BD V1 Chap 8 : Higher Order Corrections PhL 10.27.08

8.1 Electron-Positron Scattering in 4th order. (148) 1
8.2 Vacuum Polarization (153). 2
Review of this section to mid page 159 11
Derivation of 8.30 from 8.22. 12
S-matrix expansion 13
8.3 Renormalization of External Photon Lines (161). 15
8.4 Self-mass of the Electron (162). 15
Derive 8.35: 15
Derive 8.36: 17
Derive 8.37 18
Derive page 163 equation A 18
Now to get 8.40, ... 21
Now let's derive what I will call 8.40A 22
8.5 Renormalization of the Electron Propagator (164). 23
8.6 The Vertex Correction (166). 24
Verify 8.52: 25
Verify 8.51: 26
The anomalous magnetic moment of the electron. 30
Infrared Divergence Problems 35
8.7 The Lamb Shift (177) 36

8.1 Electron-Positron Scattering in 4th order. (148)

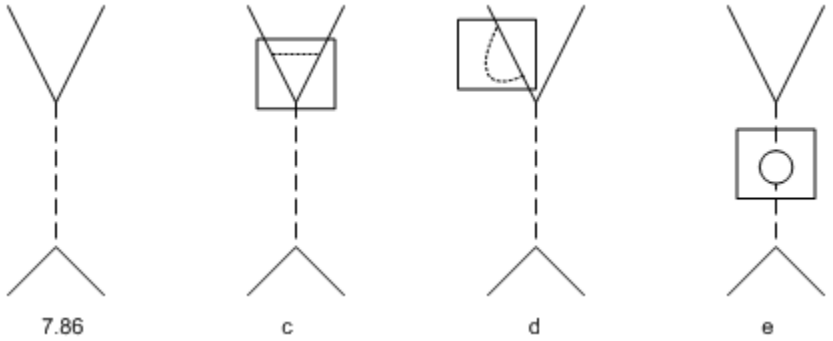
There are 18 graphs and the text discussed the 5 shown on page 148. For each one, the coordinate space S matrix element is written out. All this stuff works for me with my (-iV) rule for -i at each vertex plus the charge there. I like to put -iD for photon propagators, so I might check into that at some point. The big issue in this section is the overall **sign of the term** in order to get correct **Fermi or Bose statistics** relative to other graphs. A lot of this section deals with that, and it is a bit "fiddly". A scary example are the graphs top of page 149 which show a particular time ordering of graphs a and c on page 148. Since they are the same except for a swap of two electron lines, they must have a relative minus sign. Also, you have to have the signs right compared to the first order graphs!

After discussing each of the 5 graphs on page 148, BD comment that **disconnected graphs** can be ignored. I remember in my class an elegant proof of this fact, and it is related to renormalization, and BD hint at that theorem here, it will appear somewhere in the second volume I think.

Finally then we have an enumeration of what I would call "**the Feynman rules**: on page 151, in coordinate space. Rules 5 and 6 concern signs of the S_{fi} contributions: (1) our Fermi/Bose stuff; (2) -1 for closed fermion loop; (3) overall sign based on # positrons in initial state.

After all this, BD finally stare at **the elephant in the room**: how do you do these **horrible looking integrals** over 4-space? They won't try, but will give general discussion.

You can regard more complicated graphs as being simpler graphs with "insertions" added. This starts us off on a long topological voyage through QED that will be picked up in volume 2. Three immediate examples are given.



We see here a vertex insertion, an electron leg insertion, and a photon propagator insertion. Each of these **insertions** adds extra factors to the simpler 7.86 amplitude, and the exact terms added are shown on page 152-153.

In the next book sections, we are going to study these various "insertions" and try to see in a general sense what they do to the amplitude. The topological game is afoot.

Comment on 8.8: If we look back at graph (e)'s amplitude, it is 8.5, and BD show the Dirac space indices and you see that the inner bracket is a trace! Why is this so? If we follow the rules, we put a γ at each vertex, and a propagator on each side. We know that the Dirac space cannot cross the two photon paths shown, it is self contained. There are no external electron legs here to worry about. The spin of the internal electron loop is unmentioned. Sakurai at least mentions the trace on page 274, but I don't have that book's mindset yet, so the argument is not convincing. Really this is a new "Feynman rule" for handling closed electron loops within a diagram! The trace probably makes it covariant. So one point is that the object $I_{\mu\nu}(q)$ we deal with below is a scalar in the Dirac space sense, not a 4x4 matrix.

By way of contrast, the $\Sigma(p)$ object appearing in 8.7 *is* a Dirac 4x4 matrix.

8.2 Vacuum Polarization (153).

This cute term always refers to modifications of a photon propagator, the name from the idea that you temporarily create little e+ e- pairs. When you "polarize" cat's fur or a capacitor, you separate charges that were neutral, so that's what these fermion loops are doing, so I guess it is a reasonable term.

Right off the bat looking at 8.8 we see $k^3dk/k^2 \approx k\,dk \Rightarrow k^2$ divergence for large k. This integral **diverges quadratically**! Notice that $I_{\mu\nu}$ is the square bracket I outlined in 8.8. It is the fermion loop all by itself. We went to momentum space, but I know that is how this comes out from seeing past examples. So how are we going to fix up a huge quadratic high energy divergence of this integral -- at least it appears to diverge. [I note my own date 5.4.1975, 33 years ago, as when I last read this! I was in my last 2 years of UCB, probably the last 1 year, a little late to be learning QED from a job perspective.]

I think this loop is later called $\mathbf{\Pi_{\mu\nu}(q)}$ in volume 2, where q is the momentum feeding into one end of the Fermion loop. It gets attached to A^μ on each side, in effect, and gauge invariance says $A^\mu \rightarrow A^\mu + \partial^\mu\Lambda$ makes no difference which is $\varepsilon^\mu \rightarrow \varepsilon^\mu + k^\mu$ and this in this case we must have $q^\mu\,\Pi_{\mu\nu}(q) = 0$. This applies at least when $q^2 = 0$ meaning A^μ is a physical photon.

I verified 8.10 and agree that, were the integrals finite, you could do a variable shift and you would then have $q^\mu\,\Pi_{\mu\nu}(q) = 0$. Even if this were true, what does this show us? It would just verify gauge invariance. So hold on this question for a moment.

I now skip the cutoff general presentation in 8.11 because it makes no sense yet, I need the example that is coming first. So now we are on page 155 top. I agree with 8.12 and note that the +iε makes the integral converge for its phasor form. In order to verify 8.13, I checked the obvious factors, and now I have to show this to be true:

$$Tr[\ \gamma_\mu(\not{k}+m)\gamma_\nu(\not{k}-\not{q}+m)]\ =\ 4\ \{\ k_\mu\ (k-q)_\nu+\ k_\nu\ (k-q)_\mu - g_{\mu\nu}\ (k^2\ -\ q.k\ -\ m^2)\ \}$$

Well, the linear m terms vanish due to odd gammas. We then have two traces to do:

$$Tr[\ \gamma_\mu\not{k}\gamma_\nu(\not{k}-\not{q})]\ = k_\alpha(k-q)_\beta\ tr(\mu\alpha\nu\beta) = k_\alpha(k-q)_\beta\ 4\ \{\ \mu\alpha\ \nu\beta\ -\ \mu\nu\ \ \alpha\beta\ +\ \mu\beta\ \alpha\nu\ \}$$

$$= 4\ \{\ k_\mu(k-q)_\nu + k_\nu(k-q)_\mu\ -\ g^{\mu\nu}\ k.(k-q)\ \}$$

$$Tr[\ \gamma_\mu m\gamma_\nu(m)]\ = -m^2\ tr\ (\mu\nu)\ = 4m^2 g_{\mu\nu}$$

So my answer is this:

$$Tr = \ 4\ \{\ k_\mu(k-q)_\nu + k_\nu(k-q)_\mu\ -\ g^{\mu\nu}\ [\ \ k.(k-q)\ -\ m^2\ ]\ \}$$

and this then produces the desired result quoted above, **so 8.13 gets a red check.**

Now the next goal is to perform the d^4k integration to get 8.16, and this looks like a lot of work, so let's get started. We have never done such an integration yet in this book. Here is the expo:

$$\begin{aligned} & expi\ \{\ (k^2-m^2+i\varepsilon)z_1 + [\ (k-q)^2\ -\ m^2\ +\ i\varepsilon\ ]\ z_2\ \} \\ = \ & expi\ \{\ (k^2-m^2+i\varepsilon)z_1 + [\ (k^2-2q.k\ +\ q^2)\ -\ m^2\ +\ i\varepsilon\ ]\ z_2\ \} \\ = \ & expi\ \{\ (k^2(z_1+\ z_2)\ +\ -2q.k\ z_2\ +\ q^2 z_2\ \ -\ m^2(z_1+\ z_2)\ +\ i\varepsilon\ \ \} \end{aligned}$$

where I have factored in descending powers of k. Let's now consider their proposed $\boldsymbol{\ell}$ vector

$$\boldsymbol{\ell}^\mu\ = k^\mu - z_2/(z_1+\ z_2)\ q^\mu \qquad\qquad => d^4k\ = d^4\boldsymbol{\ell}$$

$$\boldsymbol{\ell}.\boldsymbol{\ell}\ = [\ k^\mu - z_2/(z_1+\ z_2)\ q^\mu]\ [\ k_\mu - z_2/(z_1+\ z_2)\ q_\mu]$$

$$= k^2 - 2\ z_2/(z_1+\ z_2)k.q\ +\ z_2^2/(z_1+\ z_2)^2\ q^2$$

$$\boldsymbol{\ell}.\boldsymbol{\ell}\ (z_1+\ z_2)\ = k^2(z_1+\ z_2) - 2\ z_2 k.q\ +\ z_2^2/(z_1+\ z_2)\ q^2$$

and this combination does reproduce the first two terms of our exponential. So I guess we can say

$$exp = expi\ \{\ \boldsymbol{\ell}.\boldsymbol{\ell}\ (z_1+\ z_2) - z_2^2/(z_1+\ z_2)\ q^2\ +\ q^2 z_2\ \ -\ m^2(z_1+\ z_2)\ +\ i\varepsilon\ \ \}$$

$$= expi\ \{\ \boldsymbol{\ell}.\boldsymbol{\ell}\ (z_1+\ z_2)\ +\ i\varepsilon\}\ expi\ \{\ -\ z_2^2/(z_1+\ z_2)\ q^2\ +\ q^2 z_2\ \ -\ m^2(z_1+\ z_2)\ +\ i\varepsilon\ \ \}$$

where now we "completed the square" in the exponent and I will omit this second factor for a while in computing things below, then add it back in the end. Assuming we can shift variables from k to ℓ, we are then faced with this integral, where $\alpha = z_1 + z_2 \geq 0$

$$\int d^4\ell \, \mathrm{expi}\{ \alpha \, \ell.\ell + i\varepsilon \} = \int d^4\ell \, \mathrm{expi}\{ \alpha \, g_{\mu\nu}\ell^\mu\ell^\nu + i\varepsilon \}$$

$$= \int d\ell^0 \int d\ell^1 \int d\ell^2 \int d\ell^3 \, \mathrm{expi}\{ \alpha(\ell^0)^2 \} \, \mathrm{expi}\{ -\alpha(\ell^1)^2 \} \, \mathrm{expi}\{ -\alpha(\ell^2)^2 \} \, \mathrm{expi}\{ -\alpha(\ell^3)^2 \}$$

$$= \int dy \, \mathrm{expi}\{ \alpha y^2 \} \left[\int dx \, \mathrm{expi}\{ -\alpha x^2 \} \right]^3 = \sqrt{-\pi/i\alpha} \; (\sqrt{\pi/i\alpha})^3 = +\pi^2/i\alpha^2$$

where each integral is $-\infty$ to $+\infty$. Here is how the i's are computed:

$$(-i)^{-1/2} (+i)^{-3/2} = (e^{-i\pi/2})^{-1/2} \; (e^{+i\pi/2})^{-3/2} = (e^{+i\pi/4}) (e^{-i3\pi/4}) = (e^{-i2\pi/4}) = (e^{-i\pi/2}) = -i$$

or do it an easier way

$$\sqrt{-1/i^4} = (1/i^2)\sqrt{-1} = -\sqrt{-1} = -i$$

So I think I have shown that

$$(2\pi)^{-4} \int d^4\ell \, \mathrm{expi}\{ \; (z_1 + z_2) \, \ell.\ell + i\varepsilon \} = +1/(16i\pi^2)/(z_1 + z_2)^2$$

and **this agrees with the first of the three integrals shown in 8.15**. I know they are going to use the other two integrals soon, so better do those right now. Here is one of the linear ℓ^μ integrals $\mu = 2$

$$\int d\ell^0 \int d\ell^1 \int d\ell^2 \int d\ell^3 \, \mathrm{expi}\{ \alpha(\ell^0)^2 \} \, \mathrm{expi}\{ -\alpha(\ell^1)^2 \} [\, \ell^2 \, \mathrm{expi}\{ -\alpha(\ell^2)^2 \}] \, \mathrm{expi}\{ -\alpha(\ell^3)^2 \}$$

$$= \int dy \, \mathrm{expi}\{ \alpha y^2 \} \left[\int dx \, \mathrm{expi}\{ -\alpha x^2 \} \right]^2 \int dx \; x \, \mathrm{expi}\{ -\alpha x^2 \}$$

and of course the last integral vanishes because range is even and integrand is odd, so this confirms the second of our three integrals. The third one vanishes if $\mu \neq \nu$ for the reason just shown, because then we will have two vanishing integrals multiplied together. So we need only consider $\mu = \nu$. If $\mu = \nu = 0$, then the first integral becomes

$$\int dy \; y^2 \, \mathrm{expi}\{ \alpha y^2 \} = -(1/2i\alpha)\sqrt{-\pi/i\alpha} \qquad \text{so pick up a factor } -(1/2i\alpha) \qquad \beta = -\alpha$$

If $\mu = \nu = 1$, we get instead

$$\int dx\ x^2\ \text{expi}\ \{-\alpha x^2\} = (1/2i\alpha)\sqrt{\pi/i\alpha} \qquad \text{so pick up a factor } (1/2i\alpha)$$

Thus we have shown that

$$\int d^4\ell\ \text{expi}\ \{\alpha\,\ell.\ell + i\varepsilon\}\ \ell^\mu\ell^\nu = -\delta_{\mu\nu}\ \{\ (1/2i\alpha) \text{ if } \mu=\nu=0,\ -(1/2i\alpha) \text{ if } \mu=\nu=i\ \}\ (\pi^2/i\alpha^2)$$

$$= -g^{\mu\nu}\ \ (1/2i\alpha)\ (\pi^2/i\alpha^2) = +g^{\mu\nu}\ \pi^2/2\alpha^3 = \pi^2/i\alpha^2\ * i\,g^{\mu\nu}/2\alpha$$

so that

$$(2\pi)^{-4}\int d^4\ell\ \text{expi}\ \{\alpha\,\ell.\ell + i\varepsilon\}\ \ell^\mu\ell^\nu = +1/(16i\pi^2)\ 1/\alpha^2\ * i\,g^{\mu\nu}/2\alpha$$

and setting $\alpha = z_1 + z_2$ **gives us the third integral quoted in 8.15.** I never had to stare at any contours to get these results, but that might have confirmed convergence.

Now we have to use these integrals to do the integral in 8.13 (I omit the residual expi factor for now)

$$I_{\mu\nu} = 8.13 = +4e^2 \int dz_1 \int dz_2\ (2\pi)^{-4} \int d^4\ell\ \{\ k_\mu(k-q)_\nu + k_\nu(k-q)_\mu\ - g^{\mu\nu}\ [\ k.(k-q) - m^2\ ]\ \}\ \text{expi}(\alpha\,\ell.\ell)$$

so we need to process {...} . From 8.14 we have these facts:

$$k^\mu = \ell^\mu + z_2/(z_1+z_2)\ q^\mu \qquad \text{and} \qquad (k-q)^\mu = \ell^\mu - z_1/(z_1+z_2)\ q^\mu$$

so we get

$$k_\mu(k-q)_\nu = [\ell_\mu + z_2/(z_1+z_2)\ q_\mu]\ [\ell_\nu - z_1/(z_1+z_2)\ q_\nu\ ]$$

$$= \ell_\mu\ell_\nu + z_2/(z_1+z_2)\ q_\mu\ \ell_\nu - z_1/(z_1+z_2)\ q_\nu\ \ell_\mu - z_1\ z_2/(z_1+z_2)^2\ q_\mu\ q_\nu$$

We know that the linear ℓ terms integrate to nothing, so simplify this to

$$k_\mu(k-q)_\nu \approx \ell_\mu\ell_\nu - z_1\ z_2/(z_1+z_2)^2\ q_\mu\ q_\nu$$

and the second term in our {...} just doubles this. Meanwhile, the third term is

$$-g^{\mu\nu}\ [\ k.(k-q)\ - m^2\ ]$$

The dot product is just $k_\mu(k-q)^\mu$ and again we can drop the linear terms shown above so

$$k.(k-q) = k_\mu(k-q)^\mu \approx \ell^2 - z_1\ z_2/(z_1+z_2)^2\ q^2$$

So we have

$$I_{\mu\nu} = +4e^2 \int dz_1 \int dz_2 \, (2\pi)^{-4} \int d^4\ell \;\; \mathrm{expi}(\alpha\, \ell.\ell)$$
$$\{ 2\, \ell_\mu \ell_\nu - 2\, z_1\, z_2/(z_1+z_2)^2\, q_\mu\, q_\nu \; - g_{\mu\nu} \; [\ell^2 - z_1\, z_2/(z_1+z_2)^2\, q^2 \; - m^2 \,] \}$$
$$= +4e^2 \int dz_1 \int dz_2 \, (2\pi)^{-4} \int d^4\ell \;\; \mathrm{expi}(\alpha\, \ell.\ell)$$
$$\{ [2\, \ell_\mu \ell_\nu - g_{\mu\nu}\ell^2] \; - z_1\, z_2/(z_1+z_2)^2 \, [\, 2\, q_\mu\, q_\nu \; - g_{\mu\nu} q^2] \; + g_{\mu\nu}\, m^2 \}$$

Basically we have three terms to think about in terms of our 8.15 integrals.

$$(2\pi)^{-4} \int d^4\ell \;\; \mathrm{expi}(\alpha\, \ell.\ell) \;\; \ell_\mu \ell_\nu \;\; = \tfrac{1}{2}\, g_{\mu\nu}\, (1/16\pi^2)\, /(z_1+z_2)^3$$

$$(2\pi)^{-4} \int d^4\ell \;\; \mathrm{expi}(\alpha\, \ell.\ell)\, \ell^2 \;\; = g^{\mu\nu} (2\pi)^{-4} \int d^4\ell \;\; \mathrm{expi}(\alpha\, \ell.\ell)\, \ell_\mu \ell_\nu \;\; = \tfrac{1}{2}\, g^{\mu\nu}\, g_{\mu\nu}\, (1/16\pi^2)\, /(z_1+z_2)^3$$

$$= 2 \;\; (1/16\pi^2)\, /(z_1+z_2)^3$$

Thus the first two terms of this integral add to this result

$$2 \;\; \tfrac{1}{2}\, g_{\mu\nu}\, (1/16\pi^2)\, /(z_1+z_2)^3 - g_{\mu\nu} \;\; 2 \;\; (1/16\pi^2)\, /(z_1+z_2)^3 \qquad = -\, g_{\mu\nu}\, (1/16\pi^2)\, /(z_1+z_2)^3$$

The third integral gives this:

$$(2\pi)^{-4} \int d^4\ell \;\; \mathrm{expi}(\alpha\, \ell.\ell)\, [- z_1\, z_2/(z_1+z_2)^2 \, [\, 2\, q_\mu\, q_\nu \; - g_{\mu\nu} q^2] \; + g_{\mu\nu}\, m^2 \,]$$

$$= \; [- z_1\, z_2/(z_1+z_2)^2 \, [\, 2\, q_\mu\, q_\nu \; - g_{\mu\nu} q^2] \; + g_{\mu\nu}\, m^2 \,]\, (1/16i\pi^2)\, /(z_1+z_2)^2$$

so the result is this,

$$I_{\mu\nu} = 4e^2 \int dz_1 \int dz_2 \;\; \{ \;\; - g_{\mu\nu}\, (1/16\pi^2)\, /(z_1+z_2)^3$$
$$+ \; [- z_1\, z_2/(z_1+z_2)^2 \, [\, 2\, q_\mu\, q_\nu \; - g_{\mu\nu} q^2] \; + g_{\mu\nu}\, m^2 \,]\, (1/16i\pi^2)\, /(z_1+z_2)^2 \quad \}$$

$$= (1/4\pi^2) e^2 \int dz_1 \int dz_2 \;\; \{ \;\; - g_{\mu\nu}\, /(z_1+z_2)^3$$
$$+ \; [- z_1\, z_2/(z_1+z_2)^2 \, [\, 2\, q_\mu\, q_\nu \; - g_{\mu\nu} q^2] \; + g_{\mu\nu}\, m^2 \,]\, (-i)\, /(z_1+z_2)^2 \quad \}$$

$$= (1/4\pi^2) e^2 \int dz_1 \int dz_2 \, (1/(z_1+z_2)^2 \,) \;\; \{ \;\; - g_{\mu\nu}\, /(z_1+z_2)$$
$$+ \; [- z_1\, z_2/(z_1+z_2)^2 \, [\, 2\, q_\mu\, q_\nu \; - g_{\mu\nu} q^2] \; + g_{\mu\nu}\, m^2 \,]\, (-i) \quad \}$$

but we have to tack on

$$\text{expi}\ \{ - z_2^2/(z_1+ z_2)\ q^2 + q^2 z_2\ - m^2(z_1+ z_2) + i\varepsilon\ \}$$

to this result, since that is left over from our completing the square. We can rewrite this tack on as

$$\text{expi}\ \{ + z_1 z_2/(z_1+ z_2)\ q^2 - m^2(z_1+ z_2) + i\varepsilon\ \}$$

and this agrees with the expo shown in 8.16 (ignore the m_i business for now).

If I now compare my result to their result, there are problems. I need to show that the following things are equal:

$$\text{me} = (1/4\pi^2)e^2\ \{\ - g_{\mu\nu}/(z_1+ z_2)\ - i\ [- z_1\ z_2/(z_1+ z_2)^2\ [\ 2\ q_\mu\ q_\nu\ - g_{\mu\nu}q^2]\ + g_{\mu\nu}\ m^2\ ]$$
$$\text{they} = -i\alpha/\pi\ \ [.....]$$

where [....] refers to their square bracket in 8.16. We know that $\alpha = e^2/4\pi$ so

$$(1/4\pi^2)e^2 = 1/\pi\ (e^2/4\pi) = \alpha/\pi \qquad \text{so we have}$$

$$\text{me} = \alpha/\pi\ \ \{\ - g_{\mu\nu}/(z_1+ z_2)\ - i\ [- z_1\ z_2\ /(z_1+ z_2)^2\ [\ 2\ q_\mu\ q_\nu\ - g_{\mu\nu}q^2]\ + g_{\mu\nu}\ m^2\ ]$$
$$\text{they} = -i\alpha/\pi\ \ [.....]$$

or

$$\text{me} = \{\ - g_{\mu\nu}/(z_1+ z_2)\ - i\ [- z_1\ z_2/(z_1+ z_2)^2\ [\ 2\ q_\mu\ q_\nu\ - g_{\mu\nu}q^2]\ + g_{\mu\nu}\ m^2\ ]$$
$$\text{they} = -i\ [.....]$$

But now write $1 = -i*i$ so

$$\text{me} = -i\ \{\ - ig_{\mu\nu}/(z_1+ z_2)\ + [- z_1\ z_2\ /(z_1+ z_2)^2\ [\ 2\ q_\mu\ q_\nu\ - g_{\mu\nu}q^2]\ + g_{\mu\nu}\ m^2\ ]$$
$$\text{they} = -i\ [.....]$$

or

$$\text{me} = \{\ - ig_{\mu\nu}/(z_1+ z_2)\ + [- z_1\ z_2\ /(z_1+ z_2)^2\ [\ 2\ q_\mu\ q_\nu\ - g_{\mu\nu}q^2]\ + g_{\mu\nu}\ m^2\ ]$$
$$\text{they} = [.....]$$

"My" first term $- ig_{\mu\nu}/(z_1+ z_2)$ agrees with one of their terms, as does the $g_{\mu\nu}\ m^2$ term. Here are the other terms:

$$\text{me} = \{\ [- z_1\ z_2\ /(z_1+ z_2)^2\ [\ 2\ q_\mu\ q_\nu\ - g_{\mu\nu}q^2]\ ]$$

$$\text{they} = [2(g_{\mu\nu}q^2 - q_\mu q_\nu)\ z_1 z_2/(z_1+ z_2)^2 - g_{\mu\nu}\ q^2\ z_1 z_2/(z_1+ z_2)^2\ ]$$

$$= z_1 z_2/(z_1+ z_2)^2\ \ \{\ 2(g_{\mu\nu}q^2 - q_\mu q_\nu) - \ g_{\mu\nu}\ q^2\ \}$$

$$= z_1 z_2/(z_1+ z_2)^2\ \ \{\ (g_{\mu\nu}q^2 - 2q_\mu q_\nu)\ \ \}$$

$$= - z_1 z_2/(z_1+ z_2)^2\ \ \{\ (2q_\mu q_\nu - g_{\mu\nu}q^2)\ \ \}$$

$$= \text{me}$$

and this **concludes our full verification of (8.16),** apart from the m_i^2 stuff. We do this step as indicated in 8.11, replacing m with m_i in some mysterious was (so far).

Status at this point: I computed $I_{\mu\nu}$ and got a result, and then I compute $\overline{I}$ of 8.11 and the result so far is 8.16.

Now, they point out that the $g_{\mu\nu}q^2 - q_\mu q_\nu$ term satisfies $q^\mu \Pi_{\mu\nu}(q)$, and now they are going to claim that the last three terms vanish. To this end, they copy the terms of interest into A at top of page 156. Then slide $z_i \rightarrow \lambda z_i$ gives trivially the form B. Then form C is pretty easy to show, I did it on scratch, where it pays to let $\alpha = z1+z2$ and ignore common factors. Now once we have C, we are told to go back the other way now and replace $z_i \rightarrow z_i/\lambda$ which causes λ to vanish from the exponent and causes the integration λ to also vanish, and we are left with $i\lambda\, \partial/\partial\lambda$ [something independent of λ] and so C = 0.

So fine, we have shown that the last [...] bracket in 8.16 integrates to nothing due to a tricky scaling argument. The value in the phase made all the difference here.

What is the general theorem going on here? I am getting uncomfortable with it. Why can't I do this by a similar argument: (integrals from 0 to ∞)

$$\int dx\, e^{-ax} = \int dx\, \lambda\, e^{-a\lambda x} = \int dx\, \lambda\, (1/-ax)\, \partial_\lambda\, e^{-a\lambda x} = \lambda\partial_\lambda\, (1/-a) \int dx/x\, e^{-a\lambda x}$$

$$= \lambda\partial_\lambda\, (1/-a) \int dx/x\, e^{-ax} = 0$$

But I know this integral equals 1/a, not 0. The reason this argument fails here is that the dx/x integral does not converge at the lower endpoint, so changing order of ∂_λ and integration not justified I suppose. Something about uniform convergence. You would think there would be a similar problem with the integral shown in 8.17 -- the integration seems to diverge near $z_i = 0$. For example, if we did this in polar coordinates, we would have $dz_1 dz_2 = dxdy = rdrd\theta$ and $(x+y)^3 = r^3(\cos\theta+\sin\theta)^3$ and we have dr/r^2 divergence in this region, and the exponential does not help at all in this region.

So I must conclude that the argument BD use to show 8.17 = 0 is bogus.

Let's move on for now and see if we don't get a clue about the above problem. Here is my proof of 8.18

$$\int dx/x\, \delta(1 - a/x) = \int dx/x\, \delta([x-a]/x) = \int dx/x\, \delta(g(x)) \qquad g(x) = (x-a)/x$$

$g'(x) = a/x^2$ $\qquad$ $g'(a) = 1/a$ $\qquad$ $\Rightarrow \delta(g(x)) = \delta(x-a)/(1/a) = a\,\delta(x-a)$ so continue the above,

$$= \int dx/x\, a\, \delta(x-a) = a/a = 1$$

The steps in E and F are pretty clear. When we do the z_i scaling, we get $\delta(1-\alpha)$ where $\alpha = z_1 + z_2$ so those factors all go away giving 8.19.

Now we come into the cutoff stuff for the first time. What the hell is 8.11 trying to say? We define the barred I as shown where we add those extra terms. I guess in this case we would say:

$$m_0 = m \; m_i = M_i \qquad i = 1,2,3...$$

$$c_0 = 1 \quad c_i = C_i(M_i)^2$$

So our first example of doing this is to have just a single $C_1 = -1$ and rest = 0, so we have

$$c_0 = 1 \quad c_1 = -1 \qquad M_1 = M$$

$$\bar{I}_{\mu\nu} = I_{\mu\nu}(m^2) - I_{\mu\nu}(M^2)$$

so this is the famous idea of "the subtraction". Now there are lots of steps they are not mentioning that I have to do if I want to verify 8.20.

(1) What about the z integrations. We have $\delta(1-z_1- z_2)$ so the hit is $z_2 = 1 - z_1$ so the double integration is really a single integration across this diagonal line segment joining (1,0) and (0,1). There can be no contributions for either $z_i > 1$, so that explains the upper endpoint in 8.20

(2) I proved the following useful integral by breaking into two integrals, setting y = ax in the first, and then y = bx in the second, then recombining them. The logs become ln(b/a) which can be removed from the integral. Integral x = 0,∞.

$$\int dx \ln(x) [a e^{-ax} - b e^{-bx}] = \ln(b/a)$$

(3) But we also know that

$$\int dx/x[e^{-ax} - e^{-bx}] = \int dx \, \partial_x (\ln(x)) [e^{-ax} - e^{-bx}] = - \int dx \ln(x) \, \partial_x [e^{-ax} - e^{-bx}] + \text{parts}$$

The parts vanish at the x=0 endpoint because limit xln(x) = 0. This is the main point! So we then have

$$\int dx/x[e^{-ax} - e^{-bx}] = - \int dx \ln(x) [(-a) e^{-ax} - (-b)e^{-bx}] = \int dx \ln(x) [a e^{-ax} - be^{-bx}]$$

Combining these two, we find

$$\int dx/x[e^{-ax} - e^{-bx}] = \ln(b/a)$$

We can set $x = \lambda$, $a = i(q^2 z_1 z_2 - m^2)$ $b = i(q^2 z_1 z_2 - M^2)$ and

$$\ln(b/a) = \ln(i(q^2 z_1 z_2 - M)^2 / i(q^2 z_1 z_2 - m)^2 = \ln(M^2 / (m^2 - q^2 z_1 z_2))$$

Then when we do the dz_2 integration and rename $z_1 = z$, so $z_1 z_2 \rightarrow z(1-z)$, we get the result shown on line C on page 157.

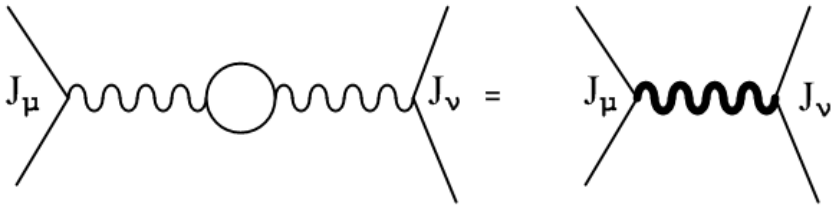
Now write $\ln(M^2/ (m^2 - q^2 z_1 z_2)) = \ln((M^2/m^2)/ (1 - q^2 z_1 z_2/m^2))$

$= \ln (M^2/m^2) - \ln (1 - q^2 z_1 z_2/m^2))$

We can do the first term integral $\int dx\, x(1-x) = 1/6$, take that out front, then a 6 on the second term.

I think this subtraction method is an alternative to just putting in a low-end cutoff on the $d\lambda$ integral, and it is a "covariant method" because nothing is frame dependent. The point here is that by adding in the subtraction term, we have tamed the integral's divergence. Then later when we add things together, they will surely show that the M^2 dependence goes completely away, but you have to add several graphs to see this. So in 8.20 we have "exposed" the M^2 dependence in a simple, dimensionless, covariant manner.

OK, now we compute the photon propagator with this lowest level vacuum polarization added in. The way you do this is precisely as shown in 8.21 . This is really how $I_{\mu\nu}$ is defined in 8.8. so I am very happy with 8.21. Now here is why they toss out the $q_\mu q_\nu$ term in the renormalized propagator: $J_\mu \; J_\nu$



So any term like kq^μ or $Kq^\mu q_\nu$ vanishes at the left vertex because $\partial_\mu J^\mu = 0$, assuming J^μ is a conserved current. In the Dirac electron theory, it is a conserved current, but the authors are planning ahead here.

Very good, we are getting on with the renorm program now. As they say, we have computed the α correction to the photon propagator! For small q^2 we get a renormalization factor Z_3 as shown, I dimly remember this stuff now. And we have our first renormalization idea, namely

$e_R^2 = e^2 Z_3$ e = "bare charge"

Now we come to 8.26 where I have noted "slight fudge here" in my old notes. By he way, log means ln in this book, now sure why they did that. Easier to read log I suppose.

Now, look at Z_3 in (8.23). Although M^2 is "large", we still need to think of $Z_3 = 1 + \varepsilon$, it is close to 1. We cannot think of Z_3 as some huge number. The reason is this. The correct 8.26 says this:

$e^2 [Z_3 - \alpha/(15\pi m q^2)] = e^2 Z_3 [1 - \alpha/(Z_3 15\pi m q^2)] \approx e^2 Z_3 [1 - \alpha/15\pi m q^2)]$

$\approx e^2 Z_3 [1 - Z_3\alpha/15\pi m q^2)] = e_R^2[1 - \alpha_R/15\pi m q^2)]$ where error is then order α^2

Aside: When $q^2 << m^2$, the integral in 8.22 becomes $-\int dz\, z^2(1-z)^2 = -1/30 q^2/m^2$ (Maple), so we get

$2\alpha/\pi * -1/30 q^2/m^2 = -\alpha/15\pi\, q^2/m^2$ = the second term in 8.26

So, we plead ignorance of the value of α, and we claim that α_R is the thing that is 1/137, then 8.26 has a term which makes a physical prediction! I agree with 8.27 because I know that $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$ from my TK notes (proven somewhere in Jackson, a Gauss theorem thing).

Once we have 8.27, we see that our potential has some value at r = 0 and this will be picked up only by ℓ=0 wavefunctions and we get a shift from the naive H atom predictions. This could even be the Darwin term on page 52, but they don't claim that. I have not verified 8.28 this time through, but have checks from an earlier time. The prediction here is - 27MHz stabilization of the ℓ=0, and Uehling predicted it in 1935. But it gets swamped by other effects on the order of 1000 MHz described on page 60 of BD1, having to do with vacuum fluctuations of the EM field. Poor Uehling must have been disappointed. On page 159 BD do comment that the -27 MHz contribution has been verified as being part of the total shift picture, since measurements now have things (1964) down to 0.3 MHz precision, so maybe Uheling is happy again.

Review of this section to mid page 159. The single loop electron bubble $I_{\mu\nu}$ is given in 8.8 and we see that the d^4k integral diverges quadratically for large k. This is because each electron propagator is 1/k. One electron has k, the other has k - q, which is q-k for its positron sense, and k + (q-k) = q, the momentum feeding in and out of the loop. The offending integration region occurs where the electron has large +k, the positron large -k (and vice versa). I suppose digital space might tame such an integral.

In 8.12 the z integrals appear allowing us to do the d^4k integration. For large k, where our quadratic divergence was, I would say the z integrals are controlled by their small-z end. For large z the phase spins fast and contributes nothing, so I think our problem is now going to be at small z. Looking at the first term in 8.16, you see that the main term of interest goes as

$dz_1 dz_2\, z_1 z_2/(z_1+z_2)^4$ * phase $\approx$ $r dr d\theta\, r^2 SC/(rC+rS)^4 \sim dr/r$ = log div at r=0

at the low end. Were we to think of this as dxdy, we would write in polar coordinates is as shown on the right which shows a log divergence at the low end. The trick shown in 8.18 lets us replace $z_1 + z_2$ with the single new variable λ. and then we clearly see our divergence transferred to λ since get dλ/λ in 8.19. So this low-end log divergence is just a remapping of our original large-k quadratic divergence in k-space.

Then in 8.20 we do a "subtraction" to remove this log divergence, where M is a "large mass" -- that is all we are told. If it were 1000 times the electron mass, we would have log = 14 in 8.22 which would cancel roughly the 3π shown there, and the correction term is still order α.

The main physical result of this section is 8.26 where that q^2/m^2 extra term appears in the photon propagator altered by our fermion bubble. Think of the left side of 8.26 as follows:

$X = \alpha(1 + a\alpha + b\alpha)$ and define: $\alpha_R \equiv \alpha Z_3 = \alpha(1+a\alpha) = \alpha + a\alpha^2$

$\Rightarrow \quad X = \alpha_R + b\alpha^2$

where all the time we are thinking of both α and α_R as << 1. Now we want to expand α in terms of α_R. Here is one way to do that. We have this equation we can solve for α

$$a\alpha^2 + \alpha - \alpha_R = 0 \qquad \Rightarrow \qquad \alpha = [-1 + (1+4a\alpha_R)^{1/2}] / 2a$$

$$\Rightarrow \alpha = [-1 + 1\,(1 + 2a\alpha_R - \tfrac{1}{2}\, a^2\alpha_R^2 + ...)]/2a \approx \alpha_R - \tfrac{1}{4}\, a\alpha_R^2 + ... = \alpha_R(1 - \tfrac{1}{4}\, a\alpha_R)$$

Then we have

$$X = \alpha_R + b\alpha^2 = \alpha_R + b\,\alpha_R^2(1 - \tfrac{1}{4}\, a\alpha_R)^2 = \alpha_R + b\,\alpha_R^2(1 - 1/2\, a\alpha_R + a^2\alpha_R^2/16)$$

$$= \alpha_R + b\,\alpha_R^2 - \tfrac{1}{2}\, ba\alpha_R^3 + ... = \alpha_R [\, 1 + \alpha_R\, b - \tfrac{1}{2}\, ba\alpha_R^2 + ...] = \alpha_R [\, 1 + \alpha_R\, b + O(\alpha_R^2)...]$$

In this last result, the M-dependent constant "a" appears in the $O(\alpha_R^2)$ term, but we drop this term if we are looking at X only computed through order α_R^2. Then as claimed, M appears nowhere in the result. We have

$$X \approx \alpha_R [\, 1 + \alpha_R\, b] \qquad \alpha_R = 1/137$$

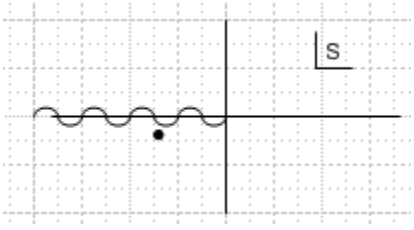
where $b = -(1/15\pi)(q^2/m^2)$. This is a hard prediction that the photon propagator is altered by exactly this numerical amount for small q^2 which is the regime of Coulomb physics. With some arm-waving, we might replace $q^2 = -q \bullet q = \nabla^2$ in r-space, and then the correction term on the far right of 8.26 becomes as shown in 8.27 in r-space. Then an energy level shift would be

$$\Delta E = \langle\psi|\, \Delta V|\psi\rangle = \int d^3x\, \psi^*(x)\, [\, K\, \delta^3(x)]\, \psi(x) = K\, |\psi(0)|^2 \qquad K = e_R^2\alpha_R/(15\pi m^2).$$

which is what the LHS of 8.28 says. Then you have to install your $\ell = 0$ wavefunction to get an actual shift, and we get the famous lowering of the hydrogen 2S state by 27 MHz. On page 57 this shift is called the Lamb shift because Lamb measured it in 1947. Lamb measured that the 2S actually was shifted up by about 1000 MHz, not down by -27 MHz, but there are various contributions and the - 27 MHz is one of them. Perhaps at the end of our current chapter we will see where the -27 fits in.

I have to admit the cutoff M is *very ugly*, but I also admit that we are getting a physical prediction here which after the dust settles is in agreement with measurement. Uehling in 1935 was the first to compute this -27 MHz contribution.

Derivation of 8.30 from 8.22. Think of the log (ln) argument as s in this picture:



where $s = 1 - (q^2/m^2)z(1-z)$. For small q^2, s is positive and real so we are on the positive real axis and the log has no imaginary part. But ln(s) develops an imaginary part when s goes negative, to the point indicated by my dot which is below the axis due to the $m^2 - i\varepsilon$. We then have $s = |s|\, e^{-i\pi}$ so we then have $\ln(s) = \ln|s| - i\pi$ and we have our imaginary part "$-\pi$", but only when $s < 0$, and this means

$$s < 0 \Rightarrow \quad z(1-z) > m^2/q^2 \qquad \text{hence } -i\pi\theta\,[\, z(1-z) - m^2/q^2\,]$$

So, if we are for some reason interested in the imaginary part of the square bracket in 8.22, multiplied by the bare propagator shown outside that bracket, the result is given by 8.30 equation A, so I have now verified this including its sign.

Now how do we do the integral over z? I threw this into Maple and had trouble until I carefully assumed a specific range for $a \equiv m^2/q^2$. Otherwise it produced an obviously wrong answer.

```
restart;
assume (a>0,a<0.25);
f := z*(1-z)*Heaviside(z*(1-z)-a);
```

$$f := z\,(1-z)\,\text{Heaviside}(z\,(1-z)-a\sim)$$

```
g :=int(f,z=0..1);
```

$$g := \frac{1}{4}\sqrt{1-4\,a\sim} - \frac{1}{12}(1-4\,a\sim)^{\left(\frac{3}{2}\right)}$$

```
factor(g);
```

$$-\frac{1}{12}\sqrt{1-4\,a\sim}\,(-2-4\,a\sim)$$

The result here is seen to be

$$\text{integral} = 1/6\ \sqrt{1\text{-}4a}\ (1+2a) \qquad a = m^2/q^2$$

$$= 1/6\ \sqrt{1\text{-}4\ m^2/q^2}\ (1 + 2\ m^2/q^2)$$

And then 2 * 1/6 = 1/3 and finally we have **verified 8.30.**

S-matrix expansion where λ denotes the order (set to 1 in the end)

$$(1 + \Sigma_{n=1}\ \lambda^n S_n)^\dagger\ (1 + \Sigma_{n=1}\ \lambda^n{}_n) = 1$$

$$\Rightarrow\ \Sigma_{n=1}\lambda^n\ (S_n{}^\dagger + S_n)\ + (\Sigma_{m=1}\ \lambda^m S_m{}^\dagger)(\ \Sigma_{n=1}\ \lambda^n S_n)\ = 0$$

Consider in general a double sum of this form:

$$\Sigma_{m=1}\ \Sigma_{n=1}\ \lambda^{m+n}\ f\,[m,n]$$

Let k = n+m and let ℓ = n-m so that n = (k+ℓ)/2 and m = (k–ℓ)/2. Then this sum can be reordered as follows:

$$\Sigma_{k=2}\,\lambda^k \sum_{\ell=-a\ \text{step}\ 2}^{a} f[(k-\ell)/2, (k+\ell)/2] \qquad \text{where } a = k-2$$

Here are some of the different k terms in this sum:

k = 2 $\ell = 0$ f [1,1]
k = 3 $\ell = -1,1$ f [2,1] + f [1,2]
k = 4 $\ell = -2,0,2$ f [3,1] + f [2,2] + f [1,3]
and so on

In our application, we have f [m,n] = $S_m^\dagger S_n$. So here is our equation above:

$$\Sigma_{k=1}\lambda^k (S_k^\dagger + S_k) = -\ \Sigma_{k=2}\,\lambda^k \sum_{\ell=-a\ \text{step}\ 2}^{a} S^\dagger{}_{(k-l)/2}\, S_{(k+l)/2}]$$

For k=1, the RHS is 0 and we get

$$(S_1^\dagger + S_1) = 0$$

For k = 2, we get

$$(S_2^\dagger + S_2) = -\,S^\dagger{}_1 S_1$$

For k = 3 we get

$$(S_3^\dagger + S_3) = -\,\{\, S^\dagger{}_2 S_1 + S^\dagger{}_1 S_2\}$$

For k = 4 we get

$$(S_4^\dagger + S_4) = -\,\{\, S^\dagger{}_3 S_1 + S^\dagger{}_2 S_2 + S^\dagger{}_1 S_3\}$$

and this is what BD 8.33 is saying, verified, and red check please. Notice that each side of each equation is Hermitian.

The subscript on S (their superscript) indicates the number of photon vertices since we get a power of e from each such vertex. Suppose we have an initial e^+e^- state as "i". Then $S^1 = 0$ because it cannot go into a single photon by energy conservation. For such an initial state, 8.33b then says that $S_2^\dagger = -S_2$ which is anti-hermitian. For 2-vertex graphs, we are talking 7.86 and the graphs on that page, so we expect the S shown in 7.86 to be anti Hermitian, and the overall "i" makes that true.

Next, we don't have any 3rd order graphs because each internal photon has 2 ends, see page 148 graphs, so we next look at the 4th order 8.33d which for us says

$$(S_4^\dagger + S_4) = -\,\{\, S^\dagger{}_2 S_2\}$$

What on earth are they trying to say here? If you look at figure 8-1(e), you have a 4th order amplitude which *looks like* two 2nd order graphs stacked one on top of the other. So maybe we should interpret $S^\dagger{}_2 S_2$ as the product of these two graphs, each of which is a second order e^+e^- second order amplitude.

The reason we busted ass to get 8.30 is that this is the REAL part of 8.22 and somehow this is associated with the Hermitian RHS of the above equation.

They do suggest one idea, that you should be able to "prove" that the QED S matrix is unitary, and they will do this in QFT.

Comment: Having looked at a few notes, I now remember more about the notions of unitarity and analyticity. An amplitude like S has cuts in the s-plane for each production threshold. In this case, you can interpret things like $S_4^\dagger + S_4$ as the difference in amplitude above and below the cut, which is the discontinuity across the cut. This in term is the imaginary part of S, so when an equation like the above has a non-vanishing RHS, it means the amplitude S_{fi} must have an imaginary part which is given by the RHS. I forget the details for now, this was a huge subject, I have my own notes, and there is a lot on this in various books I own such as Eden at all.

8.3 Renormalization of External Photon Lines (161).

The closed electron loop modified our propagator as shown in 8.22. If we have an external photon leg, it is on shell and has $q^2 = 0$ so the second term makes no contribution, and we have just a Z_3 multiplier showing in 8.22. The reasonable thing to do is put a $\sqrt{Z_3}$ factor on each end of this photon leg, realizing that it does have an "other end" somewhere miles away, as indicated by the picture on page 161. So at each end we get $e_R = \sqrt{Z_3}\, e$. Given this fact, we then don't want to "do it again" by worrying about loops in external photon lines. The authors hint that if you had multiple loops, their only effect is to shift Z_3 some more, so this notion really handles "all graphs" on the external leg. We shall see this again in BD2.

Final comment: from here on out, e_R shall be called "e" and the bare charge shall be called e_0.

8.4 Self-mass of the Electron (162).

Derive 8.35: Here is the starting position.

$$-i\Sigma(p) = e^2\, 1\, (1/2\pi)^4 \int d^4k\ \ i/(k^2-\lambda^2)\, \gamma_\nu\, (\not p-\not k+m) i/(\,(p-k)^2-m^2)\, \gamma^\nu$$

where I have stuck in two numerator i's and compensated. We can see a **linear divergence** at large k, which is not as bad as the quadratic divergence in $I_{\mu\nu}$ of the vacuum polarization. Elevate the two squared denominator factors according to 8.12, a standard no-brainer integral which says that $\int_0^\infty dz\, e^{iaz} = i/a$. We then have

$$\Sigma(p) = ie^2(1/2\pi)^4 \int d^4k\ \ \int_0^\infty dz_1 \int_0^\infty dz_2\ \ \exp i[(k^2-\lambda^2)z_1 + ((p-k)^2-m^2)z_2]\, \gamma_\nu(\not p-\not k+m)\, \gamma^\nu$$

I would sure be inclined at this point to use the theorem that $\gamma_\nu \not p \gamma^\nu = -2\not p$ so the last factor becomes

$\gamma_\nu(\not p-\not k+m)\, \gamma^\nu = -2\not p + 2\not k + 4m$, take out a 2 and get

$$\Sigma(p) = 2ie^2(1/2\pi)^4 \int d^4k\ \ \int_0^\infty dz_1 \int_0^\infty dz_2\ \ \exp i[(k^2-\lambda^2)z_1 + ((p-k)^2-m^2)z_2]\, (-\not p+\not k+2m)$$

Now we mimic what we did last time, but first we need to write out the phasor phase:

$$\begin{aligned} & \exp i[(k^2-\lambda^2)z_1 + ((p-k)^2 - m^2)z_2] \\ = \; & \exp i[(k^2-\lambda^2)z_1 + (k^2 - 2p.k + p^2 - m^2)z_2] \\ = \; & \exp i[(k^2-\lambda^2)z_1 + (k^2 - 2p.k + p^2 - m^2)z_2] \\ = \; & \exp i[k^2(z_1 + z_2) - 2p.k\, z_2 + p^2 z_2 - m^2 z_2 - \lambda^2 z_1 + i\varepsilon] \end{aligned}$$

This is very similar to what we had last time, with the difference that here $\lambda \neq m$. Notice that we don't replace $p^2 = m^2$ because we want to allow for this electron line to be off shell in the middle of some diagram. So let's try the same idea for completing the square as last time, ie,

$$\ell^\mu = k^\mu - z_2/(z_1+ z_2)\, p^\mu \qquad \Rightarrow d^4k = d^4\ell$$

$$\ell.\ell = [\, k^\mu - z_2/(z_1+ z_2)\, p^\mu][\, k_\mu - z_2/(z_1+ z_2)\, p_\mu]$$

$$= k^2 - 2\, z_2/(z_1+ z_2) k.p + z_2^2/(z_1+ z_2)^2\, p^2$$

$$\ell.\ell\,(z_1+ z_2) = k^2(z_1+ z_2) - 2\, z_2 k.p + z_2^2/(z_1+ z_2)\, p^2$$

and this combination does reproduce the first two terms of our exponential. So I guess we can say

$$\exp = \exp i\, \{\, \ell.\ell\,(z_1+ z_2) - z_2^2/(z_1+ z_2)\, p^2 + p^2 z_2 - \lambda^2 z_1 - m^2 z_2 + i\varepsilon \,\}$$

$$= \exp i\, \{\, \ell.\ell\,(z_1+ z_2) + i\varepsilon\} \exp i\, \{ - z_2^2/(z_1+ z_2)\, p^2 + p^2 z_2 - \lambda^2 z_1 - m^2 z_2 + i\varepsilon \,\}$$

$$= \exp i\, \{\, \ell.\ell\,(z_1+ z_2) + i\varepsilon\} \exp i\, \{p^2 z_1 z_2/(z_1+z_2) - \lambda^2 z_1 - m^2 z_2 + i\varepsilon \,\} \qquad \text{// trivial}$$

$$= \exp i\, \{\, \ell.\ell\,(z_1+ z_2) + i\varepsilon\} \exp i\, \{\text{other}\,\}$$

We now use the integral we used before which said

$$(2\pi)^{-4} \int d^4\ell \exp i\, \{\, (z_1+ z_2)\, \ell.\ell + i\varepsilon\} = +\, 1/(16i\pi^2)\, /(z_1+ z_2)^2$$

But let's back up now to see where we are:

$$\Sigma(p) = 2ie^2(1/2\pi)^4 \int_0^\infty dz_1 \int_0^\infty dz_2 \exp i\, \{\text{other}\,\} \int d^4k \exp i\, \{\, \ell.\ell\,(z_1+ z_2) + i\varepsilon\}\, (-\not{p}+\not{k}+2m)$$

where other as shown above does not include k so can be moved to the left of the d^4k. Now we have to do something about our $\not{k}$ sitting there on the right

$$k^\mu = \ell^\mu + z_2/(z_1+ z_2)\, p^\mu$$

$$\Rightarrow \quad (-\not{p}+\not{k}+2m) = -\not{p} + \not{\ell} + z_2/(z_1+z_2)\not{p} + 2m = -\not{p}\, z_1/(z_1+z_2) + \not{\ell} + 2m$$

According to 8.15 integral number 2, the ℓ^μ term will integrate to nothing, leaving us with:

$$\Sigma(p) = 2ie^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \, \text{expi}\{\text{other}\} (1/2\pi)^4 \int d^4k \, \text{expi}\{ \ell.\ell\,(z_1+z_2) + i\varepsilon\} [-\not{p}\, z_1/(z_1+z_2) + 2m]$$

$$= 2ie^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \, \text{expi}\{\text{other}\} [-\not{p}\, z_1/(z_1+z_2) + 2m] (1/2\pi)^4 \int d^4k \, \text{expi}\{ \ell.\ell\,(z_1+z_2) + i\varepsilon\}$$

$$= 2ie^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \, \text{expi}\{\text{other}\} [-\not{p}\, z_1/(z_1+z_2) + 2m] \; 1/(16i\pi^2)/(z_1+z_2)^2$$

$$= 2ie^2 \, 1/(16i\pi^2) \int_0^\infty dz_1 \int_0^\infty dz_2 \, 1/(z_1+z_2)^2 [-\not{p}\, z_1/(z_1+z_2) + 2m] \, \text{expi}\{\text{other}\}$$

$$= e^2/(8\pi^2) \int_0^\infty dz_1 \int_0^\infty dz_2 \, 1/(z_1+z_2)^2 [-\not{p}\, z_1/(z_1+z_2) + 2m] \, \text{expi}\{\text{other}\}$$

$$= \alpha/(2\pi) \int_0^\infty dz_1 \int_0^\infty dz_2 \, 1/(z_1+z_2)^2 [-\not{p}\, z_1/(z_1+z_2) + 2m]$$
$$* \, \text{expi}\{ p^2 z_1 z_2/(z_1+z_2) - \lambda^2 z_1 - m^2 z_2 \}$$

and miraculously, this agrees exactly with 8.35.

Derive 8.36: As before, we want to get a single variable to replace z_1+z_2 to expose problems, so we insert

$$1 = \int_0^\infty d\gamma/\gamma \, \delta(1 - [z_1+z_2]/\gamma)$$

to get

$$\Sigma(p) = \alpha/(2\pi) \int_0^\infty dz_1 \int_0^\infty dz_2 \, 1/(z_1+z_2)^2 \{\int_0^\infty d\gamma/\gamma \, \delta(1 - [z_1+z_2]/\gamma)\} [-\not{p}\, z_1/(z_1+z_2) + 2m]$$
$$* \, \text{expi}\{ p^2 z_1 z_2/(z_1+z_2) - \lambda^2 z_1 - m^2 z_2 \}$$

$$= \alpha/(2\pi) \int_0^\infty d\gamma/\gamma \int_0^\infty dz_1 \int_0^\infty dz_2 \, 1/(z_1+z_2)^2 \{\delta(1 - [z_1+z_2]/\gamma)\} [-\not{p}\, z_1/(z_1+z_2) + 2m]$$
$$* \, \text{expi}\{ p^2 z_1 z_2/(z_1+z_2) - \lambda^2 z_1 - m^2 z_2 \} \qquad \text{// moved } \int_0^\infty d\gamma/\gamma \text{ to the left}$$

Now at this point, scale both $z_i \rightarrow \gamma z_i$ to get

$$= \alpha/(2\pi) \int_0^\infty d\gamma/\gamma^3 \int_0^\infty \gamma \, dz_1 \int_0^\infty \gamma \, dz_2 \, 1(z_1+z_2)^2 \{\delta(1 - [z_1+z_2])\} [-\not{p}\, z_1/(z_1+z_2) + 2m]$$
$$* \, \text{expi}\gamma\{ p^2 z_1 z_2/(z_1+z_2) - \lambda^2 z_1 - m^2 z_2 \}$$

Now we are restricting integration to the line segment $\delta(1 - [z_1+z_2])$ which we know is in the z square, so now is the time we can lower the z endpoints to get (and cancel some γ's)

$$= \alpha/(2\pi) \int_0^\infty d\gamma/\gamma \int_0^1 dz_1 \int_0^1 dz_2 \, 1(z_1+z_2)^2 \{\delta(1 - [z_1+z_2])\} [-\not{p}\, z_1/(z_1+z_2) + 2m]$$
$$* \, \text{expi}\gamma\{ p^2 z_1 z_2/(z_1+z_2) - \lambda^2 z_1 - m^2 z_2 \}$$

$= \alpha/(2\pi) \int_0^\infty d\gamma/\gamma \int_0^1 dz_1 \int_0^1 dz_2 \{\delta(1 - [z_1+z_2])\} [- \not{p} (1-z_2) +2m]$
$* \exp i\gamma \{ p^2(1-z_2)z_2 - \lambda^2(1-z_2) - m^2 z_2 \}$ // set $z_1 + z_2 = 1$ and $z_1 = (1-z_2)$

$= \alpha/(2\pi) \int_0^\infty d\gamma/\gamma \int_0^1 dz [- \not{p} (1-z) +2m] * \exp i\gamma \{ p^2(1-z)z - \lambda^2(1-z) - m^2 z \}$
// do the z_1 integration and rename $z_2 \rightarrow z$

$= \alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) +2m] \int_0^\infty d\gamma/\gamma \exp i\gamma \{ p^2(1-z)z - \lambda^2(1-z) - m^2 z \}$
// move $\int_0^\infty d\gamma/\gamma$ to the right

$= \alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) +2m] \quad J(p,m,\lambda,z)$ // define J

and we get another score here, since this matches 8.36 exactly. We see again the log divergence in the γ integration at the low γ end

Derive 8.37. I derived this earlier, see back about 10 pages.

Comment: With $I_{\mu\nu}(q^2)$ we were shown how to do all this calculation stuff, now we are having a "test" to see if we learned anything! So far I have passed.

Derive page 163 equation A. Now we do the same "subtraction" gimmick as before using 8.37. From the above expression for $\Sigma(p)$, we extract the integral and call it J, so we get

$$\bar{J} = J(p,m,\lambda,z) - J(p,m,\Lambda,z)$$

where the cutoff large Λ is installed for the photon mass, which seems a bit odd. Last time we put it in for the electron mass. But OK, we get

$$\bar{J} = \int_0^\infty d\gamma/\gamma \; [\exp i\gamma \{ p^2 z(1-z) - m^2 z - \lambda^2(1-z)\} - \exp i\gamma \{ p^2 z(1-z) - m^2 z - \Lambda^2(1-z)\}$$

$$= \log [p^2 z(1-z) - m^2 z - \Lambda^2(1-z) / p^2 z(1-z) - m^2 z - \lambda^2(1-z)]$$

$$\approx \log [\Lambda^2(1-z) / - p^2 z(1-z) + m^2 z + \lambda^2(1-z)] \qquad \textbf{// agrees with A}$$

$$= [\Lambda^2(1-z)/m^2 z^2 / (- p^2 z(1-z) + m^2 z + \lambda^2(1-z))/m^2 z^2]$$

$$= \log [\Lambda^2(1-z)/m^2 z^2] + \log [m^2 z^2 / - p^2 z(1-z) + m^2 z + \lambda^2(1-z)]$$

So where does the extra term $\lambda^2(1-z)$ come from in the numerator of the second log as seen in B ? I guess I could do this instead:

$$= [\Lambda^2(1-z)/[m^2 z^2 + \lambda^2(1-z)] / (- p^2 z(1-z) + m^2 z + \lambda^2(1-z))/ [m^2 z^2 + \lambda^2(1-z)]]$$

$$= \log [\Lambda^2(1-z)/[m^2 z^2 + \lambda^2(1-z)]] + \log [m^2 z^2 + \lambda^2(1-z) / - p^2 z(1-z) + m^2 z + \lambda^2(1-z)]$$

Then in the first log you say $\lambda << m$, the assumed tiny photon mass is smaller than electron mass. Then you would get

$$= \log [\Lambda^2 (1-z)/[m^2 z^2]] + \log [m^2 z^2 + \lambda^2 (1-z) / - p^2 z (1-z) + m^2 z + \lambda^2 (1-z)]$$

and then we get agreement with B. I suppose you can add in whatever negligible terms you want anywhere you want, seems a bit artificial, but maybe it helps a later integration.

If we make this J replacement inside Σ, we get

$$\Sigma(p) = \alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) + 2m] \quad J(p,m,\lambda,z)$$

$$= \alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) + 2m] \log [\Lambda^2 (1-z)/[m^2 z^2]]$$
$$+ \alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) + 2m] \log [m^2 z^2 + \lambda^2 (1-z) / - p^2 z (1-z) + m^2 z + \lambda^2 (1-z)]$$

and this **agrees with B.**

Now write $\log [\Lambda^2 (1-z)/[m^2 z^2]] = \log(\Lambda^2/m^2) + \log [(1-z)/z^2]$ and we have 3 terms for Σ above:

$$\text{term1} = \alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) + 2m] \log(\Lambda^2/m^2)$$

$$= \alpha/(2\pi) \log(\Lambda^2/m^2) \int_0^1 dz [- \not{p} (1-z) + 2m]$$

$$= \alpha/(2\pi) \log(\Lambda^2/m^2) [- \not{p} \int_0^1 dz (1-z) + 2m \int_0^1 dz]$$

$$= \alpha/(2\pi) \log(\Lambda^2/m^2) [- \not{p} ½ + 2m]$$

$$= \alpha/(4\pi) \log(\Lambda^2/m^2) [- \not{p} + m + 3m]$$

$$= 3m\alpha/(4\pi) \log(\Lambda^2/m^2) - \alpha/(4\pi) \log(\Lambda^2/m^2) [\not{p} - m]$$

so our "first term" gives the first *two* terms in C. Our second term is this:

$$\alpha/(2\pi) \int_0^1 dz [- \not{p} (1-z) + 2m] \log [(1-z)/z^2]$$

and we go compute

$$\int_0^1 dz(1-z) \log [(1-z)/z^2] = 5/4$$
$$\int_0^1 dz \log [(1-z)/z^2] = 1$$

so our "second term" gives

$$\alpha/(2\pi) [- 5/4 \not{p} + 2m]$$

so our first three terms then give

$$3\alpha/(4\pi)m \log(\Lambda^2/m^2) \; - \; \alpha/(4\pi) \log(\Lambda^2/m^2) [\not{p} - m] + \alpha/(2\pi) [-5/4 \not{p} + 2m]$$

so why are they throwing out this finite term I just found??? We have to assume that $\log(\Lambda^2/m^2) >> 1$ and then we can ignore it. OK, so we will ignore it and then we have obtained 8.38 because the residual integral is what we have left above

$$\Sigma(p) = 3\alpha/(4\pi)m \log(\Lambda^2/m^2) \; - \; \alpha/(4\pi) \log(\Lambda^2/m^2) [\not{p} - m] + \alpha/(2\pi) [-5/4 \not{p} + 2m] \text{ (ignore)}$$
$$+ \alpha/(2\pi) \int_0^1 dz [-\not{p}(1-z) + 2m] \log [m^2z^2 + \lambda^2(1-z) / -p^2z(1-z) + m^2z + \lambda^2(1-z)]$$

which is 8.38 where we ignore the ignore term.

Now it is time to do this last integral in the limit where we can ignore λ. We get

$$\alpha/(2\pi) \int_0^1 dz [-\not{p}(1-z) + 2m] \log [m^2z^2 / -p^2z(1-z) + m^2z]$$
$$= \alpha/(2\pi) \int_0^1 dz [-\not{p}(1-z) + 2m] \log [m^2z / -p^2(1-z) + m^2] \qquad \text{// } \textbf{agrees with D}$$

and notice that the $\lambda^2(1-z)$ I never liked in the numerator is now gone away so makes no difference.

Now define $I1 = \int_0^1 dz(1-z) \log [m^2z / -p^2(1-z) + m^2]$ and throw this into Maple to get

```
I1 :=Int((1-z)*ln(m^2*z / (m^2 - p^2*(1-z))), z=0..1);
```

$$I1 := \int_0^1 (1-z) \ln\left(\frac{m^2 z}{m^2 - p^2(1-z)}\right) dz$$

```
I1 :=int((1-z)*ln(m^2*z / (m^2 - p^2*(1-z))), z=0..1);
```

$$I1 := \frac{1}{2} \frac{-\ln((p-m)(p+m))p^4 + p^2m^2 + m^4\ln((p-m)(p+m)) - p^4 + \ln(-m^2)p^4 - m^4\ln(-m^2)}{p^4}$$

Then we do hand algebra on this I1 to get

$$I1 = \tfrac{1}{2} (m^2 - p^2)/p^2 \; \{ \; (m^2 + p^2)/p^2 * \log[(m^2 - p^2)/m^2 + 1 \}$$

so the $\not{p}$ term in our integral becomes:

$$\alpha/(2\pi) \int_0^1 dz [-\not{p}(1-z) + 2m] \log [m^2z / -p^2(1-z) + m^2]$$
$$= -\not{p}\alpha/(2\pi) \; I1$$

$$= -\alpha/(4\pi) \not{p} \; (m^2 - p^2)/p^2 \; \{ \; (m^2 + p^2)/p^2 * \log[(m^2 - p^2)/m^2] + 1 \}$$

and this agrees with the **second term in E**.

Now define I1 = $\int_0^1 dz \log [m^2 z / - p^2 (1-z) + m^2]$ and throw this into Maple to get

```
I2 :=Int((1-0)*ln(m^2*z / (m^2 - p^2*(1-z))), z=0..1);
```

$$I2 := \int_0^1 \ln\left(\frac{m^2 z}{m^2 - p^2 (1-z)}\right) dz$$

```
I2 :=int((1-0)*ln(m^2*z / (m^2 - p^2*(1-z))), z=0..1);
```

$$I2 := \frac{(-m^2 + p^2)(-\ln((p-m)(p+m)) + \ln(-m^2))}{p^2}$$

and hand algebra gives

I2 = $(m^2 - p^2)/p^2$ * log $[(m^2 - p^2)/m^2]$

so the other term in our integral becomes

$\alpha/(2\pi) \int_0^1 dz$ [- ⧸p (1-z) +2m] log $[m^2 z / - p^2 (1-z) + m^2]$

= $\alpha/(2\pi)$ 2m I2

= $m\alpha/\pi$ * $(m^2 - p^2)/p^2$ * log $[(m^2 - p^2)/m^2]$

and this replicates the **first term in E**.

So, D and E and equation 8.39 are now verified. Notice that λ is gone and played no role at this point.

Now to get 8.40, we can see at once that the first two terms are just copied down from C above, so the third term is what we want to show here. It seems clear that we want to replace $p^2 = m^2$ in the various denominators of E, so we get

E = $m\alpha/\pi$ * $(m^2 - p^2)/p^2$ * log $[(m^2 - p^2)/m^2]$
- $\alpha/(4\pi)$ ⧸p $(m^2 - p^2)/p^2$ { $(m^2 + p^2)/p^2$ * log$[(m^2 - p^2)/m^2]$ + 1 }

= $\alpha/(4\pi)$ * $(m^2 - p^2)/p^2$ [m * 4log $[(m^2 - p^2)/m^2]$ – ⧸p * { $(m^2 + p^2)/m^2$ * log$[(m^2 - p^2)/m^2]$ + 1 }]

Now set $p^2 = m^2$ in denominators and the $(m^2 + p^2)$ factor to get

= $\alpha/(4\pi)$ * $(m^2 - p^2)/m^2$ [m * 4* log $[(m^2 - p^2)/m^2]$ – ⧸p * { 2* log$[(m^2 - p^2)/m^2]$ + 1 }]

Now write the outside factor as

$(m^2 - p^2) = - (p^2 - m^2) = - (⧸p + m) (⧸p - m) = -2m (⧸p - m)$

where we have approximated by saying $\bar{u}\,\not{p} \approx \not{p}\,m$, since we know this whole thing presses against $\bar{u}$ on the left side as shown in 8.7. So at this point we have

$$= -\alpha/(4\pi) * 2(\not{p} - m)/m \; [\; m \; * \; 4* \log [(m^2 - p^2)/m^2] \; - m * \{ 2* \log[(m^2 - p^2)/m^2] \; + 1 \} \;\;]$$

where we have now also replaced $\not{p} = m$ in the second term. As for the "1" in this second term, the log is presumably large in magnitude as we go to our limit, so we can drop the 1. We then have

$$= -\alpha/(4\pi) * 2(\not{p} - m)/m \; [\; m \; * \; 4* \log [(m^2 - p^2)/m^2] \; - m * 2* \log[(m^2 - p^2)/m^2] \;\;]$$

$$= -\alpha/(4\pi) * 2(\not{p} - m)/m \; [\; m \; * \; 2* \log [(m^2 - p^2)/m^2] \;]$$

$$= -\alpha/(4\pi) * (\not{p} - m) * \; 4 * \log [(m^2 - p^2)/m^2] \;]$$

and finally we reproduce the last term in 8.40. This is a very tricky path to follow. We don't want to say $\not{p} = m$ everywhere because then this entire term disappears.

Now let's derive what I will call 8.40A which is for a different limit, namely $p^2 = m^2$ and we now want to keep the λ stuff in 8.38. In this case the log argument in 8.38 would be

$$\log [\; m^2z^2 + \lambda^2(1-z) \; / \; \lambda^2(1-z) + m^2z^2 \;]$$

Well, I **give up** on this. One reason is that I don't think the numerator λ^2 term should exist in page 163 A,B and C. For our limit here, this term suddenly becomes important. Secondly, I tried setting $\alpha = p^2 - m^2$ and making Maple do the integral over z, and I used a multiplying (1+z) from $[2m - \not{p}(1-z)]$, but the results don't seem to do what they say. The integrals are a mess, but doable, and you have to hunt through for the leading terms. I now realize that BD are taking all these results from the paper or book referenced on page 164, and there are probably assumptions not stated by BD. So let's just accept their claim as follows:

8.40A: If you are in the limit $m^2 - p^2 << m\lambda$, then the last term in 8.40 is different.
The last two terms become:

$$\Sigma_{\texttt{last two terms}} \; = \; \alpha/(4\pi) * (\not{p} - m) * \; \{ \log\Lambda^2/m^2 - 2 \; \log [m^2/\lambda^2] \}$$

Comment: is there any BD errata?

bjorken drell errata => 405 hits
bjorken drell errata "relativistic quantum mechanics" 130 hits

Again I find nothing, authors sites nada, McGraw Hill nada.

Now the closing comment on this section which I now understand better, see comment earlier. When an electron carries off shell $p^2 > m^2$, it can of course generate photons, and then we get the idea that we are

over a production threshold, so S ought to acquire an imaginary part as usual. You see that happening in 8.40 where the log in this case gets an imaginary part $i\pi$, etc etc.

8.5 Renormalization of the Electron Propagator (164).

Line A comes from 8.7 and the definition of Σ. The second line is far from obvious, however simple it might look. You have to first remember that Σ is order α, so we are allowed to add any α^2 terms or higher we want. So let $a = -i(\not{p}-m)$ and $b = -i\Sigma$, and keep ordering correct, then write

$S = 1/a + 1/a\ b\ 1/a + 1/a\ b\ 1/a\ b\ 1/a + ...$

$= 1/a\ [\ 1 + b/a + b/a\ b/a + ...] = 1/a\ \{\ 1/(1-b/a) = 1/a\ \{\ a/[a-b]\ \} = 1/[a-b] = i\ [\ ia - ib\ ]$

$= i/[\ \not{p} - m - \Sigma\ ]$ hence 8.41 to order α

where we added an infinite number of terms, did the usual sum, and out pops the result. Now we see why that -i factor was tacked onto the definition of Σ, then Σ itself is a mass shift!

Question: why do we get a mass shift for the electron, but we did not get a mass shift for the photon? We could think of 8.21 as the start of a similar infinite series and say the sum of that series was

$1/(q^2 - I)$ where I is a matrix in the $\mu\nu$ sense and 1 means $g_{\mu\nu}$

in some sense. I can see several reasons. As 8.20 shows, $I_{\mu\nu}(q=0) = 0$, so even if you could add up the series in some way, you would have no mass shift. Surely gauge invariance is also at play here, and of course we know that renormalized photons still to the speed of light and must be massless.

We continue and define seemingly obscure objects like δm and Z_2 and C(p). I have verified all the algebra on this page except for 8.44 which I will do now. Looking at 8.43, you see that Z_2 (we don't yet know what Z_2 will be used for) is "close to 1", differing by order α. And here we see the need for both the high and low end cutoffs Λ and λ ! Recall that the UV divergence of Σ is tamed by doing the subtraction at mass Λ, which removes the low-end γ log divergence via 8.37. It seems that in the $\not{p}=m$ limit, we have another IR divergence which λ is taming, as originally introduced in 8.34. So somehow they both work their way into this Z_2 object.

Now, the first equality in **8.44** I call A follows directly from 8.41 with no approx. But then we set $Z_2 = 1$ in both places in the denominator. The claim is that when you do this, you only change the result by order α^2 , though I have not proven that to be the case. And so we arrive at 8.44 and then we realized that δm is a mass increase on the electron due to our little photon bubble diagram we are doing. So now we are talking mass renormalization.

BD comment that in a purely classical model of electron radius a somehow, it should have a mass that is $mc^2 + e^2/4\pi a = m + \alpha/a$, so we would classically be inclined to say $m_{ph} = m_{bare} + \alpha/a$. For point electron, we have an obvious linear divergence here in the physical mass. In our Dirac point particle case, we get only a log Λ divergence. Remember that if we take $\Lambda \rightarrow \infty$, we are removing the subtraction (fast phase shows this) and returning to the bare theory. BD claim that someone actually figured out why the

divergence is linear classically and logarithmic in QED, our friend **Viki Weisskopf**. I just read up on him, died 2002, worked pretty hard in this QED area in fact. But I know his name from Blatt and Weisskopf which is a 1952 Theoretical Nuclear Physics book. Not on the Russian site, and I don't have it.

Now we get a list of sort of random topics:

(1) On page 165 we see the first comment on what the size of something like δm might be. If you set the cutoff much larger than the mass of the universe, you *still* get δm << m because α is so small.

(2) You could reformulate the Dirac theory as shown in 8.45 where you now put m_{ph} in your propagator. But then you have to include a propagator correction graph of the form 166 top, which of course just compensates. I seem to remember using such graphs somewhere in the past. (**mass counterterm**)

(3) We shall from now on use "m" to mean the renormalized physical mass

(4) As you get to the important $\not{p} \approx m$ limit, the propagator we just found in 8.44 becomes 8.46. Here, we are now using the corrected physical mass m, and C(p) = 0 in this limit. So this tells us that you ought to take each propagator and slide $\sqrt{Z_2}$ each direction to a vertex where it combines with the charge, just as we did with $\sqrt{Z_3}$.

(5) For external electron lines, I think it is the same as for photon lines: you imagine you have done all your corrections to an external leg, then that makes a $\sqrt{Z_2}$ by the same "distant source" argument. But I think both this and the internal propagator $\sqrt{Z_2}$ factors will all be cancelled by a factor we shall soon learn about from the vertex correction graphs.

(6) We know that a propagator is associated with a product of fields. We saw this back in 6.48, and we will be very familiar with this when we get to QFT. This, the $\sqrt{Z_2}$ really renormalizes the field or wave function. But we are used to "normalizing wavefunctions" from regular old QM. For example, 8.47 shows how you might start with a zeroth order wavefunction φ_n which is itself properly normalized to 1. But then in perturbation theory, you get corrections to this wavefunction to various orders, and the first order correction is shown as the sum. You can see this in Schiff: page 246 (31.8) shows you the first order wavefunction correction (see also 31.2) and 31.10 shows the same thing we have here in BD.

So as you add a wavefunction correction, the coefficient of the original φ_n zeroth order term in the wavefunction has to decrease a little so that the adjusted wavefunction ψ_n can be normalized to 1. It is easy to show exactly what they are saying

The analogy BD make here is that an external QED leg is a field or a φ thing, and when you add up all its little bubble corrections, it becomes a ψ thing, and you get a $\sqrt{Z}$ factor on the leg as in 8.47.

8.6 The Vertex Correction (166).

This is a very long 10 page section with lots of math detail. The vertex graph of interest is shown page 167 with the expression in 8.49 which I have checked. Page 167 figure has all electron lines so is easiest to understand in terms of the momentum 4 vector signs. The "second order vertex part".

Our 8.49 is the most convergent we have seen yet, doing $k^3dk/k^4 = dk/k = \ln(k_{max})$, so only log divergent.

The object Λ_μ is a 4x4 matrix so lives in Dirac space. Since everything is covariant, it has the same transformation properties as γ_μ which we recall transforms as a 4-vector under Dirac transformations. Equation 8.50 is a powerful symmetry statement which I suspect applies in all orders: $\bar{u}\,\Lambda_\mu\,u$ at zero momentum transfer $q = p'-p = 0$ must be proportional to $\bar{u}\,\gamma_\mu\,u$ and the proportionality is $(Z_1^{-1}- 1)$ where Z_1 is simply defined this way, and is a function of $p^2= m^2$ and also λ^2. There are no other 4-vectors available since as they say $p_\mu \sim \gamma_\mu$ when you are in the sandwich: $p_\mu = mv_\mu = m\,\bar{u}\,\gamma_\mu\,u$. This IS the momentum of a Dirac particle. (see lemma below giving a better proof of this fact)

Now we come to the powerful claims (8.51) and (8.52) which I need to verify.

Verify 8.52: I think 8.52 comes first:

$$\partial/\partial p^\mu \{ 1/(\not{p}-m) \} = \partial/\partial p^\mu \{ (\not{p}+m)/(p^2- m^2) \} = (\not{p}+m)\,\partial/\partial p^\mu\{1/(p^2- m^2)\} + \gamma_\mu /(p^2- m^2)$$

$$\partial/\partial p^\mu\{1/(p^2- m^2)\} = - 1/(p^2- m^2)^2 * 2p_\mu$$

$$\partial/\partial p^\mu \{ (\not{p}+m)/(p^2- m^2) \} = - (\not{p}+m)\,1/(p^2- m^2)^2 * 2p_\mu + \gamma_\mu /(p^2- m^2)$$

$$= 1/(p^2- m^2)^2 \{ - 2p_\mu (\not{p}+m) + (p^2- m^2)\,\gamma_\mu \}$$

STOP. Comment: for the first time I looked for old notes on this BD1 book and found some at once in a manila folder labeled QED/Radiation. I see what happened "last time" when I ran into the above identity. I got it exactly to the point shown above. You wonder how your are going to get rid of the p_μ to get the RHS of 8.52. The trick is going to be this relation:

$\not{p}\gamma_\mu = - \gamma_\mu\not{p} + 2p^\mu$ which comes from $\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu = 2g_{\mu\nu}$

Then the above numerator becomes

$$\{ - 2p_\mu (\not{p}+m) + (p^2- m^2)\,\gamma_\mu \}$$
$$= \{ - 2p_\mu (\not{p}+m) + \gamma_\mu (p^2- m^2) \}$$
$$= \{ - 2p_\mu (\not{p}+m) + \gamma_\mu (\not{p}-m) (\not{p}+m) \}$$
$$= \{ - [\not{p}\gamma_\mu+ \gamma_\mu\not{p}] (\not{p}+m) + \gamma_\mu (\not{p}-m) (\not{p}+m) \}$$
$$= \{ - [\not{p}\gamma_\mu+ \gamma_\mu\not{p}] + \gamma_\mu (\not{p}-m) \} (\not{p}+m)$$
$$= \{ - [\not{p}\gamma_\mu] + \gamma_\mu (-m) \} (\not{p}+m)$$
$$= \{ - [\not{p}\gamma_\mu] + \gamma_\mu (-m) \} (\not{p}+m)$$
$$= \{ - [\not{p}] + (-m) \}\,\gamma_\mu (\not{p}+m)$$
$$= \{ - (\not{p}+m) \}\,\gamma_\mu (\not{p}+m)$$
$$= - (\not{p}+m)\,\gamma_\mu (\not{p}+m)$$

So I might call this thing a lemma, namely

$$-(\not p+m)\,\gamma_\mu\,(\not p+m) \;=\; -\;2p_\mu\,(\not p+m) + (p^2-m^2)\,\gamma_\mu$$

Then the rest is easy,

$$\partial/\partial p^\mu\ \{\ 1/(\not p-m)\ \} = 1/(p^2-m^2)^2\ \{\ -\ 2p_\mu\,(\not p+m) + (p^2-m^2)\,\gamma_\mu\ \}$$

$$= 1/(p^2-m^2)^2\ \{\ -\ (\not p+m)\,\gamma_\mu\,(\not p+m)\ \}$$

$$= -\ 1/(\not p-m) * \gamma_\mu\ * 1/(\not p-m) \qquad \text{which is 8.52}$$

$$\partial/\partial p^\mu\{\frac{1}{\not p-m}\} = -\frac{1}{\not p-m}\ \gamma_\mu\ \frac{1}{\not p-m}$$

or

$$\partial/\partial p^\mu\{\frac{i}{\not p-m}\} = i\,\frac{i}{\not p-m}\ \gamma_\mu\ \frac{i}{\not p-m}$$

Verify 8.51: Start with Σ taken from 8.34 and use nicer notation now

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\ \frac{-i}{k^2-\lambda^2}\ \gamma_\nu\ \frac{i}{\not p-\not k-m}\ \gamma^\nu \qquad \text{// which is 8.34}$$

Now apply the derivative $\partial/\partial p^\mu$ to both sides:

$$-i\,\partial/\partial p^\mu\,\Sigma(p) \;=\; (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\ \frac{-i}{k^2-\lambda^2}\ \gamma_\nu\ i\ \frac{i}{\not p-\not k-m}\ \gamma_\mu\ \frac{i}{\not p-\not k-m}\ \gamma^\nu$$

$$-\,\partial/\partial p^\mu\,\Sigma(p) \;=\; (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\ \frac{-i}{k^2-\lambda^2}\ \gamma_\nu\ \frac{i}{\not p-\not k-m}\ \gamma_\mu\ \frac{i}{\not p-\not k-m}\ \gamma^\nu$$

$$= \Lambda_\mu(p,p) \qquad \text{// as shown in 8.49}$$

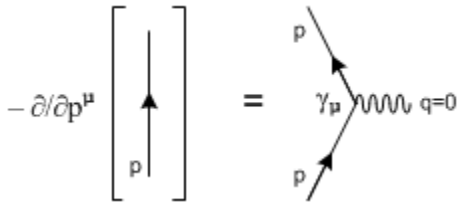
This is "the Ward identity" which we have shown here only to one order of α, but which we shall later see is true to all orders.

By the way, we can use our same lemma above to show that

$$p^\mu = p^\mu\ \bar u u = \bar u\ p^\mu\ u = \tfrac{1}{2}\,\bar u\ 2p^\mu\ u = \tfrac{1}{2}\,\bar u\ \{\ \not p\gamma_\mu + \gamma_\mu \not p\}\ u = \tfrac{1}{2}\,\bar u\ \{\ m\gamma_\mu + \gamma_\mu m\}\ u = m\ \bar u\gamma_\mu\ u$$

Then suppose 8.50 had a term of the form Ap^μ. We would write that as $Am\ \bar u\gamma_\mu\ u$ and then we would absorb Am into the definition of Z_1 .

Here is what 8.52 says graphically



Differentiation is the same as adding a zero-energy (zero-4-momentum) photon vertex with no charge e, just the γ_μ to match the ∂_μ.

So what are we going to do with this Ward identity. Let's gather together three pieces of information we have so far developed:

(1) $\Sigma(p) = \delta m - [\, Z_2^{-1} - 1 + C(p)\,]\,(\not{p} - m)$ 8.42

(2) $\partial/\partial p^\mu\, \Sigma(p) = -\,\Lambda_\mu(p,p)$ 8.51 (Ward)

(3) $\bar{u}\,\Lambda_\mu\, u = (Z_1^{-1} - 1)\,\bar{u}\,\gamma_\mu\, u$ 8.50

Now, 8.40 says that C(p) = a function of p, in fact $C(p) = \alpha/4\pi\ 4\log[(m^2-p^2)/m^2]$, *but* when we go very close to $p^2 = m^2$ we get the alternate version 8.40A which says $C(p) \equiv 0$ because the last term is independent of p and is absorbed into Z_2 as shown in 8.43. Going very close to this limit and then differentiating, we get $\partial_\mu C(p) = \partial_\mu 0 = 0$. And Z_2 as shown in 8.43 does not depend on p^2. So we can say from (1) above:

$$\partial/\partial p^\mu\, \Sigma(p) = -\,(Z_2^{-1} - 1)\,\partial/\partial p^\mu\, \not{p} = -\,(Z_2^{-1} - 1)\,\gamma_\mu = -\,\Lambda_\mu(p,p)$$

Comparison of this last result to (3) above shows that

$$Z_1 = Z_2$$

which we later will see is true to all orders. This is perhaps also called the Ward identity.

Why is this true in our second order calculation? We have Feynman expressions for both sides of (2), from which we proved (2). We have a "most general possible" expression shown in 8.50 for $\bar{u}\,\Lambda_\mu\, u$ based on "symmetry". Then by doing various integrals, we evaluated our $\Sigma(p)$ to have the form 8.42. Putting these three pieces together tells us this rather amazing fact that $Z_1 = Z_2$.

Let's restate the result two lines back:

$$\Lambda_\mu(p,p) = (Z_2^{-1} - 1)\,\gamma_\mu = (Z_1^{-1} - 1)\,\gamma_\mu = \text{a 4x4 Dirac matrix}$$

I agree with all the equations top of page 168, including the definition of $\Lambda_\mu^{\,c}(p',p)$, and I do imagine it is finite if we regulate the infrared cutoff λ, and I am glad we are not computing this all right now!

Now, back on page 157 we found that the $q^2 = 0$ photon propagator gets multiplied by Z_3 when you include the vacuum polarization correction. And we could associate a $\sqrt{Z_3}$ with each external leg. Then we could absorb all these factors into the bare charge and renormalize it to $\sqrt{Z_3}\, e$.

Then on page 164 we found that near $p^2 = m^2$ the electron propagator picks up a factor of Z_2. We decided to shunt off a $\sqrt{Z_2}$ to each vertex, and also a $\sqrt{Z_2}$ to each external leg. So this too renormalizes the charge at a vertex. So the nature of this correction is very similar to that of Z_3 and the photon propagator.

Now what have we learned about the vertex correction? According to our 8.55, we find that close to p = p' and q=0 we have

adjusted vertex = $\gamma_\mu + \Lambda_\mu = Z_1^{-1}\gamma_\mu$ // so apply $1/Z_1$ at a vertex

Now, we know that what meets at a vertex (recall Chew vertex) is two electrons and one photon. Thus, our total vertex adjustment is

$$\sqrt{Z_3}\sqrt{Z_2}\sqrt{Z_2} / Z_1 = \sqrt{Z_3}$$

and as claimed, the vertex correction cancels the electron propagator correction.

Example of How the above works out.

BD now show what I have just stated above, but they do so in their usual obscure manner, no doubt with an eye toward BD2. Let's go through their list: [notice that we have temporarily reverted to the notation that e = bare charge, but m is still the renormalized mass

(a) this is the usual basic vertex

(b) this is the adjustment due to the vertex part, and (a) + (b) = $\gamma_\mu + \Lambda_\mu = Z_1^{-1}\gamma_\mu$ as I show above.

(c) This one is less clear. First of all, there are two graphs, not one. The expression shown is for one of the two graphs, and later we will multiply by 2. Just doing the Feynman rules for one graph in (c) we get:

$$(-ie\gamma^\mu)\frac{i}{\not p - m}(-i\Sigma(p)) = (-i)(-ie\gamma^\mu)\frac{i}{\not p - m}\left[\,\delta m - (Z_2^{-1} - 1)(\not p - m)\,\right] \qquad \text{// from 8.42}$$

$$= \delta m \frac{1}{\not p - m}(-ie\gamma^\mu) - (Z_2^{-1} - 1)(-ie\gamma^\mu)$$

Page 164 8.41 shows the "Feynman rule" for adding our Σ object, and this goes back to 8.7. You need to put in the combination $-i\Sigma(p)$ for the bubble here, not $\Sigma(p)$ by itself. Recall that this definition of Σ causes the desired form shown at the end of 8.41 where no i's appear.

(d) Looking back at 8.45 on page 165, we see that using renormalized mass means we have a new kind of "interaction" of the form $-\delta m\,\psi$ which behaves in the theory like $+e\not A\psi$. It is a real vertex, and when you put it in, you define new propagators on sides which are internal to the graph. If you put in $(-ie\gamma_\mu)$ for the

normal vertex, then I guess you should put in (-i [-δm]) = +iδm for a mass counterterm vertex. This rule is confirmed bottom of page 287! I have not formally shown this, but think I could. Now we see this coming into play in graph (d) . Again we have two graphs that contribute identically, so later multiply by 2. The Feynman rules for our graph here are:

$(i\delta m)\,(-ie\gamma_\mu)\,\dfrac{i}{\not p - m}$ which is what they shown in (d)

(e) Here only one graph. The renormalized photon propagator is the bare times Z_3 as I noted a few paragraphs above. The corrective part is then $(Z_3\text{-}1)\,(-ie\gamma_\mu)$ due to the graph shown.

Now what about the Feynman "rules" for the external lines? Consider again the photon case. The photon propagator is multiplied by Z_3. For an internal photon propagator, we do $\sqrt{Z_3}$ to each vertex and absorb that into the renorm charge. To maintain this rule at a vertex which ties to an external photon, we have to shove off a factor of $\sqrt{Z_3}$ onto the external photon leg. If we attach the leg to a distant source, then that $\sqrt{Z_3}$ hits the distant vertex and all is well. So in that case, we treat the external photon leg as a fully renormalized photon propagator. If we treat it as such and it is an external leg, then we have installed a full Z_3 for this thing because the leg is fully renormalized with all bubbles, but we should only have installed just the $\sqrt{Z_3}$ factor. THAT is why we have to divide by $\sqrt{Z_3}$ for an external photon leg.

In our age 168 graphs we see this happening with the Coulomb photon. It gets a full factor of Z_3 as shown in (e). That is to say, the sum of (a) + (e) is our approximation to the full bore photon propagator on that leg, and we have installed a full factor of Z_3 for this sum. It is sort of "overcounting" in a sense, and we only wanted to install half of this factor because we have only one vertex. I like it.

The theory does the same thing with the electron legs. This Z_2 full factor is shown in 8.44 in the numerator and you imagine the same idea of dividing by $\sqrt{Z_2}$ for each external electron leg. When we come to the list shown in 8.57, however, it is not very obvious how this is working out. We don't directly see a factor of full Z_2 being added that we have to divide out. This has to do with order α stuff, and recall that

$Z_2 = 1 + \varepsilon \qquad Z_2^{-1} = 1\text{-}\varepsilon \qquad [Z_2^{-1} - 1] = -\varepsilon = [\,1 - Z_2]$

So here is my algebra for 8.58. All ε_i are order α and we ignore higher orders.

$Z_1 = 1 + \varepsilon_1$
$Z_2 = 1 + \varepsilon_2$
$Z_3 = 1 + \varepsilon_3$

[in line A] $= 1 + (1 - Z_1) - 2(1 - Z_2) + (Z_3\text{-}1)$ // twice using the rule just noted above
$= 1 - (Z_1 - 1) + 2\,(Z_1 - 1) + (Z_3\text{-}1)$
$= 1 - \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$

Meanwhile we have (always throw out higher order terms like $\varepsilon_2\varepsilon_3$ or ε_3^2)

$$1/[\, Z_2\sqrt{Z_3}\,] = 1/[(1+\varepsilon_2)(1+\varepsilon_3/2)] = (1-\varepsilon_2)(1-\varepsilon_3/2) = (1 - \varepsilon_2 - \varepsilon_3/2)$$

then all of line A is equal to

$$(1 - \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3)\,(1 - \varepsilon_2 - \varepsilon_3/2) = 1 - \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 - \varepsilon_2 - \varepsilon_3/2 = 1 - \varepsilon_1 + \varepsilon_2 + \varepsilon_3/2$$

Meanwhile, the claimed result is this on line C

$$Z_1^{-1}\, Z_2\, \sqrt{Z_3} = (1-\varepsilon_1)(1+\varepsilon_2)(1 + \varepsilon_3/2) = 1 + \varepsilon_3/2 - \varepsilon_1 + \varepsilon_2$$

and this agrees with my line A calculation above. I have no idea where BD got the totally obscure expression shown in line B! My old notes skipped completely over this obscurity.

Comment: Above I gave a quick arm-waving argument for why the renormalization of the vertex offsets the renormalization of the electron propagator. One makes $1/Z_1$ and the other makes Z_2 and since these are equal, the effect is zero.

BD then do a specific example -- the graphs shown on page 168. We used the Feynman rules for each graph, wrote the amplitude for each graph, then we added them up and confirmed the arm-waving result above, at least to first order in α. This was a very good thing for them to do, though it cost them and me a lot of time to work through in detail. I see now that the odd form in line B will appear in volume 2 and that is why they are parking it here and I continue to ignore it.

We have now seen in a primitive low order way what is going to happen in a heavier duty way in the last chapter of volume 2 which is entitled Renormalization and consumes about 100 pages of that volume! I do think it is good to get a hint of an idea in a simple case before you undertake the heavy lifting.

The anomalous magnetic moment of the electron.

Another mid section subject change of BD.

Our only earlier calculation of an observable was the 27 MHz Lamb shift *component* arising from the modification of the photon propagator by the lowest order fermion loop graph, the vacuum polarization. This propagator modification adds a $\delta^3(\mathbf{r})$ piece to the potential $V(\mathbf{r}) = 1/r$ of the H atom, and this causes a shift in S state energies because they have amplitude at r=0. This 27 MHz is the shift of the 2S below the 2P, again, just a component as we shall see later.

Here they are going to show that our vertex correction graph(s) are going to cause a correction to the magnetic moment of the electron in the Dirac theory. It took me a while to understand even what this means, see separate doc "mag moment of electron".

On page 11 1.26 we see $H' = \boldsymbol{\alpha}\bullet\mathbf{A} + e\Phi = e\,\gamma^0\,\gamma^\mu\,A_\mu$. If we do first order perturbation theory with this H', we find that

$$E' = \int d^3x\, \psi^\dagger\, H'\, \psi = \int d^3x\, J^\mu A_\mu \qquad \text{where } J^\mu = \bar{\psi}\,\gamma^\mu\,\psi$$

If we do the Gordon decomposition of this current (see op cit) then, when we take the non-rel limit, we find a term in the energy E' which is of the form $-\mu_\beta\, \boldsymbol{\sigma}\bullet\mathbf{B} = -2\,\mu_\beta\, \mathbf{S}\bullet\mathbf{B}$ and we find that the electron has a g-factor of g=2.000000000.

But this is only first order perturbation theory. As in non-rel QM, you have higher order contributions to the energy E. When you do this in RQM, the perturbation theory is the theory of the Feynman graphs, and those energy difference denominators from the non-rel QM perturbation theory show up as "propagators" in QED. So, when you do higher orders in the perturbation theory, you are not surprised to find new energy terms of the form - $\Delta\mu\ \boldsymbol{\sigma} \bullet \mathbf{B}$ and this causes g to depart from the first order value of 2.

In the present situation, we are going to find that the vertex correction graph causes an effective change in the "current" which starts off just being $\bar{\psi}\ \gamma^{\mu}\ \psi$, but then we will be finding that the higher order graph causes a change $\gamma^{\mu} \rightarrow \gamma^{\mu} + \Lambda^{\mu}$ and when we decompose this new piece in the Gordon way, we get a change to the g factor. In particular, we will find that

$$j^{\mu}{}_{\mathtt{altered}} = \text{const} * \bar{u}(p',s)\ \{\ (p^{\mu} + p'^{\mu})/2m + i\ (1+f)\ q_{\nu}\sigma^{\mu\nu}/2m\ \}\ u(p,s)$$

then we conclude that the gyromagnetic ratio has been altered as well in this way

$$g_{\mathtt{altered}} = g(1+f).$$

When you first see this idea, you wonder what on earth the above $q_{\nu}\sigma^{\mu\nu}$ term in the electron current has to do with a magnetic moment that you measure in a non-rel experiment. But we know that $\sigma^{ij} = \Sigma^{k} = (\sigma_{k},\sigma_{k})$ the spin matrix. I show in my doc that the proportionality between magnetic moment and spin is precisely determined by this spin term in the Gordon current. So it is a little less of a mystery now than when I first started.

So the program BD are going to follow is this: study the integral 8.49 and find in it a piece such that when we add the new graph and make $\gamma^{\mu} \rightarrow \gamma^{\mu} + \Lambda^{\mu}$, we get the form shown above appearing, and we conclude that our alteration of g is (1+f).

In a basic sense, this Λ^{μ} integral is the worst one yet we have encountered because there are three propagators in a single loop. I am not going to try to follow the details of this calculation, but I did prove (see separate doc) the important Feynman formula at the top of page 170. We apply that formula here with n=3 because we have three propagator denominators. This method is different from that used in earlier work where we just elevated each propagator onto an exponential. Here the three propagators are combined into a polynomial power factor in the denominator. How this is done is, in fact, shown in the lengthy footnote, where the three denominators are shown explicitly with f(k) holding the rest.

Footnote: Just want to learn the method here. The first red check is just a copy of 8.49 as just noted. The second is where we apply Feynman's formula with n=3, so we have $[...]^{3}$ on the bottom. Recall in our previous calculations that we "completed the square" in an exponent as in 8.14 where we went from k to $\boldsymbol{\ell}$ as an integration variable, just doing a shift. Here, we complete the square in the denominator by doing a similar shift, and of course the price paid is a very messy object "c". The shift seems to be this

$$k = \boldsymbol{\ell} + p'z_{2} + pz_{3} \qquad \text{// since we now have } f(k) = f(\boldsymbol{\ell} + p'z_{2} + pz_{3})$$

and $\boldsymbol{\ell}$ is then renamed k. I am unclear then how the $d^{4}k$ integral is done. Perhaps we move $d^{3}k$ to the left, and do the dk^{0} first and this picks up a residue from the triple poles at the usual locations. Once this is done, we still have to do the $d^{3}k$ integrations with a big mess up top, and they don't comment on how this is done. As before, we have to do a subtraction at some large mass Λ and that is how Λ appears. We also

have the IR cutoff λ and the net result is three messy terms shown in 8.60. Each of the last two terms is a triple z_i integral with the delta function present.

We assume this Λ_μ is sandwiched between the usual spinors $\bar{u}(p')$ and $u(p)$ which allows simplifications. At top of 171 they claim that the numerator of the second term in 8.60 can be written as 8.61, and we see now the $[\not{q},\gamma_\mu]$ term sitting there which is our desired $q_\nu\sigma^{\mu\nu}$ type animal. They claim that the z integrations have actually been done, in that 1949 Feynman paper and elsewhere, and they are very ugly things.

But, the integrals are not too bad to do in the limits of very large or very small q^2. So, it is the small q^2 one that is of main interest to us.

The first Λ term in 8.60 corresponds to the Z_1^{-1} term in 8.55 which is absorbed into renormalization. The other two integrals in 8.60 are for the residual piece of 8.55 which is called $\gamma_\mu + \Lambda_\mu^c$ and we now need to stare at the small q^2 result 8.62.

There are two distinct things to realize in this result:

(1) recall from 8.26 that we found this kind of adjustment there from vacuum polarization:

$$\gamma_0 [\ 1 - \ \alpha/15\pi\ q^2/m^2\ ] \ = \gamma_0 [\ 1 - \ 1/5\ \alpha/3\pi\ q^2/m^2\ ]$$

We thought of this as a correction to the photon propagator, but an easier way to think of this is as an adjustment to the factor γ_0 sandwiched between the spinors.

Well, our vertex correction diagram causes more of this type of correction so we have to add them together. The correction shown in 8.62 is this:

$$\gamma_0 [\ 1 - \ \alpha/8\pi\ q^2/m^2\ ] \ = \gamma_0 [\ 1 - \ 3/8\ \alpha/3\pi\ q^2/m^2\ ]$$

So that is why you see them talking about adding the -1/5 with the -3/8.

(2) The second item is that spin term on the right of 8.62 of the form

$$\alpha/(8\pi m)\ q^\nu[\gamma_\nu, \gamma_\mu] = \alpha/(8\pi m)(2/i)\ q^\nu\sigma_{\nu\mu} = \alpha/(4\pi m)\ iq^\nu\sigma_{\mu\nu}$$

When we add this extra term to the main term γ_μ with the Gordon, we get

$$j^\mu{}_{\texttt{altered}} = \text{const} * \bar{u}(p',s)\ \{\ (p^\mu + p'^\mu)/2m + (1+f)\ i\ q_\nu\sigma^{\mu\nu}/2m\ \}\ u(p,s)$$

where $f = \alpha/2\pi$, which is an amazingly simple result. This means that

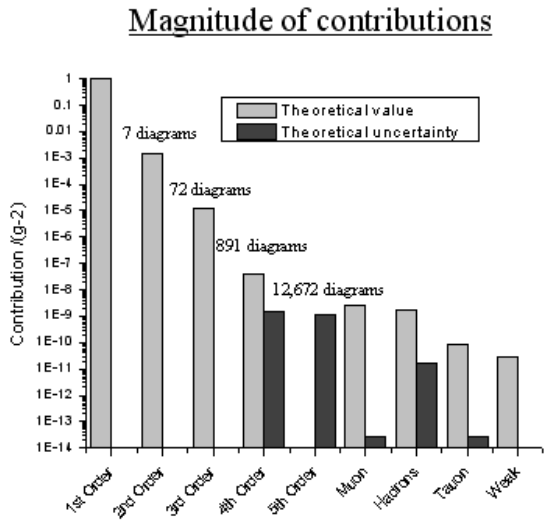
$$g = 2\ *\ (1+\alpha/2\pi)\ \approx 2*\ (1.001162)\ = (2.002323)$$

Measurement is this:

Electron g_e 2.002 319 304 3622 uncertainty = 0.000 000 000 0015

BD comment that two separate authors in 1957 computed the next order and their result is stated on page 172. Schwinger gets first credit in 1948 for the f = α/2π result. Wiki cleans that *two* more levels have now been completed!

I found a nice PPT by Chris Vo and scanned it, very good. Here are some interesting facts:



Bibliography

•First order g-2: J. Schwinger *Phys. Rev.* **73**, 416L (1948)

•Second order: C. M. Sommerfield, *Phys. Rev.* **107**, 328 (1957); *Ann. Phys. (N. Y.)* **5**, 26 (1958); A. Petermann, *Helv. Phys. Acta* **30**, 407 (1957).

•Third order: S. Laporta and E. Remiddi, *Phys. Lett. B* **265**, 182 (1991); **356**, 390 (1995); **379**, 283 (1996); S. Laporta, *Phys. Rev. D* **47**, 4793 (1993); *Phys. Lett. B* **343**, 421 (1995)

•4th order numerical: T. Kinoshita, M. Nio *hep-ph/0507249* (2005)

Experimental g-2: Brian C. Odom *Fully Quantum Measurement of the Electron Magnetic Moment,* Ph. D thesis, Harvard University (2004)

So calculations were done 1948. 1957, 1991-1996, and 2005. The last guy's work is described here

http://books.google.com/books?id=bhuBDAcc2zQC&pg=PA218&lpg=PA218&dq=kinoshita+anomalous+electron+magnetic+moment&source=web&ots=X1cTQBjwW6&sig=xvP4k5BHzlDy9WyD5oS4rH2lrkI&hl=en&sa=X&oi=book_result&resnum=1&ct=result#PPA219,M1

in a Google book QED by the author. This is a 1990 very long book that talks about all the precision measurements and calculation of QED, and shows the 4th order work! This guy has really systematized the integrals and the divergences into Kirchhoff diagrams as I might have done.

The most precise experiment uses a "Penning Trap" which is a combination of axial B field and some E fields to get confinement of electrons

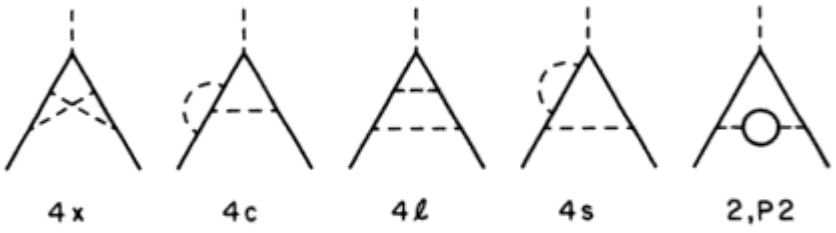


Figure 2: Examples of fourth-order vertex diagrams contributing to C_2.



Figure 3: Examples of sixth-order vertex diagrams contributing to C_3.

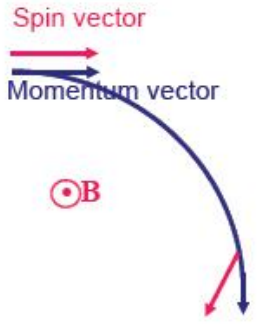
The anomalous magnetic moment

a_μ(theo) = a_μ(QED)+a_μ(had)+a_μ(weak) (+ a_μ(new physics) ?)

a_μ(QED) = 11 658 470.6 (0.3) $\times 10^{-10}$
a_μ(had) = 694.9 (8.) $\times 10^{-10}$ (based on e+e-)
a_μ(had) = 709.6 (7.) $\times 10^{-10}$ (based on τ)
a_μ(weak)= 15.4 (0.3) $\times 10^{-10}$

The Principle of the muon g-2 measurement

Non-relativistic case



$$\omega_c = \frac{eB}{m}$$

$$\omega_s = \frac{g}{2}\frac{eB}{m}$$

$$\omega_a = \omega_s - \omega_c = \frac{g}{2}\frac{eB}{m} - \frac{eB}{m} = \left(\frac{g-2}{2}\right)\frac{eB}{m} \Rightarrow \boxed{\omega_a = a\frac{eB}{m}}$$

This last is very good to see. The electron rotates at ω_c, while the spin rotates at ω_s so there is a beat frequency you can measure. I have now a few PDF's on these general subjects.

Comment: the anomalous moment is a low-energy effect in some sense. We always talk in the above about q = p'-p → 0. We first go non-rel with p' and p. Then we end up with an expectation value of something measured in the state u(p) so we have set p' = p. I guess we really take this q→0 limit first and get our <Σ> matrix thing, then do non-rel to get <σ>. So perhaps this is the ultimate in "low energy physics" experiments, as opposed to "high energy physics".

One more wiki:

The most precise **measurement** of α comes from the anomalous magnetic dipole moment, or g−2 ("g minus 2"), of the electron.[3] To make this measurement, two ingredients are needed:

1) A precise measurement of the anomalous magnetic dipole moment, and
2) A precise theoretical calculation of the anomalous magnetic dipole moment in terms of α.

As of February 2007, the best measurement of the anomalous magnetic dipole moment of the electron was made by Gabrielse et al.[4] using a **single electron** caught in a Penning trap. The difference between the electron's cyclotron frequency and its spin precession frequency in a magnetic field is proportional to g−2. An extremely high precision measurement of the quantized energies of the cyclotron orbits, or *Landau levels*, of the electron, compared to the quantized energies of the electron's two possible spin orientations, gives a value for the electron's spin g-factor:

g/2 = 1.001 159 652 180 85 (76),

a precision of better than one part in a trillion. (The digits in parentheses indicate the uncertainty in the last listed digits of the measurement.)

Infrared Divergence Problems

Another mid-section topic change. In the previous chapter on page 122 we computed Bremsstrahlung occurring as part of Coulomb scattering, and before that we did Coulomb scattering on page 106 with the Mott cross section. In practice, you cannot stop Brem from occurring with Coulomb scattering so you have to consider both processes together when you do an experiment measuring soft photons. You have to add amplitudes for elastic and inelastic scattering.

So here we have this same issue. The vertex graph we are playing with now has IR problems and this occurs when you do Coulomb scattering, you cannot stop it. So you have to add all the right cross sections together to get something that you can measure. So now we have to add all these things:

Coulomb elastic scattering
Coulomb with Brem
Coulomb with vertex-bridging extra photon

So BD now go through 5 long pages to show that when you add these things up, the IR divergences cancel. In 8.65 they show what the vertex correction does to elastic scattering, and in 8.67 they quote the earlier result what Brem does. In each case you get a multiple of the simple elastic scattering. So they just want to add these two guys and show the cancellation.

But alas, the IR cutoffs in the two cases were done in this book by different methods, so one or the other has to be redone. So they do all this horrible stuff and end up with 8.81 for the total result and you see that nothing blows up anymore at the IR end. That is to say, in 8.65 the vertex contribution blows up as log $1/\lambda$, and in 8.67 the Brem blows up as $1/k_{min}$, and there you see the two incompatible methods!

One side effect of changing IR cutoff methods is that the our vac pol number gets changed some more which brings in a 5/6. In these long 5 pages they show how Σ changes due to the change in cutoff method, amount δΣ, and this causes a corresponding $\delta\Lambda_c$ as in 8.76, and there you see an extra -5/6 we have to add in. The final result for $\Lambda_\mu{}^c$ is shown in 8.87.

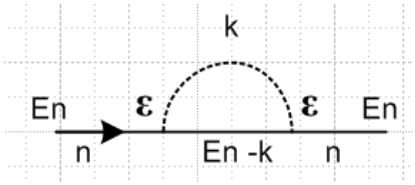
8.7 The Lamb Shift (177).

We simply add our vac pol result to our vertex graph result and that is 8.82. Recall that the -1/5 is from the vac pol, the -3/8 from the vertex by our first method, and the +5/6 comes from our conversion to the k_{min} cutoff method on the vertex correction graph.

Now how do we get to 8.83? First, insert the Coulomb A^μ as shown and replace $\gamma^0\gamma^0 = 1$ on the left side. Now Coulomb scattering is elastic, something that always seems a little strange. The charge source is infinitely massive M, so there is no recoil and so no energy loss. But of course there has to be some recoil so the scattering electron can change direction, so how do we resolve this contradiction? Well assume a direct backscatter worst case of the electron so $\Delta p = 2p_e$. OK, so the massive object recoils with this momentum. But then $E = (\Delta p)^2/2M \rightarrow 0$. If electron goes super relativistic, OK, γ will build up enough to overcome any M, but then we just make a larger M. That is the definition of Coulomb scattering. It is this large M limit. So the large M absorbs the momentum but no energy and the process is elastic. This means that $q^0 = 0$ and so q.q = **-q•q** and that explains what is going on in 8.83 at the start of the inner paren. Finally, we have $i\,\sigma_{\mu\nu}q^\nu \rightarrow i\,(i\,\gamma_0\gamma_i)q^i = +\gamma^i q^i = \boldsymbol{\gamma\bullet q}$. Thus, the first term went negative, but the last term stayed with a plus sign. So I am happy with 8.83.

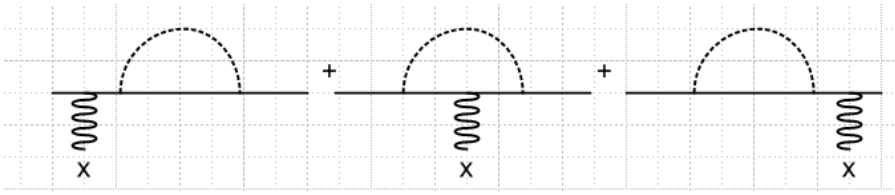
I agree with equations A and 8.84, math done in pencil at bottom of page 177. So 8.84 is our Lamb shift for a wavefunction that has amplitude at r=0, but we have to deal with the k_{min} problem.

This k_{min} section is very strange. They start with an estimate in 8.85 of the energy shift due to all soft photons below $k_{min} \sim m\,(Z\alpha)^{3/2}$, say. I think all the states are Dirac states like |n> and α really means the 4x4 α matrices. If you go back to page 11, we have H' = -e **α•A** . So here is what 8.85 is talking about, where **α • A** becomes **α • ε**



which is to say we are really making a computation about Σ for soft photons, then we will use the Ward which says $\Lambda_\mu = \partial/\partial p^\mu \Sigma$. So this is a soft photon mass renormalization calculation, or whatever you want to call it. Because photons are soft, Bethe used non-rel 2nd order perturbation theory as in 8.85. When you to non rel, you get to replace **α** with diagonal matrix **p**/m 1 where recall that $v = \bar{\psi}\,\alpha\,\psi$ and all that. This gets us to 8.86, but to get there you have to do some polarization sums and then the d^3k integral, and I have not attempted this. They argue away the first term in 8.86 as being part of mass renormalization , and they are left with just the second term. Relative to ΔE, k_{min} is large now, we get equation A on page 179, and then 8.87 itself, where he has installed $\bar{E}$ as an average energy difference somehow, which allows the sum to be done some other how and we get 8.88.

Well, now I retract that comment about Ward. I now think we should think of the above picture and our 2nd order perturbation theory in terms of these Feynman graphs where each has q = 0 so the Coulomb photon has no effect when things are very soft. This photon in hydrogen is going to the proton.



So these graphs include contributions from (c) and (b) on page 168. We are getting the total soft photon energy shift of a state "n" due to all three of these graphs added together. Then we need to add that to our formal calculation for $k > k_{min}$ and we end up with 8.89.

But now the result depends on this average $\bar{E}$ thing. We end of with the preponderance of the Lamb shift (the 1000 MHz) being in the first term in 8.89, but nothing much is said about the size of things. Then n in 8.89 refers to the principle quantum number in hydrogen.

Comment: This section is bad news in many respects.

(1) We learn nothing of the experimental facts of the Lamb shift, which I summarized in my spin-orbit document as

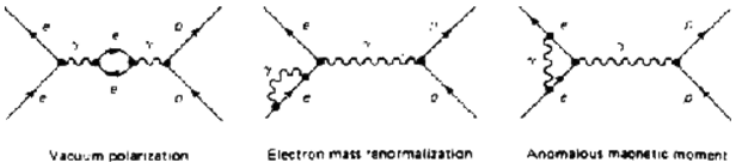


Figure 5: Feynman loop diagrams showing some effects that contribute to the Lamb shift.

Table 3 shows how much each of these contribute to the splitting of $2s_{1/2}$ and $2p_{1/2}$.

Effect	Energy contribution
Vacuum polarization	-27 MHz
Electron mass renormalization	+1017 MHz
Anomalous magnetic moment	+68 MHz
Total	+1058 MHz

Table 3: Contribution of different effects to the energy splitting of $2s_{1/2}$ and $2p_{1/2}$ in hydrogen. Numbers are given in units of frequency $\nu = E/h$.

and here you see that the big term is associated with mass renormalization, and the picture makes it seem this is associated only with the type (c) graphs on page 168

(2) they make no connection between Bethe's calculation and the graphs.

(3) all kinds of vague features are shown, and no number is presented as a result.

I did stumble onto an audio recording of a quite old **Bethe** talking about how he first did this little calculation involving "electron self-energy". **Sakurai** shows Lamb in two book sections, and the large soft photon piece can be done without QED and this is how Bethe got his 1000 MHz. Bethe died 2005.

So in some sense, QED as practiced in BD accounts for a minor part of the Lamb shift, and you have to rely on non-rel classical stuff to get the bulk, so maybe that is why BD don't do much with it. They did say in their introduction that they were not going to do much with bound state problems like hydrogen, and they short changed that whole part accordingly. Sakura is better on these things.

Wrap-up page 180. The claim is that you can compute to arbitrarily high order in α if you work hard enough, and that renormalization is fully controlled and results can be predicted well, so QED is a success. I might add however that no-one has computed α or m_e or mass ratios, even 44 years after BD was published.

I think the rest of BD1 involves bringing in other particle types, such as pions.