Bras Kets and Coordinate Transformations PhL 10.1.08

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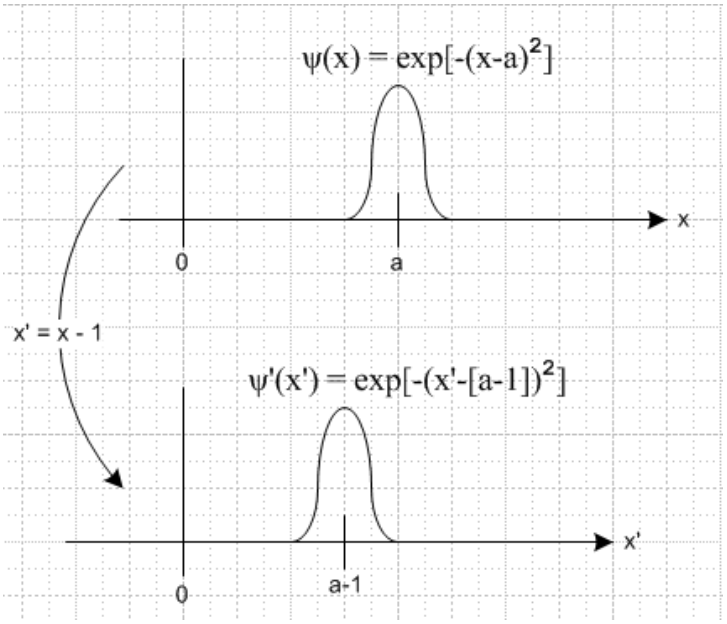
1. A simple 1D example using translation.

First, here are some pictures showing the unprimed then the primed reference frame:



The unprimed observer sees $\psi(x) = \exp[-(x-a)^2]$ as his wavefunction and notes that it peaks at x=a.

The primed observer sees $\psi'(x') = \exp[-(x'-[a-1])^2]$ as his wavefunction and notes that it peaks at x'=a-1.

The transformation is that $x' = T(-1)\,x = x - 1$, a simple translation by 1 unit.

Before doing anything, notice that

$\psi'(x') = \exp[-(x'-[a-1])^2] = \exp[-(x'+1-a)^2] = \exp[-(x-a)^2] = \psi(x)$

$\psi'(x') \neq \psi(x')$ hence the need for the prime on ψ' to distinguish its different functional form

In bra-ket notation we would say the following:

Observer in the unprimed frame has this situation

$$|\psi\rangle = |x\rangle\langle x|\psi\rangle = \psi(x)\ |x\rangle$$

Observer in the primed frame has this situation

$$|\psi'\rangle = |x'\rangle\langle x'|\psi'\rangle = \psi'(x')\ |x'\rangle$$

Now we could talk about T as an operator in our HS: (while T acts on a point in space)

$$T\ |x\rangle = |Tx\rangle = |x'\rangle = |x-1\rangle \qquad \Rightarrow \qquad \langle x|\ T^{\dagger} = \langle x'|$$

$$T\ |\psi\rangle = |\psi'\rangle$$

Then we could do some HS algebra as follows:

$$\psi'(x') = \langle x'|\psi'\rangle = \langle Tx|T\psi\rangle = \langle x|T^{\dagger}T|\psi\rangle$$

Now what do we do? Is a translation unitary? To find out, we can look to see what happens to our basis functions:

$$\langle x\ |\ T^{\dagger}T|y\rangle = \langle Tx|Ty\rangle = \langle x-1|y-1\rangle = \delta(x-y) = \langle x|y\rangle$$

and this is true for all basis functions on either side, so conclude that yes, $T^{\dagger}T = 1$ and T is unitary. So we complete our algebra line above:

$$\psi'(x') = \langle x'|\psi'\rangle = \langle Tx|T\psi\rangle = \langle x|T^{\dagger}T\psi\rangle = \langle x|\psi\rangle = \psi(x)$$

Question: what about the "extra-bracket" prime notation?

$$\psi'(x') = '\langle x|\psi\rangle'$$

with the idea that $|x\rangle'$ is a ket that the primed observer uses. Then we say

$$|x\rangle' = T\ |x\rangle = |\ x-1\rangle = |x'\rangle$$
$$|\psi\rangle' = T\ |\psi\rangle$$

This notation lets you distinguish the kets and bras that the two observers deal with. In a way, perhaps you can think of $|x\rangle' = T\ |x\rangle$ as a "passive" notation, and $|x'\rangle = T\ |x\rangle$ as an "active" notation. In the passive notation, there are two different HS's and one state vector ψ, while in the active notation there is only one HS but the two observers see different vectors in this HS called $|\psi\rangle$ and $|\psi'\rangle$. I feel pretty strongly that these two points of view are equivalent and either notation is OK. Books I have don't ever

take this "passive" point of view. I guess I will keep my primes inside the kets and bras and go with the active notation idea.

2. The analogous rotational example

I won't bother with the pictures, but we can just repeat the algebra. All the x references should be bolded, but I will leave them unbolded.

Observer in the unprimed frame has this situation

$$|\psi\rangle = |x\rangle\langle x|\psi\rangle = \psi(x)\ |x\rangle$$

Observer in the primed frame has this situation

$$|\psi'\rangle = |x'\rangle\langle x'|\psi'\rangle = \psi'(x')\ |x'\rangle$$

Now we could talk about R as an operator in our HS:

$$R\,|x\rangle = |x'\rangle = |\,R\,x\rangle \qquad \Rightarrow \qquad \langle x|\,R^{\dagger} = \langle x'|$$

$$R\,|\psi\rangle = |\psi'\rangle$$

Then we could do some HS algebra as follows:

$$\psi'(x') = \langle x'|\psi'\rangle = \langle Rx|R\psi\rangle = \langle x|R^{\dagger}R\psi\rangle$$

Now what do we do? Is a rotation unitary? To find out, we can look to see what happens to our basis functions:

$$\langle x\,|\,R^{\dagger}R|y\rangle = \langle\,R\,x|\,R\,y\rangle = \langle x|y\rangle$$

and this is true for all basis functions on either side, so conclude that yes, $R^{\dagger}R = 1$ and R is unitary. So we complete our algebra line above:

$$\psi'(x') = \langle x'|\psi'\rangle = \langle Rx|R\psi\rangle = \langle x|R^{\dagger}R\psi\rangle = \langle x|\psi\rangle = \psi(x)$$

3. Boost of the Dirac Spinor wavefunction

Before trying anything, look at what the "wave mechanics" says on page 30 of BD 1. We start with a plane wave as follows:

$$\psi^{r}(x;p) = w^{r}(p)\,\exp[-i\varepsilon_{r}p.x]$$

Suppose we take our unprimed frame to be the rest frame, so we have p =(m,**0**),

$$\psi^{r}(x;p) = w^{r}(p)\, \exp[-i\varepsilon_{r} p.x]$$

and we take our primed frame as a frame boosted by B relative to the rest frame, and p' = Bp :

$$\psi'^{r}(x';p') = w^{r}(p')\, \exp[-i\varepsilon_{r} p'.x']$$

The exponential part stays the same because p.x is a world scalar, but the spinor part changes. We find that

$$w^{r}(p') = S(B)\, w^{r}(p) \qquad \text{or} \qquad w^{r}(p')_{i} = S(B)_{ij}\, w^{r}(p)_{j}$$

and these are the columns shown on page 30. We can then write:

$$\psi'^{r}(x';p') = w^{r}(p')\, \exp[-i\varepsilon_{r} p'.x'] = S(B)\, w^{r}(p)\, \exp[-i\varepsilon_{r} p'.x'] = S(B)\, w^{r}(p)\, \exp[-i\varepsilon_{r} p.x]$$

$$= S(B)\, \psi^{r}(x;p)$$

so we arrive at this rule:

$$\psi'^{r}(x';p') = S(B)\, \psi^{r}(x;p) \qquad *$$

which we can compare to page 19 equation 2.10. The new thing we see in our explicit example is that all four-vectors including labels get transformed as you move to the primed frame, namely, p' in this case.

4. How to express this Dirac spinor boost in bra-ket notation?

Suppose we just naively try to write

$$\psi^{r}(x;p) = \langle x \mid \psi^{r}(p) \rangle \qquad \Rightarrow \qquad \psi^{r}{}_{i}(x;p) = \langle x \mid \psi^{r}{}_{i}(p) \rangle$$

Then we would get, since x' = Bx,

$$\psi'^{r}(x';p') = \langle x' \mid \psi'^{r}(p') \rangle = \langle x \mid B^{\dagger} \mid \psi'^{r}(p') \rangle$$

The only way to make this come out right is if we have this ket transformation rule:

$$B^{\dagger} \mid \psi'^{r}{}_{i}(p') \rangle = S(B)_{ij} \mid \psi^{r}{}_{j}(p) \rangle$$

because only if this is true do we get equation * shown above when we close with $\langle x|$. We can rewrite this rule as

$$B^{\dagger} \mid \psi'^{r}{}_{i}(Bp) \rangle = S(B)_{ij} \mid \psi^{r}{}_{j}(p) \rangle$$

and I would then presume that this rule applies to all Lorentz transformations

$$\Lambda^{\dagger}|\ \psi'^{r_i}(\Lambda p)\rangle\ = S(\Lambda)_{ij}\ |\ \psi^{r_j}(p)\rangle$$

In a previous section when we dealt with a scalar wave function and rotation, we got

$$R^{\dagger}\ |\psi'\rangle\ =\ |\psi\rangle$$

which is the analogous result. We act on a primed frame wavefunction with a transformation operator, and we get as a result a linear combination of unprimed frame wavefunctions, and in this last case that linear combination is quite simple.

Now, one detail that we must bring out here. The bra $\langle x|$ is a full 4-vector bra, not the 3D $\langle \mathbf{x}|$ we used in our rotation example. I would imagine that for such 4-vector kets we have these relationships:

$$\langle x|x'\rangle\ = (2\pi)^4\delta^4(x - x') \qquad 1 = \int d^4x\ |x\rangle\langle x|$$

which is covariant. With this normalization, I think we can consider the following:

$$(2\pi)^4\delta^4(x_1' - x_2') = \ \langle x_1'|x_2'\rangle\ = \langle Bx_1|Bx_2\rangle\ = \langle x_1|B^{\dagger}B|x_2\rangle\ = ???$$

This makes us ask if $\delta^4(x_1' - x_2') = \delta^4(x_1 - x_2)$. We suspect this is true because it looks covariant.
Here is a proof: (assume that $x_i' = B\ x_i$)

$$\delta^4(x_1 - x_2)\ = \int d^4p\ e^{-ip.(x1-x2)}$$

$$\delta^4(x_1' - x_2')\ = \int d^4p\ e^{-ip.(x1'-x2')} = \int d^4p\ \exp[\ -i\ B^{-1}p.(x1-x2)] \quad (\text{ because exponent is a scalar})$$

$$= \int d^4p''\ \exp[\ -i\ p''.(x1-x2)] \quad (\text{ change variable of integration to } p'' = B^{-1}p)$$

$$= \int d^4p\ \exp[\ -i\ p.(x1-x2)] \qquad (\text{rename dummy integration variable})$$

$$= \delta^4(x_1 - x_2) \quad (\text{from first line above})$$

Therefore we can continue our line above to say:

$$(2\pi)^4\delta^4(x_1' - x_2') = \ \langle x_1'|x_2'\rangle\ = \langle Bx_1|Bx_2\rangle\ = \langle x_1|B^{\dagger}B|x_2\rangle$$
$$= (2\pi)^4\delta^4(x_1 - x_2) = \ \langle x_1|x_2\rangle$$

and therefore we conclude that

$$B^{\dagger}B = 1$$

so the boost operator in this Hilbert Space is in fact unitary, something I have been waiting a long time to see. We could repeat the argument for a rotation or for any LT and get the same result. Thus, we reach our desired conclusion:

5. The Unitarity Theorem: In the infinite-dimensional Hilbert Space spanned by *covariant* kets $|x^\mu>$, the operator Λ representing any Lorentz Transformation is unitary. In particular, we have $B^\dagger B = 1$ for boosts. We know that in the 4x4 vector representation, B is NOT unitary, and it probably is non-unitary as well in the 4x4 Dirac representation, but as an operator in our covariant Hilbert Space, B is unitary! Obviously, unitarity of an operator depends on the space you are working in, which is to say, it depends on the definition of the inner product of the Hilbert Space in which one "works", and it depends on the basis functions selected in this space which in effect act as indices on the B operator viewed as a matrix.

6. More details on the Dirac wavefunction?

Look at BD page 95 equation A which I repeat here:

$$\psi_r(\mathbf{x},x^0;\mathbf{p},p^0) = \sqrt{m/E}\,(2\pi)^{-3/2}\, w^r(\mathbf{p})\exp[\,-i\varepsilon_r(p^0x^0 - \mathbf{p}\bullet\mathbf{x})\,]$$

where for now, think of p as just a label, and this is really a coordinate space wavefunction which is a plane wave for electron with momentum **p**. I have exposed the time and energy coordinates on the LHS.

A. Orthogonality.

Our first question might be "orthogonality" of two plane waves at equal times. We can then write

$$\int d^3x\, \psi_r^\dagger(\mathbf{x},x^0;\mathbf{p},p^0)\,\psi_s(\mathbf{x},x^0;\mathbf{p}',p'^0) = \sqrt{m/E}\,(2\pi)^{-3/2}\sqrt{m/E'}\,(2\pi)^{-3/2}$$

$$w^{r\dagger}(\mathbf{p})w^s(\mathbf{p}')\int d^3x\ \exp[\,+i\varepsilon_r(p^0x^0 - \mathbf{p}\bullet\mathbf{x})\,]\exp[\,-i\varepsilon_s(p'^0x^0 - \mathbf{p}'\bullet\mathbf{x})\,]$$

The first thing we do is the $\int d^3x$ integration which yields $(2\pi)^3\,\delta^3(\varepsilon_r\mathbf{p} - \varepsilon_s\mathbf{p}')$. Regardless of the signs ε_r and ε_s, we know that **p** and **p**' have the same magnitude due to this "pin", so we know that E = E' and the same for the p^0's. We can write the delta as $\delta^3(\varepsilon_r\varepsilon_s\mathbf{p} - \mathbf{p}')$ and then we get

$$\int d^3x\, \psi_r^\dagger(\mathbf{x},x^0;\mathbf{p},p^0)\,\psi_s(\mathbf{x},x^0;\mathbf{p}',p'^0) = (m/E)\, w^{r\dagger}(\mathbf{p})w^s(\varepsilon_r\varepsilon_s\mathbf{p})\,\delta^3(\varepsilon_r\mathbf{p} - \varepsilon_s\mathbf{p}')$$

Then we look at 3.11 on BD1 page 31 and think of $\varepsilon'_r = 1$ and $\varepsilon'_s = \varepsilon_r\varepsilon_s$ to get

$$w^{r\dagger}(\mathbf{p})w^s(\varepsilon_r\varepsilon_s\mathbf{p}) = (E/m)\,\delta_{rs}$$ // slight fudge, hope OK

and we finally end up with

$$\int d^3x\ \psi_{\mathbf{r}}^{\dagger}(\mathbf{x},x^0;\ \mathbf{p},p^0)\ \psi_{\mathbf{s}}(\mathbf{x},x^0;\ \mathbf{p'},p'^0)\ = \delta_{\mathbf{rs}}\ \delta^3(\varepsilon_{\mathbf{r}}\mathbf{p}\ -\varepsilon_{\mathbf{s}}\mathbf{p'})$$

Notice that the labels r and s here are not labeling the 4 components of ψ, but are indicating which of four different 4-component spinors we are talking about. If we show the spinor components, we have this

$$\int d^3x\ \Sigma_{\mathbf{i}}[\psi_{\mathbf{r}}^{*}(\mathbf{x},x^0;\ \mathbf{p},p^0)]_{\mathbf{i}}\ [\psi_{\mathbf{s}}(\mathbf{x},x^0;\ \mathbf{p'},p'^0)]_{\mathbf{i}}\ = \delta_{\mathbf{rs}}\ \delta^3(\varepsilon_{\mathbf{r}}\mathbf{p}\ -\varepsilon_{\mathbf{s}}\mathbf{p'})$$

where i is not the spinor component label. Notice that on page 19, BD avoid using component indices on the spinors like ψ(x).

B. How do we write this in bra-ket notation?

We might first just try this, imitating what we did before,

$$[\psi_{\mathbf{s}}(\mathbf{x},x^0;\ \mathbf{p'},p'^0)]_{\mathbf{i}}\ = \langle \mathbf{x}\ |\ \psi^{\mathbf{s}}{}_{\mathbf{i}}(x^0;\ \mathbf{p'},p'^0)\rangle$$

Then the above becomes:

$$\int d^3x\ \Sigma_{\mathbf{i}}\ \langle \psi^{\mathbf{r}}{}_{\mathbf{i}}(x^0;\ \mathbf{p},p^0)|\ \mathbf{x}\rangle\ \langle \mathbf{x}\ |\ \psi^{\mathbf{s}}{}_{\mathbf{i}}(x^0;\ \mathbf{p'},p'^0)\rangle\ = \delta_{\mathbf{rs}}\ \delta^3(\varepsilon_{\mathbf{r}}\mathbf{p}\ -\varepsilon_{\mathbf{s}}\mathbf{p'})$$

and we can continue to thing this way

$$\int d^3x\ |\ \mathbf{x}\rangle\langle \mathbf{x}\ |\ \ = 1\ \ = \int d^3x\ |\ \mathbf{x},\ x^0\rangle\langle \mathbf{x},\ x^0\ |$$

and we end up with

$$\Sigma_{\mathbf{i}}\ \langle \psi^{\mathbf{r}}{}_{\mathbf{i}}(x^0;\ \mathbf{p},p^0)\ |\ \psi^{\mathbf{s}}{}_{\mathbf{i}}(x^0;\ \mathbf{p'},p'^0)\rangle\ = \delta_{\mathbf{rs}}\ \delta^3(\varepsilon_{\mathbf{r}}\mathbf{p}\ -\varepsilon_{\mathbf{s}}\mathbf{p'})$$

which seems not unreasonable. If we just assume equal time, we might compact this down to be

$$\Sigma_{\mathbf{i}}\ \langle \psi^{\mathbf{r}}{}_{\mathbf{i}}(p)\ |\ \psi^{\mathbf{s}}{}_{\mathbf{i}}(p')\rangle\ = \delta_{\mathbf{rs}}\ \delta^3(\varepsilon_{\mathbf{r}}\mathbf{p}\ -\varepsilon_{\mathbf{s}}\mathbf{p'})$$

C. How do we handle transformations

From section 4 above, we thing the correct rule this:

$$\Lambda^{\dagger}|\ \psi'^{\mathbf{r}}{}_{\mathbf{i}}(\Lambda p)\rangle\ = S(\Lambda)_{\mathbf{ij}}\ |\ \psi^{\mathbf{r}}{}_{\mathbf{j}}(p)\rangle$$

This S thing is messing with the component indices, not with the spinor label r. If we think only of the r=1,2 components, you are thinking non-rel simple spin, and you expect a rotation to shuffle them around.

The HS algebra for this situation was given in Section 4 above, and we use the same $<\mathbf{x}, x^0| = <x^\mu|$ basis vectors as we used above in this section. We learned that in this space, Λ is in fact unitary.

**********OBS **********

I want to keep these two sections, am moving them here since they are "right on topic". In the first I tried to make the Dirac stuff fit into a "direct product" type basis, but I could not make it fly. That was called Plan A. Then with the help of the above, I was able to make Plan B fly, which is really just what has been done above. The structure seems to be a nested Hilbert space and not a direct product Hilbert space. The blue text I think is not used and also possibly wrong, so I leave it blue.

Plan A. Explanation of $\psi'(x') = S(a)\,\psi(x)$ in the matrix formalism : Plan A, fails! ????

The quoted equation comes from BD1 page 19 equation 2.10. I will write this out in Dirac components:

$$\psi'_r(x') = S(a)_{rs}\,\psi_s(x)$$

I think things are going to work out, but the complication is that the "matrix mechanics" base states for our Dirac theory are the following direct product states

$$\{\ |x^\mu\rangle \otimes |u_r\rangle\ \}$$

where x^μ transforms according to the vector representation of the LG, while u_r transforms according to the Dirac representation of the LG. Thus, we get the following complicated situation ($\Lambda = \Lambda\otimes\Lambda$) ,

$$\Lambda\ \{\ |x^\mu\rangle \otimes |u_r\rangle\ \} = \Lambda\ |x^\mu\rangle \otimes \Lambda\ |u_r\rangle \quad = \Lambda^\mu{}_\nu\ |x^\nu\rangle \otimes D_{rs}(\Lambda)\ |u_s\rangle$$

which I might compact down to read

$$\Lambda|\ x^\mu;\ u_r\rangle = D_{rs}(\Lambda)\ |\ \Lambda^\mu{}_\nu\ x^\nu;\ u_s\rangle$$

which we can compress further by saying

$$\Lambda|\ x;\ u_r\rangle = D_{rs}(\Lambda)\ |\ x'\ ;\ u_s\rangle$$

Our wavefunction can now be written as

$$\psi_r(x) = \{\ <x^\mu| \otimes <u_r|\ \}\ |\psi\rangle \quad = <x;\ u_r\ |\psi\rangle$$

So the conjecture here is that we somehow want to have a basis that is a "direct product representation" of two representations of the Lorentz group. This conjecture is wrong, but let's continue.

Now let's define some "primed basis states" just as we did earlier in this document by saying:

$$|x\rangle' \quad = \Lambda\ |x\rangle \ = |\Lambda x\rangle = |x'\rangle \qquad \text{and} \qquad '\langle x| \ = \langle \Lambda x| \ = \langle x'|$$

$$|u_r\rangle' = \Lambda|u_r\rangle = D_{rs}(\Lambda)\,|u_s\rangle \qquad \text{and} \qquad '\langle u_r| = D^{\dagger}_{rs}(\Lambda)\langle u_s| = \langle u_s|\,\Lambda$$

so combining things we get

$$\{\,|x^{\mu}\rangle' \otimes |u_r\rangle'\,\} = \Lambda^{\mu}{}_{\nu}\,|x^{\nu}\rangle \otimes D_{rs}(\Lambda)\,|u_s\rangle = \Lambda\,\{\,|x^{\mu}\rangle \otimes |u_r\rangle\,\}$$

or in our compact notation

$$|x;u_r\rangle' = \Lambda|x;u_r\rangle = D_{rs}(\Lambda)\,|x';u_s\rangle$$

The bra version of the above would be

$$\{\,'\langle x^{\mu}| \otimes '\langle u_r|\,\} = \langle x^{\nu}|\,\Lambda^{\dagger\mu}{}_{\nu} \otimes D^{\dagger}_{rs}(\Lambda)\langle u_s| = \{\,\langle x^{\mu}| \otimes \langle u_r|\,\}\,\Lambda^{\dagger}$$

$$'\langle x;u_r| \quad = \quad D^{\dagger}_{rs}(\Lambda)\langle x';u_r| \quad = \quad \langle x;u_r|\,\Lambda^{\dagger}$$

Then let's try to assign meaning to $\psi'_r(x')$ as follows:

$$\psi'_r(x') = \{\,'\langle x^{\mu}| \otimes '\langle u_r|\,\}\,|\psi\rangle = '\langle x;u_r|\,\psi\rangle = D_{rs}(\Lambda^{\dagger})\langle x';u_s|\,\psi\rangle = D_{rs}(\Lambda^{\dagger})\,\psi_s(x')$$

$$\psi'_r(x') = \{\,'\langle x^{\mu}| \otimes '\langle u_r|\,\}\,|\psi\rangle = \{\,\langle x^{\mu}| \otimes \langle u_r|\,\}\,\Lambda^{\dagger}|\psi\rangle$$

and this is the wrong answer because we have x' on the RHS. So let's try instead

$$\psi'_i(x') = \{\,'\langle x^{\mu}| \otimes '\langle u_i|\,\}\,|\psi\rangle' = \{\,\langle x^{\mu}| \otimes \langle u_i|\,\}\,\Lambda^{\dagger}\,|\psi\rangle'$$

where I have replaced the component index now with i, anticipating r will be needed soon. Now let's use Section 4 above which states that

$$\Lambda^{\dagger}|\,\psi'^{r}{}_{i}(\Lambda p)\rangle = S(\Lambda)_{ij}\,|\,\psi^{r}{}_{j}(p)\rangle$$

where ψ^r is one of the four plane-wave Dirac equation solutions. If we were to make an arbitrary linear combination ψ of these, I guess we would conclude that the above is also true for ψ with no r superscript:

$$\Lambda^{\dagger}|\,\psi'_i(\Lambda p)\rangle = S(\Lambda)_{ij}\,|\,\psi_j(p)\rangle$$

and we could perhaps rewrite this as

$$\Lambda^{\dagger}|\,(\,u_i, \psi'(\Lambda p)\rangle = S(\Lambda)_{ij}\,|(\,u_j, \psi\,(p)\,)\rangle$$

where the u_i is a $u_i{}^T = (1,0,0,0)$ for example, which picks off the component of the spinor ψ that we want. But alas, as before, we end up with our "nested Hilbert Space" structure, not the direct product structure. I

see no way to make that $< u_i|$ to the left of $\Lambda^\dagger$ [that is, in $\{ < x^\mu | \otimes < u_i| \}$] become usable. The direct product structure is just plain wrong, even though I now know the action of $\Lambda^\dagger$ on a spinor ket.

Plan B. Explanation of $\psi'(x') = S(a)\,\psi(x)$ in the matrix formalism: Plan B

The quoted equation comes from BD1 page 19 equation 2.10. I will write this out in Dirac components:

$$\psi'_i(x') = S(a)_{ij}\psi_j(x)$$

First, take note of this fact:

$$x' = \Lambda x \qquad x = \Lambda^{-1}x' \qquad \Lambda\,|x> = |x'> \qquad <x|\,\Lambda^\dagger = <x'|$$

Let's try representing $\psi_s(x)$ like so:

$$\psi_i(x) = <x \mid \psi_i>$$

$$\psi'_i(x') = <x' \mid \psi'_i> = <x \mid \Lambda^\dagger\,\psi'_i> = S_{ij}(\Lambda) <x \mid \psi_j> = S_{ij}(\Lambda)\,\psi_j(x)$$

where we used Section 4 above (where ψ is a linear combination of the ψ^r)

$$\Lambda^\dagger|\,\psi'_i(\Lambda p)> = S(\Lambda)_{ij} \mid \psi_j(p)>$$

So what makes this work is to think of the bra state being not ψ but ψ_i. I sure thought the Plan A approach would be consistent with this. The equation $\Lambda^\dagger \mid \psi'_i> = S_{ij}(\Lambda) \mid \psi_j>$ says that this relationship has nothing at all to do with spacetime coordinate x^μ , it is an internal symmetry, it is "spin". No matter what we close with on the left, we get the same thing. In my direct product approach, I am tying space and spin together in some wrong way. You would think I could write the above as

$$\Lambda \mid \psi_i> = D_{ij}(\Lambda) \mid \psi_j> \qquad =>$$

$$\Lambda \mid \mathbf{u}_i \bullet \boldsymbol{\psi} > = D_{ij}(\Lambda) \mid \mathbf{u}_j \bullet \boldsymbol{\psi} > \qquad =>$$

where I pick off the right component of the "vector" $\boldsymbol{\psi}$ with some basis vectors like (1,0,0,0) . Then this would become [here and in the rest of this section, ij are represented by rs, just component indices, not spinor labels.]

$$<x|\ \Lambda \mid \mathbf{u}_r \bullet \boldsymbol{\psi} > = D_{rs}(\Lambda) <x| \mathbf{u}_s \bullet \boldsymbol{\psi} > = <x' \mid \mathbf{u}_r \bullet \boldsymbol{\psi} >$$

$$<x|\ \Lambda \mid (u_r, \boldsymbol{\psi}) > = D_{rs}(\Lambda) <x|\ (u_s, \boldsymbol{\psi})> = <x' \mid\ (u_r, \boldsymbol{\psi}) >$$

So you could say we have a little 4x4 HS embedded inside the ket world. Then consider:

$$\Lambda \mid \boldsymbol{\psi}> = \mid \boldsymbol{\varphi} > \qquad \text{means} \qquad \Lambda \mid \psi_r> = D_{rs}(\Lambda) \mid \psi_s> = \mid \varphi_r>$$

I suppose you could say this

$$|\boldsymbol{\psi}\rangle = \Sigma_{\mathbf{r}}\, \mathbf{u}_{\mathbf{r}}\, |\psi_{\mathbf{r}}\rangle \quad \text{so that} \quad \Lambda\,|\boldsymbol{\psi}\rangle = \Sigma_{\mathbf{r}}\, \mathbf{u}_{\mathbf{r}}\, \Lambda\,|\psi_{\mathbf{r}}\rangle = \Sigma_{\mathbf{r}}\, \mathbf{u}_{\mathbf{r}}\, D_{\mathbf{rs}}(\Lambda)\,|\psi_{\mathbf{s}}\rangle = \Sigma_{\mathbf{r}}\, \mathbf{u}_{\mathbf{r}}\, |\varphi_{\mathbf{r}}\rangle = |\boldsymbol{\varphi}\rangle$$

so I suppose $\Lambda\,|\boldsymbol{\psi}\rangle$ is a well defined operation, again having nothing to do with x. Then we should be able to pull the various constant things $u_{\mathbf{r}}^{\mathbf{T}}$ off to the left to get

$$\mathbf{u}_{\mathbf{r}}\bullet\langle x|\ \Lambda\,|\boldsymbol{\psi}\rangle = D_{\mathbf{rs}}(\Lambda)\,\mathbf{u}_{\mathbf{s}}\bullet\langle x|\boldsymbol{\psi}\rangle = \mathbf{u}_{\mathbf{r}}\bullet\langle x'\,|\,\boldsymbol{\psi}\rangle$$

Now this looks a little like our direct product formalism, BUT, as we operate with Λ to the left, this Λ has no effect on $\mathbf{u}_{\mathbf{r}}$. In the direct product world, that was not the case. Can write the above as

$$(\mathbf{u}_{\mathbf{r}}, \langle x|\ \Lambda\,|\boldsymbol{\psi}\rangle) = D_{\mathbf{rs}}(\Lambda)\,(\,\mathbf{u}_{\mathbf{s}}\,, \langle x|\boldsymbol{\psi}\rangle) = (\mathbf{u}_{\mathbf{r}}, \langle x'\,|\,\boldsymbol{\psi}\rangle)$$

Somehow it is nested HS's and not a direct product HS. Notice the difference:

$\psi'_{\mathbf{r}}(x') = (\mathbf{u}_{\mathbf{r}}, \langle x|\ \Lambda^{\dagger}\,|\,\boldsymbol{\psi}\rangle) = \langle x|\ \Lambda^{\dagger}\,|\,(u_{\mathbf{r}},\boldsymbol{\psi})\rangle$ Plan B

$\psi_{\mathbf{r}}(x) = (\mathbf{u}_{\mathbf{r}}, \langle x\,|\,\boldsymbol{\psi}\rangle) = \langle x|\,(u_{\mathbf{r}},\boldsymbol{\psi})\rangle$ Plan B

$\psi'_{\mathbf{r}}(x') = \{\ \langle x^{\boldsymbol{\mu}}\,|\otimes\langle u_{\mathbf{r}}|\ \}\,\Lambda^{\dagger}\ |\psi\rangle$ Plan A

$\psi_{\mathbf{r}}(x) = \{\ \langle x^{\boldsymbol{\mu}}\,|\otimes\langle u_{\mathbf{r}}|\ \}\ |\psi\rangle$ Plan A

In Plan B, we have a nested HS structure which is the right structure, either HS nested in the other. In Plan A we have a direct product HS structure which is the wrong structure. We might bail it out by saying that $\Lambda = \Lambda \otimes 1$, but then we don't really have a true direct product situation. I think in Plan B, it is best to think of the $(u_{\mathbf{r}},\boldsymbol{\psi})$ as nested inside the <..> space, because only then do we know how to act with the operator $\Lambda^{\dagger}$.