Feynman's Integral 8.59 p 170 of BD1 PhL 11.1.08

Comments

All integrals on this page are from 0 to ∞.

When I first saw this integral, I tried for about 2 hours and then gave up on deriving it. Here is how I kept starting off:

$$1/a_1a_2 = \int dz_1 \int dz_2\ 1/(a_1z_1 + a_2z_2)^2\ \delta(z_1 + z_2 - 1)$$

$$= (1/a_1a_2) \int dz_1 \int dz_2\ 1/(z_1 + z_2)^2\ \delta(z_1/a_1 + z_2/a_2 - 1) \qquad // \text{ scale } z_i \rightarrow z_i/a_i$$

$$= (1/a_1a_2)\ I(a_i)$$

I then took the approach of trying to show why the resulting integral $I(a_i)$ was independent of the a_i. If I could show that, then I could just set all $a_i = 1$ and do the integral directly (though it was unclear I could do the integral for many variables). I tried taking the derivative of the RHS but that led to a δ' sitting in there, which I tried undoing by a parts integration, but that just led to a mess. I then tried elevating the delta function into an exponential, and that also led nowhere. At that point, I decided I was "missing some simple fact" and, since Halloween, let's go to Marriott and look up the Feynman reference given on BD page 171.

I found Marriott in a state of major renovation. The inner core is empty, all periodicals are in some robotic machine called the ARC. [email sent out just now on that subject] I got the 1949 Feynman article referenced on page 171 from the ARC, and the volume had lots of articles by Feynman Schwinger and others on the new QED stuff, but I did not see my integral in any of these articles. They use **A** in place of $\not{a}$ and A in place of a. The problem is doing the Feynman graph integrals with their various denominator factors from propagators, and that is where this integral comes into play. I was amazed at the amount of math detail in these old pages, thousands of symbols per page in densepack tiny print. This was Phys Rev, before it broke out into A,B,C,D.

Having failed on this, I wandered over to QC680 .J38 and looked at Jauch and Rorlich "The Theory of Photons and Electrons, another BD reference. This was 1955 and then a 1972 second edition by the surviving Rorlich (second on shelf, first in ARC). In an appendix, my integral appeared along with an induction proof which I copied. Then just before leaving I looked at a QED + QCD book by Grozin, and this had the *simple* proof I was looking for, no induction needed.

Proof of Simple Case

As I suspected, I was indeed missing a very simple fact which is this,

$$\int dx\ x^n e^{-ax} = n!/a^{n+1} \qquad (1)$$

an old friend. So let's consider the RHS of our equation above

$$J = \int dz_1 \int dz_2 \; 1/(a_1 z_1 + a_2 z_2)^2 \; \delta(z_1 + z_2 - 1)$$

The *key step* is to use the above integral (1) with n=1 to "elevate" the denominator factor into an integral representation:

$$1/(a_1 z_1 + a_2 z_2)^2 = \int dx \; x \; \exp(-[a_1 z_1 + a_2 z_2]x) = \int dx \; x \; \exp(-[a_1 z_1]x) \exp(-[a_2 z_2]x)$$

and you can see how this tends to "factorize" things, separating a_1 from a_2. We then have

$$J = \int dx \; x \int dz_1 \int dz_2 \; \exp(-[a_1 z_1]x) \exp(-[a_2 z_2]x) \; \delta(z_1 + z_2 - 1)$$

Now is the time to "scale" variables by doing $z_i' = z_i \; x$ to clean up the variables in the exponents, and we then get

$$J = \int dx \; x \int dz'_1 \int dz'_2 \; (1/x)^2 \; \exp(-[a_1 z'_1]) \exp(-[a_2 z'_2]) \; \delta(z'_1/x + z'_2/x - 1)$$

$$= \int dx \; x \int dz'_1 \int dz'_2 \; (1/x)^2 \; \exp(-[a_1 z'_1]) \exp(-[a_2 z'_2]) \; \delta(z'_1 + z'_2 - x)(x) \quad \text{// adjust } \delta$$

$$= \int dx \int dz'_1 \int dz'_2 \; \exp(-[a_1 z'_1]) \exp(-[a_2 z'_2]) \; \delta(z'_1 + z'_2 - x) \quad \text{// cancel x's}$$

$$= \int dx \int dz_1 \int dz_2 \; \exp(-[a_1 z_1]) \exp(-[a_2 z_2]) \; \delta(z_1 + z_2 - x) \quad \text{// remove primes}$$

$$= \int dz_1 \int dz_2 \; \exp(-[a_1 z_1]) \exp(-[a_2 z_2]) \int dx \; \delta(z_1 + z_2 - x) \quad \text{// move } \int dx \text{ to the right}$$

$$= \int dz_1 \int dz_2 \; \exp(-[a_1 z_1]) \exp(-[a_2 z_2]) \quad \text{// do the x integral}$$

$$= \int dz_1 \exp(-[a_1 z_1]) \int dz_2 \; \exp(-[a_2 z_2]) \quad \text{// result is fully factored}$$

$$= 1/a_1 * 1/a_2 \qquad \text{QED}$$

In the above, the scaling keeps z_1 and z_2 on an equal footing, unlike my initial scaling shown at the start of this writeup.

So we can regard the above as showing the basic idea for doing this kind of integral. Let's now do the full integral shown by Grozin, of which the Feynman integral on top of BD page 170 is a simple case. I will go in the "forward direction" instead of the reverse direction done above.

Start with as many factors as you want on the LHS:

$$\text{LHS} = [n_1! / (a_1^{n1+1})] \; [n_2! / (a_2^{n2+1})] \; ..... \; [n_k! / (a_k^{nk+1})]$$

Throw each one into its integral representation using (1) above:

$$\text{LHS} = \left(\int dx_1\, x_1^{n1} e^{-a1*x1} \right) \left(\int dx_2\, x_2^{n2} e^{-a2*x2} \right) ..... \left(\int dx_k\, x_k^{nk} e^{-ak*xk} \right)$$

Combine all the exponentials together

$$\text{LHS} = \int dx_1 \int dx_2 .... \int dx_k \;\; x_1^{n1} x_2^{n2} ... \; x_k^{nk} \; e^{-a1*x1-a2*x2 .... \; -ak*xk}$$

Now just insert $1 = \int dx\, \delta(x_1 + x_2 + ... - x)$ as part of the integrand above:

$$\text{LHS} = \int dx_1 \int dx_2 .... \int dx_k \;\; x_1^{n1} x_2^{n2} ... \; x_k^{nk} \; e^{-a1*x1-a2*x2 .... \; -ak*xk} \int dx\, \delta(x_1 + x_2 + ... - x)$$

and then shift the dx integral all the way to the left

$$\text{LHS} = \int dx \;\; \int dx_1 \int dx_2 .... \int dx_k \;\; x_1^{n1} x_2^{n2} ... \; x_k^{nk} \; e^{-a1*x1-a2*x2 .... \; -ak*xk} \; \delta(x_1 + x_2 + ... - x)$$

Now scale all integration x_i variables so that $x_i = x\, z_i$ where x that new integration variable which we know is a positive number:

$$\text{LHS} = \int dx \;\; \int dz_1 \int dz_2 .... \int dz_k \; (x)^k \;\; * \; z_1^{n1} z_2^{n2} ... \; z_k^{nk} \; (x)^{n1+n2+..nk}$$
$$* \; e^{-[a1*z1+a2*z2 .... \; +ak*zk]x} \;\; * \; \delta(xz_1 + xz_2 + ... - x)$$

Now remove x from the delta function $\delta(xz_1 + xz_2 + ... - x) = \delta(z_1 + z_2 + ... - 1)/x$ and combine powers of x to get

$$\text{LHS} = \int dx \;\; \int dz_1 \int dz_2 .... \int dz_k \; * \; z_1^{n1} z_2^{n2} ... \; z_k^{nk}$$
$$* \; e^{-[a1*z1+a2*z2 .... \; +ak*zk]x} \;\; * \; \delta(z_1 + z_2 + ... - 1) \;\; x^{k-1+n1+n2+...nk}$$

So next, move the δ to the left and the dx integral to the right to get

$$\text{LHS} = \int dz_1 \int dz_2 .... \int dz_k \; * \; z_1^{n1} z_2^{n2} ... \; z_k^{nk}$$
$$\delta(z_1 + z_2 + ... - 1) \; * \int dx \;\; e^{-[a1*z1+a2*z2 .... \; +ak*zk]x} \;\; * \; x^{k-1+n1+n2+...nk}$$

and we see that the x integral is now of our form (1) above. Let's define these temporary names

$$k\text{-}1+n_1+n_2+...n_k = N$$

$$a_1*z_1+a_2*z_2 .... + a_k*z_k = Q$$

$$\int dx\ e^{-[Q]x} * x^{N} = (N)! / Q^{N+1}$$

and our result is

$$\text{LHS} = N! \int dz_1 \int dz_2 .... \int dz_k * z_1^{n1} z_2^{n2} ... z_k^{nk} / [Q]^{N+1} \delta(z_1 + z_2 + ... -1)$$

So we have now shown that

$$[n_1! / (a_1^{n1+1})] [n_2! / (a_2^{n2+1})] ..... [n_k! / (a_k^{nk+1})]$$

$$= N! \int dz_1 \int dz_2 .... \int dz_k * z_1^{n1} z_2^{n1} ... z_k^{nk} / [Q]^{N+1} \delta(z_1 + z_2 + ... -1)$$

which we can rewrite as

$$[1/ (a_1^{n1+1})] [1 / (a_2^{n2+1})] ..... [1/ (a_k^{nk+1})]$$

$$= N! / (n_1!\ n_2!.... n_k!) * \int dz_1 \int dz_2 .... \int dz_k * z_1^{n1} z_2^{n2} ... z_k^{nk} / [Q]^{N+1} \delta(z_1 + z_2 + ... -1)$$

where $N = k\text{-}1+n_1+n_2+...n_k$ and $Q = a_1*z_1+a_2*z_2.... +a_k*z_k$.

Finally, define some new integers $m_i = n_i+1$ for i = 1 to k. then

$$N = k\text{-}1+n_1+n_2+...n_k = m_1+m_2+...m_k - 1$$

which removes term k from N. Then we have

$$[1/ (a_1^{m1})] [1 / (a_2^{m2})] ..... [1/ (a_k^{mk})]$$

$$= (m_1+m_2+...m_k - 1)! / \{ (m_1-1)! (m_2-1)! ... (m_1-1)! \}$$

$$* \int dz_1 \int dz_2 .... \int dz_k * z_1^{m1-1} z_2^{m2-1} ... z_k^{mk-1} / [Q]^{m1+m2+..+mk} \delta(z_1 + z_2 + ... -1)$$

And things are more compact if we write (n-1)! = Γ(n) and result is then

$$1/(a_1^{m1} a_2^{m2} ....a_k^{mk}) = \Gamma(m_1+ m_2+..m_k)/ (\Gamma(m_1)\ \Gamma(m_2)..\ \Gamma(m_k)) \int dz_1 \int dz_2 .... \int dz_k$$

$$* z_1^{m1-1} z_2^{m2-1} ... z_k^{mk-1} /[\ a_1*z_1+a_2*z_2.... +a_k*z_k\ ]^{m1+m2+..+mk} \delta(z_1 + z_2 + ... -1)$$

which is **our final and most general result**. Now if we set all $m_i = 1$, we get

$$1/(a_1 a_2 \ldots a_k) = (k-1)! \int dz_1 \int dz_2 \ldots \int dz_k \; 1/[\, a_1 * z_1 + a_2 * z_2 \ldots + a_k * z_k \,]^k \; \delta(z_1 + z_2 + \ldots - 1)$$

which is 8.59 on page 170 of BD.