BD V1 Chap 9 : The Klein-Gordon Equation PhL 11.4.08

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9.1 Introduction (184)

In Chapter 1 recall that we threw out the KG equation for ψ because we could not get a conserved current with a positive definite $\rho = \psi^*\psi$. Historically I think, after this throwing out, the Dirac theory appeared, and we got into that notion of negative energy states being positive ones going backwards in time. This same idea saves the KG equation, and it becomes a viable model for spin 0 particles.

There are no long-term stable spin-0 bosons, but the π and K particles are not too bad. They have lifetimes that are long compared to the "natural time unit" $\hbar/mc^2 = 10^{-23}$ sec and so we can still regard them as legs on Feynman diagrams in a KG-photon theory. Equation 9.2 shows some reactions involving these π and K with p and n particles. Then 9.3 shows decay modes of the charged π and K and the lifetimes against these decays are long because these decays are weak interactions. The neutrals in the reactions of 186 A go faster but the lifetimes are still long enough for us to make a model here.

In this chapter we are going to look at the QED-like interactions between $\pi^\pm$ particles and photons, and I think the theory here can almost be considered part of "QED". We still have a small coupling constant α so perturbation theory should work. We could then look at applications like pion-photon scattering, or $\pi^+\pi^-$ production from photons or annihilation into same, or just scattering or Bhabha-like scattering. The idea is that we maintain the photon from the "real" QED, and we try to find a "model" for our spinless (but chargeful) pions or kaons that lets the photon interact with them in a consistent relativistic manner. So, we shall somehow replace the Dirac electron with a spinless particle.

9.2 The Klein-Gordon Propagator (186)

In a relativistic theory of charged particles, you need a model for your *electric* 4-current j^μ and as such, it must be a a *real* current and a *conserved* current $\partial_\mu j^\mu = 0$ so that the model has a well defined charge $Q = \int d^3x\, j^0$. The question is: what is that model for the current?

In **non-rel SE theory**, we construct a real 3-current as follows (set $\hbar = 1$)

$$\mathbf{j} = 1/(2im)\,[\,\psi^*(\nabla\psi) - (\nabla\psi^*)\,\psi]$$

We want to see what ρ is, such that $\nabla\bullet\mathbf{j} + \partial_t\rho = 0$. We find that

$$\nabla\bullet\mathbf{j} = 1/(2im)\,[\,\psi^*\nabla^2\psi - \psi\nabla^2\psi^*]$$

and then we use $H = -\nabla^2/2m + V$ to replace $\nabla^2 = 2m\,(V-H)$, that is, we use the non-rel SE. The V terms cancel and we get

$$\nabla\bullet\mathbf{j} = 1/i\,[\,-\psi^*H\psi + \psi\,H\psi^*\,] = i\,[\,\psi^*H\psi - \psi\,H\psi^*\,] = i\,[\,\psi^*H\psi - \psi\,(H\psi)^*\,]$$

where we assume V is real. We then replace $H\psi = i\partial_t\psi$ in both places to get

$$\nabla\bullet\mathbf{j} = i\,[\,\psi^*(\,i\partial_t\psi) - \psi\,(i\partial_t\psi)^*\,] = i\,[\,\psi^*(\,i\partial_t\psi) + i\,\psi\,(\partial_t\psi)^*\,] = -\partial_t(\psi^*\psi)$$

$$= -\,[\,\psi^*(\,\partial_t\psi) + \psi\,(\partial_t\psi)^*\,] = -\partial_t(\psi^*\psi)$$

and at this point we can identify $\psi^*\psi = \rho$, it is positive definite, and all is well.

Notice that we COULD define a 4-current based on our 3-current above

$$j^\mu = i\,[\,\psi^*(\partial^\mu\psi) - (\partial^\mu\psi^*)\,\psi]/2m$$

but then we find that

$$j^0 = i\,[\,\psi^*(\partial^0\psi) - (\partial^0\psi^*)\,\psi]/2m \neq \psi^*\psi$$

so it is not even close! We have time derivatives on the left and not on the right. So, in the non-rel SE theory, you do not regard ($j^0 = \psi^*\psi$, $\mathbf{j} = 1/(2im)\,[\,\psi^*(\nabla\psi) - (\nabla\psi^*)\,\psi]$) as a 4-vector. This is no surprise really, since it is non rel theory.

But now we are going to claim that j^μ shown above (but not including the 1/2m) is the right conserved current for the relativistic KG theory. The way this works is pretty amazing I think. Because the above combination of time derivatives for j^0 occurs so often, it gets a special symbol,

$$j^0 = i\,[\,\psi^*(\partial^0\psi) - (\partial^0\psi^*)\,\psi] = \psi^*\, i\overleftrightarrow{\partial}^0\,\psi$$

So for the moment, let's "try out" the above as a charge density.

Aside: Why is this current conserved?

$$-i\partial_\mu j^\mu = \partial_\mu[\,\psi^*(\partial^\mu\psi) - (\partial^\mu\psi^*)\,\psi]$$

$$= (\partial_\mu\psi^*)(\partial^\mu\psi) + \psi^*(\partial_\mu\partial^\mu\psi) - (\partial_\mu\partial^\mu\psi^*)\,\psi - (\partial^\mu\psi^*)(\partial_\mu\psi)$$

$$= \psi^*(\partial_\mu\partial^\mu\psi) - (\partial_\mu\partial^\mu\psi^*)\,\psi$$

$$= -m^2\,\psi^*\psi + m^2\,\psi^*\psi = 0$$

where we use the KG equation which says

$$(\partial_\mu \partial^\mu + m^2)\,\psi = 0$$

and this equation was shown way back on page 6 top.

Now let's "try out" the following plane wave forms for the $\pi^\pm$ wavefunctions:

$$f^\pm_{\mathbf{p}}(x) = (2\pi)^{-3/2}\ 1/\sqrt{2\omega}\ e^{\mp\, ip.x}$$ f^+ is a positive energy solution, f^- negative energy

where by ω we mean E, but for some reason they use ω here. It looks more like a photon leg in this regard than a Dirac particle leg. The constant is shown to get properly normalization as we see in a moment. Now we do some computation, just going blind for the moment:

$$\int d^3x\ f^\pm_{\mathbf{p}'}(x)^*\ i\overleftrightarrow{\partial}^0\ f^\pm_{\mathbf{p}}(x) =$$

$$= (2\pi)^{-3}\ 1/\sqrt{2\omega}\ 1\sqrt{2\omega'}\ \int d^3x\ e^{\pm ip'.x}\ i\overleftrightarrow{\partial}^0\ e^{\mp\, ip.x}$$

$$= (2\pi)^{-3}\ 1/\sqrt{2\omega}\ 1\sqrt{2\omega'}\ e^{\pm i\omega' t}\ i\overleftrightarrow{\partial}^0\ e^{\mp\, i\omega t} \int d^3x\ e^{\mp\, ip'\bullet x}\ e^{\pm ip\bullet x}$$

where we use the fact that non-time-dependent objects move through $\overleftrightarrow{\partial}^0$. I think we can do the delta function first to get

$$= (1/2\omega)\ [e^{\pm i\omega t}\ i\overleftrightarrow{\partial}^0\ e^{\mp\, i\omega t}]\ \delta^3(\mathbf{p}-\mathbf{p}')$$

and then we evaluate

$$e^{\pm i\omega t}\ i\overleftrightarrow{\partial}^0\ e^{\mp\, i\omega t} = [\ i\ (\mp\ i\omega) - i\ (\pm i\omega)]\ e\ e\ = [\ \pm\ \omega \pm \omega\ ]\ 1\ = \pm\ 2\omega$$

and thus we have shown that

$$\int d^3x\ f^\pm_{\mathbf{p}'}(x)^*\ i\overleftrightarrow{\partial}^0\ f^\pm_{\mathbf{p}}(x) = \pm\ \delta^3(\mathbf{p}-\mathbf{p}')$$

If we have opposite signs on the two f's, then we get

$$\int d^3x\ f^\pm_{\mathbf{p}'}(x)^*\ i\overleftrightarrow{\partial}^0\ f^{\mp}_{\ \mathbf{p}}(x) = (2\pi)^{-3}\ 1/\sqrt{2\omega}\ 1\sqrt{2\omega'}\ e^{\pm i\omega' t}\ i\overleftrightarrow{\partial}^0\ e^{\pm i\omega t} \int d^3x\ e^{\mp\, ip'\bullet x}\ e^{\mp\, ip\bullet x}$$

$$= (1/2\omega)\ [\pm\ \omega\ \mp\ \ \omega]\ \delta^3(\mathbf{p}+\mathbf{p}')\ = 0$$

So we learn that, if we use this strange weighting factor $i \overleftrightarrow{\partial}^0$ inside our usual normalization integral, we find that the plane waves of the same energy are properly normalized, and those of opposite energy are orthogonal. I don't think I have ever seen such an arrangement before where we have some operator like this in the middle of our normalization integral. No explanation has yet been given, but here comes more motivation.

Comment: It seems that we have a strange completeness relation in this theory. It would be this:

$$1^+ = \int d^3x\, |x\rangle\, i \overleftrightarrow{\partial}^0 \langle x|$$

$$1^- = -\int d^3x\, |x\rangle\, i \overleftrightarrow{\partial}^0 \langle x|$$

$$\begin{aligned}\langle\psi|\varphi\rangle &= [\langle\psi+| + \langle\psi-|\,][\,|\varphi+\rangle + |\varphi-\rangle] \\ &= \langle\psi+|\,\varphi+\rangle + \langle\psi-|\,\varphi-\rangle \\ &= \langle\psi+|\,1^+|\,\varphi+\rangle + \langle\psi-|\,1^-|\,\varphi-\rangle \\ &= \int d^3x\, \langle\psi+|x\rangle\, i \overleftrightarrow{\partial}^0 \langle x|\,\varphi+\rangle - \int d^3x\, \langle\psi-|x\rangle\, i \overleftrightarrow{\partial}^0 \langle x|\,\varphi-\rangle\end{aligned}$$

$$\langle x|p+\rangle = f_{\mathbf{p}}^{+}(x) = 9.5$$

$$\langle p+|p'+\rangle = \delta^3(\mathbf{p}-\mathbf{p}')$$
$$\langle p-|p'-\rangle = -\delta^3(\mathbf{p}-\mathbf{p}')$$
$$\langle p+|p'-\rangle = 0$$

$$1^+ = \int d^3x\, |p\rangle\langle p|$$

$$1^- = -\int d^3x\, |p\rangle\langle p|$$

Neither Messiah nor Sakurai comment on the above nor even show this formalism for KG. A web search turns up this comment

Abstract. We use the theory of pseudo-Hermitian operators to address the problem of the construction and classification of positive-definite invariant inner-products on the space of solutions of a Klein–Gordon-type evolution equation. This involves dealing with the peculiarities of formulating a unitary quantum dynamics **in a Hilbert space with a time-dependent inner product.** We apply our general results to obtain possible Hilbert space structures on the solution space of the equation of motion for a classical simple harmonic oscillator, a free Klein–Gordon equation and the Wheeler–DeWitt equation for the FRW-massive-real-scalar-field models.

So here someone is at least talking about the issue. I see that there are papers talking about this strange inner product. It is strange because $\langle\psi|\psi\rangle$ is not a norm, it is not positive definite, it is not the charge density. The charge density is $\rho = \psi^* \, i \overleftrightarrow{\partial}^0 \, \psi$ and can have either sign. OK, I will let this subject ride, but at least I have noticed that it is strange.

Suppose we write a positive (or negative) energy linear combination as in 9.7,8. Then we get

$$Q = \int d^3x\, j^\circ = \int d^3x\, \varphi^* \, i \overleftrightarrow{\partial^0} \, \varphi$$

$$= \int d^3x \; \left[\int d^3p' \, f^{\pm}_{\mathbf{p}'}(x) a_{\pm}(p') \right]^* i \overleftrightarrow{\partial^0} \left[\int d^3p \, f^{\pm}_{\mathbf{p}}(x) a_{\pm}(p) \right]$$

$$= \int d^3p' \, a^*_{\pm}(p') \int d^3p \, a_{\pm}(p) \int d^3x \, f^{\pm *}_{\mathbf{p}'}(x) \, i \overleftrightarrow{\partial^0} f^{\pm}_{\mathbf{p}}(x)$$

$$= \int d^3p' \, a^*_{\pm}(p') \int d^3p \, a_{\pm}(p) \{ \pm \delta^3(\mathbf{p}-\mathbf{p}') \}$$

$$= \pm \int d^3p \, |a_{\pm}|^2 (p)$$

so now we associate positive energy states with positive charge (like π^+ pion), and negative energy states with negative charge (negative pion). So this is looking pretty good if we think of the negative energy states somehow as anti-π^+ particles.

Propagator. I want to parallel my chapter 6 notes on this somehow. Let's look back at our comparison of the non-rel scalar to the Dirac stuff:

$$(i\partial_t - H_o)\, G_o(x,x') = \delta^4(x-x') \qquad H_o = \hat{p}^2/2m \text{ for free particle}$$
$$(i\partial_t - H)\, G(x,x') = \delta^4(x-x') \qquad H = H_0 + V$$

$$\gamma^0(i\partial_t - H_o)\, S_F(x,x') = \delta^4(x-x') \qquad H_o = \boldsymbol{\alpha} \bullet \hat{\mathbf{p}} + \beta m = \gamma^0(\boldsymbol{\gamma} \bullet \hat{\mathbf{p}} + m) \text{ for free particle}$$
$$(\hat{p} - m)\, S_F(x,x') = \delta^4(x-x')$$

In both these cases we had $(i\partial_t - H_o)$ on the LHS. In the KG case, however, we have

$$H_0 = \sqrt{p^2 + m^2} \qquad \text{where } p = |\mathbf{p}|$$

and SE would say

$$i\partial_t \psi = \sqrt{p^2 + m^2}\, \psi$$

but this would make for a pretty messy Green's function definition. So instead, we try this

$$(i\partial_t \psi)\,(i\partial_t \psi) = \sqrt{p^2 + m^2}\,\sqrt{p^2 + m^2} = p^2 + m^2$$

which is sort of the product of two SE equations. We then have

$$-\partial_t^2\psi = -\nabla^2\psi + m^2\psi$$

$$[\,(\partial_t^2 - \nabla^2) + m^2\,]\psi = 0$$

$$(\Box + m^2)\,\psi = 0 \qquad \text{// compare to } (\not{p}-m)\psi = 0 \text{ for Dirac } \psi$$

and this IS the KG equation, which we could just assume as our starting point. This last comparison suggests we think of the operator $(\Box + m^2)$ the way we think of $(\not{p}-m)$. Again: $\Box = -\hat{p}^2$ so

$$-(\Box + m^2) = (\hat{p}^2 - m^2)$$
$$(-i\partial\!\!\!/ - m) = (\hat{\not{p}} - m)$$

where we add a - on the LHS to make the RHS's more comparable. So this suggests the following comparison between propagator definitions:

$$(\hat{p}_x - m)\,S_F(x,x') = \delta^4(x-x') \qquad S_F(x-x')$$
$$(\Box_x + m^2)\,\Delta_F(x,x') = -\delta^4(x-x') \qquad \Delta_F(x-x')$$

so this makes me happy with 9.9 including the sign. They have the places of x and x' reversed which is fine.

Now in the usual way we know we can expand both $\Delta_F(x-x')$ and $\delta^4(x-x')$ as in 6.41, which is to say, we get

$$(\Box_x + m^2)\int d^4p\; e^{-ip\cdot x}\,\Delta_F(p)\,(2\pi)^{-4} = -\int d^4p\; e^{-ip\cdot x}\,(2\pi)^{-4}$$

We then move $(\Box_x + m^2)$ to the right and $\partial_\mu\partial^\mu \rightarrow -ip_\mu * -ip^\mu = -p^2$ so we get

$$(2\pi)^{-4}\int d^4p\; e^{-ip\cdot x}\,(p^2 - m^2)\,\Delta_F(p) = +(2\pi)^{-4}\int d^4p\; e^{-ip\cdot x}$$

and this tells us the desired result that

$$\Delta_F(p) = 1/(p^2 - m^2) \qquad \text{compare:} \qquad S_F(p) = 1/(\not{p}-m) = (\not{p}+m)/(p^2 - m^2)$$

so in some sense maybe we can follow through all our old work replacing $(\not{p}+m)$ with 1.

Derive 9.11. When we transform back to coordinate space and deal with the poles, the numerator makes not big difference. So let's try to just apply our existing results 6.44 and 6.45 on page 94 to this KG situation

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\; e^{ip\bullet(x'-x)}\; e^{-iE(t'-t)}\,(1/2E) \qquad t' > t \quad (6.44\ KG)$$

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\; e^{ip\bullet(x'-x)}\; e^{+iE(t'-t)}\,(1/2E) \qquad t' < t \quad (6.45\ KG)$$

where E>0 in both cases, and in both cases I have just replaced ($\not{p}$+m) → 1. If we call the energy ω, then both the above can be expressed in one equation. Notice that

$-iE(t'-t) = -iE\,|\,t'-t\,|$ $\quad t' > t$ first equation

$+iE(t'-t) = -iE\,|\,t'-t\,|$ $\quad t' < t$ second equation

so we get

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p/(2\omega)\, e^{ip\bullet(x'-x)}\, e^{-i\omega|t'-t|}$$

which agrees with equation A on page 188. On the other hand, we can rewrite our two equations above like this:

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\, e^{ip\bullet(x'-x)}\, e^{-iE(t'-t)}\,(1/2E) \qquad t' > t \quad (6.44\ KG)$$

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\, e^{ip\bullet(x'-x)}\, e^{+iE(t'-t)}\,(1/2E) \qquad t' < t \quad (6.45\ KG)$$

or

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\, e^{+ip.x}\, e^{-ip.x'}\,(1/2E) \qquad p = (E,\mathbf{p}) \qquad t' > t \quad (6.44\ KG)$$

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\, e^{-ip.x}\, e^{+ip.x'}\,(1/2E) \qquad p = (E,\mathbf{p}) \qquad t' < t \quad (6.45\ KG)$$

where in the second line we changed the integration variable $\mathbf{p} \rightarrow -\mathbf{p}$ which then changed the sign of the phase of the factor $e^{ip\bullet(x'-x)}$ in the second integral to make everything work right. Now * the left phasor to get

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\,(e^{-ip.x})^* e^{-ip.x'}\,(1/2E) \qquad t' > t \quad (6.44\ KG)$$

$$\Delta_F(x'-x) = -i\,(2\pi)^{-3}\int d^3p\,(e^{+ip.x})^*\, e^{+ip.x'}\,(1/2E) \qquad t' < t \quad (6.45\ KG)$$

and then make the identification

$$\Delta_F(x'-x) = -i\int d^3p\, f^+_{\mathbf{p}}{}^*(x) f^+_{\mathbf{p}}(x') \qquad t' > t \quad (6.44\ KG)$$

$$\Delta_F(x'-x) = -i\int d^3p\, f^-_{\mathbf{p}}{}^*(x)\, f^-_{\mathbf{p}}(x') \qquad t' < t \quad (6.45\ KG)$$

which we can combine to get

$$\Delta_F(x'-x) = -i\,\theta(t'-t)\int d^3p\, f^+_{\mathbf{p}}{}^*(x) f^+_{\mathbf{p}}(x') \;- i\,\theta(t-t')\int d^3p\, f^-_{\mathbf{p}}{}^*(x)\, f^-_{\mathbf{p}}(x')$$

or

$$\Delta_F(x'-x) = -i\,\theta(t'-t)\int d^3p\, f^+_{\mathbf{p}}(x')\, f^+_{\mathbf{p}}{}^*(x) \;- i\,\theta(t-t')\int d^3p f^-_{\mathbf{p}}(x')\, f^-_{\mathbf{p}}{}^*(x)$$

except they have the f factors cosmetically swapped. We can compare this with 6.48 or 6.28 for the non-rel case which is in fact a closer comparison. There we sum over n which is the label of Ham solutions, and here that sum is n = ± energy solutions. Looking at the above, we say our f^{+} positive energy solutions only propagate into the future due to the θ(t'-t) where we go from t to t' in the future, and of course the negative energy solutions only go into the past.

So, I have now derived both A and B of (9.11) on page 188.

Now let's **derive (9.13)** with words instead of equations. Here are the steps:

Insert 9.7 for $\varphi^{+}(x)= \int d^3p\, a_{+}(p)\, f_p^{+}(x)$ on the far right of 9.13, and insert 9.11 for the Δ_F. At this point, then, we have $\int d^3p \int d^3p'$. The second term of Δ_F will yield zero because the $i\overleftrightarrow{\partial}^0$ operator in the sandwich between f's of variable x will kill from 9.6C. The first term of Δ_F makes $+\delta(p-p')$ according to 9.6B which kills say our $\int d^3p'$ and we have only $\int d^3p$ left with the $a_{+}(p)$ sitting in there and the other wave function of the propagator (the x' one) . But this combination gives us $\varphi^{+}(x')$, QED.

In **deriving 9.14,** it is exactly the same but we pick up a minus sign from -δ in 9.6B and this extra minus appears in the result. Of course in this case it is the second term of Δ_F that does not vanish.

Now let's do some more comparison with our old results, to which I now add our new KG line:

$\theta(t'-t)\varphi' = iG_0.\varphi$	how G_0 would move φ into the future	// see 6.1 with G_o not G
$\theta(t'-t)\psi' = iS_F\,\gamma^{o}.\varphi$	how S_F would move ψ into the future	// see 6.49
$\theta(t'-t)\varphi' = i\Delta_F\, i\overleftrightarrow{\partial}^0.\varphi$	how Δ_F would move ψ into the future	// see 9.13

where recall the "." means a d^3x integration. So, where we used to have γ^{o} in the S_F case, we now have this thing $i\overleftrightarrow{\partial}^0$ in the Δ_F case. Otherwise, very much the same.

Summary to this point:

1. In the KG theory, we have a conserved 4-current

$j^{\mu} = i\,[\,\varphi^*(\partial^{\mu}\varphi) - (\partial^{\mu}\varphi^*)\,\varphi]$ compare: $j^{\mu} = \psi^{\dagger}\gamma^0\gamma^{\mu}\psi$ Dirac

whose 3-current has the same form as in the non-rel SE! The 0^{th} component is this (which we would call the charge density (probability density) ρ)

$j^0 = i\,[\,\varphi^*(\partial^0\varphi) - (\partial^0\varphi^*)\,\varphi] = \varphi^*\, i\overleftrightarrow{\partial}^0\varphi$ compare: $j^0 = \psi^{\dagger}\psi$ Dirac

We are able to write down normalized wavefunctions as in 9.5 for the KG particles and then 9.6 shows how solutions of different energy are orthogonal, in a manner similar to 3.9b for Dirac. In fact, we can just replace γ^0 there with $i\overleftrightarrow{\partial}^0$ here. Of course in KG we don't have 4-component spinors, we just have single component plane wave functions f. Also, we include the plane waves in our f, where they are not included in 3.9b so no δ function. So things are a little different.

2. We then define a propagator in analogy with the Dirac theory and come up with a Green's function called Δ_F which tells us the amplitude for a KG particle to move from x to x'. The momentum space propagator is simply $1/(p^2-m^2)$ which is like the Dirac one but without the $(\not{p}+m)$ up top. Our Huygens' Principle like p79 6.1 for non-rel, or like 6.49 for Dirac, is 9.13 which has the extra $i\overleftrightarrow{\partial}^0$. Our propagator does "all the right things" in time. Our KG equation was noted above and has positive and negative energy solutions. We interpret the negative energy solutions as the anti-particles as in Dirac.

So things look a little weird here, but everything looks like a consistent model. Interesting how the γ^0 provides the time index for j^0 in Dirac, but $i\overleftrightarrow{\partial}^0$ provides it for KG. I guess there is a negative energy sea, but now we have bosons, so how does that work out? Why don't the + states all fall down? Nothing has been said about a "sea" by BD.

9.3 Adding E&M to KG. (188)

Our KG equation is just $(p^2- m^2)\varphi = 0$ so 9.15 seems the obvious thing to try "the minimal substitution" and we get 9.16. Note that $p^\mu = i\partial^\mu$.

Why is the current shown in 9.17 conserved?

I know there is a fast Noether type way to show this, but I don't know how to do that yet, so we are left with brute force only. Let's mimic 1.12 but using the full KG equation which is now 9.16. Start with

$$\psi^* [(i\partial_\mu - eA_\mu)^2- m^2]\psi - \psi [(-i\partial_\mu - eA_\mu)^2- m^2]\psi^* = 0$$

This is 0 because each term is 0 by our KG equation. The m^2 terms cancel so we have

$$\psi^* (i\partial_\mu - eA_\mu)^2\psi - \psi (-i\partial_\mu - eA_\mu)^2\psi^* = 0$$

$$\Rightarrow \quad \psi^* \{ (i\partial_\mu)^2 - (i\partial_\mu) eA^\mu - eA^\mu(i\partial_\mu) + e A_\mu A^\mu\} \psi$$

$$- \psi \{ (i\partial_\mu)^2 + (i\partial_\mu) eA^\mu + eA^\mu(i\partial_\mu) + e A_\mu A^\mu\} \psi^* = 0$$

where ∂_μ always acts on everything to its right. The $A_\mu A^\mu$ terms cancel and we are left with

$$\Rightarrow \quad \psi^* \{ (i\partial_\mu)^2 - (i\partial_\mu) eA^\mu - eA^\mu(i\partial_\mu) \} \psi$$

$$- \psi \{ (i\partial_\mu)^2 + (i\partial_\mu) eA^\mu + eA^\mu(i\partial_\mu) \} \psi^* = 0$$

$$\Rightarrow \quad \psi^* \{ (i\partial_\mu)^2 \psi - e(i\partial_\mu A^\mu) \psi - eA^\mu(i\partial_\mu \psi) - eA^\mu(i\partial_\mu \psi) \}$$

$$- \psi \{ (i\partial_\mu)^2 \psi^* + e(i\partial_\mu A^\mu) \psi^* + eA^\mu(i\partial_\mu \psi^*) + eA^\mu(i\partial_\mu) \psi^* \} = 0$$

In each line we may combine the last two terms with a factor of 2. The second terms on each line are the same, so we can combine them with a factor of 2. We then have

$$\psi^* (i\partial_\mu)^2 \psi - \psi (i\partial_\mu)^2 \psi^* - 2e \psi^* \psi(i\partial_\mu A^\mu) - 2 e\psi^* A^\mu(i\partial_\mu \psi) - 2 \psi eA^\mu(i\partial_\mu \psi^*) = 0$$

$$[\psi^* (i\partial_\mu)^2 \psi - \psi (i\partial_\mu)^2 \psi^*] - 2e \psi^* \psi (i\partial_\mu A^\mu) - 2 e A^\mu [\psi^*(i\partial_\mu \psi) + \psi (i\partial_\mu \psi^*)] = 0 \qquad (*)$$

I don't see any further simplifications. So I have shown that (*) is true.

Now let's just assume the current in 9.17 and see if $\partial_\mu j^\mu$ gives the above equation. We get

$$\partial_\mu j^\mu = \partial_\mu \{ \psi^*(i\partial_\mu - eA_\mu) \psi - \psi (i\partial_\mu + eA_\mu) \psi^* \}$$

$$= \partial_\mu \{ \psi^*(i\partial_\mu \psi) - eA_\mu \psi^* \psi \}$$
$$- \partial_\mu \{ \psi(i\partial_\mu \psi^*) + eA_\mu \psi \psi^* \}$$

$$= \{ (\partial_\mu \psi^*)(i\partial_\mu \psi) + \psi^* \partial_\mu(i\partial_\mu \psi) - e(\partial_\mu A_\mu) \psi^* \psi - eA_\mu (\partial_\mu \psi^*) \psi - eA_\mu \psi^* (\partial_\mu \psi) \}$$
$$- \{ (\partial_\mu \psi)(i\partial_\mu \psi^*) + \psi \partial_\mu(i\partial_\mu \psi^*) + e(\partial_\mu A_\mu) \psi \psi^* + eA_\mu (\partial_\mu \psi) \psi^* + eA_\mu \psi (\partial_\mu \psi^*) \}$$

$$= (\partial_\mu \psi^*)(i\partial_\mu \psi) + \psi^* \partial_\mu(i\partial_\mu \psi) - e(\partial_\mu A_\mu) \psi^* \psi - eA_\mu (\partial_\mu \psi^*) \psi - eA_\mu \psi^* (\partial_\mu \psi)$$
$$- (\partial_\mu \psi)(i\partial_\mu \psi^*) - \psi \partial_\mu(i\partial_\mu \psi^*) - e(\partial_\mu A_\mu) \psi \psi^* - eA_\mu (\partial_\mu \psi) \psi^* - eA_\mu \psi (\partial_\mu \psi^*)$$

We can cancel the first terms on each line, and combine that last three to get

$$= [\psi^* \partial_\mu(i\partial_\mu \psi) - \psi \partial_\mu(i\partial_\mu \psi^*)] - 2e(\partial_\mu A_\mu) \psi^* \psi - 2eA_\mu (\partial_\mu \psi^*) \psi - 2e A_\mu (\partial_\mu \psi) \psi^*$$

Now multiply each term by i to get

$$= [\psi^* i\partial_\mu(i\partial_\mu \psi) - \psi i\partial_\mu(i\partial_\mu \psi^*)] - 2e(i\partial_\mu A_\mu) \psi^* \psi - 2eA_\mu (i\partial_\mu \psi^*) \psi - 2e A_\mu (i\partial_\mu \psi) \psi^*$$

$$= [\psi^* (i\partial_\mu)^2 \psi) - \psi (i\partial_\mu)^2 \psi^*)] - 2e(i\partial_\mu A_\mu) \psi^* \psi - 2eA_\mu [(i\partial_\mu \psi^*) \psi + (i\partial_\mu \psi) \psi^*]$$

and in fact this duplicates (*) above and thus **we have shown that $\partial_\mu j^\mu = 0$ by brute force.**

Note that we could write the current in 9.17 as follows:

$$j^\mu = \varphi^* i \overleftrightarrow{\partial}^\mu \varphi - 2eA^\mu \varphi^* \varphi$$

$$j^0 = \varphi^* i \overleftrightarrow{\partial}^0 \varphi - 2eA^0 \varphi^* \varphi$$

and **thus we get result (9.18)** for the conserved charge, which certainly looks strange to me. Again, come back here when Noether is known.

Now look at 9.16. Take all the stuff other than -□ and put it on the RHS, negate both sides so we get $+m^2$ on the LHS, and then define V as shown in 9.19C. Given this choice of sign for V, and given the minus sign in the Green's definition 9.9, **we end up with 9.19B** as our "integral equation" in this KG world. This is just our usual Green's business, compare

$\psi' = \varphi' + G_0 V \psi$		6.14	integral equation for ψ, driven by V and G_o
$\Psi' = \psi' + S_F\, e\not{A}\, \Psi$	(3)	**6.53**	
$\Phi' = \varphi + \Delta_F V \Phi$	(3)	**9.19**	// from raw notes

so here we have just "done it one more time".

Equation 9.20 comes by inserting 9.11 for Δ_F into 9.19, seems pretty obvious, where we make these shuffles: $x' \rightarrow x$ and $x \rightarrow y$, so for example $t'-t \leftrightarrow t - y_0$.

I don't really understand why 9.20 is so compelling for BD to make the red underlined statement about antiparticles existing. Yes, our KG theory has both objects.

For the π^0 , you just assume φ is real, you get Q = 0 and the propagator is the same, 9.11.

Comment: I note that for the first time, as shown in 9.19 B with C, we have two photons coupling to the same point in spacetime, so this KG theory is going to have more than one kind of vertex and we will need Feynman rules for each kind of vertex. I am eager to see if there are any physical predictions from the KG theory that have been confirmed. The hadronic confusion of course will be present.

Comment: Both ∂_μ operators in the potential V act on everything to the right. You don't just have an isolated $(\partial_\mu A^\mu)$ in that first term. So V is a differential operator, not just a function.

9.4 Scattering Amplitudes (190)

We are just continuing down the pipe with our "new theory" for KG particles. I just added a comment above on the nature of the "inner product" for KG theory. Now consider:

$$S_{fi} = \langle \varphi_{f+} | \Phi_i \rangle = \langle \varphi_{f+} | 1^+ | \Phi_i \rangle = \langle \varphi_{f+} \int d^3x\, |x\rangle\, i \overleftrightarrow{\partial^0} \langle x | \Phi_i \rangle$$

$$= \int d^3x \langle \varphi_{f+} |x\rangle\, i \overleftrightarrow{\partial^0} \langle x | \Phi_i \rangle$$

$$= \int d^3x\, f^+(p_f)^*\, i \overleftrightarrow{\partial^0}\, \Phi_i(x)$$

and this is where 9.21 A comes from on page 190. The result 9.21 B here results from these steps

(1) start with 9.21 A

(2) replace Φ by the integral equation for Φ shown in 9.20, which involves a d^3p and a d^4y integral. This obviously produces the $\delta^3(\mathbf{p}-\mathbf{p}')$ term, but then the second term looks like this:

$$-i \int d^3x\, f_{\mathbf{p}f}^{+}(x)^{*}\, i\overleftrightarrow{\partial}^{0}_{\mathbf{x}} \int d^3p\, f_{\mathbf{p}}^{+}(x) \int d^4y\, \theta(t-y_0)\, f_{\mathbf{p}}^{+}(y)^{*}\, V(y)\Phi(y)$$

$$= -i \int d^3p \{ \int d^3x\, f_{\mathbf{p}f}^{+}(x)^{*}\, i\overleftrightarrow{\partial}^{0}_{\mathbf{x}}\, f_{\mathbf{p}}^{+}(x) \} \int d^4y\, \theta(t-y_0)\, f_{\mathbf{p}}^{+}(y)^{*}\, V(y)\Phi(y)$$

$$= -i \int d^3p \{ \delta^3(\mathbf{p}-\mathbf{p}_f)\} \int d^4y\, \theta(t-y_0)\, f_{\mathbf{p}}^{+}(y)^{*}\, V(y)\Phi(y)$$

$$= -i \int d^4y\, \theta\, f_{\mathbf{p}}^{+}(y)^{*}\, V(y)\Phi(y) \qquad \text{// limit y into infinite future}$$

[**Comment**: I think this has the same sign for the $f_{\mathbf{p}}^{-}$ case. You get one minus sign by using the second term in 9.11, but then you get $-\delta^3(\mathbf{p}-\mathbf{p}_f)$ above instead of +, so signs cancel.]

and there we have 9.21B which now looks just like our Dirac equation

$S_{fi} = \delta_{fi} - i\, \varphi_f^{\dagger}\, V\, \psi_i^{(+)}$ form 5 6.34 see note above

So the name of this game is pretty basic: Start from the KG equation, add the EM field, push extra terms on to the RHS as potential V. Create a propagator Green's for KG, turn all the usual gears, and we end up with a perturbation theory expansion for S_{fi} . The big difference is that our V is very different from the V of Dirac QED.

So the rest of section 9.4 discusses the meaning of swinging legs around in the graphs top of page 190, but we have talked about this ad infinitum in the Dirac theory and it is all the same comments just being repeated. The positive and negative energy solutions. The π^- is tagged as the anti-particle in this discussion, although of course either would do.

We are now ready to do some QED "applications" in the KG theory world!

9.5 Low-order Scattering processes. (191).

Coulomb Scattering of π^+ ·

We start of course with Coulomb scattering of a π^+. Let's just do it from 9.21

$$S_{fi} = -i \int d^4y\, f_{\mathbf{p}'}^{+}(y)^{*} \{+ie [\partial_\mu A^\mu + A^\mu \partial_\mu] \}\, f_{\mathbf{p}}^{+}(y)$$

where we keep only the order e term from the V in 9.19C, since doing lowest order only.

I think the idea here is to do parts with the ∂_μ of the first term. Then we get

$$- \partial_\mu f_{p'}{}^{+}(y)^* \sim - \partial_\mu e^{+ipf.y} \sim - i\, p'_\mu$$

$$\partial_\mu f_p{}^{+}(y) \sim \partial_\mu e^{-ip.y} \sim -i\, p_\mu$$

so the two momenta add. Throw in the norm factors and we get

$$S_{fi} = -i(+ie) \int d^4y\, f_{p'}{}^{+}(y)^* \{\partial_\mu A^\mu + A^\mu \partial_\mu\}\, f_p{}^{+}(y)$$

$$= -i(+ie) \int d^4y\, f_{p'}{}^{+}(y)^* \{(-i)\,[p'_\mu + p_\mu]\, A^\mu\}\, f_p{}^{+}(y)$$

$$= (-i)^2(+ie) \int d^4y\, f_{p'}{}^{+}(y)^* \{[p'_\mu + p_\mu]\, A^\mu\}\, f_p{}^{+}(y)$$

$$= -ie \int d^4y\, f_{pf}{}^{+}(y)^* (p'_\mu + p_{i\mu})\, A^\mu\}\, f_{pi}{}^{+}(y)$$

$$= -ie \int d^4y\, (p'_\mu + p_\mu)\, A^\mu\} e^{-i(p\,-\,p').y} * \text{norm}$$

$$= -ie\, (p'_\mu + p_\mu) \int d^4y\; A^\mu(y)\, e^{-i(p\,-\,p').y} * \text{norm}$$

$$= -ie\, /(\sqrt{2\omega}\,\sqrt{2\omega'}\,)(2\pi)^{-3}\; (p'_\mu + p_\mu) \int d^4y\; A^\mu(y)\, e^{+iq.y}$$

$$= -ie\, /(\sqrt{2\omega}\,\sqrt{2\omega'}\,)(2\pi)^{-3}\, (p'_\mu + p_\mu)\, A^\mu(q)$$

And then we install our Coulomb $A^\mu(y)$ and compute $A^\mu(q)$ [see page 100 B]

Derive 9.26A. Square the amplitude including both factors of e. The momentum sum is just an energy sum since μ = 0 only. One of the 2πδ() becomes just T, and divide it out to get rate, while the other remains to do energy conservation. In our continuum normalization, it is as if you took $V = (2\pi)^3$ as you can see from page 187 top. Then phase space is just d^3p as from page 101. The flux is $v/V = v/(2\pi)^3$ where v is the velocity difference which in this case is just v of the initial particle.

Derive 9.26B. Write $d^3p = p^2dp\, d\Omega = pEdE\, d\Omega \rightarrow pE\, d\Omega$ with the energy delta. We then have

$$pE/v = pE/\beta = (\gamma\beta m)(\gamma m)/\beta = E^2$$

and q^2 comes from page 106 top, so we should get this:

$$d\sigma/d\Omega = E^2\, (2\pi)^4\, Z^2e^4(2E)^2\, (2\pi)^{-6}\, (2E)^{-2}\, 1/16\; 1/p^4\; 1/\sin^4(\theta/2) \qquad \alpha = e^2/4\pi$$

$$= E^2\, (2\pi)^{-2}\, Z^2e^4\; 1/16p^4\sin^4(\theta/2) = 1/4\pi^2\, (4\pi\alpha)^2\, Z^2 1/16p^4\sin^4(\theta/2)$$

$= E^2\ (\alpha^2 Z^2/4)/\ p^4 \sin^4(\theta/2)\ \ = (E/p)^2\ (\alpha^2 Z^2/4)/\ p^2 \sin^4(\theta/2)$

$= \ \alpha^2 Z^2/\ [4\beta^2 p^2 \sin^4(\theta/2)\ ]$

and this agrees exactly with the Mott formula on page 7.22 except here we don't have the extra numerator factor, so we can claim this is a "spin effect". That is, spin-1/2 makes you have the Dirac theory, not the KG theory.

Does this agree with experiment?? It is really just the Rutherford formula, so I guess it must.

Coulomb Scattering of π^-

Amplitude has a minus sign because the $[\partial_\mu A^\mu + A^\mu \partial_\mu]$ factor his opposite phase signs. But we can also interpret this as just negating both of the p^μ. The S_{fi} in 9.21 has same sign for + and - cases, as noted in the comment above. So same cross section. Again, this is what Rutherford non-rel predicts.

"Compton" Scattering

Derive 9.29. Here I quote the perturbation expansion from Chap 6 notes

$$S_{fi} = \delta_{fi}\ - i\ \varphi_f^\dagger\ V\ \varphi_i\ - i\ \varphi_f^\dagger\ V\ G_0 V\ \varphi_i\ - i\ \varphi_f^\dagger\ V\ G_0 V\ G_0 V\ \varphi_i\ + \ldots \quad 6.33b$$

and in our current context, $^\dagger = *$, which we write as [write out V as its two terms in 9.19 C]

$$S_{fi} = - i\ \varphi_f^*\ V_2\ \varphi_i\ - i\ \varphi_f^*\ V_1\ \Delta_F V_1\ \varphi_i$$

where the term $\varphi_f^*\ V_1\ \varphi_i$ is blocked by momentum conservation and also only couples in one photon so is not part of our Compton process. Let's now insert the potential pieces to get

$$S_{fi} = - i\ \varphi_f^*\ [\ -e^2 A_\mu A^\mu]\ \ \varphi_i\ - i\ (ie)^2\ \varphi_f^*\ [\ (\partial_\mu A^\mu) + A^\mu \partial_\mu]\ \ \Delta_F\ [\ (\partial_\mu A^\mu) + A^\mu \partial_\mu]\ \varphi_i$$

$$= \ ie^2\ \varphi_f^*\ [\ A_\mu A^\mu]\ \ \varphi_i\ -\ (ie)^2\ \varphi_f^*\ [\ (\partial_\mu A^\mu) + A^\mu \partial_\mu]\ \ i\Delta_F\ [\ (\partial_\mu A^\mu) + A^\mu \partial_\mu]\ \varphi_i$$

$$= \ ie^2\ \varphi_f^*\ [\ A_\mu A^\mu]\ \ \varphi_i\ +\ (ie)^2\ \varphi_f^*\ i[\ (\partial_\mu A^\mu) + A^\mu \partial_\mu]\ \ i\Delta_F\ i[\ (\partial_\mu A^\mu) + A^\mu \partial_\mu]\ \varphi_i$$

where in the last two lines I am shuffling the i's around to get a **9.29 match**. Remember in my matrix notation for coordinate space the d^4x integrations are implied since no "." appear. So the above does agree with 9.29, I consider that equation now derived.

Now we have that same issue of the four possible A^μ terms but we only keep the one that corresponds to our process.

Derive 9.30. Just a good exercise to keep doing these things. Let's start with the quadratic term

$$ie^2\ \varphi_f^*\ [\ A_\mu A^\mu]\ \ \varphi_i\ = ie^2\ (2\pi)^{-3/2}\ \ 1/\sqrt{2\omega'}\ \ e^{+ip'.x}\ [A_\mu A^\mu]\ (2\pi)^{-3/2}\ \ 1/\sqrt{2\omega}\ \ e^{+ip.x}$$

$$= ie^2\ (2\pi)^{-3}\ 1/\sqrt{2\omega'}\ \ e^{+ip'.x}\ [\varepsilon_\mu\ e^{-ik.x}\ \varepsilon'^\mu\ e^{-ik'.x}]\ (2\pi)^{-3}\ 1/\sqrt{2\omega}\ \ e^{+ip.x}\ 1/\sqrt{2k}\ 1/\sqrt{2k'}$$

where I use the momentum labels from page 194 bottom. Now write in the integral and group factors to the left:

$$= ie^2(2\pi)^{-6}\ 1/[\sqrt{2\omega'}\ \sqrt{2\omega}\ \sqrt{2k}\ \sqrt{2k'}\ ]\ \varepsilon_\mu\varepsilon'^\mu\ \ \int d^4x\ e^{+ip'.x}\ [e^{-ik.x}\ e^{-ik'.x}\ e^{+ip.x}\ ]$$

$$= ie^2(2\pi)^{-6}\ 1/[\sqrt{2\omega'}\ \sqrt{2\omega}\ \sqrt{2k}\ \sqrt{2k'}\ ]\ \varepsilon_\mu\varepsilon'^\mu\ (2\pi)^4\ \delta^4(\text{momentum conservation})$$

$$= (+i)\ ie^2(2\pi)^{-6}\ 1/[\sqrt{2\omega'}\ \sqrt{2\omega}\ \sqrt{2k}\ \sqrt{2k'}\ ]\ (-i)\varepsilon_\mu\varepsilon'^\mu\ (2\pi)^4\ \delta^4(\text{momentum conservation})$$
$$= (+i)^2\ e^2(2\pi)^{-6}\ 1/[\sqrt{2\omega'}\ \sqrt{2\omega}\ \sqrt{2k}\ \sqrt{2k'}\ ]\ \{-i\varepsilon_\mu\varepsilon'^\mu\ \}(2\pi)^4\ \delta^4(\text{momentum conservation})$$

We get another identical term by adding the amplitude where the two photons connections are reversed at the vertex, that is, where $A_\mu A^\mu = [\varepsilon'_\mu\ e^{-ik'.x}\ \varepsilon^\mu\ e^{-ik.x}]$ compared to what is shown above. So we get the factor of 2 shown in 9.30 and which BD comment on. So, I have derived exactly the last term in 9.30.

Now do the other term. We have

$$(ie)^2\ \int d^4y \int d^4z\ \varphi_f^*(y)\ i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_y\ \ i\Delta_F(y\text{-}z)\ \ i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_z\ \varphi_i(z)$$

$$= (ie)^2\ \int d^4y \int d^4z\ \varphi_f^*(y)\ i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_y\ \int d^4q/(2\pi)^4\{e^{-iq.(y-z)}\ i/q^2\}\ \ i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_z\ \varphi_i(z)$$

$$= (ie)^2\ \int d^4q/(2\pi)^4\ \int d^4y \int d^4z\ \varphi_f^*(y)\ i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_y\ \{e^{-iq.(y-z)}\ i/q^2\}\ \ i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_z\ \varphi_i(z)$$

where I have Fourier elevated Δ_F according to 9.10 on page 187, but I have omitted the mass for now to save space. Now we need to deal with the ∂_μ stuff.

$(\partial_\mu A^\mu)_y\ \rightarrow$ - $i(p'_\mu A^\mu)$ - in phase, - from *, - from parts

$(A^\mu\partial_\mu)_y\ \rightarrow A^\mu\ (-iq)_\mu$

$i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_y\ \rightarrow + (q + p')_\mu A^\mu$

Then for the right grouping we get

$(\partial_\mu A^\mu)_z\ \rightarrow$ - $i(q_\mu A^\mu)$ + in phase, , - from parts

$(A^\mu\partial_\mu)_z\ \rightarrow A^\mu\ (-ip)_\mu$

$i[\ (\partial_\mu A^\mu) + A^\mu\partial_\mu]_z\ \rightarrow + (q + p)_\mu A^\mu$

So our long expression above becomes

$$= (ie)^2 \int d^4q/(2\pi)^4 \int d^4y \int d^4z\, \varphi_f^*(y)\, (q + p')_\mu A^\mu(y)\, \{e^{-iq.(y-z)}\, i/q^2\}\, (q + p)_\nu A^\nu(z)\, \varphi_i(z)$$

$$= (ie)^2 \int d^4q/(2\pi)^4\, i/q^2\, (q + p')_\mu\ (q + p)_\nu \int d^4y \int d^4z\, \varphi_f^*(y)\, \varepsilon'^\mu\, e^{-ik'.y}\, \{e^{-iq.(y-z)}\}\, \varepsilon^\nu\, e^{-ik.y}\, \varphi_i(z)$$

where I have left out the photon norm factors for the moment, and I have chosen the leftmost graph in bottom page 194 such that the left graph vertex is $\varepsilon^\nu\ e^{-ik.y}\ \varphi_i(z)$. Now we can do our two spatial integrations. We get (including the φ_f and φ_i phasors)

$$\int d^4y \rightarrow (2\pi)^4\, \delta^4(k' + q - p') \qquad\qquad \int d^4z \rightarrow (2\pi)^4\, \delta^4(k + p - q)$$

so we can think of q = p+k as shown in the graph. Then $\int d^4q/(2\pi)^4 \rightarrow 1$ against one of these, so

$$= (ie)^2\, [\text{ leg norms}]\, (2\pi)^4\, \delta^4(\text{ pcons})\{\ i/q^2\, (q + p')_\mu\ (q + p)_\nu\ \varepsilon'^\mu\, \varepsilon^\nu\ \}$$

The bracket becomes

$$\{\ i/(p+k)^2\, \varepsilon'^\mu\, (p+k+p')_\mu\ \varepsilon^\nu\, (k+p+p)_\nu\ \} = \{\ i/(p+k)^2\, \varepsilon'^\mu\, (p'+k'+p')_\mu\ \varepsilon^\nu\, (k+p+p)_\nu\ \}$$

$$= \{\ \varepsilon\ .\ (2p+k)\ \ i/(p+k)^2\, \varepsilon'.(2p'+k')\ \}$$

Now reinstall the mass I omitted from the Δ_F propagator and this becomes

$$= \{\ \varepsilon\ .\ (2p+k)\ \ i/[(p+k)^2 - m^2]\ \ \varepsilon'.(2p'+k')\ \}$$

and when we put in those leg norm factors, we have derived the first term in 9.30! We see this rule

Rule: "add the momenta on the two sides of the photon, then dot into ε. "

Using this same rule, we can quickly obtain the second term in 9.30, and it has a relative + sign since Bose. **This concludes my derivation of 9.30.**

The gauge invariance claims made in 9.31 and 9.32 must be true, but you would have to do some work to actually verify them in 9.30. For example, if $\varepsilon \rightarrow \varepsilon + \lambda k$, you get contributions from all three terms in 9.30 and you would have to rationalize the propagator denominators and add things all together, and only then would you find that the λk contributions are 0. You get to use k.k = 0, but no other tricks at this level, you would have to show it. But we know it must be true in each order of α because it is built into the theory from the beginning. I will learn a lot more about this later I suspect.

Now if we use the transverse-only polarization choice, we get ε.k = 0 and ε'.k' = 0. This also tells use that in the lab frame for the initial π^+ particle, ε.p = 0 and ε.p' = 0. In this lab frame, then,

$\varepsilon.(2p+k) = 0$ kills off the first term

$\varepsilon.(2p'-k) = 0$ kills off the second term

so in the lab frame, S_{fi} reduces to just the last term and we get

$|S_{fi}|^2 /VT = (2\pi)^4\delta^4(..)(\text{factors})^2 [\ 4\ (\varepsilon.\varepsilon')^2\ ]\ d^3p'd^3k'$ = rate per volume

$= (2\pi)^4\ \delta$ (energy only) $(\text{factors})^2[\ 4\ (\varepsilon.\varepsilon')^2\ ]\ E'p'dE'\ d\Omega_{p'}$

$= (2\pi)^4\ (\text{factors})^2[\ 4\ (\varepsilon.\varepsilon')^2\ ]\ E'p'\ d\Omega_{p'}$

Somehow this comes out being A on page 195, but now we are working in the continuum normalization so I have to go through the steps again. The denominator in A is just k'^2 from 7.70 page 129, so it is just coming from our various norm factors and the E'p' shown. I skip this detail for now.

Then for non-rel, the denominator is just 1 and we get page 131 D which is the Thomson once again. So I don't think I would have any trouble showing A and 9.33 on page 195, but am not doing them for right now.

The point is that we have really done the full relativistic $\pi^+\ \gamma \rightarrow\ \pi^+\ \gamma$ "Compton" scattering in our KG theory for spinless charged particles. The result is simpler than the Klein-Nishina formula given on page 131 which is a lab spin-averaged result. Has this been experimentally tested? Why on comments?

9.6 Higher-order Processes (195)

This section lists off the Feynman rules for our KG scattering. Unfortunately, the rules as stated seem a bit ambiguous, such as Rule #1. I could clarify if I wanted. No minus signs for loops. Simpler propagator. Experiment says pions are bosons, so does the spin-statistics theorem somewhere. So you tend to add graphs and not have any relative minus signs. Three examples are given, one a little obscure regarding the decay of a K^+ where their point is that the amplitude is symmetric in the two product pions. Other examples are the scattering processes shown page 197 and 198, like electron and Bhabha. On page 197 they said "no filled negative energy sea" for pions, but they don't say what happens to such a sea which the reader might be thinking should exist. Probably this all relates to some horrible anti-Hermitian operator stuff. Then finally on page 198 we get the comment I was waiting for which is that these particles also have hadronic interactions, so experiments are not always obvious to interpret.

9.7 Nonrelativistic reduction and interpretation of the KG equation. (198).

The KG says this

$(\Box + m^2)\varphi = 0 \qquad (\partial_\mu\partial^\mu + m^2)\varphi = 0 \qquad (\partial_0^2 - \nabla^2 + m^2)\varphi = 0$

$\partial_0^2\ \varphi = (\nabla^2 - m^2)\varphi\ = \ddot{\varphi}$

We are told to define

$\xi \equiv \dot{\phi}$ => $\dot{\xi} = \ddot{\varphi}$

Then we have

$\dot{\xi} = (\nabla^2 - m^2)\varphi$ 9.40

But then this function ξ is never used! Instead, we now define θ and χ as shown in 9.41. The unusual thing is that each is a linear combination of φ and ξ, if you will. So we are linearly combining our original KG wavefunction φ with a $\dot{\phi}$. Why are we doing this?

(1) This produces a perfect SE as shown in 9.47 with a Hamiltonian as in 9.48 where the wavefunction is now the 2-vector (θ, χ). All algebra is checked, see hint at top of page in pencil (which I added just now). In this SE of course we have a linear time derivative. The second time derivative got swept under the rug! Of course things like $\dot{\theta}$ now include $\ddot{\varphi}$, but our wavefunctions of interest is now θ, not φ. This all ties in with that strange norm time operator I suspect.

(2) In the limit of no KE (particles at rest), this equation has a positive energy solution $(\theta,0)$ with E = m, and a negative energy solution $(0,\chi)$ with E = -m. This result was anticipated in the discussion earlier on the page surrounding equations 9.41→9.44 which gave motivation for the definitions of θ and χ as in 9.41.

(3) The only problem is that H_o is not Hermitian, which means $U = e^{-iHot}$ is not unitary, which means that $f(t) = \langle\psi(t)|\psi(t)\rangle$ varies in time which says $\int d^3x\, \psi^*\psi \neq$ constant, etc. But we already know that the KG theory has this issue so no surprise to find it here in our spinor reduction method.

Now we are going to attempt a FW transformation to remove the odd terms, as was done on page 47 for the Dirac equation. We want to find a rotation angle of some sort that cancels out the odd terms which couple θ and χ together. So let's make a little analogy here between the two cases:

$H = \boldsymbol{\alpha}\bullet\mathbf{p} + \beta m$ = odd + even

$$H_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}[-\nabla^2/2m] + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[-\nabla^2/2m + m] = \text{odd} + \text{even}$$

$= \rho\ [-\nabla^2/2m\ ] + \eta\ [-\nabla^2/2m + m]$ $\rho = i\sigma_y$ $\rho^2 = -1$

$= \rho\ (p^2/2m) + \eta\,(p^2/2m + m)$ $\eta = \sigma_z$ $\eta^2 = 1$

The analogy is not perfect, so we need to drop back into the logic on page 47. We need to kill off the ρ term with e^{iS} and a candidate here is this:

$\exp[iS] = \exp[i\eta\rho\theta]$ $\eta\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$ Pauli

Then we can use our general rotation formula on the rotation matrices TK page which says

$R_{\hat{\mathbf{n}}}(\varphi) = \exp[i\varphi \hat{\mathbf{n}} \bullet \mathbf{J}] = \exp(i\varphi/2 \hat{\mathbf{n}} \bullet \boldsymbol{\sigma}) = \cos(\varphi/2) + i\, \hat{\mathbf{n}} \bullet \boldsymbol{\sigma} \sin(\varphi/2)$

so then we have

$\exp[iS] = \exp[i\theta\sigma_x] = \cos(\theta/2) + i\, \sigma_x \sin(\theta/2)$

Thus we have [at the end I will take S→-S as per 9.52, causing θ → -θ, but for now do it this way]

$\exp[-iS]\, H_o \exp[+iS]$

$= [\cos(\theta/2) - i\, \sigma_x \sin(\theta/2)]\ [\rho\ (p^2/2m) + \eta\ (\ p^2/2m + m\ )]\ [\cos(\theta/2) + i\, \sigma_x \sin(\theta/2)]$

which is like D on page 47. Rewrite the above as

$= [\cos(\theta/2) - i\, \sigma_x \sin(\theta/2)]\ [i\sigma_y\ (p^2/2m) + \sigma_z\ (\ p^2/2m + m\ )]\ [\cos(\theta/2) + i\, \sigma_x \sin(\theta/2)]$
$= [C - i\, \sigma_x\, S]\ [i\sigma_y\ a + \sigma_z\ b]\ [C + i\, \sigma_x S]$

$= [C - i\, S\, \sigma_x]\ [iaC\sigma_y + bC\sigma_z - a\, S\sigma_y\sigma_x + ibS\sigma_z\sigma_x]$

$= [C - i\, S\, \sigma_x]\ [iaC\sigma_y + bC\sigma_z + ai\, S\sigma_z - bS\sigma_y]$ // $\sigma_x\sigma_y = i\sigma_z$ etc

$= [C - i\, S\, \sigma_x]\ [(iaC - bS)\sigma_y + (bC + iaS)\sigma_z]$

$= [C(iaC - bS)\, \sigma_y + C\,(bC + iaS)\, \sigma_z] - iS\,[(iaC - bS)\, \sigma_x\, \sigma_y + (bC + iaS)\, \sigma_x\, \sigma_z]$

$= [C(iaC - bS)\, \sigma_y + C\,(bC + iaS)\, \sigma_z] - iS\,[(iaC - bS)\, i\sigma_z - (bC + iaS) i\sigma_y]$

$= \{\ C(iaC - bS) + iS(bC + iaS)\, i\ \}\ \sigma_y + \{\ C\ (bC + iaS) - iS\,(iaC - bS)\, i\ \}\ \sigma_z$

$= \{\ C(iaC - bS) - S(bC + iaS)\ \}\ \sigma_y + \{\ C\ (bC + iaS) + S(iaC - bS)\ \}\ \sigma_z$

Now finally we can see the condition that would kill off the "odd" term which is σ_y

$C\,(iaC - bS) - S(bC + iaS) = 0$

$ia(C^2 - S^2) - b(2SC) = 0$

$ia\cos(2\theta) - b\sin(2\theta) = 0$

$\Rightarrow \tan(2\theta) = ia/b = i\,(p^2/2m)/(\ p^2/2m + m\ )$

$\Rightarrow \theta = \tfrac{1}{2} \tan^{-1} i[(p^2/2m)/(\ p^2/2m + m\ )] = \tfrac{1}{2}\, i \tanh^{-1}[(p^2/2m)/(\ p^2/2m + m\ )]$ //Sch p 31 8.96

Now as noted above, we take $\theta \rightarrow -\theta$ so that $S \rightarrow -S$ so my assumed starting position starts with 9.52. Then we finally get

$$\theta = -\tfrac{1}{2} i \tanh^{-1}[(p^2/2m)/(p^2/2m + m)]$$

which agrees with 9.51.

Note: Since θ comes out being *imaginary*, S is anti-Hermitian $S = -S^\dagger$, and e^{iS} is not unitary.

Now let's get all the signs correct before doing the next step. We have shown that

$$H_0' = e^{iS}He^{-iS} = \{ C(iaC + bS) + S(bC - iaS) \}\, \sigma_y + \{ C(bC - iaS) - S(iaC + bS) \}\, \sigma_z$$

and choosing the angle as above kills off the σ_y coefficient. The coefficient of σ_z is then

$$C(bC - iaS) - S(iaC + bS) = b(C^2 - S^2) - ia2SC$$

$$= b\cos(2\theta) - ia\sin(2\theta)$$

Let's now make our triangle based on the corrected result above which is

$$\tan(2\theta) = -ia/b$$

so imagine a standard triangle with $y = -ia$, $x = b$. Then we would say that

$$\sin(2\theta) = -ia/r \qquad \cos(2\theta) = b/r$$

where

$$r^2 = x^2 + y^2 = b^2 - a^2 = (p^2/2m + m)^2 - (p^2/2m)^2 = p^2 + m^2 \qquad \text{//amazing}$$

Then our coefficient is

$$b(b/r) - ia(-ia/r) = (b^2 - a^2)/r = r$$

and we end up with the grand result that

$$H_0' = e^{iS}He^{-iS} = \sqrt{p^2 + m^2}\ \sigma_z = \sqrt{p^2 + m^2}\ \eta = E\eta \qquad p = |\mathbf{p}|$$

and this is the famous decoupled result, just like 4.1 for the Dirac world. So we have found a "representation" of the problem with 2-component spinors and everything makes total sense and in fact H_o' is now Hermitian and thus all the results 9.53 → 9.60 are completely conventional, we have the usual probability and so on. No approximations have been made, we just found a unitary transformation which took us from the (θ,χ) representation to one that is decoupled. Of course we still have this funny stuff 9.41 going on.

Comment: In the Dirac case, we started with a 4x4 world and did FW which decoupled the lower 2x2 world from the upper 2x2 world. We ended up with equations for the upper 2x2 "positive energy " 12 states world with terms like spin-orbit interaction when fields were added.

In the Klein Gordon case, things are very different. We start of with a 1x1 world (after all, it is a spinless particle S = 0), but we then complexify instead of simplify to a 2x2 world by making linear combinations of φ with $\dot{\varphi}$. Then the upper 1x1 world of this 2x2 world is our new non-rel FW playground, analogous to the upper 2x2 world of the Dirac 4x4 world. I am wondering what things like spin-orbit might appear here, but of course we don't expect to get spin-orbit itself since there is no pion spin.

So, the next 5 pages repeat the above with the EM field turned on, and here we will get into approximations as we did with the Dirac stuff. I will continue here after Nov 16, hopefully. Signing off 11.7.08. // Resuming 11.18.08 after Cape Cod trip, and after reviewing Dirac FW and this KG chapter.

Our E&M starting point is 9.48. In other words, we go through the same complexification process as with the no-fields case, getting to a 2x2 space using 9.41. But now we replace ∇^2 by $-\pi^2$ and we end up with an adjusted Ham as in page 202 B where I have shown the unit matrix on the $e\Phi$ term. Again, they make a comment about the object ξ such that 9.39 becomes p 202 D, but again we never used this thing, I am mystified as to why they keep mentioning it. We use only the θ and χ as shown in 9.41.

Now let's just repeat what we did above but now with π installed, we are just restating B = 9.48,

$$H_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} [\pi^2/2m] + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [\pi^2/2m + m] + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e\Phi = \text{odd} + \text{even}$$

$$= \rho \quad [\pi^2/2m] + \quad \eta \quad [\pi^2/2m + m] \quad + e\Phi \qquad \rho = i\sigma_y \qquad \eta = \sigma_z$$

$$= \beta m + \mathcal{E} + \mathcal{O} \qquad \mathcal{E} = (\pi^2/2m)\,\eta + (e\Phi)\,1 \qquad \mathcal{O} = (\pi^2/2m)\,\rho \qquad \beta = \eta$$

and we are now "set up" as in 4.2 p48 but now we have new things for $\mathcal{E}$ and $\mathcal{O}$. Also, we should replace ψ with our new 2-component object Φ to get a new version of p49 A, and our 9.48 H_o is the original H of 4.2. The claim is that we just do exactly the same algebra that we did on page 49 and we end up at 4.4 but we have to replace, in that result, $\mathcal{E}$ and $\mathcal{O}$ and β by the three new objects shown above. In other words, we are now at the point in our FW process where

$$H_o' = \beta m + \mathcal{E}' + \mathcal{O}'$$

Notice that S has the same form it did before, the computations top page 50 are the same as well.

Now look at 4.4 where we see there are 5 terms which I have labeled $\mathcal{E}'$. Call these terms 1,2,3,4,5. We want to correlate these terms of 4.4 with various terms in 9.62. So

term 0 = $\beta m = \eta m$ = first term in 9.62

term 1 = $\beta\mathcal{O}^2/2m = \eta(\pi^2/2m)^2\,\rho^2 = \eta\,\pi^4/8m^3\,(-1) = -(\pi^2/8m^3)\eta$ = third term in 9.62

term 2 = order $1/m^7$ so we ignore it

term 3 = $\mathcal{E} = (\pi^2/2m)\,\eta + (e\Phi)$ = second and fourth term in 9.62

term 4 = $-1/8m^2\ [(\pi^2/2m)\ \rho,\ [(\pi^2/2m)\ \rho,\ (\pi^2/2m)\ \eta + (e\Phi)]\ ]$

$= (-1/8m^2)\ (1/2m)^2[\pi^2\rho,\ [\pi^2\rho,\ (\pi^2/2m)\ \eta + (e\Phi)]\ ]$

$= (-1/8m^2)\ (1/2m)^2[\pi^2\rho,\ [\pi^2\rho,\ (e\Phi)]\ ]$ // throwing out a relative 1/m term

$= (-1/8m^2)\ (1/2m)^2[\pi^2\rho,\ \rho[\pi^2,\ (e\Phi)]\ ]$ // since $[\rho,1] = 0$

$= (-1/8m^2)\ (1/2m)^2\ \rho^2[\pi^2,\ [\pi^2,\ (e\Phi)]\ ]$ // since π commutes through other stuff

$= (+1/8m^2)\ (1/2m)^2\ [\pi^2,\ [\pi^2,\ (e\Phi)]\ ]$ // $\rho^2 = -1$

$= (+1/32m^4)\ [\pi^2,\ [\pi^2,\ (e\Phi)]\ ]$ = fifth term in 9.62

term 5 = involves time derivative on $\mathcal{O}$ but we consider "static fields" here so contributes 0

So I have now derived all portions of the equation set labeled 9.62.Remember that π includes a derivative so its commutator is something to worry about as shown.

I agree that the first (...) grouping is in fact the first three terms of the expansion of $\sqrt{m^2+\pi^2}$ which is the energy you expect when you have momentum π including the field momentum.

Now compare 1.34 p 12 with 9.62. In our non-rel limit, we see $\pi^2/2m$ in 1.34 for the φ positive-energy non-rel solution, and we saw how, if you assume a uniform B field, this creates an orbital L•B type term in 1.35. I did the math for this in my chap 1 notes, it arises from the two cross terms in π^2.

We of course have exactly this same term as the second term in 9.62 which applies to the upper 1x1 of our 2x2 KG space. Of course m is now the pion mass, not the electron pass.

Now we have a typical BD mystery statement: something about the orbital g factor being reduced by m/E. I would say the KG and Dirac orbital g factors are both the same and are both -1, given the correct mass and charge for the KG particle.

Of course in our KG case we don't have a spin term like that shown in 1.34.

They claim that the last term in 9.62 is a Darwin type term similar to what we got in the Dirac case. The only similarity I see is that it does involve the potential Φ and so acts as some kind of Φ correction to the straight $e\Phi$ term, and these types of corrections are "Darwin terms".

So I was asking what "kinds of terms appear" when you do the FW process in the KG case. The answer is that you get a spin-orbit term that is the same as the Dirac one, and you get some kind of Darwin term which is a little different, and you get no term representing spin, ie, no σ•B type term since the KG particle has no spin.

Now starting bottom p203 and ending top page 205, we have a sort of summary of how you do physics in this limit of the KG world. We literally take 9.62 as our operative Hamiltonian in the 2x2 sense. We can define a positive energy solution as in 9.63, write a SE for it as in 9.64 (where now the matrix η no longer appears, we are in the upper 1x1 space now). We have the traditional prob density 9.65 and matrix element of energy 9.66 where they carefully define H'(x,e) along the way.

Now we consider the negative energy solutions and we are going to cast them into anti-particle solutions! We define them also with $E_n > 0$ as 9.68 shows. Now when you use the lower 1x1 part of the KG space of 9.62, the two terms shown there with e change sign relative to the other terms due to the η matrix, so easiest to think of $e \rightarrow -e$. Thus we end up with 9.69 showing H'(x,-e) and we can think of that as an anti-particle solution. If ψ^+ is a π^+, then the other is π^-.

BD use a scalar spatial function ψ and a spinor function Φ in this section. Thus, in 9.71 where we see Φ appear, the H' must be the matrix thing 9.62 including its η matrix. But in 9.66, we have the ψ^+ type function, so the H' there is 9.67 without the matrix. Yes, I agree that you can install the η matrix as in 9.72,3 to get results shown. This makes E>0 for both particle and antiparticle, and makes the charges opposite.

"Proceeding further" along these lines starting mid page 205, we can think about "π-mesic atoms" and calculations regarding same. Such objects do exist for a while. You just use good old QM methods as with regular atoms, with our FW Hamiltonian such as 9.67. That is, we think of perturbation theory and H' and transition rates and all that stuff.

Now, we arrived at the expressions 9.72,3 via our FW reduction (complexification if you will to the 2x2 space). Maybe these equations are true not just in the NR limit, maybe they are more general and apply at all energies. This is the *ansatz* stated bottom of page 205 (the word "ansatz" was used often in my era as "hypothesis", but I have not heard it used lately).

On page 206 we derive some interesting factoids *assuming* η should be inserted in 9.72,3:

• The **energy** of a general state here called $\omega_{\mathbf{p}}$ [state is labeled by "p", label is "n" in 9.72; we are assuming now a *free-particle* general state because we make use of 9.76 which I think is only true for free particles] seems to be a "scalar" relative to our FW transformation e^{iS}, which we recall is a 2x2 non-unitary matrix transformation. The expression for $\omega_{\mathbf{p}}$ is form invariant, you can compute the same number in either "frame", either "representation". In the starting representation, the 2x2 H is given by p202 B, while in the FW representation it is H' = 9.62 which contains only "even" terms. Of course this is an approximation, whereas H is presumably exact.

• the **charge** of a state is also such a scalar. In the 2x2 world, you have $\rho = \Phi^\dagger \eta \Phi$ which seems a little more reasonable that the time derivative form. As the third line of 9.78 shows, this is just $\theta^*\theta - \chi^*\chi$, which goes with the idea of θ being the particle and χ the oppositely-charged anti-particle. Then when you install the expressions for θ and χ, lo and behold you obtain the previous result

$$j^0 = i\,[\,\varphi^*(\partial^0\varphi) - (\partial^0\varphi^*)\,\varphi\,] = \varphi^*\, i \overleftrightarrow{\partial}^0 \varphi \qquad \text{compare:} \qquad j^0 = \psi^\dagger\psi \quad \text{Dirac}$$

because there are time derivatives buried in the θ and χ functions.

• Although we showed the above bullet items to be true for general free particle Φ, BD claim it is true with "interactions" are present, such as an EM field. The bullets are true for any number of FW iterations, as suggested by equations J. The idea then as shown in 9.80 is to generalize the whole notion of "matrix element" to include the η matrix.

Somehow, there is some underlying math stuff going on here which BD either don't know about, or did not want to confuse is with. I would imagine KG theory has progressed a bit even since 1964. The time derivative in the norm relates to antiparticles.

So finally I think I am done with this long K-G chapter! Have 24 pages of notes here.

****

Note on *Dirac computer algebra*: there is something called FORM which I never heard of. There is a web site, and there is no windows exe version, but perhaps a Cygwin version. Academia has stayed in the UNIX world I think, not a bad choice, and maybe again now that costs are down.

the author of the symbolic manipulation package FORM[2] which was used to check many of the trace calculations. Samples of these calculations can be found in appendix D.

Finally, I would like to thank Kaston Leung for his assistance in proof reading an early version of the manuscript, and James Fuite for proof reading the final manuscript.

Doug Gingrich


```
*************************************************************
FORM by J.Vermaseren,version 3.1(Oct 13 2002)
Run at: Sat Jun 11 15:26:21 2005
    *
    * Equation 8.130.
    *
    OFF STATS;
    INDICES mu, nu;
    SYMBOLS m, M, E, Ep;
    VECTORS pi, pf, Pi, Pf;
    *
    LOCAL L = ((g_(1,pi) + m)/(2*m) * g_(1,mu)
              * (g_(1,pf) + m)/(2*m) * g_(1,nu))/2;
    LOCAL H = ((g_(2,Pi) + M)/(2*M) * g_(2,mu)
              * (g_(2,Pf) + M)/(2*M) * g_(2,nu))/2;
    LOCAL Mfi2 = L*H;
    *
    TRACE4,1;
    TRACE4,2;
    *
    ID Pf = Pi + pi - pf;
    ID Pi.Pi = M^2;
    ID pi.pi = m^2;
    ID pf.pf = m^2;
    ID Pi.pi = M*E;
    *
    ID Pi.pf = M*E;
    ID E/M = 0;
    *
    BRACKET m;
    *
    PRINT;

   L =
       + m^-2 * ( 1/2*pi(mu)*pf(nu) + 1/2*pi(nu)*pf(mu)
       - 1/2*d_(mu,nu)*pi.pf )
```

This is all from a May 2006 book by Gingrich called practical QED which does not use field theory but follows BD. He does KG first, then it is easier to see how spin affects things.