4}\,4\alpha^2\,m^4/(2m^2)\,F\,(\theta/2)d\Omega_3\;8a^4$$

$$= \alpha^2\,m^4/(2m^2)\,F(\theta/2)d\Omega_3\;8(1/16m^4)$$

$$= 1/4\;\alpha^2\,F(\theta/2)/m^2\,d\Omega_3$$

Notice that $p^2dp = p\,(pdp) = p\,(EdE)$ when you do it exactly, and we used

$$pE/(2\beta E^4) = pE/(2pE^3) = 1/(2E^2) \approx 1/(2m^2)$$

The CMS result is then

$$d\sigma_{CMS} = 1/4\;\alpha^2\,F(\theta/2)/m^2\,d\Omega_3 = 1/4\;\alpha^2\;\tfrac{1}{2}\,m^4/p^4\,\{\,1/S^4 + 1/C^4 + 1/S^2C^2\,\}/m^2\,d\Omega_3$$

$$= 1/8\;\alpha^2\,m^2/p^4\,\{\,1/S^4 + 1/C^4 + 1/S^2C^2\,\}\,d\Omega_3$$

where I have *not* added a ½ for identical particles. Meanwhile:

Goldstein gives us his CMS result on page 84 which I will write here as

Goldstein: $\sigma(\theta) = \tfrac{1}{4}\,e^4/(4E^2)\;1/S^4$

where Goldstein's "e" is in esu units, let's say. Here is the math on all that stuff

$$e_{HL}^2 = 4\pi\,e_{esu}^2 \qquad \alpha = e_{HL}^2/(4\pi)$$

$$\Rightarrow \qquad e_{esu}^4 = (e_{HL}^2/4\pi)^2 = \alpha^2$$

So we would translate Goldstein to say

$$d\sigma = \tfrac{1}{4}\,\alpha^2/(4E^2)\; 1/S^4\; d(\cos\theta)2\pi \;\; = \tfrac{1}{4}\,\alpha^2/(4E^2)\; 1/S^4\; d\Omega$$

Then of course $E^2 = p^2/2m$ so G then says

$$d\sigma = \;\; \tfrac{1}{4}\,\alpha^2/(4E^2)\; 1/S^4\; d\Omega = 1/16\;\alpha^2/(E^2)\; 1/S^4\; d\Omega \;\; = 1/16 \;\; \alpha^2/(p^4)\; (2m)^2 1/S^4\; d\Omega$$

$$= \;\tfrac{1}{4}\,(\alpha^2 m^2/p^4) * 1/S^4 * d\Omega$$

whereas my result says

$$d\sigma_{CMS} = \;\; 1/8\;(\alpha^2\; m^2/p^4)\;\{\; 1/S^4 + 1/C^4 \;\; + 1/S^2C^2\;\}\; d\Omega_3$$

so we are not too far apart. I will ignore the leading fraction and comment only on the rest. The classical calculation does not know about "interference" which we are here getting between the two Feynman quantum mechanics diagram amplitudes. If the particles were not identical, you would only have the one diagram and you would only get the $1/S^4$ term, and this is the basis on which G did his calculation.

Note added: if we write $r_0 = \alpha/m$ then my formula above can be written

$$d\sigma_{CMS} = \;\; 1/8 * (r_o^2\;\beta^4)\;\{\; 1/S^4 + 1/C^4 \;\; + 1/S^2C^2\;\}\; d\Omega_3$$

and NOW if I add the identical particle ½ I get

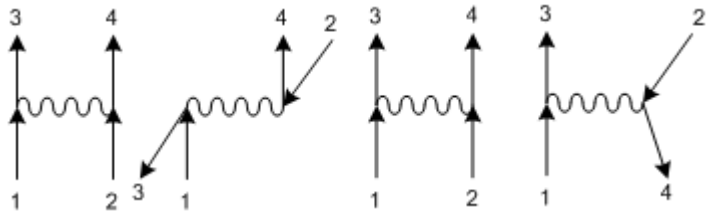
$$d\sigma_{CMS} = \;\; 1/16 * \;\; (r_o^2\;\beta^4)\;\{\; 1/S^4 + 1/C^4 \;\; + 1/S^2C^2\;\}\; d\Omega_3$$

and this agrees exactly with Sakurai page 258 (4.347) except for sign of the interference term! I have traced backwards on this and I cannot find a sign error, so maybe his result is wrong?

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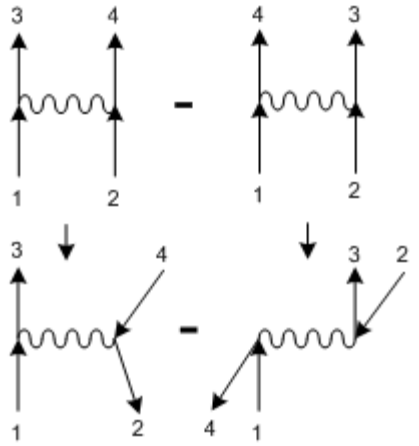
**B. Electron-positron scattering (Bhabha 1935)**

Things are a little hazy here on how we get to the diagrams on top of page 139. Suppose we start with just the left diagram on page 136. Consider:




In the left case, I have "swung around" one electron on each end of the photon, and in the right case, I have swung around both legs on the same end of the photon, so you get different graphs from doing this. Obviously I could make another version of the above picture with 3↔4 everywhere.

In order to make use of a substitution rule, I think we need to start with the exact set of starting graphs and do the same swings on both. So consider this, where in each graph we swing 2 and 4:




This is how BD get the page 139 drawings. We know that for a swung-around leg, you use the negative 4-momentum. BD have given these negated vectors new names, and these appear in the pictures:

$p_4 \rightarrow -q_1$
$p_2 \rightarrow -q_1'$

The actual physical particle momenta of the q-legs on page 139 top figure are $+q_1$ and $+q_1'$, so the labeling is little strange. For example, in the second picture, the photon has $p_1+q_1$. They show negatives so you know where to put v's in the matrix element $S_{fi}$. In any even, the conservation story is $p_1 + q_1 = p_1' + q_1'$.

So fine, we get 7.86 to match our former 7.82, only the overall minus sign needs a comment. In 6.56 on page 96 we see that $\varepsilon_f$ sitting there. In 6.56 we need to think of $\psi$ and $\Psi$ as one of the two electron lines, and we think of $\mathbb{A}$ as coming from the other line. The $\varepsilon_f$ then goes with the line we chose. If we twist the leg for this $\psi$, the sign should change. But if we twist the other leg, somehow $\mathbb{A}$ probably changes. Remember it is $(-i\mathbb{A})$ that appears, and I guess this becomes $(+i\mathbb{A})$ if you twist the legs that make it. So OK, by either argument we get an overall plus sign on 7.86 and on goes our red check.

BD then give a tortured argument regarding the Fermi antisymmetry for our new diagrams, including a hole theory interpretation. This is stretching things more than I care to do, and I know in QFT all this stuff goes away, but nice of them to attempt it. Their sentences just don't make sense to me, and I don't need the results. I know that swing-arounds work.

Now comes the important fact: we take the Møller result and make our substitutions. But how exactly do we do that here? In my notation above we get

$p_1 \rightarrow p_1$
$p_2 \rightarrow -p_2 > 0$ // >0 just means in future light cone
$p_3 \rightarrow p_3$
$p_4 \rightarrow -p_4 > 0$

Now consider what we had in the Møller case:

$p_1 = (E, \mathbf{p_1})$ $1.2 = E^2 + p^2$

$p_2 = (E, -\mathbf{p}_1)$ $\quad 1.3 = E^2 - p^2\cos\theta$

$p_3 = (E, \mathbf{p}_3)$ $\quad 1.4 = E^2 + p^2\cos\theta$

$p_4 = (E, -\mathbf{p}_3)$ $\quad 2.3 = E^2 + p^2\cos\theta \quad = 1.4$

$\quad 2.4 = E^2 - p^2\cos\theta \quad = 1.3$

$\quad 3.4 = E^2 + p^2 \quad = 1.2$

$1.2 = 2E^2$

$1.3 = E^2(1-\cos\theta) = 2E^2\sin^2(\theta/2)$

$2.3 = E^2(1+\cos\theta) = 2E^2\cos^2(\theta/2)$

Let's modify this by negating the 2 and 4 4-vectors to get

$p_1 = (E_1, \mathbf{p}_1)$ $\quad -1.2 = E_1^2 + \mathbf{p}_1 \bullet \mathbf{p}_3$ $\quad *$

$-p_2 = (E_1, -\mathbf{p}_3)$ $\quad *$ $\quad 1.3 = E_1E_3 - \mathbf{p}_1 \bullet \mathbf{p}_3$

$p_3 = (E_3, \mathbf{p}_3)$ $\quad -1.4 = E_1E_3 + \mathbf{p}^2$ $\quad *$

$-p_4 = (E_3, -\mathbf{p}_1)$ $\quad *$ $\quad -2.3 = E_1E_3 + \mathbf{p}^2 \quad = -1.4 \quad *$

$\quad 2.4 = E_1E_3 - \mathbf{p}_1 \bullet \mathbf{p}_3 \quad = 1.3$

$\quad -3.4 = E_3^2 + \mathbf{p}_1 \bullet \mathbf{p}_3 \quad = -1.2 \quad *$

$-2.3 = 2E^2$

$1.3 = E^2(1-\cos\theta) = 2E^2\sin^2(\theta/2)$

$-1.2 = E^2(1+\cos\theta) = 2E^2\cos^2(\theta/2)$

Comments: We had to modify things a lot because now the initial state contains particles $p_1>0$ and $-p_4 > 0$, recall that we "swung around" legs 2 and 4. So if we assign $\mathbf{p}_1$ to $p_1$, then $-p_4$ will have $-\mathbf{p}_1$ (not $p_2$). So I made those changes in red above (lines with *). Notice that these three equalities have NOT changed:

$2.3 = 1.4$

$2.4 = 1.3$

$3.4 = 1.2$

So I think I have to go back now to the dot product results of the Moller calculation and recompute things from that point forward, since the θ kinematics are different. So go back to here:

$2.3 = -2E^2$

$1.3 = E^2(1-\cos\theta) = 2E^2\sin^2(\theta/2)$

$1.2 = -E^2(1+\cos\theta) = -2E^2\cos^2(\theta/2)$

$T_3 + (1\leftrightarrow 2) = 16\, a^4(1.2)^2/[\ 1.3\ 2.3\ ]$ // interference terms

$T_1 + (1\leftrightarrow 2) = 8\, a^4\ [\ \{(1.2)^2 + (2.3)^2\}/(1.3)^2 + \{(1.2)^2 + (1.3)^2\}/(2.3)^2\ ]$

My plan is to keep the above $T_i$ items exactly as they were in Moller, not changing anything at all. But then the substitutions are made when I replace the dot products with the new-world functions of θ/2 . For example, in Moller world we had $1.2 = 2E^2$ , but in Bhabha world we have $1.2 = -2E^2\cos^2(\theta/2)$. The

minus signs are handled here rather than changing them in the dot product $T_i$ expressions. I think this is OK, and the result comes out right below.

So here are my physics package items, but now I have to recompute them as follows ($2E^2$ all cancel)

$$(1.2)^2/ [\ 1.3\ 2.3\ ] = C^4/[\ -S^2]$$

$$[\ \{(1.2)^2 + (2.3)^2\}/(1.3)^2 + \{(1.2)^2 + (1.3)^2\}/(2.3)^2\ ] = [\ \{\ C^4 + 1\}/S^4 + \{\ C^4 + S^4\}\ ]$$

I think I just add these together and get

$$16\ a^4\ C^4/[\ -S^2] \ +\ 8\ a^4\ \ [\ \{\ C^4 + 1\}/S^4 + \{\ C^4 + S^4\}\ ]$$

$$8\ a^4\ \{\ -\ 2\ C^4/S^2\ \ +\ \ \{\ C^4 + 1\}/S^4 + \{\ C^4 + S^4\}\ \}$$

=>

$$g(\theta/2) = \{\ -\ 2\ C^4/S^2\ \ +\ \ \{\ C^4 + 1\}/S^4 + \{\ C^4 + S^4\}\ \ \}$$

Two of these three terms agree with 7.87! What about the third term? Yes, it is also correct, but you have to compute it out a bit:

$$they = \tfrac{1}{2}\ (1 + \cos^2\theta)\ = \tfrac{1}{2}\ (1 + [C^2\text{-}S^2]^2)\ = \tfrac{1}{2}\ (\ 1 + C^4 + S^4\ -\ 2C^2S^2\ )$$

$$=\ \tfrac{1}{2}\ (\ [\ C^2 + S^2]^2 + C^4 + S^4\ -\ 2C^2S^2)\qquad = \tfrac{1}{2}\ (\ 2C^4 + 2S^4) = C^4 + S^4$$

So I have now completely derived 7.87 for Bhabha scattering in the ER limit. The formula is NOT the same as the Moller formula because the kinematics is different, though the angle is defined in exactly the same way, as that between 1 and 3. Note that here the interference terms of $T_3$ make the $-\ 2\ C^4/S^2$ term in 7.87 which is very different from the central interference term in 7.84.

Recall that the Coulomb scattering of positrons and electrons was the *same* for the 1 vertex graphs we used there. We would expect to get this same result in a low-energy limit of our Moller and Bhabha results, but we do have the "bound state complication" with positronium to deal with. I would guess that as long as you stay way above 10 eV or so, you can ignore this more or less.

If I had infinite motivation, I would compute the non-rel Bhabha and compare it to the Moller above, but I think we have done enough on this subject!

**7.10 Polarization in Electron Scattering (140)**

Page 140 takes us back to Coulomb scattering of electrons, and the trick is to replace the occurrence of $u(p_i, s_i)$ by $\Sigma(s_i)\ u(p_i, s_i)$ as shown in Equation A. You then go ahead and sum on the $s_i$ (without the ½ averaging factor, since we are not averaging now), as well as the $s_f$, and the index linkage into a trace happens just as before, but you have this extra $\Sigma(s_i)$ matrix in the trace, just in front of the corresponding energy projector. When the $s_i$ sum is performed, you get nothing for the $-s_i$ value, so that is why you don't need to insert it twice as noted in their footnote.

So what does this extra matrix do? The ½ term replicates our former result including the averaging ½ factor for it, so anything new must come from the $\gamma_5 \not{s}_i$ contribution. We have these kinds of terms:

$\gamma_5 \not{s}_i \not{p}_i \gamma^0 \not{p}_f \gamma^0$ = 0 because $\gamma_5$ counts as 4 gamma, so total number is 9 which is odd
$\gamma_5 \not{s}_i \not{p}_i \gamma^0 \gamma^0 = \gamma_5 \not{s}_i \not{p}_i$ = 0 by 7.18
$\gamma_5 \not{s}_i \gamma^0 \not{p}_f \gamma^0 = -\gamma_5 \not{s}_i \not{p}_f \gamma^0 \gamma^0 + 2E_f \gamma_5 \not{s}_i \gamma^0 = 0 + 0$
$\gamma_5 \not{s}_i \gamma^0 \gamma^0 = \gamma_5 \not{s}_i = 0$ since 5 gammas

So yes, the $\gamma_5$ term yields zero as they claim. The conclusion is that you get the same Mott cross section whether your initial beam is unpolarized, or is polarized with some $s_i$. Somehow this does not surprise me, but I suppose somehow the spin can link with the orbital L and cause a difference, and they hint that in higher order terms there is a difference. Keep in mind that in this example just completed, we have summed on $s_f$.

The next example is NOT sum on the final spins (in effect) and to compute scattering with a specific initial and a specific final polarization. For initial spin state, we of course have s.s = -1 which we know from going the rest frame where s = (0,**s**) and p = (m,**0**) and so know s.p = 0 as well. So in general we know all these things for a particle with spin:

$s.s = -1$ $\qquad$ $p.p = m^2$ $\qquad$ $s.p = 0$

Writing these things out a bit, we have in some arbitrary frame:

$-1 = (s^0)^2 - \mathbf{s}\bullet\mathbf{s} = (s^0)^2 - \mathbf{s}^2$ $\qquad$ $s^0 E - \mathbf{s}\bullet\mathbf{p} = 0$

From the right equation we solve to get

$s^0 = (p/E)\, \mathbf{s}\bullet\hat{\mathbf{p}} = \beta\, \mathbf{s}\bullet\hat{\mathbf{p}} = \mathbf{s}\bullet\boldsymbol{\beta}$

Then from the left equation we find that

$s^2 = 1 + (s^0)^2 = 1 + (\mathbf{s}\bullet\boldsymbol{\beta})^2 = 1 + s^2 (\hat{\mathbf{s}}\bullet\boldsymbol{\beta})^2$

$\Rightarrow\ s^2 = 1/[1 - (\hat{\mathbf{s}}\bullet\boldsymbol{\beta})^2]$

Remember that $s^2 \neq 1$ when particle moves, that is what this is saying! Now define

Now define $s_R$ as the spin 4-vector which has $\hat{\mathbf{s}}_R \bullet \boldsymbol{\beta} = +\beta$. Then

$s_R = (s^0_R, s_R \hat{\boldsymbol{\beta}})$

Then yes,

$s_R^2 = 1/[1 - \beta^2] = s_L^2 = \gamma$ $\qquad$ $s^0_R = \mathbf{s}_R \bullet \boldsymbol{\beta} = s\beta = \beta s_R$

Now define $s_L = -s_R$ (4-vectors) so we have

$s_L = -(s^0_R, s_R \hat{\boldsymbol{\beta}}) = (-s^0_R, -s_R \hat{\boldsymbol{\beta}}) = (s^0_L, s_L \hat{\boldsymbol{\beta}})$

The relation $s^2 = 1/[1 - (\hat{\mathbf{s}} \bullet \boldsymbol{\beta})^2]$ does not depend on the overall sign of 4-vector s, so it is the same and we have $s_R = s_L = \gamma$. But of course now $\hat{\mathbf{s}}_L \bullet \boldsymbol{\beta} = -\beta$ and this is the negative helicity spin state. If you insert $s_R$ or $s_L$ into 7.89 center equation, you of course find that $u(p_i, s_{iR})$ is an eigenstate of $\Sigma(s_{iR})$, and they are just giving these u spinors the name "helicity eigenstates". Fine.

We can of course define our "helicity polarization" as shown in 7.95, and this brings us to the end of page 141 which was a bit time consuming in its micro detail.

**Page 142:** Assume the initial beam is all in the R helicity state, and then I agree with the form for $P_R$ as shown in the first line. The second line shows the two numerator terms, but they have combined the denominator terms into one trace where we get the $\gamma_5$ terms cancelling, so I like the second equals sign.

Now what about the third equals sign? First, in the numerator we combine the two terms to get

$$\mathrm{Tr}[\, \gamma_o(\, 1 + 5\not{s}_i)(\not{p}_i + m)\, \gamma_o(5\not{s}_f)(\not{p}_f + m)\, ]$$

The claim is that the "1" term vanishes. This is similar to what we did earlier but let's do it again:

show $\mathrm{Tr}[\, \gamma_o\, (\not{p}_i + m)\, \gamma_o(5\not{s}_f)(\not{p}_f + m)\, ] = 0$

There are 4 terms to consider.

$\gamma_o(\not{p}_i\, \gamma_o 5\not{s}_f \not{p}_f$ = 9 gammas = 0

$\gamma_o\, (\not{p}_i\, \gamma_o(5\not{s}_f)$ = 8 gammas, so ponder a bit

$= -\not{p}_i(5\not{s}_f) + 2E_i\, \gamma_o\, 5\not{s}_f = 0 + 0 = 0$ due to special $5\gamma\gamma$ rule.

$\gamma_o\, \gamma_o(5\not{s}_f)$ = 7 gammas = 0.

So our simplified numerator then becomes ½ the numerator shown after the last equals.

Denominator: The claim here is that this vanishes:

$$\gamma_o(5\not{s}_i)\, (\not{p}_i + m)\, \gamma_o(\not{p}_f + m)$$

This has a similar structure to our previous term that vanished. Only the linear m terms are of concern and we have for example

$\gamma_o(5\not{s}_i)\, \not{p}_i \gamma_o = 5\not{s}_i)\, \not{p}_i = 0$

$\gamma_o(5\not{s}_i)\, \gamma_o(\not{p}_f$ = same idea as we got last time = 0

Now let's do the numerator trace in some kind of shorthand.

$0\ 5\ r\ (i + m)\ 0\ 5\ R\ (f + m)$

$= + 0\ r\ 5(i + m)\ 0\ 5\ R\ (f + m)$ // move $\gamma_5$ through two $\gamma$'s

$= + 0\ r\ (-i + m)5\ \ 0\ 5\ R\ (f + m)$ // move 5 right one more

$= \ 0\ r\ (i - m)0\ 5\ 5\ R\ (f + m)$ // move 5 right one more

$= \ 0\ r\ (i - m)\ 0\ R\ (f + m)$ // $5^2 = 1$

So this gets rid of all the 5's. The leading term has 6 gammas, and the next down has 4. Here is the leading term:

$0\ r\ i\ \ 0\ R\ f\ \ = -\ 0\ r\ 0\ i\ \ R\ f\ + 2E_i\ \ 0\ r\ R\ f$

$-\ 0\ r\ 0\ i\ \ R\ f = \ + 0\ 0\ r\ i\ \ R\ f\ -\ 2r^0\ 0\ i\ \ R\ f$

So get

$0\ r\ i\ \ 0\ R\ f\ = r\ i\ \ R\ f + 2E_i\ \ 0\ r\ R\ f -\ 2r^0\ 0\ i\ \ R\ f$

Then we can ponder the 4-traces:

$r\ i\ \ R\ f\ = 4\ (0 - r.R\ i.f + r.f\ i.R)\ = 4\ (-\ r.R\ i.f + r.f\ i.R)$

$0\ r\ R\ f\ = 4(r^0\ 0 - R^\circ\ 0 + f^0\ r.R)\ = \ 4(f^0\ r.R)$

$0\ i\ \ R\ f = 4(i^0 0\ \ - R^0 i.f + f^0\ i.R)\ = \ 4(-\ R^0 i.f + f^0\ i.R)$

So this term is making a good mess:

$0\ r\ i\ \ 0\ R\ f\ = 4\ (-\ r.R\ i.f + r.f\ i.R) + 2\ i^0\ \ 4(f^0\ r.R) - 2r^0\ 4(-\ R^0 i.f + f^0\ i.R)$

Now in the CMS, we have $E_i = E = i^0$ and $E_f = E = f^0$ so that simplifies a bit :

$0\ r\ i\ \ 0\ R\ f\ = 4\ (-\ r.R\ i.f + r.f\ i.R) + 2\ E\ \ 4(E\ r.R) - 2r^0\ 4(-\ R^0 i.f + E\ i.R)$

It still looks very messy. Look at some of the dot products:

$r.R = \ s_{iR}.s_{fR} = \ (s_{iR}{}^0, \mathbf{s}_{iR}).(s_{fR}{}^0, \mathbf{s}_{fR})\ = s_{iR}{}^0\ s_{fR}{}^0 - \mathbf{s}_{iR} \bullet \mathbf{s}_{fR}$

$= \ \beta_i\gamma_i\ \beta_f\gamma_f - \gamma_i\gamma_f \cos\theta$

$= \gamma_i\gamma_f\ (\beta_i\beta_f - \cos\theta)$

$= \gamma^2(\beta^2 - \cos\theta)$

$i.f = E^2 - p^2\cos\theta$

$r.f = (s_{iR}{}^0, \mathbf{s}_{iR}).(E,\mathbf{p}_f) = \beta\gamma E - \gamma p\cos\theta$

OK, enough of this for now! They provide no checking intermediate steps, so my chances of screwing up are pretty high along the way. Conclusion: I know how to derive 7.97, but I am not going to grind through it today. You just write out all the dot products in CMS and assemble all the pieces.

As they point out from having the final result for $P_R$, for high-rel limit P=1 and electron just keeps its spin, there is no change in helicity state as it whips by the Coulomb potential.

As for 7.99, I have a simple proof. We know from 7.96 that $P_L = P_R$ because $s_{iL} = -s^{iR}$. Therefore

$P = (p_L P_L - p_R P_R) / (p_L + p_R) = (p_L - p_R) P_L / (p_L + p_R) = p\, P_L/1 = p\, P_L$

The rest of this section is of little interest to me right now. Here is a summary:

(1) For electron in state u(p,s) you can define $\cos\alpha \equiv \langle \boldsymbol{\sigma} \bullet \mathbf{n} \rangle$ where n is a unit vector having the form of a rest spin. In 7.101 this cosα is related to $\hat{\mathbf{s}} \bullet \mathbf{n}$ . When you use helicity states, this cosα comes out being +1 and -1 for the two helicity states and they call it cosδ in this case. The point seems to be that you can represent the polarization P by the cosα if you want. This entire discussion does nothing at all for me, I think it has a special application for the authors.

(2) If you are only interested in high-rel limit, you can just remove $\not{s}$ from your projection operators and then do the calculation. OK, this is just a little time-saving tool if you are forced to do polarization calculations all day long.

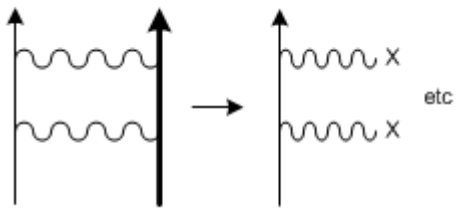
Comment: polarization is a prediction of this QED theory, so one ought to do experiments to check it, and not just do unpolarized scattering. This book section shows how one does these calculations, so good that they included it.

End of Chapter 7, it was a long voyage!

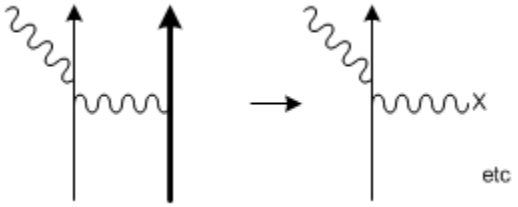
**Reading through the problems at the end of the chapter.**

1. Concerns the Dirac particle wavefunction normalization.

2. Figures 7.6 and 7.7 show two-photon exchange between an electron and proton. We wrote down the amplitude, but the integral was too hard to do so BD did not continue it to get a 2nd order cross section. In this problem, we are supposed to take the limit that $M \to \infty$ for the proton, and we should then obtain the double-Coulomb graph on page 108 right side. This is "second Born". A reasonable reduction to do.

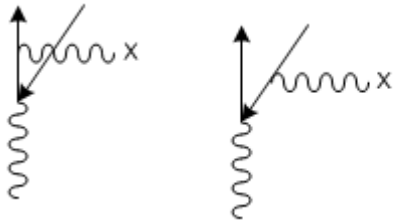



3. Bremsstrahlung limit idea: show left gives right in m << M limit




4. On page 122 in 7.57 we wrote the $S_{fi}$ for Coulomb bremsstrahlung , but then we took the small k limit in going to the cross section (Bethe-Heitler) and showing it was proportional to the Coulomb scatter cross section. Here we are asked to do the full calculation for any k. I see this problem is circled, so maybe I did it once, but I would never know if I got the right answer!

5. Here we consider the swing arounds of the page 122 Brem graphs, and get graphs like this

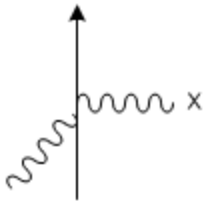



where you have an incoming single photon that makes an e+ e- pair in the presence of a Coulomb potential. So BD still call this a Bethe-Heitler formula, even though it is not for Brem -- it is for pair-production as shown!

6. In the gamma-pair-production of section 7.8 we got a fairly simple form 7.80 for the $d\sigma/d\Omega$. This problem asks us to do the $\theta$ integration to get a full result, then check the two limits on page 135. We would do the polarization sum of $(\varepsilon_1.\varepsilon_2)^2$ as before, so this is just doing three integrals. A problem for Maple.

7. This asks to do Compton scattering in the lab frame. I had a fairly covariant result when I did this above, and then I specialized to CMS, so here we are supposed to instead specialize to the lab situation where $E_2 = m$. Does not seem too hard.

8. Here we are supposed to compute the following situation:




where an electron absorbs a photon. This is a swing-around of Brem on page 122 graphs. The problem asks us to think of the electron as starting off in a bound state in H like atom. I have never done this kind of problem, but in the limit they give, maybe you just do it with all plane waves so you can ignore the bound state. In general you would replace the incoming electron plane wave with a "Coulomb wave function" as noted above.

9. This problem asks us to reconsider the spin computation tools of page 143 using something other than the helicity choice for spin basis.

10. This asks us to derive 7.97, something I did not do in the raw notes. When they say "a short calculation", you can imagine perhaps 10 pages, of which I did maybe 2 starting in the raw notes.

So OK, I understand all these problems and could do them if I had to. I think we can move on.