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Chapter 1: The Dirac Equation (2)

1.1 Formulation of Relativistic Quantum Theory (2) . Authors write out 6 basic facts that we all know the relativistic theory must respect.

1.2 Early Attempts (4). The idea $H = \sqrt{p^2 + m^2}$ is non-local and had negative energy solution issues. The idea that $H^2 = \mathbf{p}^2 + m^2$ gives the Klein-Gordon equation as the SE but results in a non-positive-definite probability density. So both these ideas were rejected.

1.3 The Dirac Equation (6) Dirac then tried an NxN matrix/spinor idea and found a solution for N=4. His H has the form $i\hbar\partial_t\psi = (\hbar c/i)\, \boldsymbol{\alpha} \bullet \nabla\psi + \beta mc^2\psi$ and is linear in ∇, and has $\rho = \psi^\dagger \psi \geq 0$ as required. We know now that the 4-spinors transform according to a certain LG representation different from the vector one with the same dimensionality 4. We call this the "Dirac representation" below. [Dirac 1928]

1.4 Nonrelativistic Correspondence (10). We first find the four "at rest" solutions, 2 of positive energy and 2 of negative energy. The velocity operator in the Dirac Space is shown to be $\mathbf{v}_{op} \equiv c\boldsymbol{\alpha}$ and we then have a conserved current with $\mathbf{j} = \psi^\dagger \mathbf{v}_{op} \psi$ and $j_0 = \psi^\dagger\psi$.

We then add an EM field in the usual covariant manner and do a non-relativistic limit for a Dirac particle in arbitrary but slow motion, looking only at the positive energy solutions. We find that the upper two spinor components dominate, and by doing an approximation for the lower components and feeding that back into the upper spinor Dirac equation, we end up with a term appearing "out of nowhere" in the 2x2 reduced Hamiltonian which interacts with **B** in a way suggesting the particle has an intrinsic magnetic moment. Thus the notion of "spin" $\mathbf{S} \equiv (\hbar/2)\, \boldsymbol{\sigma}$ falls out of this theory. By then assuming a uniform **B** field, we find that the Ham has the term $(\mathbf{L} + 2\mathbf{S}) \bullet \mathbf{B}$ and we suddenly have the basics of atomic spectroscopy falling out of the theory. So the Dirac theory concerns spin-1/2 particles like electrons. The orbital **L** has a classical derivation (with Thomas precession factor of 2 correction), but the **S** term is just out of the blue.

Chapter 2: Covariance of the Dirac Equation (16)

2.1 Covariant form of the Dirac Equation (16). The original 4x4 matrices α_i and β are replaced by the four matrices γ^μ which at least has a covariant "look" to them. Then notation $\gamma^\mu a_\mu = \not{a}$ (slash notation) is introduced, and then our Dirac equation reduces to $(\not{p} - mc)\,\psi(x) = 0$, very compact.

2.2 Proof of Covariance (18). We learn here that the Dirac equation will be form invariant and will appear as $(\not{p}' - mc)\,\psi'(x') = 0$ in a new frame if we transform by $\psi'(x') = S(a)\psi(x)$. Here S(a) is a 4x4 matrix that provides the *Dirac* representation of the Lorentz Group. We get $S = \exp[- (i/4)\,\sigma_{\mu\nu}\,\omega^{\mu\nu}]$ where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$ are the Dirac generator matrices (only 6 are distinct) and $\omega^{\mu\nu}$ represents an arbitrary combined boost/rotation set of parameters. If you want to *pick out* the isolated boosts and rotations, you can write $S = \exp[- (i/4)\,\omega\,\sigma_{\mu\nu}(I_n)^{\mu\nu}]$ for n = 1,2,3,4,5,6. In this form, ω is your boost or rotation parameter, and it turns out that $i\,I_n$ are the generator matrices of the LG *vector* representation (I explain this in my Lorentz Group document, follows from $\Lambda = e^{\omega}$). So using this S, we get form invariance, and our proof of covariance is complete. If course things like x^u transform in the usual *vector* Lorentz manner with $a^\mu{}_\nu$ so that x' = ax.

2.3 Space Reflection (24). If we want to also have our form invariance under a parity inversion, we find that $S = \gamma^0$ does the trick. It is noted that the first two at-rest Dirac solutions have +1 "intrinsic parity", while the negative energy solutions have -1 "intrinsic parity". Time reflection is deferred to a later chapter. It is noted that the equation $S^{-1} = \gamma_0 S^\dagger \gamma_0$ is valid for both Lorentz transformations and parity. Something like unitarity, but not quite.

2.4 Bilinear Covariants (25). The 4x4 Dirac space can be spanned by 16 4x4 "basis matrices" which can be partitioned as shown on page 25. They are chosen this way because each partition has a certain well-defined transformation property when sandwiched between $\bar{\psi}$ and ψ, such as "scalar" or "pseudoscalar" (this one is the famous γ^5) or "tensor" (these of course are the $\sigma_{\mu\nu}$).

Chapter 3: Solutions of the Dirac Equation for a Free Particle (27)

3.1 Plane Wave Solutions (28). A lot of stuff is packed into this small section, none of it can be ignored.

(1) In order to find the solutions to the Dirac equation for a particle with arbitrary momentum **p**, we take advantage of our knowledge that $\psi'(x') = S\psi(x)$ and we compute S for the boost which takes our particle from a frame where **p** = **0** to one where **p** = **p**, and we then apply S to the rest-frame unit spinors to find what the general-**p** solutions are. After much technical gyration, we end up with our four solutions being a certain phasor times the four columns of the matrix 3.7. These columns are called $w^r(p)$ and the phasor is $\exp(-i\varepsilon_r\, p_\mu x^\mu/\hbar)$ where $\varepsilon_r = +1$ for the first two spinors, and -1 for the negative energy spinors 3 and 4.

(2) We then take note that these solution spinors $w^r(p)$ solve the Dirac equation in momentum space which is shown in 3.9, namely that $(\varepsilon_r \not{p} - mc)\,w^r(p) = 0$. The ε_r factor appears because

$$\hat{p}_\mu \exp(-i\varepsilon_r\, p_\mu x^\mu/\hbar) = +i\hbar\,\partial^\mu \exp(-i\varepsilon_r\, p_\mu x^\mu/\hbar) = +\,\varepsilon_r p_\mu \exp(-i\varepsilon_r\, p_\mu x^\mu/\hbar)$$

and we see that the negative energy solutions have negative **p** as well as negative E.

(3) Equation 3.9b states the covariant orthogonality of the four spinor solutions $w^{r}(p)$ where the left w has a bar on it. The γ^0 = diag(1,-1) which this bar implies causes the ε_r factor shown on the right. Since this equation is a scalar on both sides, we need only show it is true in the rest frame which is easy. So the RHS is $\varepsilon_r \delta_{r,r}$ and we find that if r ≠ r', two spinors are orthogonal. Think of $\langle \varphi_r | \varphi_{r'} \rangle = \delta_{r,r'}$

(4) Equation 3.9c is the statement of "completeness" for the spinors. They span the space. The ε_r appears here so that it can cancel the ε_r in the previous equation when you "prove" completeness. The sum is r = 1,2,3,4 and the RHS is $\delta_{\alpha,\beta}$ where α and β are the component indices of the spinors. Think of
$1 = \Sigma_r \, | \varphi_r \rangle\langle \varphi_r |$ and $\delta_{ij} = \Sigma_r \langle i | \varphi_r \rangle\langle \varphi_r | j \rangle$.

(5) Equation 3.11 is a non-covariant statement of no-bar-orthonormality where the ε_r now appear in the **p** arguments of the spinors. There is no bar over the left w now. When both spinors are positive energy, we are talking probability density and we get the Lorentz contraction factor on the right in our boosted frame. And we see that, without this bar, a spinor of positive energy and momentum +**p** is orthogonal in this sense to a negative energy spinor of momentum -**p** .

(6) In the 2x2 world we know the Levitt idea that $\hat{\mathbf{s}} \bullet \boldsymbol{\sigma} \, |\theta,\varphi\rangle = +1 \; |\theta,\varphi\rangle$ where $|\theta,\varphi\rangle$ is a state of "spin up" in the θ,φ direction. Here, the vector $\hat{\mathbf{s}}$ points in the $|\theta,\varphi\rangle$ direction and is called the spin polarization vector. In general one can say $\hat{\mathbf{s}} \bullet \boldsymbol{\sigma} \, |\theta,\varphi\rangle = \lambda \; |\theta,\varphi\rangle$ where λ = ± 1. We are familiar with the idea that the 2-spinor (1,0) has $\hat{z} \bullet \boldsymbol{\sigma} \, |0,0\rangle = +1 \; |0,0\rangle$ and is "spin up", while u = (0,1) has λ = -1 and is "spin down".

(7) To learn about spin, we should look in the rest frame to find some $s^{\mu} = (0,\mathbf{s})$ where $\mathbf{s} = \hat{\mathbf{s}}$ = the spin polarization (BD don't bother with the unit vector hat). And of course $p^{\mu} = (mc,0)$ so we have $p_{\mu} \, s^{\mu} = 0$ and this will be true in any frame. Also, we have $s_{\mu} s^{\mu} = -1$ as a second world scalar. Recall that the Lorentz group has two Casimirs and I am guessing they can be taken as $s_{\mu} s^{\mu}$ and $p_{\mu} \, p^{\mu}$

(8) In the 4x4 world the matrix $\boldsymbol{\Sigma}$ = diag(**σ**,**σ**). So if we go to the rest frame, it is very clear that the r = 1,3 are spin up with λ = + 1 and spinors r = 2,4 are spin down λ = -1. This is so because, for example, the spinor r=1 has the form (1,0,0,0).

(8) With an eye to the upcoming hole theory interpretation of the negative energy solutions, the four spinors u(p, s=±u_z) and v(p,s=±u_z) are *defined* in terms of the $w^{r}(p)$ as shown in 3.16. Here, $u_z^{\mu} = (0,\hat{\mathbf{z}})$ in the rest frame, so this is a particular value for s^{μ}. Here u are the positive energy spinors r = 1,2 and v are the negative energy ones r = 3,4. The unusual definition is that you would expect maybe $w^{3}(p)$ to have the same spin sense as $w^{1}(p)$, based on item (6) above, but it is defined oppositely. This I think means that v(p,s) has opposite momentum *and* spin compared to u(p,s). I think in hole theory this is what we want. The key thing to remember is that these u and v are just defined as shown, you can define something however you want.

3.2 Projection Operators for Energy and Spin

The energy projectors $\Lambda_{\pm}(p)$ are shown in 3.18. If you have an arbitrary linear combination of spinors in a solution, this will project out the part that has the indicated energy. The projector is covariant in the same way the momentum-space Dirac equation 3.9a is covariant. When you go to a primed frame, you need only prime the p.

$$\Lambda_{\pm}(p) = (\pm \not{p} + mc)/(2mc) \qquad // \ 3.18$$

The spin projectors $\Sigma(s)$ as shown in 3.10 are covariant in exactly this same way. We obtain an expression for this spin projector by first determining the spin of the four basic spinors $w^{r}(p)$ [+.–.–.+]. That is, we verify that $\Sigma(s = -u_z)\ w^3(p) = w^3(p)$, for example.

We then change the order of things so instead of talking r = 1,2,3,4 we talk r = 1,2,4,3 and we name these u(p,s), u(p,-s), v(p,s), v(p,-s). Then we have $\Sigma(s)y(p,s) = y(p,s)$ for y = u and v.

For the spin stuff, we start in the rest frame, find a covariant form, and then that applies in all frames.

$$\Sigma(s) = \tfrac{1}{2}\ (1 + \gamma_5 \not{s}\ ) \qquad // \ 3.19$$

Finally, we can make combination energy/spin spinors such as those shown page 35. These are chosen to project the $w_r(p)$ for r = 1,2,3,4. The projection operators are useful later when we do calculations.

3.3 Physical Interpretation of Free-particle Solutions and Packets (35)

This section contains four "activities" or items of interest (explorations) which I now describe. As usual there is much math detail which I suppress in these meta notes.

(1) The first activity of this section is to construct a ***wave packet*** **from only positive** energy solutions of the Dirac equation, and that packet is called called $\psi^{+}(x^{\mu})$. The amount of u(p,s) is called b(p,s) which is the r=1 component, and of course the amount of u(p,-s) is b(p,-s) which is the r=2 component [see 3.16]. So this thing has a sum over spins s, and an integral over d^3p with a factor that causes momentum space normalization to be nice as shown in 3.24.

One we have this wave packet defined and normalized as in 3.24, we can go compute things with it. The main item of interest is to compute the total current J^{μ}. After using the technical "**Gordon decomposition**" and the orthogonality of the spinors. we end up after much work with the following:

$$J^{\mu} = c \int d^3x\ \ \bar{\psi}^{(+)}\ \gamma^{\mu}\ \psi^{(+)} \ = \int d^3p (p^{\mu} c^2/E)\ \ \Sigma_{\pm s}\ |b(p,s)|^2$$

$$J^{0} = < \psi^{\dagger (+)} \mid c\ \ |\psi^{(+)}> \ = c \int d^3p\ \Sigma_{\pm s}\ |b(p,s)|^2 \qquad = c \qquad // \text{ normed in 3.24}$$

$$\mathbf{J} = < \psi^{\dagger (+)} \mid c\ \gamma^0\ \boldsymbol{\gamma}\ |\psi^{(+)}> \ = \int d^3p (\mathbf{p}\ c^2/E)\ \ \Sigma_{\pm s}\ |b(p,s)|^2 \qquad = < c\ \gamma^0\ \boldsymbol{\gamma}> = <c\ \boldsymbol{\alpha}\ > = <\mathbf{v}>$$

Recall in E&M that $\mathbf{j}(\mathbf{x},t) = q\mathbf{v}\delta(\mathbf{x}-\mathbf{v}t)$ for a moving point charge and $\mathbf{J} = q\mathbf{v}$ for the total current after you integrate over all space. We find a similar result here for the "probability current" (think q=1), and we conclude that the total current is the group velocity of the packet as computed above, and the total probability is "1". This group velocity is the expectation value of the Dirac velocity operator $\mathbf{v} = c\boldsymbol{\alpha}$,

exactly as we would expect. Normally with any kind of wave packet, this is the thing you always compute -- the group velocity -- so I guess we are happy to see it come out being reasonable.

(2) The second activity of this section is to **generalize the wave packet** to include r = 3,4 as well as r = 1,2 spinor components. Now we add amplitudes d(p,±s) and things become pretty messy. As before, we write out the current J^{μ} and the claimed result (I did not verify it in detail) a given in 3.31 and just above 3.31 where we see that once again, total probability is "1".

There is an interesting new feature now caused by the cross terms between the b(p,±s) and the d(p,±s). We suddenly get exponential factors $\exp(\pm 2ix_o p_o/\hbar) = \exp(\pm 2iEt/\hbar)$. The frequency here is very large, because $E > mc^2$ we have $\omega > 2mc^2/\hbar = 2 \times 10^{21}$ Hz !!! This is the mass energy associated with two electrons, and in some weird way, this cross term represents pair creation from the vacuum which we shall see is just the hole theory mechanism. The fact is that the total current **J** includes these high-frequency components and to these components the famous word **zitterbewegung** was attached by Schrodinger in 1930 (born Austria).

(3) The third activity is to construct a wave packet with a standard 3D spatial **gaussian** shape of radius d at t=0. [We now have two meanings for the symbol d.] You find that such a packet, even at t=0, must contain a mixture of b and d coefficients, as shown in 3.33. The d contribution becomes significant only when p increases to p ~ mc (to make the spinor v be significant) *and* when d ~ $\hbar$/p (to make the d contribution expo in 3.33 be significant). Together these imply that you must confine such that d ~ $\hbar$/mc and this latter ratio is called the **Compton wavelength**. So, if you confine to Compton or smaller, you will have significant d contributions, and you will then have the zitter effects.

(3) The fourth activity is a thought experiment called the **Klein Paradox**. Put an electron in a 1D box with steep walls V_o as shown page 41 and look at the right wall. After you solve the problem with the usual boundary conditions made happy, you find the following strange thing: if you make the wall V_o have a height which exceeds mc^2, you get a negative transmitted current, and a reflected current larger than the incident current. Authors defer till later the interpretation of this result in terms of hole theory. I think in order to penetrate that high wall, we create somehow a hole/electron pair, and these new particles explain the paradox.

Chapter 4: The Foldy-Wouthuysen Transformation (46)

4.1 Introduction (46). The idea here is to do a set of iterative unitary transformations on our $H\psi = \hat{E}\psi$ problem to get to a new and valid description $H'\psi' = \hat{E}\psi'$ in which the lower spinor components are more and more *decoupled* from the upper two components (which means the "odd" matrix terms in H' are made smaller and smaller in magnitude). The iterations in essence form a power-series expansion in the smallness parameter v/c which here appears in the form 1/m (as explained in raw notes). We are doing this expansion with an interest in bringing out more and more detail of the NR limit of the Dirac theory. As we do it -- as we decouple the component pairs -- H' in general gets more complicated as new "terms" appear. But it is just these "terms" that we care about. When we carry out this process with H including the EM field in the Dirac Hamiltonian (everything is done relativistically correctly), these "terms" pop out before our very eyes. We shall find one term which is the "**electron spin**" interacting with the B field as a magnetic dipole interaction (a 1/m term). We shall find two more terms that relate to "**spin orbit**

coupling" and another which is the unusual **Darwin term** (all these are $1/m^2$ terms). These result from doing three iterations of the FW transformation. I have no idea what effects are extracted by doing *more* terms, but they will surely be extremely small effects. At each iteration, the effects drop off by factor v/c.

Comment: It just happens that γ and β refer to 4x4 matrices and these same two terms are used in a different sense to describe the velocity and energy gain of a relativistic particle. Be warned.

4.1 Free-particle FW Transformation (46). If there is no EM field, we can do FW exactly in a single shot! The result is exact not just in the NR limit. We start with $H = \boldsymbol{\alpha}\bullet\mathbf{p} + \beta m$ ($\hbar$ and c = 1 starting in this chapter), and we end up with $H' = \beta\sqrt{m^2 + p^2}$ which is completely "even", so the upper and lower spinor components are 100% decoupled and the upper spinors like (1000) have energy $+\sqrt{m^2 + p^2}$ and the lower ones like (0010) have energy $-\sqrt{m^2 + p^2}$. The unitary transformation e^{iS} that does this is stated in terms of a certain angle θ which is then exactly computed. In the NR limit, we obtain $iS = \beta O/2m$ where O is the "odd terms" of the Ham, which in this case is just $\boldsymbol{\alpha}\bullet\mathbf{p}$. We could expand things in 1/m using 47 H for θ and no doubt replicate our later results with the EM field turned on, where we then turn off the field.

4.2 The FW Transformation with E&M Fields (48). Here we incorporate the EM fields "in the usual way" and the claim is made that there can be no "single shot" fully-decoupling FW solution because the fields are time dependent, though I don't quite follow this argument. Ie, I could imagine a solution angle which is a function of time perhaps, but I guess H' is not supposed to contain explicit time.

As described in my raw notes, we separate the even and odd terms in our starting H. We then write out the unitary transformation as e^{iS} where S is now time dependent. The relation between H and H' is shown in page 49C to be $H' = U^{-1}(H - i\partial_t)U$ and this is expanded in powers of 1/m (= v/c) to get the huge mess shown in p 49 E with BCH expansions done through order $1/m^3$.

We then do things iteratively, one "order of m" at a time. In the first iteration which includes order m^1 and m^0, we take $iS = \beta O/2m$ where $O = \boldsymbol{\alpha}\bullet(\mathbf{p}\text{-}e\mathbf{A})$ = the odd terms in H. With this choice of S, we clean out the odd terms, and we get $H' = \beta m + e\varphi$. But we need more "orders" to get a useful result, so we iterate.

To prepare for the second iteration, we use our zeroth-order S in equation E to compute (computations shown top of page 50) the next terms in H' to order m^{-1}. In fact, we compute all the higher order expansion terms at the same time. So we cancelled out the order m^0 odd terms, but when we add these extra order m^{-1} terms from double-expansion E, we then get H' = as shown in 4.4 which has odd terms of order 1/m. The odd terms are called O' and the even E'.

So we then do a second unitary transformation this time with some S' in order to clear out these O' terms. Of course the S' we use here is just $iS' = \beta O'/2m$. When we do this, we clear out the odd terms to order 1/m and we have H". But now we still have some odd terms of order $1/m^2$ and we then do a third FW to knock them out, and we end up with H"' shown top of page 51. We then do the usual painful acrobatics to replace the EM potentials where they appear with EM fields **E** and **B** and we end up finally with H"' as shown in 4.5.

The first term is really the square root shown in E which, were there no EM field, we would refer to as the relativistic mass increase; all the other terms are exactly the "terms" we wanted to ferret out by doing the FW stuff in the first place. We named these terms above. An interesting interpretation of the

Darwin term is stated on page 52. It is a $\nabla^2 V(r)$ term attributed to the zitter smearing out the potential over a Compton wavelength.

4.3 The Hydrogen Atom (52). This is only a thin survey of the Dirac H atom solution, I have to see my Sakurai book for details. Facts are stated without proof; the derivation is completely glossed over. On page 55 they simply quote the energy levels formula which comes out of the solution as 4.14. They then quote the spinor ground state and list off the lowest states and their energies. What the Dirac theory successfully predicts is the so-called **fine structure** which is a roughly 10 GHz energy split for H.

Then comes a quick discussion of the various 1 GHz adjustments one must make to the Dirac theory predictions. These include the nuclear spin **hyperfine structure**, and the **Lamb shift** stuff coming from higher order Feynman diagrams. They then present arm-waving explanations of the rough size of these two corrections on the last three pages of this section (58-61).

The Dirac H atom was first solved separately by Darwin and Gordon in 1928 papers, the same year of the Dirac theory itself! We are given latter day analyses in books dated 1957 (Bethe & Salpeter) and 1961 (Rose). I might have some stuff in some of my Dover atomic physics books.

Chapter 5: Hole Theory (64)

5.1 The Problem of Negative-energy Solutions (64). Why don't regular electrons all radiate down into the negative energy states? Dirac's answer is that the negative Dirac Sea *is filled up*, and the Pauli Principle prevents positive-energy electrons from falling down there. The Dirac theory predicts pair production and annihilation, predicts positron existence and anti-matter in general. Our theory however is now a multi-particle theory, unlike the classic solution of the H atom which involves a single electron.

5.2 Charge Conjugation (66) You can change the sign of "eA" in the Dirac equation by doing a sandwich transform with $C = \gamma^2 K$. In fact, they use $C = i\gamma^2 K$ because $(i\gamma^2)$ is then antidiag(1,-1,-1,1) = real. I examined $(i\gamma^2)\psi^* = \psi_c$ for ψ being a negative energy spinor and found this:

$$(i\gamma^2)[\, w^4(\mathbf{p}) \exp(-i\varepsilon_4 p^\mu s_\mu / \hbar)] \,^* = [w^1(\mathbf{p}) \exp(-i\varepsilon_1 p^\mu s_\mu / \hbar)] \qquad \varepsilon_4 = -1 \;\; \varepsilon_1 = +1$$

and the projectors show that this transform changes neither p^μ nor s^μ. There is potential confusion with regard to each 4-vector. We imagine that **p** should not change under charge conjugation, and you see that both w^1 and w^4 have the same **p**. The energy does change, but the energy is $\varepsilon_r p^0$, not just p^0, so our notation lets p^0 stay the same and the ε changes sign. Thus, the sign change of the phasor above is handled by the ε factors, and $p^\mu s_\mu$ remains fixed. As for the spin, it seems odd that s^μ is the same, when we know that w^4 is spin down and w^1 is spin up, in the rest frame. But this is explained by

$$\tfrac{1}{2}(1 + \gamma_5 \not{s}) = \tfrac{1}{2}(1 + \gamma_0 \boldsymbol{\Sigma} \bullet \mathbf{s})$$

so that γ_0 and spin-down $-\mathbf{s}$ provide a double negative for the w^4 state.

Remember that "the Dirac equation" in the form $(i\not{\nabla} - e\not{A} - m)\psi = 0$ is really $H\psi = \hat{E}\psi$ written in covariant form (with EM field added covariantly by prescription). We might write this just as $D(e)\,\psi = 0$.

Doing $(i\gamma^2)\psi^* = \psi_c$, we have shown that we must also have the equation D(-e) $\psi_c = 0$. This is why we interpret ψ_c as a positron wavefunction spinor.

There are three distinct "interpretations" going on here and they tend to get muddled.

#1: The operator $C = i\gamma^2 K$ changes a negative energy Dirac spinor ψ to a positive energy Dirac spinor ψ_c having the same p and s 4-vectors as one sees from projectors. But when the Dirac equation is included, operator C also in effect changes e to -e, a number which is not "visible" in the spinor. So overall, the operator C leaves s^μ and p^μ the same, but changes e, and this is what "charge conjugation" ought to do. We can then claim that *C acting on* a negative energy electron state gives a positive energy positron state. We do not claim that negative energy electron states "are" positron states.

#2: The fact that D(e) $\psi = 0$ and D(-e) $\psi_c = 0$ are equally valid equations says that electrons and positrons should both "exist", they are both predicted by the theory.

#3: In solid state semiconductor physics with energy bands and gaps, we are used to the notion of a "hole" being *the absence of* an electron in the energy band below the gap. The "hole" has its own properties such as mobility and positive charge, just as if it were a "real particle". In Dirac theory, a "hole" in the negative energy electron sea *would behave as* a positive energy particle of positive charge. Such a hole would have "properties" such as spin and momentum. The vague phrase "would be behave as" is what interpretation item #1 above attempts to clarify. The behavior of the hole is described by $\psi_c = (i\gamma^2)\psi^*$ where ψ is the spinor of the "missing electron in the negative energy sea". So when the "vacuum" of just the Dirac sea absorbs a $2mc^2$+ photon, we end up with a ψ_c as just described (the positron), and we also have a ψ+ which is the wave function for the electron. From |00> we now have $|\psi_c \psi_+>$. You can see how field theory makes this formalism nicer.

5.3 Vacuum Polarization (70) (I quote exactly my raw notes text here) Now we have to think of an isolated normal electron as being "surrounded" by the Dirac Sea of negative energy and negatively charged electrons (not positrons!). Does the Coulomb force couple between positive and negative energy electrons? If so, we might expect the sea electrons to be "repelled" away from the isolated positive-energy electron causing a region around our electron to have an effective positive charge distribution as illustrated in the second figure on page 70, where the first shows the charge distribution in some sense for the positive-energy electron itself. The region of repelled sea electrons seems similar to the way a charge in a dielectric would align electric dipoles, causing a region of opposite polarization charge around the charge in question. For this reason, this effect is called "vacuum polarization" meaning Dirac Sea polarization. The figures indicate some scale R for this polarization, but nothing is said about R. Perhaps the "bare charge" is really -2|e|, but when you test from far away, you see the "dressed charge" which is only -|e|, with the Dirac sea providing +|e| from its integral. These are very vague notions that get supported later in field theory. There, if you are very close with your "test charge", energies are high and you can make pairs and these are associated with the "vacuum polarization" Feynman diagram and this is in fact the major part of the Lamb shift (which seems to contradict their comment here). But in field theory, "polarization" means a slightly different thing. You actually create a particle pair, causing a charge separation which is then the "polarization". Again, the Dirac Sea model is not quite physical.

They "sidestep" the final question of why we don't "see" all that negative charge of the Sea with the comment that if it made an E field, what direction could it point in? Sure, an infinite uniform charge distribution can't make an E field.

5.4 Time Reversal and Other Symmetries (71)

(1) Parity. In parallel with the C discussion two sections above, we know that the **parity** operator is P = γ^0 (times a phase) and $\psi_P(\mathbf{p}) = P\psi(\mathbf{p}) = \psi(-\mathbf{p})$. The energy of the solution (positive or negative) does not change. Nor of course does the charge change. Only **p** changes sign. You can see this because γ^0 = diag(1,-1) and recall the form of the $w^1(\mathbf{p})$ spinor, for example. Analysis of the spin projector shows that **s** stays the same but s^0 changes sign. So both $\psi_P(\mathbf{p})$ and $\psi(\mathbf{p})$ have "spin up" in rest frame. Quantity $s_\mu p^\mu$ changes sign under P (meaning that if you do the energy and spin projectors, you find $p^\mu \rightarrow p_\mu$ and $s^\mu \rightarrow -s_\mu$) which is in agreement with a vector dotted with a pseudovector angular momentum.

(2) Time Reversal. When we consider (**Wigner**) **time reversal**, we need an operator $\mathcal{T}$ which, when we do the sandwich stuff, causes the Dirac equation with EM fields to be "form invariant". Discussion uses t' = -t. The correct operator such that $\psi_{TR}(t) = \mathcal{T}\,\psi(t)$ turns out to be $\mathcal{T} = i\gamma^1\gamma^3 K$. Time reversal takes $A_\mu(t)$ to $A^\mu(t')$ because it reverses the current which makes the vector potential **A**. Analysis of the projectors shows that the effect of $\mathcal{T}$ is to take $p^\mu \rightarrow p_\mu$ and $s^\mu \rightarrow +s_\mu$ which is similar to parity by you have a + sign on the spin. As expected, "time reversal" reverse linear and angular momentum type objects (spin in rest frame).

(3) PCT Combined. We next consider the combination of all three PCT. The operator ends up being i $e^{i\varphi}\,\gamma_5$ where $e^{i\varphi}$ is the parity phase and i is the charge conjugation phase. Let's review the previous results:

C	$p^\mu \rightarrow p^\mu$ and $s^\mu \rightarrow s^\mu$	$x^\mu \rightarrow x^\mu$	$p^\mu x_\mu \rightarrow p^\mu x_\mu$	ne → pe
P	$p^\mu \rightarrow p_\mu$ and $s^\mu \rightarrow -s_\mu$	$x^\mu \rightarrow x_\mu$	$p^\mu x_\mu \rightarrow p^\mu x_\mu$	pe → pe
T	$p^\mu \rightarrow p_\mu$ and $s^\mu \rightarrow +s_\mu$	$x^\mu \rightarrow -x_\mu$	$p^\mu x_\mu \rightarrow - p^\mu x_\mu$	pe → pe
CPT	$p^\mu \rightarrow p^\mu$ and $s^\mu \rightarrow - s^\mu$	$x^\mu \rightarrow - x^\mu$	$p^\mu x_\mu \rightarrow - p^\mu x_\mu$	ne → pe

I guess reversing a motion in time does not change the sign of its energy, think ball rolling on plane. So I put pe → pe on the T line. Only the C line does ne → pe so that persists in the final TCP line. So if we define the positron (going forwards in spacetime) to be the ψ_{TCP} then "it" is γ_5 acting on a negative electron state running backwards in spacetime. Only T changes the rest spin, so our positron also has opposite rest spin from the negative energy electron.

All these "interpretations" are very delicate and fragile and not quite precise.

Somehow this positron notion will find its way into our Feynman diagrams and this is the **Feynman-Stuckelberg** approach.

Final comment is that an interaction Lagrangian may or may not respect each of these symmetries, but it is thought that CPT together must be a symmetry, but this claim is not proven but the assumptions of a proof are stated to be proper Lorentz invariance and spin-statistics association.

Chapter 6: Propagator Theory (78)

This is a tough chapter mainly in terms of positron interpretation, not so much the math.

Part I: NON-RELATIVISTIC SCALAR MODEL

6.2 The Nonrelativistic Propagator (78). I begin my notes with a reminder of what a Green's function is all about:

$$Lu = f(x) \qquad Lg(x,x') = \delta(x-x') \qquad u(x) = \int dx'\, g(x',x)\, f(x')$$

and in our case we have these two Green's functions of interest

$$(i\partial_t - H_o)\, G_o(x,x') = \delta^4(x-x') \qquad H_o = \hat{p}^2/2m \text{ for free particle}$$
$$(i\partial_t - H)\, G(x,x') = \delta^4(x-x') \qquad H = H_0 + V$$

In our context, the Green's function of the Schrodinger equation is called a "**propagator**" because we can interpret it as the quantum amplitude for a particle to move from one spacetime point to another. The propagator is of course different if you have a potential V than if you have a **free particle**.

Authors then show that the Schrodinger equation is equivalent to **Huygen's Principle** which is this

$$\theta(t'-t)\, \psi(x') = i \int d^3x\, G(x',x)\, \psi(x) \qquad \text{// we only scatter forward in time! Causality.}$$

This section then bogs down in a complicated attempt to explain the various scattering equations which are these (where I use x-space matrix notation and where "." means a d^3x integral, else d^4x.)

$\psi' = iG.\psi$ **6.1** or **6.17**, ψ in the finite past , this is Huygens' = SE

$G = G_0 + G_0VG$ **6.12** integral equation for G, driven by V and G_o

$\psi' = iG.\varphi$ **6.13** prime means x' and t' in future
$\varphi' = iG_0.\varphi$ how G_0 would move φ into the future

$\psi' = iG.\varphi = i(G_0 + G_0VG).\varphi = (iG_0.\varphi) + G_0V(iG.\varphi) = \varphi' + G_0V\psi$, so we have

$\psi' = \varphi' + G_0V\psi$ **6.14** integral equation for ψ, driven by V and G_o

$S_{fi} = \varphi_f^{\dagger}.\psi_i^{(+)}$	form 1	**6.16a**	defines S_{fi}
$S_{fi} = \delta_{fi} + \varphi_f^{\dagger}.G_0V\psi_i^{(+)}$	form 4	**6.16b**	f1 + (3)

Among these equations we have Huygens's Principle, an integral equation for G, an integral equation for ψ, and a definition of the S matrix and one alternate form for it. I imagine one of my other books has a more formal matrix approach with a more logical presentation of these things, which in this chapter seem pretty haphazard. I have lots of books that contain scattering theory discussions. In my raw notes I gather

up all the many forms for S_{fi} which BD write down using the various limits and integral equations, including some from future sections. So they also have an order of presentation problem I think.

6.3 Formal Definitions and Properties of Green's Functions (still non-relativistic situation)

The first thrust here is to show why Huygens' Principle 6.17 is really an alternate way to write the Schrodinger equation. We apply the operator $(i\partial_t - H)$ to both sides of Huygens and we get an equality. A key element is the Heaviside function on the left of Huygens which is crucial for making things work.

In this section we wander back and forth between x-space and p-space. When we come back from p-space, we have a d^4p integral to do, and in the dp^0 integral (energy $dE \sim d\omega$) we have to run contours correctly to generate the correct Heaviside function because there are poles right smack on the real axis where the nominal dp^0 contour runs. The Heaviside is a boundary condition.

Authors then "calculate" the G_o free particle propagator in several ways. They write G(x) as a Fourier elevation of G(p), apply $(i\partial_t - H_o)$ to both sides, get δ^4 on the LHS, and conclude that
$G(p) = 1/(\omega - p^2/2m)$ where ω is $E = p^o$. That is to say, if this is what G(p) is, then RHS = δ^4 as well. When we "come back to x-space", we displace the ω pole slightly so that G(x) explicitly reveals its built-in Heaviside function, as shown in 6.26 page 86. There we see that the rest of G(x) is something that looks like plane wave orthogonality, but the times are shifted. So this is the first "form" in which they present $G_o(x)$. Notice the factor of "-i" in 6.26 which really comes from the $2\pi i$ Cauchy residue gizmo as the contour wraps the pole. So there is a tendency for grouping the i with G_o in the form iG_0, at least in this representation of G_o. In the limit that the times are equal, $iG_0 \rightarrow \delta^3(x-x')$. The explicit form for $G_o(x)$ is shown in the footnote on page 86, and it is not obvious how the limit just mentioned is realized (but I think I did that limit once).

At this point authors *return* to and continue their disorganized presentation of equations relating to scattering theory. The equations in the rest of this section are the following (I have bolded equation numbers).

$S_{fi} = i\,\varphi_f^\dagger . G . \varphi_i$		form 2	**6.30**	f1 + (0)
$G = G_0 + G_0VG$	**6.12** and **6.32**	integral equation for messy G		(2)
$S_{fi} = \delta_{fi} + \varphi_f^\dagger V G . \varphi_i$		form 6	**6.33a**	f5 + (0)
$S_{fi} = \delta_{fi} - i\,\varphi_f^\dagger V \psi_i^{(+)}$		form 5	**6.34**	see note above
$S_{fi} = \delta_{fi} - i\,\varphi_f^\dagger V (\varphi_i + G_0V(\varphi + G_0V\psi))$		form 7		f5 + (3) twice
$= \delta_{fi} - i\,\varphi_f^\dagger V \varphi_i - i\,\varphi_f^\dagger V G_0V \varphi_i - i\,\varphi_f^\dagger V G_0V G_0V \varphi_i + \ldots$			**6.33b**	

where again the "." means d^3x else d^4x. If we make the groupings iG and -iV, then all signs in all equations are +. The "form 5" shown differs from the rest in that it has no G appearing and it has no d^3x integrations, and I derive this myself in the raw notes. If we take this "form 5" and install into it several times the integral equation $\psi = \varphi + G_0V\psi$ we get the 6.33b result above which is of very special interest:

this is the series we will use in perturbation theory. For example, when we do Coulomb scattering, we will use the first term of 6.33b, namely (avoiding the forward direction so we ignore the δ_{fi})

$$S_{fi} \approx -i\,\varphi_f^{\dagger}\,V\,\varphi_i = -i\int d^4x\,\varphi_f^{\dagger}(x)V(x)\varphi_i(x)$$

We are still in the non-relativistic "scalar" wavefunction section of the book, but I have put a † (instead of a *) on the left plane wave because it will be there in the 4x4 matrix version of all this stuff.

In passing, authors show two facts:

(1) G propagates φ forward in time, and also propagates φ* backward in time, see 6.29 and thereabouts.
(2) The S matrix is unitary, see 6.37

Comments: All members of the above equation zoo have counterparts in the 4x4 Dirac theory. The space and spacetime integrations are the same, the factors of i are the same, the only real change is the imposition of the 4x4 space and some γ matrices. The thing called G_0 above is called S_F in the Dirac version.

Part II: RELATIVISTIC DIRAC SPACE MODEL

6.4 The Propagator in Positron Theory (89).

The propagator S_F can run positive energy electrons forward in time, *and* it can run negative energy electrons backwards in time. You can see these two parts of S_F in the three different expressions given for S_F on pages 94-95. The "go backwards in time" part is new compared to the non-rel theory presented earlier. We interpret the negative-energy electrons running backward in time as if they were real physical positrons going forward in time. This interpretation is *supported by* the fact that ψ_c which is the charge conjugated ψ does in fact have positive energy and positive charge due to complex conjugation, but this is only support, it does not prove anything. The fact is that if you follow the rule, your calculations give the right answer, so the interpretation must be correct.

Let's compare two situations which arise later in Chapter 7. We can compare (7.4) for electron scattering to page 107A for positron scattering

electron Coulomb scattering
initial state = $u(p_i,s_i)$ put on the right side where initial is supposed to go for S-matrix
plane wave factor going with this is $\exp(-ip_ix)$ as befits a positive energy electron

positron Coulomb scattering
initial state = $v(p_f,s_f)$ put on the right side where initial is supposed to go for S-matrix
plane wave factor going with this is $\exp(+ip_fx)$ as befits a negative energy electron

Here we use v because it is a negative energy electron. The "initial state" for this electron going backwards in time is labeled with p_f, s_f because this really is the initial state for the n-e electron going backwards in time. It is the Final state for the physical electron going forward in time and this is what we

are calculating, so we put "f" subscripts there. This n-e electron initial state gets the exp(+ip$_f$x) phase factor because it has n-e! The combined factor appears as exp[i(p$_f$-p$_i$)x] in both cases it turns out.

The "Feynman rules" are to draw an arrow on your line indicating the direction an electron flows regardless of its energy. Then the right side matrix element goes with the start of this line. This may be the Initial or the Final state for a physical particle depending on which kind it is.

Now the rest of this long section provides generalizations of the various equations we got in our non-rel propagator theory (where there was only one pole, and nothing went backwards in time). We already know that the 4x4 Dirac matrix sense is going to appear, but that is completely suppressed below except you see † on the left-side spinors. The other complication is that a γ^0 factor also appears in certain places, and here is the reason for that. Consider

$(i\partial_t - H_o)\, G_o(x,x') = \delta^4(x-x')$ $\quad H_o = \hat{p}^2/2m$ for free particle

$\gamma^0(i\partial_t - H_o)\, S_F(x,x') = \delta^4(x-x')$ $\quad H_o = \boldsymbol{\alpha} \bullet \hat{\mathbf{p}} + \beta m = \gamma^0(\boldsymbol{\gamma} \bullet \hat{\mathbf{p}} + m)$ for free particle

$(\hat{p} - m)\, S_F(x,x') = \delta^4(x-x')$

$(i\partial_t - H)\, G(x,x') = \delta^4(x-x')$ $\quad H = H_0 + V$

$\gamma^0(i\partial_t - H)\, S_F'(x,x') = \delta^4(x-x')$ $\quad H = H_0 + V = H_0 + e\varphi - \boldsymbol{\alpha} \bullet (e\mathbf{A})$ // see 4.2 p48

$(\not{p} - e\not{A} - m)\, S_F'(x,x') = \delta^4(x-x')$ $\quad = \gamma^0(\boldsymbol{\gamma} \bullet \hat{\mathbf{p}} + m + e\gamma^0\varphi - \boldsymbol{\gamma} \bullet (e\mathbf{A})) = \gamma^0(\boldsymbol{\gamma} \bullet \hat{\mathbf{p}} + m + e\not{A})$

The top three lines are for the "free" propagators, while the lower three lines for the "full" propagators. In each case, we write the S_F version in two ways on two lines. You see that the definition of S_F requires a pre-left-multiplication of the non-rel style form by γ^0, and when we do our various analogous operations, that γ^0 will show up in certain places. I then tried on my own to generalize all the non-rel equations shown above to this S_F world. You can see that the new γ° factors appear in the combination $S_F'\gamma^\circ.\psi_i$. Only some of these equations actually appear in this section, and I give those equation numbers in bold. The first equations here are Huygens' principle for the S_F case and we get γ^0 appearing in them.

$\theta\,\Psi' = \varepsilon\, iS_F'\,\gamma^\circ.\,\psi$ (0) forward in time $\theta(t'-t)$ and $\varepsilon = 1$, else both opposite

$\theta\,\psi' = \varepsilon\, iS_F\,\gamma^\circ.\psi$ (1) **6.49** and **6.50**

$S_F' = S_F + S_F\, e\not{A}\, S_F'$ (2) **6.51**

$\Psi' = \psi' + S_F\, e\not{A}\, \Psi$ (3) **6.53**

$S_{fi} = \psi_f^\dagger.\,\Psi_i^{(+)}$ form 1 defines S_{fi}

$S_{fi} = i\,\psi_f^\dagger.S_F'\,\gamma^\circ.\,\psi_i$ form 2 f1 + (0)

$S_{fi} = \delta_{fi} + i\,\psi_f^\dagger.\,S_F\, e\not{A}\, S_F'\gamma^\circ.\,\psi_i$ form 3 f2 + (2) + (1)

$S_{fi} = \delta_{fi} + \psi_f^\dagger.\,S_F\, e\not{A}\,\Psi$ form 4 f1 + (3)

$S_{fi} = \delta_{fi} + \varepsilon_f\, i\,\bar{\psi}_f^s\, e\not{A}\,\Psi$ form 5 **6.56** special derivation

$S_{fi} = \delta_{fi} + \varepsilon_f \varepsilon_i \bar{\psi}_f^s \, e\not{A} S_F' \, \gamma^0 . \, \psi_i$ form 6 f5 + (0)

$S_{fi} = \delta_{fi} - \varepsilon_f \, i \, \bar{\psi}_f^s \, e\not{A} \, \psi_i - \varepsilon_f \, i \, \bar{\psi}_f^s \, e\not{A} \, S_F \, e\not{A} \, \psi_i + \ldots$ form 7 **6.57** $\varepsilon_f = 1$ f5 + (3) twice

In equations which contain the energy sign factor ε_f I have labeled the ψ_f with "s" to indicate s = 1,2,3,4 and this determines ε_f . You can think of this as s = s_f and of course the initial ψ should have an s_i label, and -- once again -- we have to be careful of the words "initial" and "final" as just remarked above. The Coulomb first order terms appear in the last equation and we would have

$S_{fi} = \delta_{fi} - i \, \bar{\psi}_f^s \, e\not{A} \, \psi_i$ // final state is a positive-energy electron with s = 1,2 **(7.1)**

$S_{fi} = \delta_{fi} + i \, \bar{\psi}_f^s \, e\not{A} \, \psi_i$ // final state is a negative-energy electron with s=3,4 **(7.23)**

The origin of this ε sign is traceable to the sign present in the projection operators

$\Sigma_{r=1,2} \, (+1) \, | \, w^r(p) >< \bar{w}^r(p) \, | = 1^+$

$\Sigma_{r=3,4} \, (-1) \, | \, w^r(p) >< \bar{w}^r(p) \, | = 1^-$

which in turn comes from the ε in the normalization (3.9b) page 30 which in turn just comes from the sign in γ^0 when you compute this normalization in the rest frame.

Chapter 7: Applications (100)

Overview: (copied from raw) This is the meat and potatoes section of this book so far, 47 jam-packed pages. Many of the basic QED cross sections are computed at least in part. The Dirac theory was 1928, and you can see that all these calculations were first done in the 1929-1935 time frame, very long ago from 2008, but still pretty new to me. I suspect computations were done differently and less conveniently in earlier times, now we always use the trace method even for a polarized cross sections, making use of the energy and spin projectors. Even so, the computations still take a lot of work. Probably computer algebra (Clifford) programs can do the traces now, and I downloaded some stuff on the subject but don't want to get sidetracked too much. In all these calculations, the photon is treated as a "particle" (really a plane wave A^μ) when a leg of a process, but otherwise as just a classical field even though it gets into propagators via the Moller potential idea. So it is pretty nice to see all these calculations done with only the Dirac theory of the electron (but with covariant E&M inclusion) and with no quantum field theory at all. Some day I will learn more about the history here, the names and geography are getting quite familiar now. As noted above, each of the major cross sections now has a name or names associated with it, which helps people remember what the process is.

7.1 Coulomb Scattering of Electrons (100) .

First, we have the *setup* for the problem in (7.1,2,3) where we use our simple first-order perturbation formula from the last chapter, we just snap on the appropriate spinors, and we normalize the plane wave

states to volume V. We do a simple integral d^3x which converts our 1/r Coulomb into $1/q^2$ where q is the momentum transfer. The spinor portion is very simple, being $\bar{u}\,\gamma^0 u$ from the fact that **A**=0 and we have only A^0. So our S matrix amplitude is as shown in (7.5). We square this, multiply by phase space, apply the usual tricks, find a rate, define the differential cross section, and we end up finally with (7.12). Here, we have summed over final spin states and averaged over initial, hence $d\bar{\sigma}/d\Omega$.

At this point, as we write out the squared spinor elements, we notice that $u\bar{u}$ appears with a spin sum, and we recognize this as the energy projector $(\not{p}+m)/2m$. Because the $u\bar{u}$ came from the two "ends" of our long initial expression, this projector acts to close a seam in the Dirac indices. The same thing happens with the other $u\bar{u}$ combination, and then the Dirac spinor indices form a closed ring, and this is the trace, so this is **how the famous trace appears**. We have then rewritten (7.12) as (7.14), and you see the isolated γ^0 guys mentioned above along with the two projectors.

7.2 Trace Theorems and Final Evaluation of the Mott cross section (103) .

Since these traces will be very common, a bunch of trace theorems are proven at this point, which I recall from olden times. I proved them all in the raw notes. Here are some of these theorems.

$\mathrm{tr}(\not{a}) = \mathrm{tr}(\not{a}\not{b}\not{c}) = 0$ = trace of any odd number of gammas

$\mathrm{tr}(\not{a}\not{b}) = 4a.b \qquad \mathrm{tr}(\gamma^\mu\gamma^\mu) = 4g^{\mu\nu}$

$$\mathrm{tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = a_1.a_2\ \mathrm{tr}(\not{a}_3\not{a}_4) - a_1.a_3\mathrm{tr}(\not{a}_2\not{a}_4) + a_1.a_4\ \mathrm{tr}(\not{a}_2\not{a}_3)$$
$$= 4\ \{\ a_1.a_2\ a_3.a_4\ -\ a_1.a_3\ a_2.a_4 + a_1.a_4\ a_2.a_3\} \qquad +12, -13, +14$$

The general idea is that you can reduce by two in each iteration no matter how many gammas you have. These theorems all have counterparts where you write the actual gamma matrices as I show on one line above.

So we then apply the appropriate theorems to our situation and do a little final juggling and we end up with the exact relativistic result for Coulomb scattering to first order in α^2, 7.22. This is called the **Mott cross section**, he did it in **1929** one year after the Dirac theory appeared, so you can see how very old all this stuff is! If you take $\beta\to 0$ it reduces to the more familiar Rutherford cross section.

I have done a lot of detail work in the raw notes on this calculation, no stone was left unturned.

Comment: In Goldstein we did Rutherford scattering and we used impact parameter and the notion of the electron having a "trajectory" which was an orbit of the central force mechanics problem. In QM, there is no trajectory, and there is no impact parameter. You cannot follow an incoming electron and see where it goes, you only get probability using plane waves. But there is still the notion of a cross section.

7.3 Coulomb Scattering of Positrons (106) .

I again define Initial as meaning the right end of any perturbation expansion term in 6.57, and I define initial to mean the initial state in the physical scattering process of real particles. Our Feynman graph for the current process is shown page 107 where Initial is a negative energy electron v $e^{+ipf.x}$.

Putting in the obvious plane waves 7.24 (which I call ψ_I) and 7.25 (ψ_F) we do indeed get Page 107 A which is identical to 7.4 for our electron result except for (1) the overall - sign which arises because of ε_r in our perturbation formula, and (2) v in place of u in both places. We get 7.26 which is the same as 7.12 except for the v's. When we now form the trace linkages and do the spin sums, so we get the negative energy projectors as shown in E. Each has a minus sign, but these signs cancel and we get F which is just like 7.14 on page 103. The only difference now is -m in place of +m in two places, but these terms involve triple gamma traces which vanish anyway, so we see that the **positron cross section is *the same* as the electron cross section (to this order).**

At first I was surprised to learn this, since the scattering center is positive in both cases, but then I looked back at Goldstein and I realized it is the same there in the non-rel classical solution.

In our first order calculation, the sign of A^μ (in the form of $A^0 = -Ze/(4\pi r)$) does not affect the answer because we have S_{fi}^2 for our rate. However, if we write out S_{fi} to second order in perturbation theory using our standard formula quoted above

$$S_{fi} = \delta_{fi} - \varepsilon_f\, i\, \bar{\psi}_f^s\, e\not{A}\, \psi_i - \varepsilon_f\, i\, \bar{\psi}_f^s\, e\not{A}\, S_F\, e\not{A}\, \psi_i + \ldots \quad \text{form 7} \qquad \mathbf{6.57} \quad \varepsilon_f = -1$$

then when we square S_{fi} we will get cross terms which are order$(A^\mu)^3$ and then we can see that the sign *might* affect the result, and in fact it does, say BD. So when higher orders are included, electron and positron scattering from a positive Coulomb potential are no longer exactly the same.

The little analysis on page 107 took me a while to comprehend. What is says in my notation is this:

$$\bar{\psi}_{Ic}\, \not{A}\, \psi_{Fc} = \bar{\psi}_F\, \not{A}\, \psi_I$$

The RHS here is basically equation A (p 107), our Feynman-rules matrix element, where $\psi_I(x) \approx v(p_f)e^{+ipf.x}$ is our negative energy electron of momentum $-p_f$ (defining *momentum* as the p in $e^{-ip.x}$). This is the calculation we just reviewed above. The LHS has a Final state being $\psi_{Ic} \approx u(p_f)\, e^{-ipf.x}$ which is our positive energy positron with momentum p_f. But this is what we used as a final state in our *electron*-Coulomb scattering. Thus, we expect our first-order positron result to be *identical* to the first-order electron result, and this is in fact true. In other words, the left calculation would have +m in those two traces, but we already noted they don't matter. Here the authors are trying to firm up the idea that the negative energy electron going backwards in time ψ_I has a connection to the positive energy positron going forward in time here $\psi_{Ic} = C\gamma^0\psi_I^*$. In the current example, we see how this connection has the matrix elements being equal as shown above.

7.4 Electron Scattering from a Dirac Proton (108) .

The idea here is this: the electron gets scattered by a classical EM field A^μ which is created by a flowing proton current J^μ. Using regular old Jackson E&M, we know $\Box A^\mu = J^\mu$ and using the usual Green's method we solve this for our desired $A^\mu(x) = \int d^4y\, D_F(x-y)\, J^\mu(y)$. It sure seems reasonable then to insert for J^μ the Dirac current of the proton treated as a pure Dirac particle, as expressed in 7.34. The resulting A^μ is called the **Møller potential.** This gives the very symmetrical looking result 7.35 where we have no integrals. What happened to the d^4x and d^4y appearing in 7.32, along with the d^4q added when D_F is Fourier-elevated? The x and y integrations each give a δ^4 because everything is plane wave. Then the q

integration kills one delta, and the other remains in 7.35 to express 4-momentum conservation. Note that the S-matrix is not a function of spacetime coordinates, it is here a purely momentum-space entity, which is a result of our assumed plane waves.

The next step is to square S_{fi} and get a rate per unit volume w_{fi} as shown in 7.36. M_{fi} is the "physics package" and is a world scalar. We associate this package with the Feynman diagram shown page 111.

Now we mimic what we did in the Coulomb scattering case, and we define a cross section

$$j_{incident}\ (\text{probability/area/sec}) * d\sigma\ (\text{area}) = \text{rate (probability/sec/volume)} * V = w_{fi}\ V$$

but this time, if beams are collinear, we use $\mathbf{j}_{inc} = \Sigma\boldsymbol{\beta}/V$ where $\Sigma\beta$ means the sum of the opposing velocities. I argue that this is correct non-relativistically, but since it makes a world-scalar as shown in 7.41, it must be correct even relativistically. We then get a general elastic scattering formula in the red box on page 113, equation 113 where we add the usual phase space factors. I have checked all these things in full detail. Every factor in 7.42 is a Lorentz scalar.

If beams are not collinear, you can use a consolation prize formula B which I did derive.

Now we want to insert into our 7.42 the computed $|M|^2$ object which we find done in 7.43. Again, pretty obvious it is a scalar. So our general answer to this problem (application) is 7.42 with 7.43 inserted, and this applies to **collinear beams** in **any frame of reference**, and of course this is only to first order in perturbation theory as indicated by the single Feynman diagram.

Next, we compute 7.42 in the experimental "lab frame" and it becomes 7.44. The notation is M = proton, m = electron, p, E and p',E' are for the initial and final electron.

We then look at a particular limit. If E of the initial electron is << M of the proton, we replicate the Mott cross section we found in the simpler Coulomb scattering analysis. This is still a relativistic result for the electron motions.

The second limit looks at E >> m, so we have very hot electrons coming in. In this case, they compute the kinematic shell C (page 115) which is a reduction of the lab frame shell in 7.44. Then they compute $|M|^2$ in this limit, and this is the one chunk of algebra I did not do (hence no check marks). Inserting this computed result into the shell, we get a complete cross section result in 7.46 which has quite a complicated angular dependence. Unfortunately, they say nothing about how accurate this is compared to experimental data. We know it is not correct because the proton is not a pure Dirac particle, but we suspect it might be not too far off for hot electrons with E >> m but also E << M so the proton looks more or less like a point particle.

I looked a little on the web for scattering data. Found http://arxiv.org/ which is the preprint archive with half million so far. Some other time! See also http://pdg.lbl.gov/ which has that **particle data book** I once had a copy of. I am now downloading the "2008 RPP" 40 MB whatever that is, coming in slowly. This thing is now 1400 pages and tells how to find data on the web, so a keeper.

The electron-proton scattering theory presented in this section does not have a particular name, but perhaps Møller scattering comes closest. His name now applies to electron-electron scattering which we will get to on page 135.

7.5 Higher order corrections to Electron-Positron Scattering (116).

In this section we look at the two Feynman diagrams for the next term in perturbation theory after the first order ("Born") term for electron-proton scattering. The authors speak of the proton with its two photon squiggles coming off as a classical **second order potential** $A_\mu(x)A_\nu(y)$ as shown page 117, because we

have not yet done QFT. The form of this double potential is conjectured to have the form 7.48 based on the requirement that things be symmetric between the electron and proton current factors in S_{fi}. [This becomes apparent in the coordinate space form of the S matrix element shown in 7.50 which I have fully checked.] The coordinate space expression 7.48 for S_{fi} goes with the coordinate space Feynman diagram on page 116. A second diagram must be added in, page 117. The argument is that (1) you need both diagrams (2) with a relative + sign so the amplitude is symmetric under exchange of the two photon source locations on the proton, since they are Bose particles. I might add that in the cross channel, you have states of two photons and this same conclusion is reached. But I am jumping way ahead in talking about S_{fi} for a cross channel being related to that of the direct channel. Recall s,t and u.

So, 7.50 (showing Fourier elevations) is just for the first term in 7.49, and we go over to momentum space to get 7.51 (all very clear, all factors correct), and that 7.51 goes with the p-space Feynman diagram on page 119. It is claimed that the integration that is left, basically over the topological loop in the diagram, is "non-trivial", by which BD mean it is completely **intractable**. Dalitz in 1951 did the large M Coulomb limit and they give a reference. We are given no comments about how large these terms might be compared to first order, and we get a passing remark that the renormalization issue of divergences comes up in the integral. This fact will play a role in the Bremsstrahlung discussion to come in the next section.

In this section the **"rules" are firmed up**. Each vertex gets $-ie\gamma^{\mu}$ and each propagator gets +i (well, photon prop gets -i). The vertex rule comes from my rule of (-iV(x)) for the potential in all those integral equations we struggled with back in Chapter 6 (see paragraph in raw notes there on this).

BD constantly spring strange phrases for things, such as $[\bar{\psi}_f(x)\, e\gamma_{\mu}\, iS_F(x\text{-}y)\, e\gamma_{\nu}\, \psi_i(y)] \equiv J_{\mu\nu}(x,y)$ being a "**second order current**", which just means an electron arrow with two vertices on it. It is hardly what I would call a "current", but OK, the term is borrowed from the first order object. And we could have an "n^{th} order current" in the obvious way. Perhaps the line itself is a "current". Perhaps a current is a photon source and it can emit photons from multiple points.

So this was a good 5-page section developing the Feynman rules with a good example.

7.6 Bremsstrahlung (116).

For the first time we now have a **photon as an external leg** in a scattering process. We have one electron in, one electron + one photon out. For the final photon, the Dirac spinor ψ_s (transforms according to the Dirac rep) is replaced by the 4-vector ε^{μ} (transforms according to the Vector rep). We compare spin worlds:

electron spin	photon spin
$s^{\mu} = (0,\mathbf{s})$ in rest frame, spacelike	$\varepsilon^{\mu} = (0,\boldsymbol{\varepsilon})$ in some frame, spacelike
$s^{\mu}s_{\mu} = -\mathbf{s}\bullet\mathbf{s} = -1$	$\varepsilon^{\mu}\varepsilon_{\mu} = -\boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon} = -1$ Coulomb gauge: ↓
$p^{\mu} = (m,0)$ in rest frame, timelike	$k^{\mu} = (k,\mathbf{k})$ in some frame, lightlike $\mathbf{k}\bullet\boldsymbol{\varepsilon} = 0$
$p^{\mu}p_{\mu} = m^2$	$k^{\mu}k_{\mu} = 0$ since massless, $m^2 = 0$
$s^{\mu}p_{\mu} = 0$ in rest frame and thus all frames	$\varepsilon^{\mu}k_{\mu} = 0$ in rest frame and thus all frames

and our photon plane wave is $A^{\mu} = N\ \varepsilon^{\mu}\ (e^{-ik.x} + e^{+ik.x})$. The idea that $\varepsilon.k = 0$ is called the **transversality condition**. In the "timeless" frame we know $\varepsilon = (0,\boldsymbol{\varepsilon})$ so in that frame we know $\varepsilon.k = -\mathbf{k}\bullet\boldsymbol{\varepsilon}$. BD choose the **transverse gauge** in which $\nabla\bullet\mathbf{A} = 0$ which for our plane wave means $\mathbf{k}\bullet\boldsymbol{\varepsilon} = 0$. Thus,

in this gauge we can regard ε.k = 0 in all frames. There are only two spin polarizations in this gauge, we are familiar with them from usual E&M, and when we later sum over polarizations, these are what we sum over.

To **normalize the photon leg**, BD require that the total EM energy in the box of the photon be $\hbar\omega = \hbar k = k$ and then we find that $N = 1/\sqrt{2kV}$. For Dirac particles, also one particle in the box.

The fact that we have $e^{-ik.x} + e^{+ik.x}$ says the photon can **go either way in time** somewhat like the electron (and unlike our prototype non-rel particle in Chapter 6), but in this application, we use $e^{-ik.x}$ only because the exiting photon is going forward in time.

Next, we learn that you **cannot "just radiate a photon" with a single vertex graph** because you cannot conserve momentum, a little calculation shows this.

So our lowest Bremsstrahlung graph requires an extra photon in addition to the radiated one. We are going to do the **Coulomb limit** in our calculation here, rather than use a real proton to provide the necessary virtual photon. Thus, our two graphs will be as shown on page 122. The argument is the same as given above, that the electron cannot know which photon vertex occurred first so include both graphs with a +.

I went through all the math details in the raw notes, so here we just summarize. The two terms in coordinate space are shown in 7.56, while in momentum space we have 7.57. Notice the $\not{\epsilon}$ objects which arise from the γ^μ of the electron current and the A^μ wavefunction of the emitted leg. And as usual we have γ^0 for the Coulomb potential because it has $A^\mu_{Coul} = \delta^\mu_0 A^0$.

In any calculation, we always have two "objects" to deal with.

The first object is "**the kinematics package**" (called a shell earlier) for the process which includes a phase space factor for each outgoing leg, and includes the various plane wave normalizing factors that appear in $|S_{fi}|^2$. For a Coulomb process, we always have just the energy delta function ("elastic") which when squared gives us a factor T, and then we talk about $|S_{fi}|^2/T$ as a "rate". When we have one or two incoming "beams", we can configure the kinematics package to give us a "cross section" dσ, which being a transverse area is a Lorentz invariant concept.

The second object is "**the physics package**" which is the spinor stuff squared, called $|M_{fi}|^2$, and this is where the spin-dependence always lies, for example.

In 7.57 we see the full S_{fi} which contains portions of both packages (but does not show phase space factors). The M_{fi} portion is clearly visible in this S_{fi}, as it always is. Were we to square this M_{fi} and do the usual spin sums and averages (including for the photon spin) , and were we to install this result into the kinematics package, the final result would be called the **Bethe-Heitler Formula** which I quote from the web in my raw notes. It has the form $d\sigma/[d\Omega_f d\Omega_k dk]$ because there are two final particles.

BD instead decide to examine the **small-k limit of the Bethe-Heitler Formula**, but they are not quite clear on how small k needs to be. When we throw out various k terms, the physics package simplifies to 7.58 where we have in large parens a certain factor which depends on momenta and photon polarization ε. We can see from this form that the M_{fi} for Bremsstrahlung is *proportional to* the M_{fi} for regular Coulomb electron scattering $d\sigma/d\Omega_f$, so the resulting Bremsstrahlung cross section is as shown in 7.59, that proportionality remaining. The extra terms here contain the phase space for the photon and our paren photon polarization factor.

We then digress momentarily from the calculation to note that this Bremsstrahlung cross section has a **log divergence as k→0** and that to fix this, you have to add the two-photon QED graphs from the two box graphs we considered in the previous section, and those shown page 124. These are corrections to the pure Coulomb electron scattering, and somehow when all this is thrown in, the divergence is said to go away. This is our first encounter with a QED divergence issue. Their explanation is good.

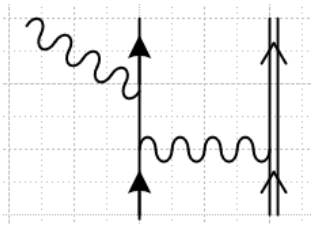
The rest of the section is just "calculation". On page 125 we learn how to do the **photon spin sum** on our special paren factor shown in 7.60, and this gives our simplified version of Bethe-Heitler shown in 7.62. The first line there you can think of as $d\sigma/[d\Omega_f d\Omega_k dk]$. If you integrate over a finite detector range of k, the next line is $d\sigma/[d\Omega_f d\Omega_k]$ and you see the potential divergence lurking. The $d\Omega_k$ integral for the photon is more or less doable, but not quite, so we have to take certain limits. The end results are shown top page 127 for the NR and ER limits. Again, we are in a small-k approximation where the resulting cross section $\mathbf{d\sigma/d\Omega_f^{Brem}}$ **is proportional to** $\mathbf{d\sigma/d\Omega_f^{e\text{-}scatter}}$. The NR and ER limits of course refer to the energy of the incoming electron, not to the photon which always has small k.

My answer to "how small is k" depends on how large those correction terms are for given k. You need to have k be small but still large enough for the corrections to be small, then this formula is meaningful.

In this section Bremsstrahlung is treated as a sample QED calculation. **In the practical world**, Bremsstrahlung is much more complicated. You have to do all sorts of corrections to get a formula which predicts what you really see. One correction we mentioned above is that you have to include the 4-vertex graphs in your elastic Coulomb electron scattering. Next, Coulomb itself is an approximation, so nuclear recoil requires a correction. Next, the incoming electron beam does not just see a simple Coulomb potential; each atom presents some kind of fancy screened Coulomb potential which is usually modeled by adding a $e^{-\mu r}$ factor where μ would depend on the target atom species. Next, the incoming electrons also interact with the orbital electrons, and you then have e-e scattering with Bremsstrahlung to worry about. Next, we only did a lowest order computation of Bremsstrahlung (first Born term). You may need higher Born terms depending on what you are doing. And there are other factors as well, I have an overview PDF.

When all is said and done, the Bremsstrahlung energy loss is dE/dx = constant * E (E of incoming beam), so you get $E(x) = e^{-x/L}$ where L is the radiation length, so you have expo loss as beam goes into a target to depth x. A physicist could spend several years being a specialist in Bremsstrahlung I think.

Comment: The above Bremsstrahlung work is done with a Coulomb source supplying the enabling photon. The "real" process here would be a 5-leg process $e + p \rightarrow e + p + \gamma$ which is not treated in this chapter. Here is one of the two diagrams which would apply.



7.7 Compton Scattering (127)

This is $e+\gamma \rightarrow e+\gamma$. Incoming photon is k,ε while outgoing is k',ε'. The physics package is the same as Bremsstrahlung with a few substitutions I list in the raw notes. The kinematics package has an extra photon leg normalizer. As for the i factors, remember $-i\not{A}$ and also -i on photon propagators, +i on electron propagators. When going to a cross section, in the lab frame where the initial electron is not moving you have $\beta_{rel} = 1$. Recall this is a factor which in theory can be as much as 2 (that would be the

case in $\gamma + \gamma \rightarrow e^+ + e^-$, a case not fully studied in this chapter, called "two photon annihilation" which Sakurai does treat).

Now, there are two graphs as shown page 128 which are added with a + sign. Again, I think of identical photons in the cross channel as requiring these two terms for **Bose symmetry** of the amplitude.

Crossing symmetry is mentioned here for the first time in BD. This Compton amplitude S_{fi} is symmetric if you swing around the two photon legs. In so doing, the spin stays put with its leg, but you have to negate each 4-momentum because you are changing the time direction of that leg. The claim then is that "the amplitude is invariant if you make these changes: $\varepsilon \rightarrow \varepsilon'$, $k \rightarrow -k'$; $\varepsilon' \rightarrow \varepsilon$, $k' \rightarrow -k$. This is compactly stated as $\varepsilon \leftrightarrow \varepsilon'$ and $k \leftrightarrow -k'$, and you can easily SEE this symmetry easily in 7.67 in both the physics and kinematics packages (prior to addition of final particle phase space).

Energy conservation gives the "**Compton condition**" 7.70 which relates k' to k and θ (photon scattering angle). Recall that in Thomson scattering, k'=k and Compton's discovery ("effect") in 1923 was that you get this wavelength shift k' < k at higher photon energies, making photons look like particles.

The **trace calculations** for the Compton physics package are messier than anything we have done so far, but there are certain critical simplifying conditions. Remember that you want to keep using the general rules $\not{a}\not{b} = -\not{b}\not{a} + 2a.b$ and $\not{a}\not{a} = a.a$ to simplify the traces. We know that k.k = k'.k' = 0 for each photon leg. We know that ε.k = ε'.k' = 0 because photons are transverse polarized in the transverse gauge we use. We know that ε.ε = ε'.ε' = -1. And because we are in the p_i lab frame, we know that $p_i.\varepsilon = p_i.\varepsilon' = 0$ since $p_i = (m,\mathbf{0})$ and $\varepsilon = (0,\boldsymbol{\varepsilon})$. Finally, we can remove p_f-containing dot products using 7.73 which arise from energy conservation and other facts already stated.

So the point is that the trace calculations are very messy, but at least they are doable because of all these conditions which simplify things. The worst trace has 8 gammas. If we look at 7.72, we see four general trace terms where I have indicated the T_1 term. This trace is laboriously computed on page 130 bottom and I did all the details. Then you know T_2 from "crossing symmetry". Trace $T_3 = T_4$ are the "cross term traces" in 7.72 and T_3 is computed on page 131, again each little step is painful -- it took me several pages of raw notes to verify all the steps.

When you combine the results of the four traces with the kinematics package in 7.72, you do in fact arrive at the ***stunningly simple*** **Klein-Nishina formula** 7.74 (1929). As k→ 0, this has a low energy limit which is the Thomson cross section shown in D, but I had to go off and write a whole separate document on Thomson scattering because that bag of information had never before been tamed by me in one place, and that is now done. **Thomson scattering** (1906) is a classical mechanically-non-rel Jackson electric dipole E&M radiation computation where a free charge just responds in sync with an incoming EM plane wave, so k' = k and $\bar{\sigma} = (8\pi/3)r_o^2$ (bar means averaged over incoming photon spin and r_o is the classical electron radius).

Equation D shows the Thomson *differential* cross section with its $\sin^2\psi$ ($\sin\psi = \boldsymbol{\varepsilon}\bullet\boldsymbol{\varepsilon}'$) dependence and in the classical calculation you then average over incoming polarization continuum and then integrate over Ω to get the total cross section. In the K-N formula, you do the usual sum over final and average over initial photon spins and the result is equation F where θ is the polar angle of the outgoing photon, assuming the incoming one was along $\hat{\mathbf{z}}$. I did this polarization sum stuff in my raw notes, BD do not show it. Equation F has its Thomson analog, see the Thomson doc.

By the way, looking at the simple K-N formula, you see the symmetry $\varepsilon \leftrightarrow \varepsilon'$ and $k \leftrightarrow -k'$ in the physics package part, but not in the kinematics package part. The kinematics $(k'/k)^2$ factor arises from the photon leg normalizers (which are symmetric) and the phase space d^3k' which applies only to the outgoing particle, and so this factor is not symmetric.

To get the total K-N cross section, you need to integrate over the final photon angle θ shown in equation F and this is done on page 132. I did this in Maple and got the exact result, and then I verified the two limits shown. Note that $r_o = \alpha/m$ in our H-L units so you see the Thomson result for the non-rel limit in equation A on page 132. In the extreme rel limit, the cross section gets smaller as ln(k)/k as k gets large.

7.8 Pair annihilation into gamma rays (132) (aka "two-photon annihilation")

Sakurai uses the latter term which is a little misleading. We are annihilating an e^+e^- pair into two photons, we are not annihilating two photons. The reverse process would be pair production. So the words production and annihilation always refer to the Dirac particles, not the photons.

The graphs are shown on page 132 and the amplitude is presented in 7.76 which I verified. Then we learn that by doing the **crossing substitutions** shown in 7.77 we can arrive at 7.76 starting from the Compton amplitude shown in 7.67. The swing-around rules of 7.77 turn one u into a v. Then we can get quickly to the cross section 7.78 by making the 7.77 changes to the Compton trace, and then we adjust the kinematics package for the **different phase space**. I did it all in full detail, all annotated in the raw notes, every single factor. There are two initial spins now so ¼ goes into the averaging. The $v\bar{v}$ causes a -1. In both Compton and Pair-Kill, we like to work in a **lab frame** where an initial electron is at rest, so $v_{rel} = \beta_+$. We then do some phase space integration to get 7.79 and then 7.80 along with A and B on page 135. Because we have 2 **identical final particles, we need a ½** in 7.81, and I argue the ½ should also be put in 7.80 but they don't do that. Reasons all given in the raw notes. I then did the non-rel limit and got result C, and I accept the high rel limit for the time being. As you increase energy, cross section goes down which does not surprise me.

So nothing dramatic here, a few new twists. The idea of using substitution rules to handle related processes is very clear and obvious hugely useful. The final result here has no special name and it is noted that **Dirac** first derived the result in **1930**.

As a side note, the graphs on page 134 show e^+e^- pair *production* in a Coulomb field, and these two graphs are "swing-arounds" of our page 122 Bremsstrahlung graphs, so we could compute this Coulomb pair production by modifying the Bremsstrahlung kinematics package and doing the subs in the physics package. BD don't do this however, perhaps it is an exercise.

Another side note concerns the pair annihilation again. In practice, the pair form positronium hydrogen-like atom and it is the overlap in the S state that causes the annihilation. Although none of my books treating the H atom say it, you can write the radial equation in a way such that the radial solutions are special functions called Coulomb Wave Functions, for the obvious reason. I never really noticed these in A&S but they are there. So in a footnote, BD comment that treating the non-rel limit with plane waves is not a great approximation for what actually happens. I have seen these Coulomb wave functions mentioned in other places, now I know what they are, see raw notes for detail.

7.9 Electron-electron and Positron-positron Scattering.

A. Electron-electron scattering (Møller)

This is a straightforward application with two graphs having a Fermi relative minus sign. The complete cross section is shown in 7.83 with unevaluated traces. If you carry through this program, you get the Moller formula, but I could not find a statement of the full result on the web in a quick search. I verified the high-rel limit to be 7.84. As with all work done so far in this chapter, this is a spin-averaged result. High rel would mean p >> m so we neglect terms containing m in the traces. All details done by me. I also did the low-energy limit and got a CMS result similar to Goldstein, but it shows interference effects from the two graphs, and I could not find this limit quoted anywhere. The graphic on page 137 is a shorthand for doing traces, but I like my shorthand better, for example.

Tr[(3+m)μ(1+m)ν(4+m)μ(2+m)ν]

On the web I frequently see a Moller formula which does not seem to agree with our high-rel result:

the energy of the beam, respectively. The Møller cross section in the CM system can be expressed as [37]

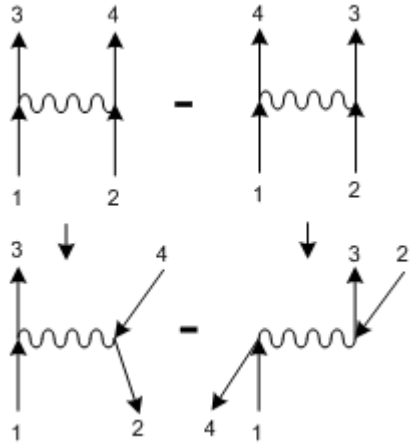
$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{r_e^2 m_e^2}{s}\left[\frac{3+\cos^2\theta_{CM}}{\sin^4\theta_{CM}}\right], \qquad (2.15)$$

where $s = 2m_e(E+m_e)$ is the square of the center of mass energy and r_e is the classical electron radius. The scattering angles in the laboratory system ($\theta_{LAB,1}$, $\theta_{LAB,2}$) and the

I could not find anything on the web to clarify this mystery.

B. Electron-positron scattering (Bhabha)

The physics package is related by crossing to the Moller physics package. Then you redo the kinematics slightly in the new 1-4 CMS (since now have different particles in the initial state), and in the high-rel limit you get the result shown in 7.87 which is different from the Moller result due to the different kinematics. You swing legs 2 and 4 on the graphs on page 136 to get the graphs on page 139. Here is my version of these graphs where I call the final particles 3 and 4 instead of 1' and 2' :



Again, I don't know the full Bhabha cross section result, but can write it with traces. In theory, you just run this into a Maple-like program and let it compute everything.

There is a program SPUR which adds onto Maple to compute gamma traces, but it is "premium content" and I cannot find a free download anywhere. // Well I was able to register and download the package, but it won't work in my maple, seems to need version 7 or newer, too bad. I found another Clifford package which works for version 5, but don't understand it at all so did not download it. I might pursue this some other time. You can see indications of version 7 in the two dirac files I installed in the lib directory.

7.10 Polarization in Electron Scattering

In all computations to this point we have summed over final spins and averaged over initial. The same general method works even if you are interested in having, say, the initial electrons polarized in some manner, the trick is to just jam in spin projectors in the right place.

This section opens by doing this in Mott scattering (electron, Coulomb). The projector picks out only the initial spin state s_i (four vector). It turns out in Mott that the result is exactly the same regardless of initial polarization because so many traces vanish! [We are still summing over final spin states.] BD say this does not carry through in higher order Born terms.

They then wander off and do some detail work on the spin 4-vector s^μ on page 141, all of which I followed. As a particular case, they then assume spin states along and against the momentum direction, which I well recall is the helicity formalism expounded by Jacob and Wick. Seems an obvious basis one might choose to get simple results. R and L circular polarization it turns out.

So the next application in this section is to do the same Mott scattering, start with all electrons R polarized, and compute the spin *asymmetry* of the result, the so called P_R net resulting polarization. This is of course all determined within the "physics package" since kinematics does not know about spin, and we end up with P_R = the trace ratio shown in 7.96. I started on the upper trace, but decided not to complete it, so I did not verify the result 7.97, just mechanical work and I have spent too much time here already.

The result is slightly generalized to the case where the initial beam is partially polarized, always in the helicity basis, result in 7.99 which seems pretty obvious.

The second last section defines a certain angle related to the spin which is simplest in the helicity basis; this is supposed to be a geometric aid, but it did nothing for me.

The last section points out that, if you only care about high-rel results, you can simplify the spin projectors right at the start.

So I am glad they threw at least something in here about doing polarized computations. We invested heavily earlier in those spin projectors, and here they come into play. The whole trace mechanism is maintained in a nice way.

Chapter 8: Higher Order Corrections (148)

8.1 Electron-Positron Scattering in 4th order. (148)

In this section we draw some of the Bhabha scattering Feynman graphs which have 4 vertices (4th order) and thus two internal photons. These are called (a) through (e) and BD have some comments on each one. One issue is the relative sign between graphs which is determined by Fermi or Bose statistics. We will

learn later that disconnected graphs can be discarded. The Feynman rules are stated carefully on page 151. You start to get a notion of topology here with "vertex correction" or "external leg correction", etc.

8.2 Vacuum Polarization (153).

This term (in lowest order) refers to having a fermion loop in the middle of a photon propagator. The integral shows a trace from the fermion loop, since the Dirac indices can't go through the photons! Think of a full bubble on the photon that will make the full photon propagator, call it $I_{\mu\nu}$ in our lowest order where the bubble is just an e+ e- pair = fermion loop. The integral has a **quadratic UV divergence**! Most of my raw notes are brute force derivations of all the integrals. Then at the top of page 157 you see for the first time the notion of **"doing a subtraction"** (like McCain's "doing a Google"). You subtract off the same integral $I_{\mu\nu}$ evaluated not at the electron mass, but at some large other mass M, and this has the effect of having the quadratic divergence cancel between the two subtracted terms and you end up with ln (M/m) floating around. In this book, BD use "**log**" instead of "ln" probably for readability.

When all the dust settles, the result of doing this integration work is that the **photon propagator is modified** by our computed $I_{\mu\nu}$ as shown in 8.21. When we install our result for $I_{\mu\nu}$, we get 8.22 which says that you have to multiply $1/q^2$ by the [..] shown. BD keep tacking on $e^2\,\bar{u}\,\gamma_0 u$ to our $1/q^2(1 + \text{stuff})$ because the combination $e^2\,\bar{u}\,\gamma_0 u$ and $1/q^2$ appear together in Coulomb scattering which we will be very interested in soon. In the final result 8.26 we take the first two terms to be a factor which renormalizes the electron charge from bare to $\mathbf{e}_R^2 = \mathbf{Z}_3\mathbf{e}^2$. The main point is that by adjusting the charge in this way, **we make that subtraction value M disappear from our result**, and we say that e_R is what we experimentally see, and we don't care about e^2, the bare charge. The suggestion is somehow that e^2 must be infinitely small, since Z_3 diverges. Not a lot is said about this unpleasant fact. If we imagine $M \to \infty$, then our subtracted term vanishes, and everything is fine (except $e_{bare} \to 0$).

The last term in 8.26 is linear in q^2 and so cancels the $1/q^2$ of the propagator and so causes an adjustment to the 1/r Coulomb potential of the form $\delta^2(\mathbf{r})$ [I do this on page 177 bottom pencil). This causes an energy shift of a bound state wavefunction which does not vanish there, and that means S orbitals in H, for example, and this we have the **- 27 MHz Uehling** 1935 **contribution to the Lamb shift**.

This section ends with comments about **unitarity** which don't seem very interesting at this point. This subject is well treated in my Eden et all book with the little bubble diagrams, and the stu variables, and the branch cuts appearing at each threshold, and discontinuity being imaginary part and all that stuff. The application here is diagram (e) on page 148 which will have a cut at $s = (2m)^2$ and so will have an imaginary part, and so on.

8.3 Renormalization of External Photon Lines (161).

The closed electron loop modified our propagator as shown in 8.22. If we have an external photon leg, it is on shell and has $q^2 = 0$ so the second term makes no contribution, and we have just a Z_3 multiplier showing in 8.22. The reasonable thing to do is put a $\sqrt{Z_3}$ factor on each end of this photon leg, realizing that it does have an "other end" somewhere miles away, as indicated by the picture on page 161. So at each end we get $e_R = \sqrt{Z_3}\, e$. Given this fact, we then don't want to "do it again" by worrying about loops in external photon lines.

8.4 Self-mass of the Electron (162).

The integral for the little photon loop on an electron is called $\Sigma(p)$ and it has a linear UV divergence and it will have and IR divergence as well. As we shall see later, the IR divergence is a bit of a different animal than the UV one, the IR one involving soft photons and what experiments measure. The method of removing the linear UV divergence is done a little differently here. First, they give the photon a mass λ, and then the subtraction is done as $\Sigma(p,m,\lambda) - \Sigma(p,m,\Lambda)$, where Λ is some huge photon mass. As usual, a bar is put over a subtracted object.

This object $\Sigma(p)$ is called the "self mass" because in the next section we shall see that if shifts the mass in the usual electron propagator. The purpose of this section is just to calculate $\Sigma(p)$.

8.5 Renormalization of the Electron Propagator (164).

As we did in 8.21 for the vacuum polarization bubble in the photon propagator, here we do for the electron self-mass bubble in the electron propagator as shown 8.41. When we do this, $\Sigma(p)$ shows up in the location of a "mass shift" so we have $1/(\not{p}-m-\Sigma(p))$, a different result from what happened with renormalizing the photon propagator. We write out $\Sigma(p)$ in terms of seemingly obscure newly defined symbols, but then we see that δm is the mass shift due to the photon loop. Then $\Sigma(p$ such that $\not{p}=m)$ = in the limit is the quantity $\delta m - (Z_2^{-1} - 1)(\not{p}-m)$ and this causes Z_2 to be in the electron propagator *numerator* in 8.44, just as we had Z_3 in the photon propagator numerator. So we get two distinct effects from this electron propagator adjustment: (1) a mass shift or renormalization δm *which houses the divergent Λ subtraction mass*; (2) a factor Z_2 in the numerator which we could treat as we did the photon case: break it into $\sqrt{Z_2}\sqrt{Z_2}$ and slide one factor to each vertex, and if an external left, we are left with $\sqrt{Z_2}$ left over on the electron leg, just as we had a $\sqrt{Z_3}$ going to the photon leg. So, Z_2 affects charge renormalization, and δm affects mass renormalization. We don't need to know yet that Z_2 in this second effect will be cancelled exactly by the vertex correction's Z_1 factor we shall see soon.

Concerning the external legs, I quote the raw notes: " Now what about the Feynman "rules" for the external lines? Consider again the photon case. The photon propagator is multiplied by Z_3. For an internal photon propagator, we do $\sqrt{Z_3}$ to each vertex and absorb that into the renorm charge. To maintain this rule at a vertex which ties to an external photon, we have to shove off a factor of $\sqrt{Z_3}$ onto the external photon leg. If we attach the leg to a distant source, then that $\sqrt{Z_3}$ hits the distant vertex and all is well. So in that case, we treat the external photon leg as a fully renormalized photon propagator. If we treat it as such (ie fully renormalized) and it is an external *leg*, then we have installed a full Z_3 for this thing because the leg is fully renormalized with all bubbles, but we should only have installed just the $\sqrt{Z_3}$ factor. THAT is why we have to divide by $\sqrt{Z_3}$ for an external photon leg. We have this $\sqrt{Z_3}$ left over that has to be removed if you want everything left to be renormalized.

So we do the above dividing for either $\sqrt{Z_2}$ on an electron leg, or $\sqrt{Z_3}$ on a photon leg. This in fact is one of the Feynman rules. We implement this rule of dividing as shown by the first factor in page 169 equation A, where there are two electron legs and one photon leg. We also see the idea clearly stated on page 166 underlined in red.

Finally, in 8.45 we see the notion of the **mass counterterm**. If you are going to *use* the fully renormalized mass for your symbol m in $\not{p}$-m, then you acquire an effective "interaction" term - $\delta m\psi$ which you have to now include on every electron. This counterterm is of order α ad 8.42 B shows.

8.6 The Vertex Correction (166).

Looking back, the photon propagator adjustment $I_{\mu\nu}(q)$ (known as vac pol) was a 4-vector tensor and a Dirac-scalar (quadratic UV). The electron propagator adjustment $\Sigma(p)$ was a 4-vector scalar and a Dirac tensor (linear UV). We now come to the vertex correction $\Lambda^{\mu}(p,p')$ which is a 4-vector and a Dirac tensor (log UV).

Now, based on a trivial symmetry argument we know that in the limit $p' \to p$, we will have

$$e\,\bar{u}(p)(\gamma_\mu + \Lambda_\mu)\,u(p) = e\,Z_1^{-1}\bar{u}(p)\,\gamma_\mu\,u(p) \qquad (1)$$

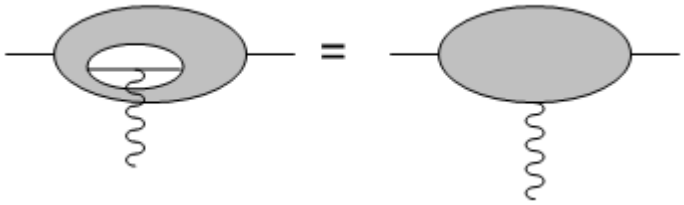
where Z_1 is some constant which causes a charge renormalization at the vertex. We can compute Z_1 without even looking at the Λ_μ integral by using the Ward identity: Based on our integrals for Λ^{μ} and Σ, these two objects are related by the Ward identity

$$\Lambda_\mu(p,p') = -\,\partial/\partial p^{\mu}\Sigma(p)$$

which in turn is based on this key fact:

$$\partial/\partial p^{\mu}\,[\,(1/(\not{p}-m)\,] = -\,(1/(\not{p}-m)\gamma^{\mu}\,(1/(\not{p}-m)$$

so it is as if this derivative inserted a vertex on an electron line. Here is my graphic interpretation of the Ward identity:



where on the left you sum over all possible insertion points on all internal electron lines. In our lowest order, this just says the vertex correction graph equals the vertex correction graph. [Somehow the minus sign goes away when you deal with factors of i on various propagators.]

OK, now let's compute $\partial/\partial p^{\mu}\Sigma(p)$ from 8.42 which is our expression for $\Sigma(p)$:

$$\partial/\partial p^{\mu}\,\Sigma(p) = -\,(Z_2^{-1} - 1)\,\partial/\partial p^{\mu}\,\not{p} = -\,(Z_2^{-1} - 1)\,\gamma_\mu = -\,\Lambda_\mu(p,p)$$

where we have done this "at $\not{p} = m$" which is what happens when you have $\bar{u}(p)\ u(p)$ sandwiched around things. The above line says

$$\bar{u}(p)(\gamma_\mu + \Lambda_\mu)\,u(p) = Z_2^{-1}\bar{u}(p)\,\gamma_\mu\,u(p)$$

and bang, comparing to (1) above, we have $Z_1 = Z_2$.

Now here is my arm-waving argument for why this fact implies a cancellation between renormalization due to electron propagator adjustment and vertex adjustment. For a single vertex with adjustment, we pick up a factor of Z_1^{-1} from the fact that $u(p)(\gamma_\mu + \Lambda_\mu)\,u(p) = Z_1^{-1}\bar{u}(p)\,\gamma_\mu\,u(p)$. But

we pick up $\sqrt{Z_2}$ from each electron line at the vertex because a vertex is where two electron propagators meet, and Z_2 is how such a propagator is adjusted. This is true even if one electron line is a leg, because in that case one $\sqrt{Z_2}$ goes to the vertex and the other $\sqrt{Z_2}$ goes to the leg, where we later divide it out. therefore, the net effect of Z_1 and Z_2 is this: factor = $\sqrt{Z_2}\sqrt{Z_2}$ $Z_1^{-1} = Z_2/Z_1$ =1. Thus, their effects exactly cancel. The photon propagator adjustment makes a $\sqrt{Z_3}$ at each vertex and that survives.

BD on page 168-169 replace my arm-waving argument with a very detailed analysis of the sum of all the graphs shown on page 168. Each of these graphs (except the bare vertex graph) is order e^3, though this might not be immediately obvious. We are evaluating these graphs at p' = p. The factors like Z_1^{-1} - 1 are linear in α as for example in page 164 C, and $\alpha = e^2/4\pi$. Also, δm is linear in α, page 164 B. In my raw notes, I show exactly how each expression in 8.57 arises from the Feynman rules for that graph, there is no rocket science here. We then add up all the graphs, and we find that Z_1 and Z_2 have exactly cancelling effects regarding charge renormalization for our vertex. I did this calculation in a special way which uses the fact that various quantities are first order in α. The expression B used by BD is totally meaningless at this point to the reader, and is not necessary to do the calculation. They wrote that "rather elaborate notation" with an eye to BD2 and QFT.

So much for the subject of charge renormalization and the Z_1, Z_2 cancellation.

The anomalous magnetic moment of the electron. In this small q limit, it turns out that the form of the adjusted vertex implies a change in the magnetic moment of the electron such that the g factor moves away from the value 2. This is something that can be measured in a Penning Trap experiment where a single electron goes around in a B field and g-2 is proportional to the rate that electron spin rotates. You can measure g-2 to about 10 decimal places, so this serves as a great testing ground to see if QED has anything to do with the real world and observations.

To compute the predicted value for g, we finally have to look at our Λ_μ integral. Using a famous integral representation of the propagators top of page 170, the integral is examined for small q and the result is shown in 8.62, where the Λ^μ_c is the residual part of Λ^μ shown in 8.55. There are two interesting facts about 8.62: (1) the $-3/8q^2$ term causes an adjustment of the 1/r potential similar to what happened from our vacuum polarization graph, where we got a -1/5 factor. So this will get into the Lamb shift. (2) the last term can be written as the last term in 8.64, from which we conclude that g' = g (1+α/2π) is how g gets adjusted. It took me a while to understand this connection between $\sigma_{\mu\nu}q^\nu$ and magnetic moment, and I wrote this all up in a separate document. Things are based on the Gordon decomposition which breaks the Dirac electron current γ^μ into two terms, where the first is a "normal" or convection current, and the second is a spin current. It is this spin current which creates the Hamiltonian term $-g\boldsymbol{\mu_B}\bullet\mathbf{B}$ in the presence of a uniform B field you can describe by an appropriate A^μ coupled to the spin current. The result is

$$g = 2 * (1+\alpha/2\pi) \approx 2* (1.001162) = (2.002323)$$

$$(g-2)/2 = g/2 - 1 = 0.001162 \quad = \text{"the anomaly"}$$

This result, first obtained by Schwinger in 1948, must have been stunning to everyone. That is because today the g_e is measured (recall Penning Trap) as follows:

Electron g_e 2.002 319 304 3622 uncertainty = 0.000 000 000 0015

and the first order Schwinger correction is correct to about 5 decimal places! People have computed the next three order levels and theory and experiment agree to about 11 decimal places. I have lots of information on this in my raw notes. So this is one of the key major predictions of QED that has been verified, and it is a relatively simple low energy experiment you can do without a huge accelerator. This result I think forced people to accept all the divergence problems as a price worth paying to be able to compute something in the real world. [Note: the precision of g_e and α are interconnected]

Infrared Divergence Problems. This section starting mid page 172 shows how the IR divergences cancel when these three processes are added up

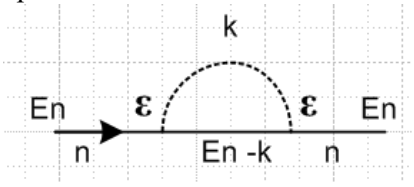
Coulomb elastic scattering
Coulomb with Brem
Coulomb with vertex-bridging extra photon

I have not said much about the IR divergences of our integrals, but they are caused by photon propagators blowing up at q=0 in the phase space. In this chapter, we have been regulating these things with a small photon mass called λ. Because our earlier work with Brem was done using the k_{min} method, and the vertex graph was done with the λ method, BD have to "redo" their vertex graph calculation to change it to the k_{min} method. When the dust cloud disperses here, we find that the Λ_c^{μ} changes by the $\delta \Lambda_c^{\mu}$ amount shown in 8.76, so that our new Λ_c^{μ} with the k_{min} method is as shown in 8.77. Two things to note here : (1) we now have $\log(1/k_{min})$ as the IR divergence in place of $\log(1/\lambda)$, (2) the Lamb shift term gets changed by -5/6. Using this result, BD add the processes shown above and demonstrate that there is no longer an IR divergence.

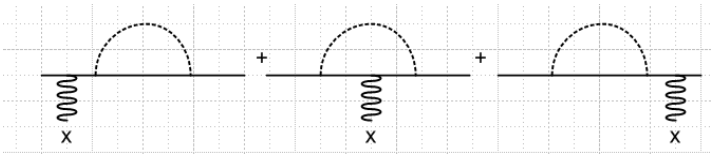
8.7 The Lamb Shift (177).

The Lamb shift basically has two components. It turns out that the components we have computed in this book are relatively small, being perhaps - 27 MHz and +68 MHz. These come from the QED calculations we have just done. The -27 MHz comes from the vacuum polarization graph, and the +68 MHz (that number may be wrong) comes from the vertex graph. All these QED corrections come from large-k photons above our k_{min} cutoff. So in this regime, we see in 8.82 how we have the three fractions +5/6 (arising from the $\delta\Lambda_c^{\mu}$ change) - 3/8 (arising from the vertex Λ_c^{μ}), and -1/5 (arising from the vacuum polarization $I_{\mu\nu}$).

But MOST of the Lamb shift, which is about 1000 MHz, comes from $k < k_{min}$ photons, and this is not a QED calculation, it is a non-relativistic QM calculation using second order perturbation theory. This section outlines how Hans Bethe did this (1947) and got the 1000 MHz. So the reader is a little disappointed to learn that this Lamb shift physical observable does not provide quite the same kind of precision QED testing that the magnetic moment provides. Still, this QM perturbation calculation can be described by this picture



which we would associate with the term "mass renormalization" or "electron self energy". It is thus the low-k non QED portion of this "graph" where $0 < k < k_{min}$. The electron is going around a proton in H, and it is doing this process all the time and that is where most of Lamb is coming from. My interpretation of the above graph is that the q=0 photon could be coming in anywhere from the Coulomb A^{μ} and has no effect on things:



but these pictures do show that we are filling in the low-k part of the problem for our QED graphs.

In the Bethe audio tape, he keeps saying he is computing the difference between the bound state electron and the free electron. I think the idea was that a free electron does not radiate (we ignore renorm graphs), whereas the bound electron is accelerating, hence radiates, hence can capture its own radiation in terms of the above graphs which for us are the Coulomb scattering correction graphs. Bethe's calculation involves a certain average energy which must be different for the 2S and 2P states of hydrogen, but our BD discussion does not get much into this and we never see the famous final formula for Lamb shift.

It is always a little hard to know what the authors knew right in 1947 say. They did not talk in terms of Feynman diagrams I suspect. Lamb's experiment was in that same year.

Chapter 9: The Klein-Gordon Equation (184)

This chapter is basically the QED theory applied to charged spinless particles instead of to Dirac particles. I don't know if the official "QED" name encompasses the KG theory, nor do I know to what extent the KG theory of QED has been experimentally verified. Pions and kaons are the only common charged spin-0 mesons, and this is where the KG theory would apply. Obviously experiments are harder to do since pions are short lived, and since they have hadronic interactions which might swamp E&M effects. All I can say is that the KG "theory" is at least relativistically viable, and the antiparticle aspect resolves the strange non-positive ρ issue. There is some underlying math here involving anti-Hermitian operators which I think is being kept quiet. There seems to be nothing like the anomalous magnetic moment that can be checked to lots of decimal places. These meta notes were written 11.18.08.

9.1 Introduction (184)

There is a Dirac equation, and there is a KG equation which is second order in the time derivative and thus "has complications". This section lists some hadronic and weak interactions involving mesons and quotes some lifetimes, argues that you can still treat such a particle as a Feynman graph "leg".

9.2 The Klein-Gordon Propagator (186)

Here we learn about the Green's function of the KG equation called Δ_F which turns out to be just

$1/(p^2-m^2)$ in momentum space, no great surprise. Amplitude is highest in any propagator when the particle in question stays close to its mass shell. A very strange feature of the KG theory shows up in this section, which I summarize as the following coordinate-space completeness relations:

$$1^+ = \int d^3x\, |x\rangle\, i \overleftrightarrow{\partial^0}\, \langle x| \qquad\qquad 1^- = -\int d^3x\, |x\rangle\, i \overleftrightarrow{\partial^0}\, \langle x|$$

which in effect appear as 9.6 in the text. This ∂^0 is tied in with the requirement of antiparticles with opposite charge such that the traditional $\psi^*\psi$ is no longer ρ as it is in non-rel QM. This time operator appears in some of the usual propagation related equations such as 9.13, but not in others.

Here is the conserved 4-current of the KG theory, and we can compare to Dirac:

$j^\mu = i\,[\,\varphi^*(\partial^\mu\varphi) - (\partial^\mu\varphi^*)\,\varphi]$ compare: $j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$ Dirac

$j^0 = i\,[\,\varphi^*(\partial^0\varphi) - (\partial^0\varphi^*)\,\varphi] = \varphi^*\, i \overleftrightarrow{\partial^0}\varphi$ compare: $j^0 = \psi^\dagger\psi$ Dirac

9.3 Adding E&M to KG (188)

We add E&M to the KG equation in the only relativistic way we know how, as shown in 9.16, the usual so-called "minimal prescription". When we treat the E&M terms as perturbations, we can kick them to the RHS of the equation and call them "the potential V" as shown in 9.19, which is then the basis for "QED of KG mesons". You see there two kinds of couplings, one being $e\partial_\mu A^\mu$ style, and the other being $e^2 A_\mu A^\mu$, so right off the bat we have two kinds of Feynman vertices in this theory, and the second one has two photons and the meson involved all at the same spacetime point. Drawings on page 190 show the usual idea of swinging legs around. The propagator is like S_F in that it takes plus energy states into the future and neg energy states into the past, and there are many similarities between the S_F and the Δ_F equations.

9.4 Scattering Amplitudes (190)

As we did with the Dirac electron, we here set up the S_{fi} perturbation expansion, everything is completely analogous to earlier work, apart from the detail of $i \overleftrightarrow{\partial^0}$ appearing in place of Dirac γ_0 as for example in 9.21 A.

9.5 Low-order Scattering processes. (191)

Coulomb Scattering of π^+. Here the e^2 new term does not contribute, and we get S_{fi} as in 9.24 where you see the new Feynman rule for our "momentum coupling". Instead of having a $\gamma_\mu A^\mu$ type rule, we have a $(p+p')_\mu A^\mu$ rule. This just falls out from the basic form 9.19. We carry this through to get a cross section, and it looks just like the Mott formula without the extra numerator "spin" factor.

Coulomb Scattering of π^- . We just negate the momenta and get 9.27 instead of 9.24, and now we are scattering π^- from our Coulomb center, and of course to this order the cross section is the same.

Compton Scattering. That is, photon-meson scattering. The graphs are shown page 194 bottom, and you see "the usual" pair of graphs and the "new" graph with the 4-particle vertex. We go through the normalization of legs and just do the calculation. First we get the S_{fi} as in 9.30 where you see the three terms. In the lab frame, only the last 4-vertex term survives, and we get the spin-0 meson version of the Klein-Nishina formula in p 195 A. It is simpler than the KN, but similar. And it has the same Thomson classical limit.

9.6 Higher-order Processes (195)

The Feynman rules are stated for the KG particles doing their QED. Graphs on page 197 show the scattering of two positive pions as the expected pair of graphs, now added and not subtracted due to Bose statistics. This is like the **electron-electron scattering** we did in regular Dirac QED. Then page 198 shows the **Bhabha scattering** idea for pions. And you can have higher order graphs as on page 199, and you can think about all the renormalization issues, but BD don't venture into that area.

The big problem is that, unlike electrons and positrons, pions have a hadronic interaction which our QED does not know about, so any experiment we do is going to be clouded by the hadronic stuff. Probably some low energy scatterings can be compared to the theory, but authors say little about that.

9.7 Nonrelativistic reduction and interpretation of the KG equation. (198)

The reduction is very strange, and again involves that $i\overleftrightarrow{\partial}^0$ issue of the normalization. We have to complicate the KG theory, bringing it up to a 2-spinor theory in order to get a "Schrodinger equation" , we have to bury some of the extra time derivatives inside newly defined spinor component wavefunctions θ and χ as in 9.41. When we do all this, we get a reasonable 2x2 Hamiltonian as in 9.48.

For free particles (mesons), we can do a FW rotation very similar to the Dirac one, and we can bring the Hamiltonian into the desired form 9.52 where the plus and minus energy solutions are decoupled, and where we can revert to our conventional $\psi^*\psi$ density, matrix elements, and all that stuff.

When the EM is added using the usual π object, we do the FW transform (one iteration) and we get rid of the odd terms and end up with 9.62 which has only even terms (but is an approx). This then forms the basis of doing regular QM with mesons, such as with pi-mesonic atoms which might be created in a collider.

The last section starting on page 205 suggests the forms shown top of that page with the η (=β) matrix as part of the definition of matrix elements. For free particles, we find that the form of the energy and charge matrix elements is the same before or after the FW transform, and we get at the end the statement that you should put matrix η inside all matrix elements in this "theory", ie, in the realm where this FW stuff makes sense for mesons.

One nice fact appears on page 206 which I point out: you can see in the last two lines of 9.78 how the spinor "complexification" ties in with the $i\overleftrightarrow{\partial}^0$ operator. The charge has to be the difference between the particles and the antiparticles. This works out differently in the Dirac theory.

Comments: I get the feeling that one should be able to treat particles of any spin somehow on the same basis, and we are not here learning how that works. The Dirac and KG theories are just separate unrelated theories, each with its underlying "equation" (be it the Dirac Equation or the KG Equation). Something is missing here, and I guess QFT will perhaps fix this up. We are seeing "projections" of some more general theory. Many more complications are going to appear before we can even ponder simplifying things:

weak and strong interactions, gravity theory, symmetries, QCD, strings, who knows what. I think this chapter says more about KG theory than any other book I possess.