

Relativistic Quantum Mechanics

David J. Miller
University of Glasgow

SUPA Graduate School
October/November 2006

http://www.physics.gla.ac.uk/~dmiller/lectures/RQM_2006.ppt

Recommended Text: **Quarks & Leptons** by F. Halzen and A. Martin
(though this is not really necessary)

Rough outline of topics

❖ Non-relativistic Quantum Mechanics

- The Schrodinger equation

❖ Relativistic Quantum Mechanics

- The Klein-Gordon Equation, The Dirac Equation, Angular Momentum and Spin, Symmetries of the Dirac Equation

❖ Quantum Electrodynamics

- Classical Electromagnetism, The Dirac Equation in an electromagnetic field

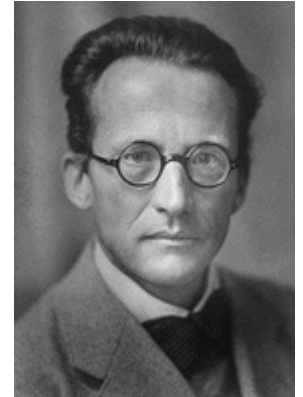
❖ Scattering and Perturbation Theory

- The QED Feynman rules, cross-sections, crossing symmetry, identical particles, the fermion propagator, decay rates

❖ Quantum Chromodynamics

- Quarks, gluons and color, renormalisation, running couplings

Non-relativistic Quantum Mechanics



Erwin Schrödinger

Consider a plane wave: $\psi(t, \vec{x}) = N e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

This plane wave has energy $E = \hbar\omega$ and momentum $p = \hbar k$

These values can be extracted using the energy and momentum **operators** defined by:

$$\hat{E} \equiv i\hbar \frac{\partial}{\partial t} \qquad \hat{\vec{p}} \equiv -i\hbar \vec{\nabla}$$

Classically we know the relation between total energy and momentum is

$$E = \frac{p^2}{2m} + V$$

Writing this in terms of operators provides us with the **Schrödinger equation**:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

But how do we interpret the Schrödinger equation and the associated wavefunction?

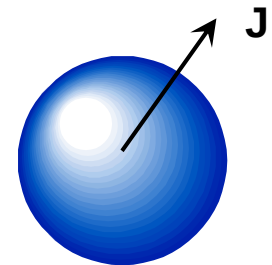
In the pre-school problems, you should have shown that the quantity $\rho = \psi^* \psi$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{with} \quad \vec{J} = \frac{\hbar}{2im} [\psi^* (\vec{\nabla} \psi) - (\vec{\nabla} \psi^*) \psi]$$

Now, integrating over a volume V:
$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \vec{\nabla} \cdot \vec{J} dV$$

and using Gauss' Law

$$\frac{\partial}{\partial t} \int_V \rho dV = - \int_A \mathbf{J} \cdot d\mathbf{A}$$



Volume V enclosed
by Area A

Any change in the total ρ in the volume must come about through a current $\mathbf{J}$ through the surface of the volume.

$\rho = \psi^* \psi$ is a conserved density and we interpret it as the **probability density** for finding a particle at a particular position.

Notice that ρ is positive definite, as required.

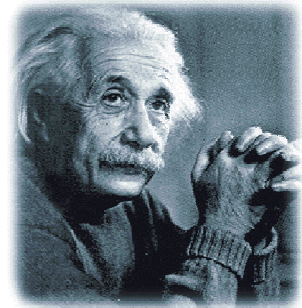
Relativistic Quantum Mechanics

The Schrödinger Equation only describes particles in the non-relativistic limit. To describe the particles at particle colliders we need to incorporate special relativity.

Let's have a quick review of special relativity

We construct a position *four-vector* as

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, \vec{x}) \quad (\mu = \{0, 1, 2, 3\})$$



An observer in a frame S' will instead observe a four-vector $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ where Λ denotes a Lorentz transformation.

e.g. under a Lorentz boost by v in the positive x direction:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The quantity $x^\mu x_\mu$ is **invariant** under a Lorentz transformation

$$x^\mu x_\mu \equiv g_{\mu\nu} x^\mu x^\nu = (ct)^2 - |\vec{x}|^2$$

note the definition of
a **covector** $x_\mu \equiv g_{\mu\nu} x^\nu$

where $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the **metric tensor** of Minkowski space-time.

$$\left[g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta} \Leftrightarrow x'^\mu x'_\mu = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta = g_{\alpha\beta} x^\alpha x^\beta = x^\mu x_\mu \right]$$

A particle's **four-momentum** is defined by $p^\mu = m \frac{dx^\mu}{d\tau}$ where τ is **proper time**,

the time in the particle's own rest frame. Proper time is related to an observer's time via $t = \gamma\tau$

Its four-momentum's time component is the particle's energy, while the space components are its three-momentum

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right)$$

and its length is an invariant, its **mass**:

$$p^\mu p_\mu = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2 c^2$$

Finally, I define the derivative $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right)$

Watch the minus sign!

Note that you will sometimes use the vector expression $\partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right)$

∂_μ transforms as $\partial_\mu \rightarrow \partial'_\mu = [\Lambda^{-1}]^\nu_\mu \partial_\nu$

$$\left[\partial x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \partial x^\nu = \Lambda^\mu_\nu \partial x^\nu \quad \text{so} \quad \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = [\Lambda^{-1}]^\nu_\mu \frac{\partial}{\partial x^\nu} \right]$$

For simplicity, from now on I will use **natural units**. Instead of writing quantities in terms of kg, m and s, we could write them in terms of c, $\hbar$ and eV:

$$c = 299\,792\,458 \text{ ms}^{-1}$$

$$\hbar = 6.582\,118\,89(26) \times 10^{-16} \text{ eV s}$$

$$1 \text{ eV} = 1.782\,661\,731(70) \text{ c}^2 \text{ kg}$$

So any quantity with dimensions $\text{kg}^a \text{ m}^b \text{ s}^c$ can be written in units of $\text{eV}^\alpha \hbar^\beta \text{ c}^\gamma$ with,

Then we omit $\hbar$ and c in our quantities (you can work them out from the dimensions).

The Klein-Gordon Equation

The invariance of the four-momentum's length provides us with a relation between energy, momentum and mass:

$$p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$$

Replacing energy and momentum with $E \rightarrow i\frac{\partial}{\partial t}$, $\vec{p} \rightarrow -i\vec{\nabla}$ gives the **Klein-Gordon** equation:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = m^2 \phi$$

(We have set $V=0$ for simplicity.)

Alternatively, in covariant notation:

$$p^\mu p_\mu = m^2 \quad \text{with} \quad p^\mu \rightarrow i\partial^\mu \quad \text{gives} \quad (\partial^2 + m^2) \phi = 0$$

↑
($\partial^2 \equiv \partial^\mu \partial_\mu$ is sometimes written as $\square$)

This has plane-wave solutions $\phi(t, \vec{x}) = N e^{-i(Et - \vec{p} \cdot \vec{x})}$

↑
normalization



Oskar Klein

This is the relativistic wave equation for a **spin zero** particle, which conventionally is denoted ϕ . Under a Lorentz transformation the Klein-Gordon operator is invariant:

$$\left(\frac{\partial}{\partial x'^{\mu}} \frac{\partial}{\partial x'_{\mu}} + m^2 \right) \phi'(x') = \left(\frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial x_{\mu}} + m^2 \right) \phi(\Lambda x) = 0$$

So $\phi'(x') = S\phi(x)$ with $\begin{cases} S \text{ real} & (\text{since } \Lambda \text{ is real}) \\ |S|^2 = 1 & (\text{Lorentz trans. preserve the norm}) \end{cases}$

Under continuous Lorentz transformations, S must be the same as for the identity, ie. $S = 1$

But for a parity inversion $(t, \vec{x}) \rightarrow (t, -\vec{x})$ it can take either sign

If $S = 1$, then ϕ is a **scalar** $\phi'(t', \vec{x}') = \phi'(t, -\vec{x}) = \phi(t, \vec{x})$

If $S = -1$, then ϕ is a **pseudoscalar** $\phi'(t', \vec{x}') = \phi'(t, -\vec{x}) = -\phi(t, \vec{x})$

Since ϕ is invariant, then $|\phi|^2$ does not change with a Lorentz transformation.

This is at odds with our previous interpretation of $|\phi|^2$ as a probability density, since densities **do** change with Lorentz transformations (the probability $P = \rho V = \text{constant}$).

We need new definitions for the density ρ and current $\mathbf{J}$ which satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{or, covariantly,} \quad \partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = (\rho, \vec{J})$$

One possible choice is:

$$\rho = i \left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi \right]$$

$$\vec{J} = -i \left[\phi^* (\vec{\nabla} \phi) - (\vec{\nabla} \phi^*) \phi \right]$$

As a four-vector,

$$j^\mu = i \left[\phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi \right]$$

 (this is the same current as before, just with a different normalisation)

Consider our plane-wave solution: $\phi(t, \vec{x}) = N e^{-i(Et - \vec{p} \cdot \vec{x})}$

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \phi &= (\nabla^2 - m^2) \phi \quad \Rightarrow \quad E^2 = m^2 + |\vec{p}|^2 \\ &\Rightarrow \quad E = \pm \sqrt{m^2 + |\vec{p}|^2} \end{aligned}$$

We have **solutions with negative energy**, and even worse,

$$\rho = i \left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi \right] = 2|N|^2 E$$

so these negative energy states have **negative probability distributions!**

We can't just ignore these solutions since they will crop up in any Fourier decomposition.



This is why Schrödinger abandoned this equation and developed the non-relativistic Schrödinger equation instead – he (implicitly) took the positive sign of the square root so that he could ignore the negative energy solutions.

Feynman-Stuckelberg Interpretation

You will see in your QFT course that positive energy states must propagate forwards in time in order to preserve causality.

Feynman and Stuckelberg suggested that negative energy states propagate **backwards in time**.

If $E < 0$, just move the sign into the time: $e^{-Et} = e^{-(-E)(-t)}$

Particles flowing backwards in time are then reinterpreted as **anti-particles** flowing forwards in time.

If the field is charged, we may reinterpret j^μ as a charge density, instead of a probability density:

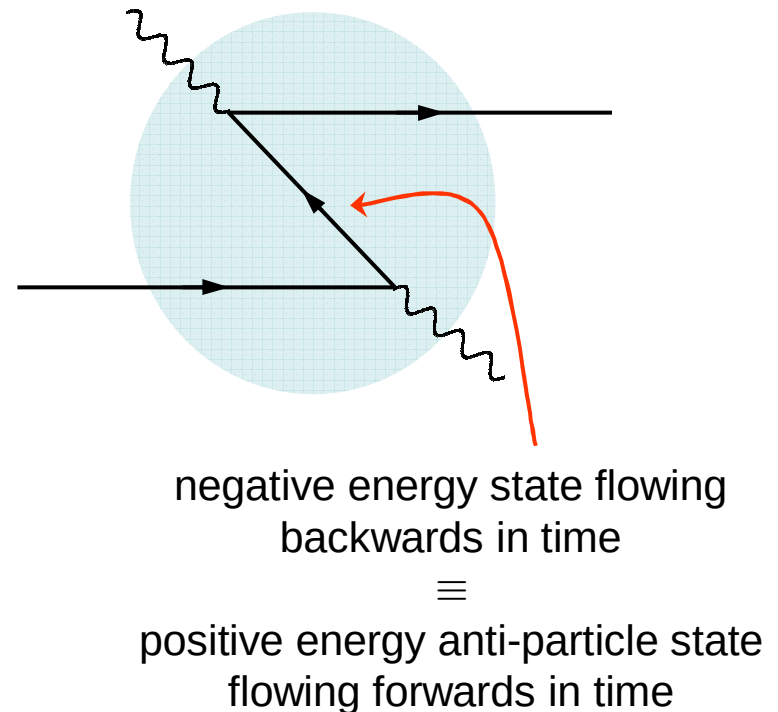
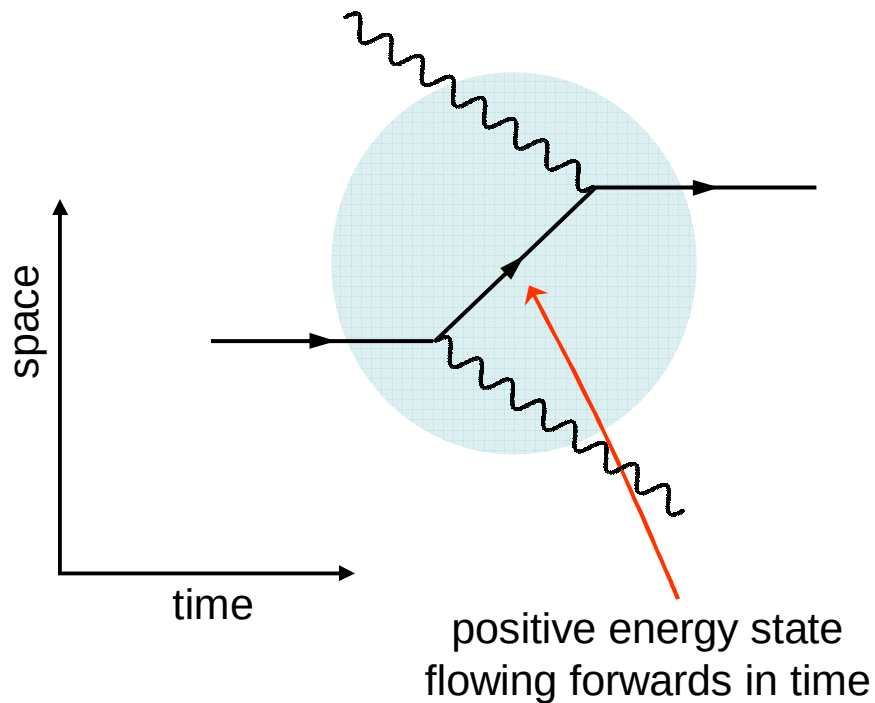
$$j^\mu = -ie [\phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi]$$

Now $\rho = j^0$, so for a particle of energy E : $j^0 = -2e|N|^2 E$

while for an anti-particle of energy E : $j^0 = +2e|N|^2 E = -2e|N|^2 (-E)$

which is the same as the charge density for an electron of energy $-E$

In reality, we only ever see the final state particles, so we must include these anti-particles anyway.



Quantum mechanics does not adequately handle the creation of particle—anti-particle pairs out of the vacuum. For that you will need **Quantum Field Theory**.

Normalization of KG solutions

The particle (or charge) density allows us to normalize the KG solutions in a box.

$\rho = 2|N|^2 E$ so in a box of volume V the number of particles is:

$$\int_V \rho dV = \int_V 2|N|^2 E dV = 2|N|^2 EV$$

So if we normalize to **$2E$ particles per unit volume**, then $N = 1$

Notice that this is a covariant choice. Since the number of particles in a box should be independent of reference frame, but the volume of the box changes with a Lorentz boost, the density must also change with a boost. In fact, the density is the time component of a four-vector j^0 .

The Dirac Equation

The problems with the Klein-Gordon equation all came about because of the square root required to get the energy:

$$E = \pm \sqrt{m^2 + |\vec{p}|^2}$$



Paul Dirac

Dirac tried to get round this by finding a field equation which was linear in the operators.

$$E = \vec{\alpha} \cdot \vec{p} + \beta m \quad \longrightarrow \quad i \frac{\partial \psi}{\partial t} = (-i \vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi$$

All we need to do is work out $\vec{\alpha}$ and β

Now, we have $E = \alpha_i p_i + \beta m$ where i and j are summed over 1,2,3

$$\begin{aligned}\Rightarrow E^2 &= \alpha_i \alpha_j p_i p_j + (\alpha_i \beta + \beta \alpha_i) m p_i + \beta^2 m^2 \\ &= \frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + (\alpha_i \beta + \beta \alpha_i) m p_i + \beta^2 m^2\end{aligned}$$

So, comparing with $E^2 = m^2 + |\vec{p}|^2$ we must have:

$$\begin{aligned}\alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} \\ \alpha_i \beta + \beta \alpha_i &= 0 \\ \beta^2 &= 1\end{aligned}$$

$\vec{\alpha}$ and β are anti-commuting objects – not just numbers!

These commutation relations **define** α and β . Anything which obeys these relations will do. One possibility, called the **Dirac representation**, is the 4×4 matrices:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$


2x2 matrices

where σ_i are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since these act on the field ψ , ψ itself must now be a 4 component vector, known as a **spinor**.

We can write this equation in a four-vector form by defining a new quantity γ^μ :

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} \equiv \beta \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

The anti-commutation relations become:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

And the Dirac Equation is: (with $p^\mu \rightarrow i\partial^\mu$)

$$(i\gamma^\mu \partial_\mu - m) \psi = 0$$

Often $\gamma^\mu \partial_\mu$ is written as $\not{\partial}$

Does the Dirac Equation have the right properties?

Is the probability density positive definite?

A appropriate conserved quantity is now $\rho = \psi^\dagger \psi$ with $\vec{J} = \psi^\dagger \vec{\alpha} \psi$ $(\psi^\dagger = (\psi^*)^T)$

In four-vector notation,

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad \text{with} \quad \bar{\psi} \equiv \psi^\dagger \gamma^0 \quad (\text{Note } \gamma^0 \gamma^0 = \beta^2 = 1)$$

Clearly $\rho = \psi^\dagger \psi > 0$ always! ✓

Does the Dirac Equation only have positive energy solutions?

Look for plane wave solutions:

$$\psi(t, \vec{x}) = u(\vec{p}) e^{-i(Et - \vec{p} \cdot \vec{x})} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} e^{-i(Et - \vec{p} \cdot \vec{x})}$$

4 component spinor   2 component spinors

Since we want the energy, it is easier to work without four-vector notation:

$$i \frac{\partial \psi}{\partial t} = (\vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi \Rightarrow E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

For a particle **at rest**, $\vec{p} = 0$

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

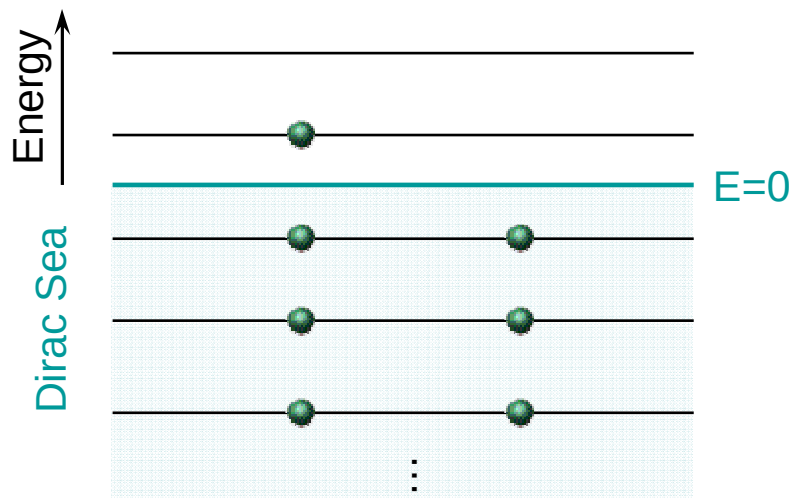
Solutions:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ with } E = m \quad \text{OR} \quad u = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ with } E = -m$$

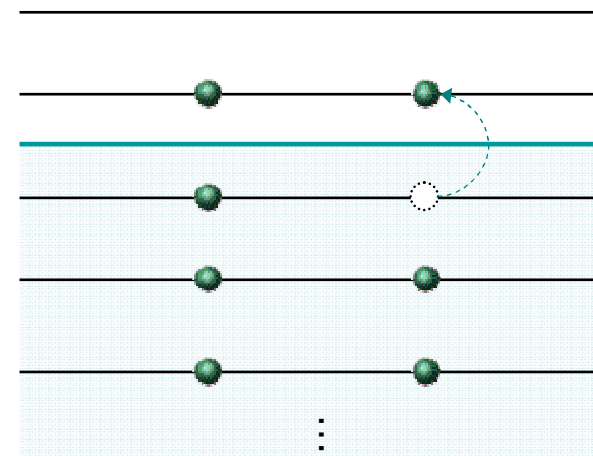
Oops! We still have negative energy solutions! ✖

Dirac got round this by using the **Pauli Exclusion principle**.

He reasoned that his equation described particles with spin (e.g. electrons) so only two particles can occupy any particular energy level (one spin-up, the other spin-down).



If all the energy states with $E < 0$ are already filled, the electron can't fall into a negative energy state.



Moving an electron from a negative energy state to a positive one leaves a hole which we interpret as an anti-particle.

Note that we couldn't have used this argument for bosons (no exclusion principle) so the **Feynman-Stuckleberg interpretation** is more useful.

A General Solution

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \Rightarrow \begin{cases} \phi = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi \\ \chi = \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \phi \end{cases}$$

Check these are compatible:

$$\phi = \left(\frac{\vec{\sigma} \cdot \vec{p}}{E + m} \right) \left(\frac{\vec{\sigma} \cdot \vec{p}}{E - m} \right) \phi = \frac{|\vec{p}|^2}{E^2 - m^2} \phi = \phi \quad \text{since} \quad E^2 = m^2 + |\vec{p}|^2$$

$$\left[\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k \Rightarrow (\vec{\sigma} \cdot \vec{p})^2 = |\vec{p}|^2 + i (\vec{p} \times \vec{p}) \cdot \vec{\sigma} = |\vec{p}|^2 \right]$$

We need to choose a basis for our solutions. Choose,

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Positive Energy Solutions, $E > 0$, are $\psi^{(1)}, \psi^{(2)}$

$$\psi^{(s)}(x) = u^{(s)} e^{-ip \cdot x} = \sqrt{E + m} \begin{pmatrix} \xi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \xi^{(s)} \end{pmatrix} e^{-ip \cdot x} \quad (s = 1, 2)$$

[Normalization choice (see next slide)]

Negative Energy Solutions, $E < 0$, are $\psi^{(3)}, \psi^{(4)}$

$$\psi^{(s+2)}(x) = u^{(s)} e^{-ip \cdot x} = \sqrt{-E + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \xi^{(s)} \\ \xi^{(s)} \end{pmatrix} e^{-ip \cdot x} \quad (s = 1, 2)$$

Typically, we write this in terms of the antiparticle's energy and momentum:

$$u^{(3,4)}(-\vec{p}) e^{-i(-p) \cdot x} \equiv v^{(1,2)}(\vec{p}) e^{ip \cdot x}$$

Conventions differ here: sometimes the order is inverted

Antiparticle spinor, normalised to make $v^{(r)\dagger} v^{(s)} = 2E \delta^{rs}$

Normalization of solutions

Notice the normalization choice $N = \sqrt{E + m}$ made for the spinors.

Just like for the KG equation, we can choose to have $2E$ particles per unit volume

$$\int_{\substack{\text{unit} \\ \text{volume}}} \psi^\dagger \psi dV = 2E$$

$$\text{But } u^{(s)\dagger} u^{(s)} = |N|^2 \left[1 + \left(\frac{\vec{\sigma} \cdot \vec{p}}{E + m} \right)^2 \right] = |N|^2 \frac{2E}{E + m}$$

So a covariant normalization is $N = \sqrt{E + m}$

Occasionally, I will instead normalize spinors to $2E$ particles per volume V

$$\text{Then } \int_V \psi^\dagger \psi dV = |N|^2 \frac{2E}{E + m} V = 2E \quad \Rightarrow \quad N = \sqrt{\frac{E + m}{V}}$$

and only set $V=1$ at the end.

Orthogonality and completeness

With the normalization of $2E$ particles per unit volume, it is rather obvious that:

$$u^{(r)} \dagger u^{(s)} = 2E \delta^{rs} \quad v^{(r)} \dagger v^{(s)} = 2E \delta^{rs}$$

This is a statement of orthogonality.

Less obvious, but easy to show, are the completeness relations:

$$\sum_{s=1,2} u^{(s)}(p) \bar{u}^{(s)}(p) = \not{p} + m$$
$$\sum_{s=1,2} v^{(s)}(p) \bar{v}^{(s)}(p) = \not{p} - m$$

Angular Momentum and Spin

The angular momentum of a particle is given by $\vec{L} = \vec{r} \times \vec{p}$.

If this commutes with the Hamiltonian then angular momentum is conserved.

$$[H, \vec{L}] = [\vec{\alpha} \cdot \vec{p}, \vec{r} \times \vec{p}] = -i\vec{\alpha} \times \vec{p}$$

This is not zero, so $\vec{L} = \vec{r} \times \vec{p}$ is **not** conserved!

But, if we define $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\vec{\alpha} \quad (= -i\gamma_1\gamma_2\gamma_3\vec{\gamma})$

$$\text{then } [H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p}, -i\alpha_1\alpha_2\alpha_3\vec{\alpha}] = 2i\vec{\alpha} \times \vec{p}$$

So the quantity $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ is conserved! $[H, \vec{J}] = 0$

$\vec{L}$ is the orbital angular momentum, whereas $\frac{1}{2}\vec{\Sigma}$ is an **intrinsic** angular momentum

Notice that our basis spinors are eigenvectors of $\frac{1}{2}\Sigma^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

with eigenvalues $\pm\frac{1}{2}$

Note that an $E < 0$ electron with spin $+\frac{1}{2} \equiv$ an $E > 0$ positron with spin $-\frac{1}{2}$

This is why we switched the labelling of the anti-particles earlier.

Helicity of massless fermions

If the mass is zero, our wave equation becomes

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \Rightarrow \begin{cases} E\phi = \vec{\sigma} \cdot \vec{p} \chi \\ E\chi = \vec{\sigma} \cdot \vec{p} \phi \end{cases}$$

Writing $\Psi_{R,L} \equiv \frac{1}{2}(\chi \pm \phi)$ then we find the equations decouple

$$E\Psi_R = \vec{\sigma} \cdot \vec{p} \Psi_R \quad \text{and} \quad E\Psi_L = -\vec{\sigma} \cdot \vec{p} \Psi_L$$

These two component spinors, called **Weyl spinors**, are completely independent, and can even be considered as separate particles!

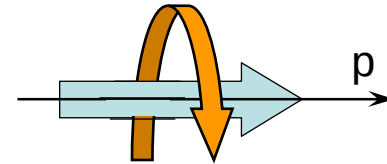
Notice that each is an eigenstate of the operator $\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$ with eigenvalues ± 1

$$\left(|\vec{p}| = E \text{ for massless state} \right)$$

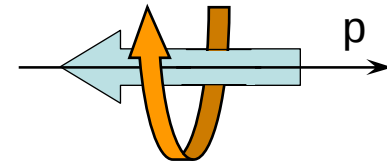

For the full Dirac spinor, we define the **Helicity** operator as $\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|}$

This is the component of spin in the direction of motion.

A particle with a helicity eigenvalue $+\frac{1}{2}$ is **right handed**



A particle with a helicity eigenvalue $-\frac{1}{2}$ is **left handed**



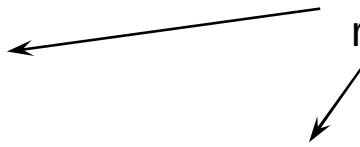
Since an antiparticle has opposite momentum it will have opposite helicity.

left handed particle



right handed antiparticle

We can project out a particular helicity from a Dirac spinor using γ matrices.

Define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  This is the Dirac representation.

and **projection operators** $P_{R/L} \equiv \frac{1}{2}(1 \pm \gamma^5) = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$

Then a spinor $P_L u$ will be left handed, while $P_R u$ will be right handed.

e.g. $P_R u = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix}$

but $\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|} \frac{1}{2} \begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix} = \frac{1}{2} \frac{1}{2} \begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix}$ so $P_R u$ is right handed.

We can make this more explicit by using a different representation of the γ matrices.

The **chiral representation** (sometimes called the **Weyl representation**) is:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Now } P_L \equiv \frac{1}{2}(1 - \gamma^5) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad P_R \equiv \frac{1}{2}(1 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

The left-handed Weyl spinor sits in the upper part of the Dirac spinor, while the right handed Weyl spinor sits in the lower part.

e.g.

$$P_R u = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

The **weak interaction** acts only on left handed particles.

Since parity transforms left handed particles onto right handed ones (and vice versa),

$$\text{i.e.} \quad P\psi_L = \psi_R \quad P\psi_R = \psi_L$$

the weak interactions is parity violating.

Also, helicity is only a **good quantum number** for **massless** particles.

If a particle has a mass, I can always move to a reference frame where I am going faster than it, causing the momentum to reverse direction. This causes the helicity to change sign.

For a massless particle there is no such frame and helicity is a good quantum number.

You will explicitly see in your QFT course that a mass term in the Lagrangian looks like

$$m\bar{\psi}\psi = m (\psi_L^\dagger \psi_R^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m (\bar{\psi}_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

so mass terms mix left and right handed states.

(chiral rep.)

Symmetries of the Dirac Equation

This notation differs in different texts.
e.g. Peskin and Schroeder would write

$$\psi(x) \rightarrow \psi'(x) = S\psi(\Lambda^{-1}x)$$



The Lorentz Transformation

How does the field $\psi(x)$ behave under a Lorentz transformation?

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \quad \partial_{\mu} \rightarrow \partial'_{\mu} = [\Lambda^{-1}]^{\nu}_{\mu} \partial_{\nu} \quad \psi(x) \rightarrow \psi'(x') = S\psi(x)$$

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \rightarrow (i\gamma^{\mu}[\Lambda^{-1}]^{\nu}_{\mu}\partial_{\nu} - m)S\psi(x) = 0$$

(γ^{μ} and m are just numbers and don't transform)

Premultiply by S^{-1} :

$$(iS^{-1}\gamma^{\mu}S[\Lambda^{-1}]^{\nu}_{\mu}\partial_{\nu} - m)\psi(x) = 0$$

$\Rightarrow$

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}_{\nu}\gamma^{\nu}$$

We can find S for an **infinitesimal proper** transformation $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$

$$\Lambda^\mu{}_\nu \gamma^\nu = S^{-1} \gamma^\mu S$$

$\uparrow$
 [antisymmetric]

write $S = 1 + \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}$ (just a parameterisation)

$$\gamma^\mu + \omega^\mu{}_\nu \gamma^\nu = \left(1 - \frac{i}{4} \sigma_{\alpha\beta} \omega^{\alpha\beta}\right) \gamma^\mu \left(1 + \frac{i}{4} \sigma_{\alpha\beta} \omega^{\alpha\beta}\right)$$

$$\Rightarrow 2i\omega^{\alpha\beta} (\delta^\mu{}_\alpha \gamma_\beta - \delta^\mu{}_\beta \gamma_\alpha) = [\gamma^\mu, \sigma_{\alpha\beta}] \omega^{\alpha\beta} \quad \text{[ignoring terms } \mathcal{O}(\omega^2)\text{]}$$

$$\Rightarrow \boxed{\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]}$$

[I jumped a few steps here]

This tells us how a fermion field transforms under a Lorentz boost.

The adjoint transforms as $\bar{\psi} \equiv \psi^\dagger \gamma^0 \rightarrow \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger \gamma^0 S^{-1} = \bar{\psi} S^{-1}$

[since $S^\dagger \gamma^0 = \gamma^0 S^{-1}$ for the explicit form of S derived above]

So $\bar{\psi}\psi$ is invariant.

And $j^\mu = \bar{\psi}\gamma^\mu\psi \rightarrow \bar{\psi}S^{-1}\gamma^\mu S\psi = \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi$ so our current is a four-vector.

Common fermion bilinears:

$$\bar{\psi}\psi \rightarrow \bar{\psi}\psi$$

scalar

$$\bar{\psi}\gamma^5\psi \rightarrow \text{Det}(\Lambda)\bar{\psi}\gamma^5\psi$$

pseudoscalar

$$\bar{\psi}\gamma^\mu\psi \rightarrow \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi$$

vector

$$\bar{\psi}\gamma^\mu\gamma^5\psi \rightarrow \text{Det}(\Lambda)\Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\gamma^5\psi$$

axial vector

$$\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi \rightarrow \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \bar{\psi}\sigma^{\alpha\beta}\gamma^5\psi$$

tensor

Parity $t \rightarrow t, \quad \vec{x} \rightarrow -\vec{x}$

A **parity transformation** is an improper Lorentz transformation $t \rightarrow t, \vec{x} \rightarrow -\vec{x}$ described by

$$[\Lambda^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Again $\Lambda^\mu{}_\nu \gamma^\nu = P^{-1} \gamma^\mu P$, so $P \gamma^0 = \gamma^0 P$ and $P \gamma^i = -\gamma^i P$ $[i = 1, 2, 3]$

Since γ^0 commutes with itself (trivially) and anticommutes with γ^i , a suitable choice is

$$P = \eta \gamma^0, \quad |\eta| = 1$$

$$P : \psi(t, \vec{x}) \rightarrow \psi_P(t, -\vec{x}) = P \psi(t, \vec{x}) = \gamma^0 \psi(t, \vec{x})$$

Can you derive the parity transformations of the bilinears given on the last slide?

You should see that η drops out, so there is no loss of generality setting $\eta = 1$

Charge Conjugation

Another discrete symmetry of the Dirac equation is the interchange of particle and anti-particle.

$$\psi \rightarrow \psi_c \equiv C\bar{\psi}^T$$

Take the complex conjugate of the Dirac equation:

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)^* \psi^*(x) &= \left(-i(\gamma^{\mu\dagger})^T \partial_\mu - m\right) (\psi^\dagger)^T \\ &= \gamma^{0T} (-i\gamma^{\mu T} \partial_\mu - m) \bar{\psi}^T \quad \left[\text{used } \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \text{ and } \bar{\psi} \equiv \psi^\dagger\gamma^0\right]\end{aligned}$$

Premultiply by $C\gamma^{0T}$ and the Dirac equation becomes:

$$(-iC\gamma^{\mu T}C^{-1}\partial_\mu - m)\psi_c = 0$$

Therefore we need C such that

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu$$

The form of C changes with the representation of the γ -matrices. For the Dirac representation a suitable choice is

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$C : \psi(t, \vec{x}) \rightarrow \psi_C(t, \vec{x}) = C\bar{\psi}^T(t, \vec{x}) = i\gamma^2\gamma^0\bar{\psi}^T(t, \vec{x})$$

How does this transformation affect the stationary solutions?

$$\begin{aligned} \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} &\longrightarrow \psi_c = C \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{imt} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt} \\ \psi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} &\longrightarrow \psi_c = C \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{imt} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{imt} \quad \text{etc} \end{aligned}$$

We have mapped particle states onto antiparticle states, as desired.

Time Reversal

A naive transformation of the wavefunction $t \rightarrow -t$, $\vec{x} \rightarrow \vec{x}$ is not sufficient for time reversal. Since the momentum of a particle, is a **rate of change**, it too must change sign.

Changing the momentum direction for a plane wave gives:

$$e^{-i(Et - \vec{p} \cdot \vec{x})} \longrightarrow e^{-i(Et - (-\vec{p}) \cdot \vec{x})} = e^{i(E(-t) - \vec{p} \cdot \vec{x})} = \left(e^{-i(E(-t) - \vec{p} \cdot \vec{x})} \right)^*$$

We must (again!) make a complex conjugation: $\psi(t, \vec{x}) \longrightarrow \psi_T(-t, \vec{x}) = T\psi^*(t, \vec{x})$

Take complex conjugation of Dirac Equation, switch $t \rightarrow -t$ and pre-multiply by T:

$$\begin{aligned} \left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla} - m \right) \psi(t, \vec{x}) &\longrightarrow T \left(-i\gamma^{0*} \frac{\partial}{\partial(-t)} - i\vec{\gamma}^* \cdot \vec{\nabla} - m \right) T^{-1} T\psi^*(-t, \vec{x}) \\ &= \left(i[T\gamma^{0*}T^{-1}] \frac{\partial}{\partial t} + i[-T\vec{\gamma}^*T^{-1}] \cdot \vec{\nabla} - m \right) \psi_T(t, \vec{x}) \end{aligned}$$

Need:

$$T\gamma^0 T^{-1} = \gamma^0, \quad T\vec{\gamma} T^{-1} = -\vec{\gamma}$$

A suitable choice is:

$$T = i\gamma^1\gamma^3 = \begin{pmatrix} -i\sigma_1\sigma_3 & 0 \\ 0 & -i\sigma_1\sigma_3 \end{pmatrix} = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$T : \psi(t, \vec{x}) \rightarrow \psi_T(-t, \vec{x}) = T\psi^*(t, \vec{x}) = i\gamma^1\gamma^3\psi^*(t, \vec{x})$$

CPT

For the discrete symmetries, we have shown:

$$\begin{aligned} C : \psi(t, \vec{x}) &\rightarrow \psi_C(t, \vec{x}) = C \bar{\psi}^T(t, \vec{x}) = i\gamma^2 \gamma^0 \bar{\psi}^T(t, \vec{x}) \\ P : \psi(t, \vec{x}) &\rightarrow \psi_P(t, -\vec{x}) = P \psi(t, \vec{x}) = \gamma^0 \psi(t, \vec{x}) \\ T : \psi(t, \vec{x}) &\rightarrow \psi_T(-t, \vec{x}) = T \psi^*(t, \vec{x}) = i\gamma^1 \gamma^3 \psi^*(t, \vec{x}) \end{aligned}$$

Doing all of these transformations gives us

$$\begin{aligned} CPT : \psi(t, \vec{x}) &\rightarrow \psi_{CPT}(-t, -\vec{x}) = i\gamma^2 \gamma^0 \gamma^0 \gamma^0 T [\gamma^0 i\gamma^1 \gamma^3 \psi^*(t, \vec{x})]^* \\ &= i\gamma^2 \gamma^0 \gamma^0 \gamma^0 (-i) \gamma^1 \gamma^3 \psi(t, \vec{x}) \\ &= \gamma^0 \gamma^1 \gamma^2 \gamma^3 \psi(t, \vec{x}) \\ &= -i\gamma^5 \psi(t, \vec{x}) \end{aligned}$$

So if $\psi(x)$ is an electron, $\psi_{CPT}(-x)$ is a positron travelling backwards in space-time multiplied by a factor $-i\gamma^5$.

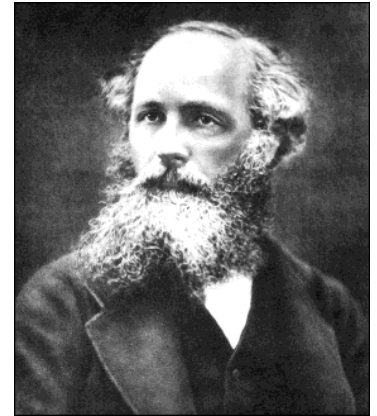
This justifies the Feynman-Stuckleberg interpretation!

Quantum Electro-Dynamics (QED)

Classical Electromagnetism

Maxwell's equations:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \rho & \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 & \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{J}\end{aligned}$$



James Clerk Maxwell

Maxwell wrote these down in **1864**, but amazingly they are covariant!

Writing $F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$ and $j^\mu \equiv (\rho, \vec{J})$ they are

$$\partial_\mu F^{\mu\nu} = j^\nu$$

[Note: the ability to write
Maxwell's Equations in this form
is **not** a proof of covariance!]

Maxwell's equations can also be written in terms of a **potential** A^μ

Writing $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

we have $\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$

Now, notice that I can change A_μ by a derivative of a scalar and leave $F_{\mu\nu}$ unchanged

$$A^\mu \rightarrow A^\mu + \lambda \partial^\mu \phi \quad \Rightarrow \quad F^{\mu\nu} = F^{\mu\nu} + \lambda \partial^\mu \partial^\nu \phi - \lambda \partial^\nu \partial^\mu \phi = F^{\mu\nu}$$

Choose λ such that

$$\partial_\mu A^\mu = 0$$

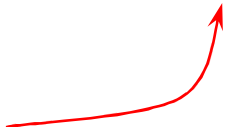
This is a **gauge transformation**, and the choice $\partial_\mu A^\mu = 0$ is known as the **Lorentz gauge**.

In this gauge:

$$\partial^2 A^\mu = j^\mu$$

The wave equation with no source, $\partial^2 A^\nu = 0$ has solutions

$$A^\mu = \epsilon^\mu e^{iq \cdot x} \quad \text{with} \quad q^2 = 0$$

polarisation
vector 

The Lorentz condition ($\partial_\mu A^\mu = 0$) $\Rightarrow$ $q_\mu \epsilon^\mu = 0$

So ϵ^μ has only 3 degrees of freedom (two transverse d.o.f. and one longitudinal d.o.f.)

We still have some freedom to change A^μ , even after our Lorentz gauge choice:

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad \text{is OK, as long as} \quad \partial^2 \chi = 0$$

Usually we choose χ such that $\vec{\nabla} \cdot \vec{A} = 0$. This is known as the **Coulomb gauge**.

$$\vec{\nabla} \cdot \vec{A} = 0 \quad \Rightarrow \quad \vec{q} \cdot \vec{\epsilon} = 0$$

So only two polarisation states remain (both transverse).

The Dirac Equation in an Electromagnetic Field

So far, this has been entirely classical. So how do we incorporate electromagnetism into the quantum Dirac equation?

We do the 'obvious' thing and replace the momentum operator $i\partial^\mu \rightarrow i\partial^\mu - eA^\mu$

charge of the electron = -e

Then $(i\gamma^\mu\partial_\mu - e\gamma_\mu A^\mu - m)\psi = 0$

Beware: conventions differ,
e.g. Halzen and Martin have
 $i\partial^\mu \rightarrow i\partial^\mu + eA^\mu$
while Peskin & Schroeder
have as above

Often we write

$$D^\mu \equiv \partial^\mu + ieA^\mu$$

D^μ is called the **Covariant Derivative**, and the Dirac Equation in an electromagnetic field becomes

$$(i\not{D} - m)\psi = 0$$

The Magnetic Moment of the Electron

We saw that the interaction of an electron with an electromagnetic field is given by

$$(i\gamma^\mu \partial_\mu - e\gamma_\mu A^\mu - m)\psi = 0$$

Writing $\psi \propto \begin{pmatrix} \chi \\ \phi \end{pmatrix}$ as before,
$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A}) \\ \vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A}) & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

[Coulomb gauge $\Rightarrow A^0=0$]

So
$$\phi = \frac{\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})}{E + m} \chi \quad \text{and} \quad \left(E - m + \frac{[\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})]^2}{E + m} \right) \chi = 0$$

Also,

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k \Rightarrow [\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})]^2 = |-i\vec{\nabla} - e\vec{A}|^2 - e(\vec{\nabla} \times \vec{A} + \vec{A} \times \vec{\nabla}) \cdot \vec{\sigma}$$

$$\vec{\nabla} \times \vec{A} \psi + \vec{A} \times \vec{\nabla} \psi = (\vec{\nabla} \times \vec{A}) \psi - \vec{A} \times (\vec{\nabla} \psi) + \vec{A} \times (\vec{\nabla} \psi) = (\vec{\nabla} \times \vec{A}) \psi = \vec{B} \psi$$

So we have,
$$\left(E - m + \frac{|\vec{p} - e\vec{A}|^2 - e\vec{B} \cdot \vec{\sigma}}{E + m} \right) \chi = 0$$

In the non-relativistic limit, $E \approx m$ and $\phi \approx \frac{\vec{\sigma} \cdot (\vec{p} - e\vec{A})}{2m} \chi \ll \chi$, so we can write the Dirac equation as approximately:

$$\frac{1}{2m} |\vec{p} - e\vec{A}|^2 \psi - \frac{e\vec{B} \cdot \vec{\Sigma}}{2m} \psi = 0$$

This is an **magnetic moment** interaction $-\vec{\mu} \cdot \vec{B}$ with $\vec{\mu} = -\frac{e}{2m} \vec{\Sigma}$

The magnetic moment $\vec{\mu}$ is composed of a contribution from the orbital angular momentum,

$$\vec{\mu}_L = -\frac{e}{2m} \vec{L}$$

and the intrinsic spin angular momentum

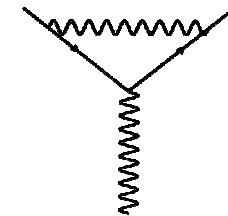
gyromagnetic ratio $\vec{\mu}_s = -g \frac{e}{2m} \vec{s}$ $\vec{s} = \frac{1}{2} \vec{\Sigma}$

The Dirac equation predicts a gyromagnetic ratio **$g = 2$**

We can compare this with experiment: **$g_{\text{exp}} = 2.0023193043738 \pm 0.0000000000082$**

The discrepancy of $g-2$ from zero is due to **radiative corrections**

The electron can emit a photon, interact, and reabsorb the photon.



If one does a more careful calculation, including these effects, QED predicts:

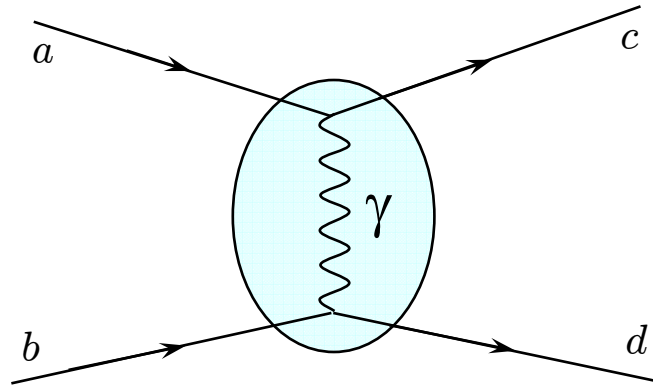
$$\frac{g-2}{2} = 1 + \frac{\alpha}{2\pi} - 0.328 \left(\frac{\alpha}{\pi}\right)^2 + 1.181 \left(\frac{\alpha}{\pi}\right)^3 - 1.510 \left(\frac{\alpha}{\pi}\right)^4 + \dots + 4.393 \times 10^{-12}$$

Theory:	$\frac{g_{\text{th}} - 2}{2} = 1159652140(28) \times 10^{-12}$	}	excellent agreement!
Experiment:	$\frac{g_{\text{exp}} - 2}{2} = 1159652186.9(4.1) \times 10^{-12}$		

The muon's magnetic moment is more interesting because it is more sensitive to new physics.

Scattering and Perturbation theory

Now we have the Dirac equation in an Electromagnetic field we can calculate the scattering of electrons (via electromagnetism) $a + b \rightarrow c + d$



We will assume that the coupling e is small, and that far away from the interaction, i.e. outside the shaded area, the electrons are free particles.

The Dirac Equation (in a field) can be written:

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^i \nabla^i \psi - m\psi - e\gamma^0 V\psi = 0 \quad \text{with} \quad V = \gamma^0 \gamma^\mu A_\mu$$

c.f. the Schrödinger equation in a potential V

[Remember $\gamma^0 \gamma^0 = 1$]

Let's assume that the state at time $t = -\infty$ is an momentum eigenstate $\Psi_{\vec{p}}$ of the free Dirac equation ($V=0$) with energy $E = \sqrt{\vec{p}^2 + m^2}$

$$\text{i.e. } H_0 \Psi_{\vec{p}} = i \frac{\partial}{\partial t} \Psi_{\vec{p}} \quad \text{with} \quad H_0 = -i \gamma^0 \gamma^i \nabla^i + m \gamma^0$$

Our Dirac equation in an external field is $(H_0 + eV) \psi = i \frac{\partial \psi}{\partial t}$

We need to solve this equation for ψ .

Now, since $\Psi_{\vec{p}}$ form a complete set, any solution must be of the form

$$\psi = \int d^3 p \frac{1}{2E_{\vec{p}}} \kappa_{\vec{p}}(t) \Psi_{\vec{p}}(x) \quad \text{with} \quad \int d^3 x \Psi_{\vec{q}}^\dagger \Psi_{\vec{p}} = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q})$$

Let's expand this in powers of e :

$$\psi = \sum_n e^n \int d^3 p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(n)}(t) \Psi_{\vec{p}}(x) \quad \left[\text{i.e. } \kappa_{\vec{p}} = \sum_n \tilde{\kappa}_{\vec{p}}^{(n)} e^n \right]$$

Let's stick this in and see what we get:

$$\left[\psi = \sum_n e^n \int d^3p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(n)}(t) \Psi_{\vec{p}}(x) \right]$$

$$(H_0 + eV) \psi = i \frac{\partial \psi}{\partial t}$$

$$\Rightarrow \sum_n e^n \int d^3p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(n)} \left(\color{red}{H_0} \color{red}{\Psi_{\vec{p}}} + eV \Psi_{\vec{p}} \right) = \sum_n e^n \int d^3p \frac{1}{2E_{\vec{p}}} \left(i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(n)}(t)}{\partial t} \Psi_{\vec{p}} + \tilde{\kappa}_{\vec{p}}^{(n)} \color{red}{i \frac{\partial}{\partial t} \Psi_{\vec{p}}} \right)$$

cancel

To order e^0 :

$$\int d^3p \frac{1}{2E_{\vec{p}}} i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(0)}(t)}{\partial t} \Psi_{\vec{p}} = 0 \quad \text{i.e. } \tilde{\kappa}_{\vec{p}}^{(0)}(t) = \tilde{\kappa}_{\vec{p}}^{(0)}(-\infty)$$

To order e^1 :

$$\int d^3p \frac{1}{2E_{\vec{p}}} i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(1)}(t)}{\partial t} \Psi_{\vec{p}} = \int d^3p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(0)} V \Psi_{\vec{p}}$$

We can now extract $\frac{\partial \tilde{\kappa}_{\vec{p}}^{(1)}}{\partial t}$ using the orthogonality of $\Psi_{\vec{p}}$: $\int d^3x \Psi_{\vec{q}}^\dagger \Psi_{\vec{p}} = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q})$

$$\Rightarrow i \frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}(t)}{\partial t} = \int d^3p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(0)} \int \Psi_{\vec{q}}^\dagger V \Psi_{\vec{p}} d^3x$$

But at time $t = -\infty$ the initial state is $\psi_i = \int d^3p \frac{1}{2E_{\vec{p}}} \kappa_{\vec{p}}(-\infty) \Psi_{\vec{p}} = \Psi_{\vec{p}_i}$,

$$\text{ie. } \tilde{\kappa}_{\vec{p}}^{(0)} \approx \kappa_{\vec{p}}(-\infty) = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{p}_i)$$

$$\Rightarrow i \frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}}{\partial t} = \int \Psi_{\vec{q}}^\dagger V \Psi_{\vec{p}_i} d^3x$$

Integrate over t: $\tilde{\kappa}_{\vec{q}}^{(1)}(t) = \tilde{\kappa}_{\vec{q}}^{(1)}(\text{zero}) - i \int_{-\infty}^t dt' \int d^3x \Psi_{\vec{q}}^\dagger V \Psi_{\vec{p}_i}$

By time $t = \infty$ the interaction has stopped. The probability of finding the system in a state

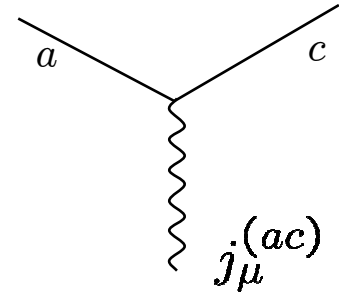
$$\psi_f = \Psi_{\vec{p}_f}(x)$$

is given by $|\kappa_{fi}|^2 \approx |e \tilde{\kappa}_{\vec{p}_f}^{(1)}|^2$ to order e^1 with:

$$\kappa_f \approx -ie \int d^4x \psi_f^\dagger V \psi_i$$

Explicitly putting in our $V = \gamma^0 \gamma^\mu A_\mu$ gives

$$\begin{aligned}\kappa_{ca} &= -ie \int d^4x \psi_c^\dagger (\gamma^0 \gamma^\mu A_\mu) \psi_a \\ &= i \int j_\mu^{(ac)} A^\mu d^4x\end{aligned}$$

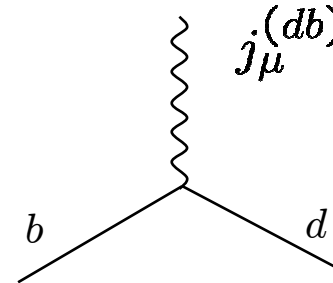


$$j_\mu^{(ac)} = -e \bar{\psi}_c \gamma_\mu \psi_a$$

OK, so now we know the effect of the field A^μ on the electron, but what A^μ does the other electron produce to cause this effect?

$$\partial^2 A^\mu = j_{(db)}^\mu = -e \bar{u}(p_d) \gamma^\mu u(p_b) e^{i(p_d - p_b) \cdot x}$$

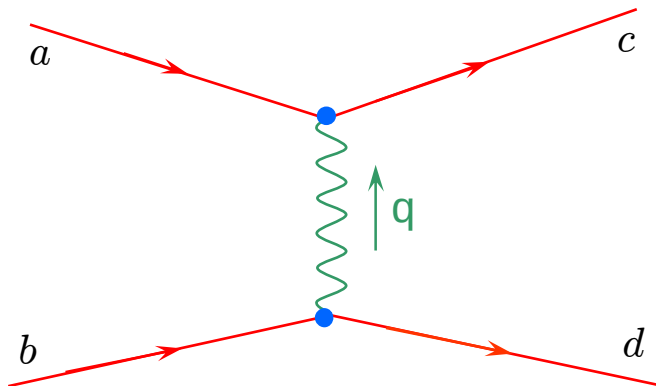
$$\Rightarrow A^\mu = -\frac{1}{q^2} j_{(db)}^\mu \quad (q = p_d - p_b)$$



$$j_\mu^{(db)} = -e \bar{\psi}_d \gamma_\mu \psi_b$$

Putting this all together:

$$\begin{aligned}
 \kappa_{fi} &= -i \int j_\mu^{(ac)} \frac{1}{q^2} j_{db}^\mu d^4x \\
 &= i \bar{u}(p_c) (-e\gamma_\mu) u(p_a) \left(-\frac{1}{q^2} \right) \bar{u}(p_d) (-e\gamma^\mu) u(p_b) \underbrace{\int e^{i(-p_a - p_b + p_c + p_d) \cdot x} d^4x}_{(2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)}
 \end{aligned}$$



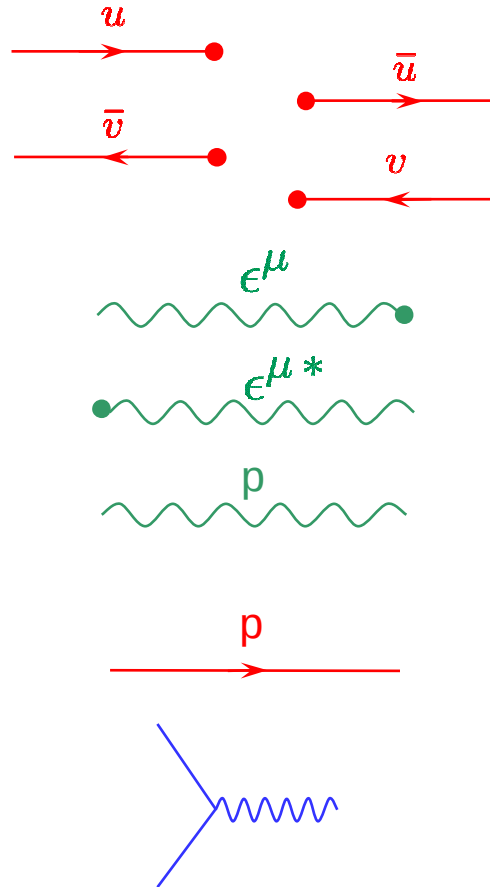
forces momentum conservation

Feynman Diagrams: The QED Feynman Rules

We can construct transition amplitudes simply by associating a mathematical expression with the diagram describing the interaction.

For each diagram, write:

- u for each incoming electron
- $\bar{u}$ for each outgoing electron
- $\bar{v}$ for each incoming positron
- v for each outgoing positron
- ϵ^μ for each incoming photon
- $\epsilon^{\mu*}$ for each outgoing photon
- $-ig_{\mu\nu} \frac{1}{p^2}$ for each internal photon
- $i \frac{\not{p} + m}{p^2 - m^2}$ for each internal electron
- $-ie\gamma^\mu$ for each vertex

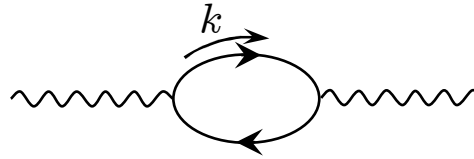


Richard Feynman

Remember that γ -matrices and spinors do not commute, so be careful with the order in spin lines. Write left to right, **against** the fermion flow.

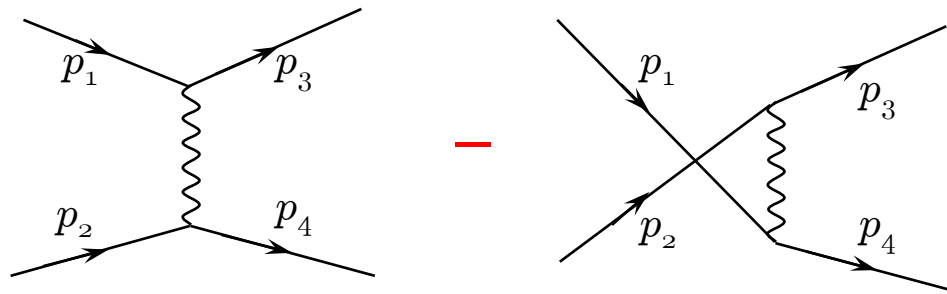
2 details:

- **Closed loops:**



Integrate over loop momentum $\int \frac{d^4 k}{(2\pi)^4}$ and include an extra factor of -1 if it is a fermion loop.

- **Fermi Statistics:** If diagrams are identical except for an exchange of electrons, include a relative – sign.

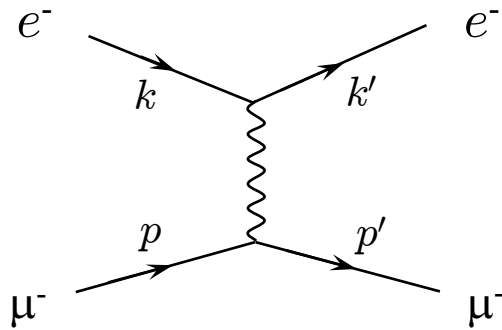


These rules provide $i\mathcal{M}$, and the transition amplitude is

$$\kappa_{fi} = -i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}$$

The probability of transition from initial to final state is $|\kappa_{fi}|^2$

An example calculation: $e^- \mu^- \rightarrow e^- \mu^-$



$$i\mathcal{M} = -e^2 \bar{u}(k') \gamma^\mu u(k) \frac{g_{\mu\nu}}{q^2} \bar{u}(p') \gamma^\nu u(p)$$

This is what we had before.

To get the total probability we must **square** this, **average over initial spins**, and **sum over final spins**.

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{q^4} \frac{1}{4} \sum_{\text{spins}} \left\{ [\bar{u}(k') \gamma^\mu u(k)] [\bar{u}(k') \gamma^\nu u(k)]^* \right\} \left\{ [\bar{u}(p') \gamma_\mu u(p)] [\bar{u}(p') \gamma_\nu u(p)]^* \right\} \\ &= \frac{e^4}{q^4} \mathcal{L}^{\mu\nu}(k, k') \mathcal{L}_{\mu\nu}(p, p') \end{aligned}$$

$$\begin{aligned} \text{But } [\bar{u}(k') \gamma^\mu u(k)]^* &= [u^\dagger(k') \gamma^0 \gamma^\mu u(k)]^\dagger & (\bar{u} \equiv u^\dagger \gamma^0) \\ &= u^\dagger(k) \gamma^\mu \dagger \gamma^0 \dagger u(k') \\ &= u^\dagger(k) \gamma^0 \gamma^\mu \gamma^0 \gamma^0 u(k') & (\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0, \gamma^{0\dagger} = \gamma^0) \\ &= \bar{u}(k) \gamma^\mu u(k') & (\gamma^0 \gamma^0 = 1) \end{aligned}$$

But don't forget that the u are 4-component spinors and the γ are 4×4 matrices:

$$\text{So } \mathcal{L}^{\mu\nu}(k, k') = \frac{1}{2} \sum_{s=1,2} \sum_{s'=1,2} \bar{u}_{\alpha}^{(s')}(k') [\gamma^{\mu}]_{\alpha\beta} u_{\beta}^{(s)}(k) \bar{u}_{\gamma}^{(s)}(k) [\gamma^{\nu}]_{\gamma\delta} u_{\delta}^{(s')}(k')$$

(summation over $\alpha, \beta, \gamma, \delta = 1, 2$)

We can simplify this using the completeness relation for spinors:

$$\sum_{s=1,2} u_{\alpha}^{(s)}(k) \bar{u}_{\beta}^{(s)}(k) = k_{\mu} [\gamma^{\mu}]_{\alpha\beta} + m \delta_{\alpha\beta} \quad \left[\begin{array}{l} \text{beware normalization here – this is only} \\ \text{true for 2E particles per unit volume} \end{array} \right]$$

$$\begin{aligned} \text{Then } \mathcal{L}^{\mu\nu}(k, k') &= \frac{1}{2} \sum_{s'=1,2} u_{\delta}^{(s')}(k') \bar{u}_{\alpha}^{(s')}(k') [\gamma^{\mu}]_{\alpha\beta} \sum_{s=1,2} u_{\beta}^{(s)}(k) \bar{u}_{\gamma}^{(s)}(k) [\gamma^{\nu}]_{\gamma\delta} \\ &= \frac{1}{2} \left(k'_{\rho} [\gamma^{\rho}]_{\delta\alpha} + m \delta_{\delta\alpha} \right) [\gamma^{\mu}]_{\alpha\beta} \left(k_{\sigma} [\gamma^{\sigma}]_{\beta\gamma} + m \delta_{\beta\gamma} \right) [\gamma^{\nu}]_{\gamma\delta} \\ &= \frac{1}{2} \text{Tr} \left[(\not{k}' + m) \gamma^{\mu} (\not{k} + m) \gamma^{\nu} \right] \end{aligned}$$

We need some trace identities!

Trace Identities

$$\text{Tr } 1 = 4$$

$$\text{Tr } \gamma^\mu \gamma^\nu = \frac{1}{2} \text{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \frac{1}{2} \text{Tr} (2g^{\mu\nu}) = g^{\mu\nu} \text{Tr } 1 = 4 g^{\mu\nu}$$

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\lambda = \text{Tr } \gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda = \text{Tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^5 = -\text{Tr } \gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda = 0$$

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa = 4 (g^{\mu\nu} g^{\lambda\kappa} - g^{\mu\lambda} g^{\nu\kappa} + g^{\mu\kappa} g^{\nu\lambda})$$

$$\text{Tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa = -4i\epsilon^{\mu\nu\lambda\kappa}$$

← (be careful with this one!)

↑
[This is true for any **odd**
number of γ -matrices]

Using these identities:

$$\mathcal{L}^{\mu\nu}(k, k') = 2 (k^\mu k'^\nu + k^\nu k'^\mu - (k \cdot k' - m^2) g^{\mu\nu})$$

So

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{q^4} 4 (k^\mu k'^\nu + k^\nu k'^\mu - (k \cdot k' - m_e^2) g^{\mu\nu}) (p_\mu p'_\nu + p_\nu p'_\mu - (p \cdot p' - m_\mu^2) g_{\mu\nu}) \\ &= 8 \frac{e^4}{q^4} ((k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') - m_e^2(p' \cdot p) - m_\mu^2(k' \cdot k) + 2m_e^2 m_\mu^2) \end{aligned}$$

If we are working at sufficiently high energies, then $p^2 \gg m_e^2, m_\mu^2$ and we may ignore the masses.

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \approx 8 \frac{e^4}{(k - k')^4} \left((k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') \right)$$

Often this is written in terms of **Mandlestam Variables**, which are defined:

$$\begin{aligned} s &= (k + p)^2 \approx 2k \cdot p \approx 2k' \cdot p' \\ t &= (k - k')^2 \approx -2k \cdot k' \approx -2p \cdot p' \\ u &= (k - p')^2 \approx -2k \cdot p' \approx -2k' \cdot p \end{aligned}$$

[Note that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \approx 0$]

Then

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \frac{s^2 + u^2}{t^2}$$

Cross-sections

So we have $|\mathcal{M}|^2$ but we are not quite there yet – we need to turn this into a cross-section.

Recall

$$\kappa_{fi} = -i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}$$

$$\begin{aligned} \Rightarrow |\kappa_{fi}|^2 &= \left[(2\pi)^4 \delta^4(p_f - p_i) \right]^2 |\mathcal{M}|^2 \\ &= (2\pi)^4 \delta^4(p_f - p_i) VT |\mathcal{M}|^2 \end{aligned}$$

$$\text{since } (2\pi)^4 \delta^4(p_f - p_i) = \int d^4x e^{i(p_f - p_i) \cdot x} = \int_{-T/2}^{T/2} dt \int_V d^3x = VT$$

But we need the transition probability per unit time and per unit volume is:

$$\frac{|\kappa_{fi}|^2}{VT} = (2\pi)^4 \delta^4(p_f - p_i) |\mathcal{M}|^2$$

The **cross-section** is the probability of transition per unit volume, per unit time $\times$ the number of final states / initial flux.

$$d\sigma = \frac{|\kappa_{fi}|^2}{VT} \times \frac{\# \text{ final states}}{\text{initial flux}}$$

Initial Flux

In the lab frame, particle A, moving with velocity $\vec{v}_A$, hits particle B, which is stationary.



The number of particles like A
in the beam, passing through
volume V per unit time is $\left\{ |\vec{v}_A| \frac{2E}{V} \right\}$

The number of particles like B
per volume V in the target is $\left\{ \frac{2E}{V} \right\}$

So the initial flux in a volume V is $\frac{1}{V^2} |\vec{v}_A| 2E_A 2E_B$

For a **collider**, where A and B are both moving, this becomes:

$$\frac{1}{V^2} |\vec{v}_A - \vec{v}_B| 2E_A 2E_B = \frac{4}{V^2} \sqrt{[(p_A \cdot p_B)^2 - m_A^2 m_B^2]}$$

final states

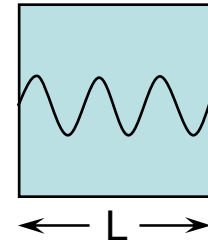
How many states of momentum $\vec{p}$ can we fit in a volume V ?

In order to not have any particle flow through the boundaries of the box, we must impose **periodic boundary conditions**.

$Lp_x = 2\pi n$ so the number of states between p_x and $p_x + dp_x$ is $\frac{L}{2\pi} dp_x$

So in a volume V we have

$$\left(\frac{L}{2\pi} dp_x\right) \left(\frac{L}{2\pi} dp_y\right) \left(\frac{L}{2\pi} dp_z\right) = \frac{V}{(2\pi)^3} d^3p$$



But there are $2EV$ particles per volume V , so

$$\text{\# final states per particle} = \frac{1}{(2\pi)^3} \frac{d^3p}{2E}$$

Note that $\int \frac{1}{(2\pi)^3} \frac{d^3p}{2E} = \int \frac{d^4p}{(2\pi)^4} 2\pi \delta(p^2 - m^2)$ so this is covariant!

Putting all this together, the **differential cross-section** is:

$$d\sigma = \frac{1}{F} |\mathcal{M}|^2 d\text{Lips}$$

where the **Flux** F is given by,

$$F = 4\sqrt{[(p_A \cdot p_B)^2 - m_A^2 m_B^2]}$$

and the **Lorentz invariant phase space** is,

$$d\text{Lips} = (2\pi)^4 \delta^4(p_c + p_d - p_a - p_b) 2\pi \delta(p_c^2 - m_c^2) 2\pi \delta(p_d^2 - m_d^2) \frac{d^4 p_c}{(2\pi)^4} \frac{d^4 p_d}{(2\pi)^4}$$

momentum conservation

on-shell conditions

integration measure

In the **centre-of-mass**, this becomes much simpler

This frame is defined by $\vec{p}_a = -\vec{p}_b$ and $E_a + E_b = \sqrt{s}$
 $\left(\begin{array}{l} \text{Remember} \\ s \equiv (p_a + p_b)^2 \end{array} \right)$

So $p_a = (E_a, \vec{p}_a)$ and $p_b = (E_b, -\vec{p}_a)$

with $|\vec{p}_a| = \frac{1}{4s} \sqrt{\lambda(s, m_a^2, m_b^2)}$ $\lambda(\alpha, \beta, \gamma) \equiv \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma$

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \quad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}$$

Then the **Flux** becomes
$$F = 4\sqrt{[(p_a \cdot p_b)^2 - m_a^2 m_b^2]}$$
$$= 4(|\vec{p}_a|E_b + |\vec{p}_b|E_a)$$

$$F = 4|\vec{p}_a|\sqrt{s}$$

Also $\vec{p}_c = -\vec{p}_d$ and $E_c + E_d = \sqrt{s}$ with relations analogous to those for p_a and p_b

The phase space measure becomes:

$$\begin{aligned}
 d\text{Lips} &= (2\pi)^4 \delta^4(p_c + p_d - p_a - p_b) \frac{1}{2E_c} \frac{d^3\vec{p}_c}{(2\pi)^3} \frac{1}{2E_d} \frac{d^3\vec{p}_d}{(2\pi)^3} \\
 &= \frac{1}{4\pi^2} \delta(E_c + E_d - \sqrt{s}) \frac{1}{4E_c E_d} d^3\vec{p}_c \\
 &= \frac{1}{4\pi^2} \delta(E_c + E_d - \sqrt{s}) \frac{1}{4E_c E_d} |\vec{p}_c|^2 d|\vec{p}_c| d\Omega \quad \left(\text{since } \frac{d|\vec{p}_c|}{d\sqrt{s}} = \frac{E_c E_d}{|\vec{p}_c| \sqrt{s}} \right) \\
 &= \frac{1}{16\pi^2} \delta(\sqrt{s} - E_c - E_d) \frac{|\vec{p}_c|}{\sqrt{s}} d\sqrt{s} d\Omega
 \end{aligned}$$

$$d\text{Lips} = \frac{1}{16\pi^2} \frac{|\vec{p}_c|}{\sqrt{s}} d\Omega$$

Putting this together:

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{CM}} = \frac{1}{64\pi^2} \frac{|\vec{p}_c|}{|\vec{p}_a|} \frac{1}{s} |\mathcal{M}|^2$$

Returning to our process $e^- \mu^- \rightarrow e^- \mu^-$

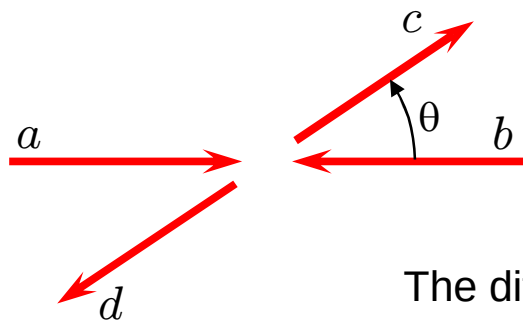
[With $m_e = m_\mu = 0$]

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \sum_{\text{spins}} \frac{1}{64\pi^2} \frac{|\vec{p}_c|}{|\vec{p}_a|} \frac{1}{s} |\mathcal{M}|^2 = \frac{1}{64\pi^2} \frac{1}{s} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{32\pi^2} \frac{1}{s} \frac{s^2 + u^2}{t^2} = \frac{\alpha^2}{2s} \frac{s^2 + u^2}{t^2}$$

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \frac{s^2 + u^2}{t^2}$$

The fine structure constant $\alpha \equiv \frac{e^2}{4\pi}$

In terms of the angle between a and c



$$t \equiv (p_a - p_c)^2 = -2p_a \cdot p_c = -\frac{s}{2}(1 - \cos \theta)$$

$$u \equiv (p_a - p_d)^2 = -2p_a \cdot p_d = -\frac{s}{2}(1 + \cos \theta)$$

The differential cross-section is:

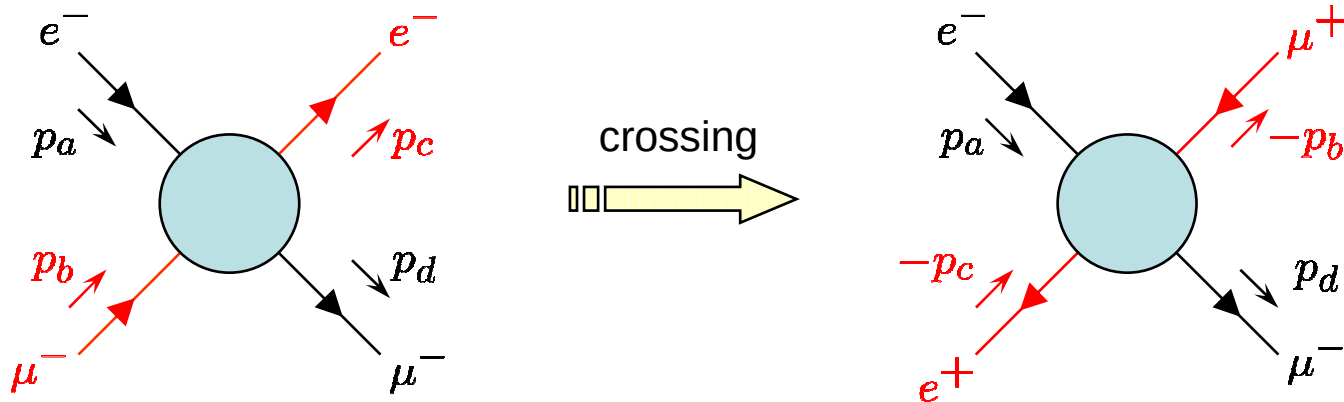
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8s} \frac{4 + (1 + \cos \theta)^2}{(1 - \cos \theta)^2}}$$

Notice that this is divergent for small angles: $\frac{d\sigma}{d\Omega} \sim \frac{4\alpha^2}{s} \frac{1}{\theta^4}$ as $\theta \rightarrow 0$

This is exactly the same divergence as is in the Rutherford scattering formula.

Crossing symmetry

Generally, in a Feynman diagram, any incoming particle with momentum p is equivalent to an outgoing antiparticle with momentum $-p$.

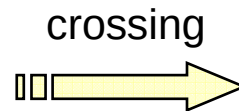


This lets us use our result for $e^- \mu^- \rightarrow e^- \mu^-$ to easily calculate the differential cross-section for $e^+ e^- \rightarrow \mu^+ \mu^-$.

$$s = (p_a + p_b)^2$$

$$t = (p_a - p_c)^2$$

$$u = (p_a - p_d)^2$$



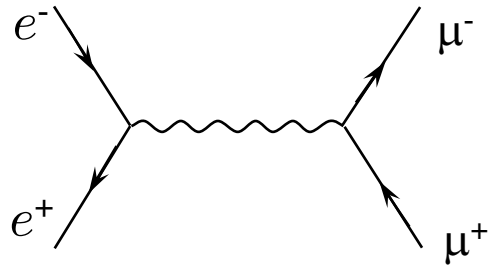
$$s = (p_a - p_c)^2$$

$$t = (p_a + p_d)^2$$

$$u = (p_a - p_d)^2$$

i.e.

$$s \leftrightarrow t$$



$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \frac{t^2 + u^2}{s^2} \Rightarrow$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \frac{t^2 + u^2}{s^2}$$

Be careful not to change the s from flux and phase space!

Writing θ as the angle between the e^- and μ^- ,

$$t = -\frac{s}{2}(1 - \cos \theta),$$

$$u = -\frac{s}{2}(1 + \cos \theta) \quad \left[\text{as before} \right]$$

$\Rightarrow$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta)$$

Notice the singularity is gone!

The total cross-section is

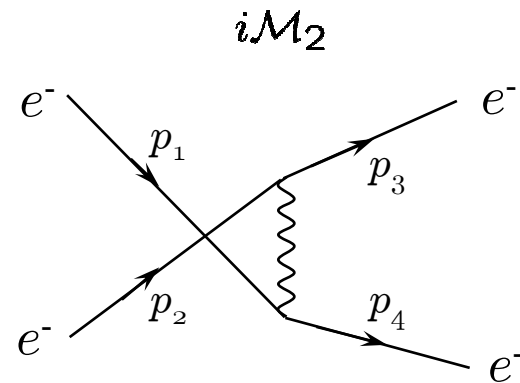
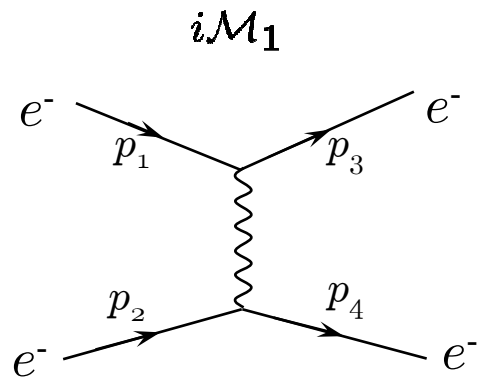
$$\sigma_{\text{Tot}} = \frac{\alpha^2}{4s} 2\pi \int_{-1}^{+1} (1 + \cos^2 \theta) d(\cos \theta) = \frac{4\pi\alpha^2}{3s}$$

Identical particles in initial or final state

So far, in the reactions we have looked at, the final state particles have all been **distinguishable** from one another. If the final state particles are **identical**, we have additional Feynman diagrams.

e.g. $e^- e^- \rightarrow e^- e^-$

[See Feynman rules]



$$ie^2 \frac{1}{t} \bar{u}(p_c) \gamma^\mu u(p_a) \bar{u}(p_d) \gamma_\mu u(p_b)$$

$$-ie^2 \frac{1}{u} \bar{u}(p_d) \gamma^\mu u(p_a) \bar{u}(p_c) \gamma_\mu u(p_b)$$

interchange of identical
fermions $\Rightarrow$ minus sign

p_c and p_d interchanged

Since the final state particles are identical, these diagrams are **indistinguishable** and must be summed **coherently**.

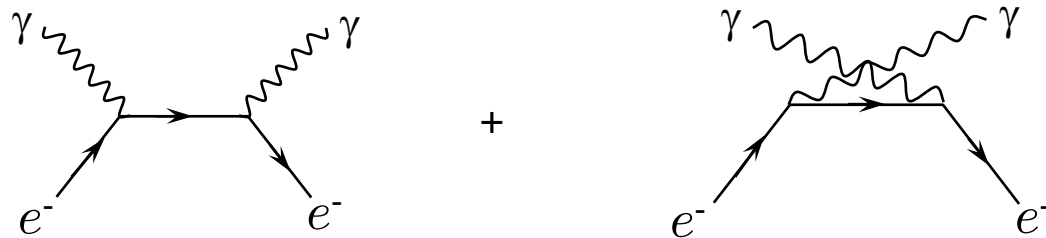
$$|\mathcal{M}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2 \Re \mathcal{M}_1 \mathcal{M}_2^*$$

We have interference between the two contributions.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left(\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + 2 \frac{s^2}{tu} \right)$$

Compton Scattering and the fermion propagator

Compton scattering is the scattering of a photon with an electron.



I just quoted the Feynman rule for the **fermion propagator**, but where did it come from?

Let's go back to the photon propagator first.

Recall the photon propagator is $-ig^{\mu\nu} \frac{1}{p^2}$

$$\epsilon^{\mu*} \quad -\frac{1}{p^2} \quad \epsilon^{\nu}$$

The $-\frac{1}{p^2}$ is the inverse of the photon's wave equation:

$$\partial^2 A^\mu = -p^2 A^\mu = j^\mu \quad \Rightarrow \quad A^\mu = -\frac{1}{p^2} j^\mu$$

The $g^{\mu\nu}$ is coming from summing the photon polarization vectors over spins:

$$\sum_{\lambda=1}^4 \epsilon_{(\lambda)}^{\mu*} \epsilon_{(\lambda)}^{\nu} = -g^{\mu\nu} \quad \longleftarrow \quad \text{this is for virtual photons}$$

So, the photon propagator is then $-i \left(-\frac{1}{p^2} \right) \sum_{\lambda=1}^4 \epsilon_{(\lambda)}^{\mu*} \epsilon_{(\lambda)}^{\nu} = -ig^{\mu\nu} \frac{1}{p^2}$

For a **massless** fermion propagator we follow the same procedure

$$\overline{u} \quad \xrightarrow{-\frac{1}{p^2}} \quad u$$

The massless fermion spin sum is

$$\sum_{s=1}^2 u_{(s)} \bar{u}_{(s)} = \not{p}$$

so the massless fermion propagator is

$$-i \left(-\frac{1}{p^2} \right) \sum_{s=1}^2 u_{(s)} \bar{u}_{(s)} = i \not{p} \frac{1}{p^2} \quad \left(\text{sometimes written } \frac{i}{\not{p}} \right)$$

But what about massive propagators?

Lets think about a **massive scalar** propagator since it is easier (we can forget the spin-sum).

$$(\partial^2 + m^2) \phi = 0$$

We can consider the mass term as a perturbation on the 'free' (i.e. massless) theory.

$$\begin{aligned}
 \partial^2 \phi &= -V \phi \quad \text{with} \quad V = m^2 \\
 \text{---} \bullet \text{---} &= \text{---} \frac{i}{p^2} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots \\
 &\quad \frac{i}{p^2} \quad \frac{i}{p^2} (-im^2) \frac{i}{p^2} \quad \frac{i}{p^2} (-im^2) \frac{i}{p^2} (-im^2) \frac{i}{p^2} \\
 &= \frac{i}{p^2} \left(1 + \frac{m^2}{p^2} + \frac{m^4}{p^4} + \dots \right) = \frac{i}{p^2} \left(\frac{1}{1 - \frac{m^2}{p^2}} \right) = \frac{i}{p^2 - m^2}
 \end{aligned}$$

So the **massive scalar** propagator is $\frac{i}{p^2 - m^2}$

The **Klein-Gordon** equation $(\partial^2 + m^2) \phi = 0$ leads to a propagator $\frac{i}{p^2 - m^2}$

The same procedure on the **Dirac equation** $(i\not{\partial} - m) \psi = 0$ gives a propagator

$$\frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2}$$

More precisely, the propagator is the momentum space Fourier transform of the wave equation's **Greens function**

Green's function S obeys: $(i\not{\partial} - m)S(x - y) = i\delta^{(4)}(x - y)$

Writing $S(x - y) = \int \frac{d^4p}{(2\pi)^4} \tilde{S}(p) e^{-ip \cdot (x-y)}$, and pre-multiplying by $(i\not{\partial} - m)$

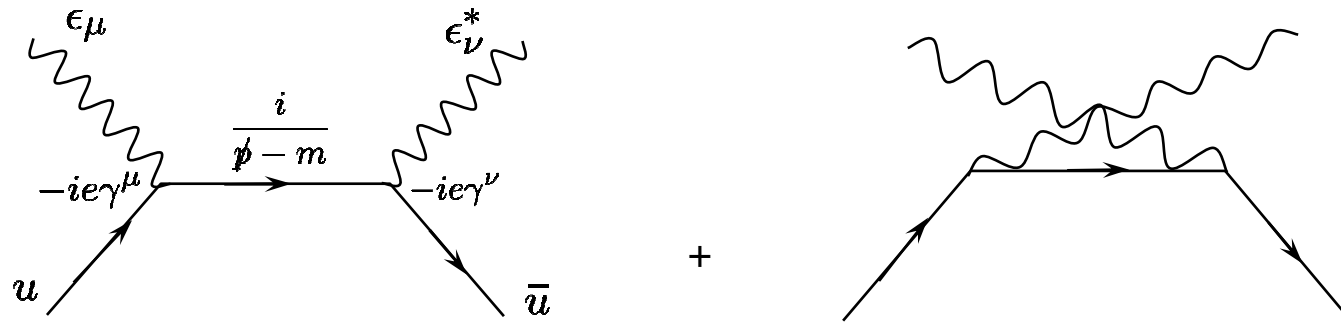
gives $\int \frac{d^4p}{(2\pi)^4} (\not{p} - m) \tilde{S}(p) e^{-ip \cdot (x-y)} = i\delta^{(4)}(x - y)$

$\Rightarrow$

$$\tilde{S}(p) = \frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2}$$

More details in your QFT course!

So now we are armed with enough information to calculate **Compton Scattering**



Putting in the Feynman rules, and following through, with $m_e = 0$

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = -2e^4 \left(\frac{u}{s} + \frac{s}{u} \right)$$

Can you reproduce this? You will need to use $\sum_{\text{spins}} \epsilon_{(T)}^{\mu*} \epsilon_{(T)}^\nu \rightarrow -g^{\mu\nu}$

Decay Rates

So far we have only looked at $2 \rightarrow 2$ processes, but what about decays?

A decay width is given by:

$$d\Gamma = \frac{|k_{fi}|^2}{VT} \times \frac{\text{\# final states}}{\text{\# of decaying particles per unit volume}}$$

This replaces the Flux.

For a decay $a \rightarrow b + c$ we have

$$\frac{|k_{fi}|^2}{VT} = (2\pi)^2 \delta^4(p_a - p_b - p_c) |\mathcal{M}|^2$$

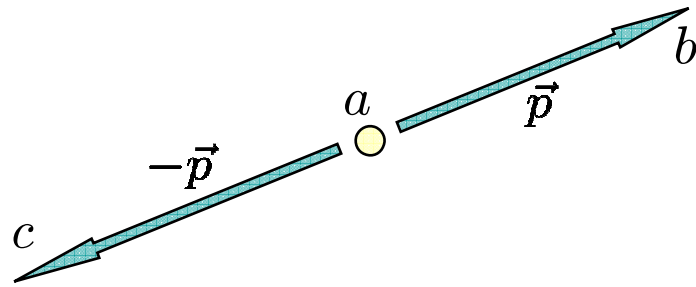
$$\text{\# final states} = \frac{1}{2E_b} \frac{d^3\vec{p}_b}{(2\pi)^3} \frac{1}{2E_c} \frac{d^3\vec{p}_c}{(2\pi)^3} = \frac{d^4p_b}{(2\pi)^4} \frac{d^4p_c}{(2\pi)^4} (2\pi) \delta(p_b^2 - m_b^2) (2\pi) \delta(p_c^2 - m_c^2)$$

$$\text{\# of decay particles per unit volume} = \frac{1}{2E_a}$$

$\Rightarrow$

$$d\Gamma = \frac{1}{2E_a} |\mathcal{M}|^2 d\text{Lips}$$

In the rest frame of particle a : $p_a = (m_a, \vec{0})$, $p_b = (E_b, \vec{p})$, $p_c = (E_c, -\vec{p})$,



with $E_b = \sqrt{|\vec{p}|^2 + m_b^2}$, $E_c = \sqrt{|\vec{p}|^2 + m_c^2}$

The decay is back-to-back

$$d\Gamma = \frac{1}{2m_a} |\mathcal{M}|^2 \frac{1}{4E_b E_c} \frac{1}{(2\pi)^2} \delta(m_a - E_b - E_c) |\vec{p}|^2 d|\vec{p}| d\Omega$$

But $|\vec{p}| d|\vec{p}| = E_b dE_b = E_c dE_c = \frac{E_b E_c}{E_b + E_c} d(E_b + E_c)$

$$\Gamma = \int \frac{1}{8m_a} |\mathcal{M}|^2 \frac{1}{(2\pi)^2} \delta(m_a - E_b - E_c) \frac{|\vec{p}|}{E_b + E_c} d(E_b + E_c) d\Omega$$

$$\Gamma = \int \frac{1}{32m_a^2 \pi^2} |\mathcal{M}|^2 |\vec{p}| d\Omega$$

Remember, to get the total decay rate, you need to sum over all possible decay processes.

$$\Gamma_{\text{Tot}} = \sum_i \Gamma_i$$

The inverse of the total width Γ_{Tot}^{-1} will give the **lifetime** of the particle:

If the number of particles = N_a then,

$$\Gamma_{\text{Tot}} = -\frac{1}{N_a} \frac{dN_a}{dt} \quad \Rightarrow \quad N_a(t) = N_a(0)e^{-\Gamma_{\text{Tot}}t}$$

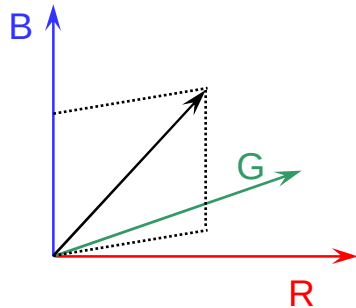
Quantum Chromo Dynamics (QCD)

Quarks, Gluons and Color

QCD describes the interaction of **quarks** and **gluons**.

It is very similar to QED, except we have 3 types of 'charge' instead of just one.

Conventionally we call these charges **red**, **green** and **blue**, and each quark can be written as a vector in "color space":



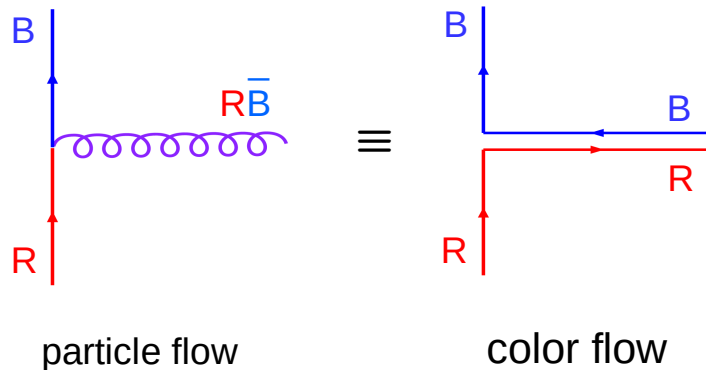
$$q = \begin{pmatrix} q^R \\ q^G \\ q^B \end{pmatrix}$$

However, QCD is symmetric under rotations in this color-space, so we can always rotate the quarks to pure color states and say they are either red, green or blue.

This symmetry is known as $SU(3)_{\text{color}}$, and parallels the $U(1)_{\text{QED}}$ symmetry of QED.

The force between the quarks is mediated by **gluons** which can also change the color of the quarks.

Since we have **3** different sorts of quark (red, green and blue), to connect them all together we naively need **$3 \times 3 = 9$** different gluons.



Since we are connecting together quarks of different color, the gluons must be colored too.

So, for example, we could have gluons:

$R\bar{B}$, $R\bar{G}$, $G\bar{R}$, $G\bar{B}$, $B\bar{R}$, $B\bar{G}$ + three orthogonal combinations of $R\bar{R}$, $G\bar{G}$, $B\bar{B}$

Conventionally these last 3 are

$$\frac{1}{\sqrt{2}} (R\bar{R} - B\bar{B}), \frac{1}{\sqrt{6}} (R\bar{R} + B\bar{B} - 2G\bar{G}), \frac{1}{\sqrt{3}} (R\bar{R} + B\bar{B} + G\bar{G}).$$

Since QCD is symmetric to rotations in color-space, the first 8 of these must have related couplings. However, the last one is a color singlet, so in principle can have an arbitrary coupling. In QCD, its coupling is zero.

$\Rightarrow$ We have 8 gluons

In order to transform one quark color-vector onto another, we need eight 3×3 matrices.

For example to turn a red quark into a blue quark we need a $R\bar{B}$ gluon represented by

$$T^{R\bar{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The above matrix is not a very convenient choice (it is actually a ladder operator). Instead we normally write T^A in terms of the **Gell-Mann λ matrices**.

$$T^A = \frac{1}{2}\lambda^A$$

These matrices are generators of the SU(3) group and obey the **SU(3) algebra**,

$$[T^A, T^B] = if^{ABC}T^C$$

SU(3) structure constants

are conventionally normalised by $\text{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$ and are **traceless**.

[Hence the removal of $\frac{1}{\sqrt{3}}(R\bar{R} + B\bar{B} + G\bar{G})$]

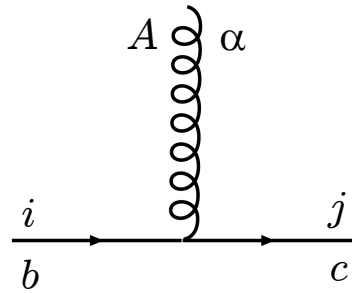
The Gell-Mann matrices are:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

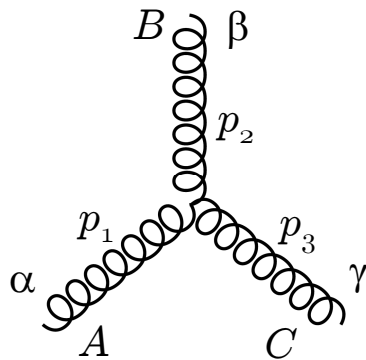
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

At a vertex between quark and gluons we need to include a factor



$$-ig_s T_{cb}^A \gamma_{ji}^\alpha$$

The **gluons also carry color**, so we must also include a gluon-gluon interaction. This is given by



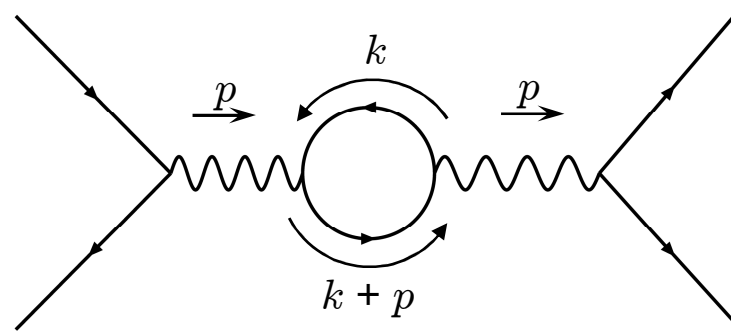
$$-g_s f^{ABC} \left((p_1 - p_2)^\gamma g^{\alpha\beta} + (p_2 - p_3)^\alpha g^{\beta\gamma} + (p_3 - p_1)^\beta g^{\gamma\alpha} \right)$$

The full QCD Feynman rules will be given to you in the Standard Model course.

Renormalisation

When we calculate beyond leading order in our perturbative expansion, we will find that we have diagrams with loops in them.

For example, the corrections to our $e^+e^- \rightarrow \mu^+\mu^-$ would include the diagram



But momentum conservation at all vertices leaves the momentum flowing around the loop unconstrained! We need to integrate over this loop momentum, and find a result proportional to

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2] [(k - p)^2 - m^2]}$$

This integral is infinite!

To see that it is infinite, let's look at this integral in the limit as $k \rightarrow \infty$. Then we can neglect the momentum p and the mass m . The integral becomes

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \int_0^\infty dk \frac{1}{k} = \log \infty - \log 0$$

Ultra-Violet (UV) singularity

this is a fake, because our approximation doesn't work for $k \rightarrow 0$

This is not really that surprising. Even in classical electromagnetism we have singularities when we go to small distances/high energies.

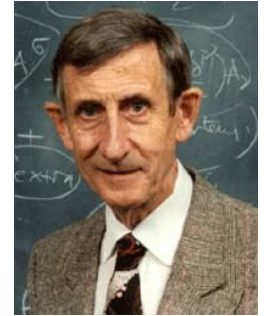
For example, in classical electromagnetism, the energy associated with a charged sphere of radius R is:

$$\frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 R} \rightarrow \infty \quad \text{as } R \rightarrow 0$$

So classically, a point charge should have infinite energy!

Are infinities really a problem?

Our theories such as QED and QCD make **predictions of physical quantities**. While infinities may make the theory difficult to work with, there is no real problem as long as our predictions of physical quantities are finite and match experiment.



Freeman Dyson

We find, that in both QED and QCD, that our physical observables are finite: they are **renormalizable** theories.

To understand this, let's think about the one-loop calculation of the electron mass

$$\text{Feynman diagram with a loop} = \text{Feynman diagram with a straight line} + \text{Feynman diagram with a loop} + \mathcal{O}(e^4)$$
$$m = m_0 + e^2 m_1 + \mathcal{O}(e^4)$$

finite $\swarrow$ $\nwarrow$ $\nwarrow$
 $-\infty$ ∞

In order for the physically measured mass to be **finite**, the **bare mass** must be **infinite** and cancel the divergence from the loop.

We absorb infinities into unmeasurable bare quantities.

In reality, what we are doing is measuring differences between quantities.

$$\begin{array}{c} Q^2 \\ \longrightarrow \bullet \longrightarrow \end{array} = \longrightarrow + \begin{array}{c} \text{wavy line} \\ \longrightarrow \end{array} + \mathcal{O}(e^4)$$

Since the loop contains a dependence on the momentum scale, Q , the mass changes with probed energy. The **difference** between two masses at different scales is:

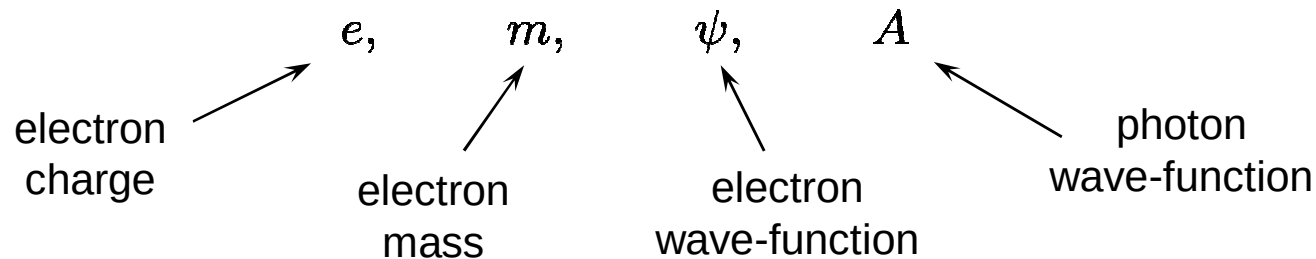
$$\begin{aligned}
 m(Q_1^2) - m(Q_2^2) &= (m_0 + e^2 m_1(Q_1^2) + \mathcal{O}(e^4)) - (m_0 + e^2 m_1(Q_2^2) + \mathcal{O}(e^4)) \\
 &= \cancel{m_0 - m_0} + e^2 \underbrace{(m_1(Q_1^2) - m_1(Q_2^2))}_{\substack{\text{infinities are the same in both } m_1\text{'s} \\ \Rightarrow \text{finite}}} + \mathcal{O}(e^4)
 \end{aligned}$$

The difference between the masses is **finite**.

Both philosophies, absorption or subtraction of singularities, are doing the same thing. We replace the **infinite bare quantities** in the Lagrangian with **finite physical ones**. This is called **renormalization**.

The beauty of QED (and QCD) is that we don't need to do this for every observable (which would be rather useless). Once we have done it for certain observables, everything is finite! This is a very non-trivial statement. We say that QED and QCD are **renormalizable**.

In QED we choose to absorb the divergences into:



Instead of writing our observables in terms of the infinite bare quantities $e_0, m_0, \psi_0, A_0,$ we write them in terms of the physical measurable 'renormalized' quantities e_R, m_R, ψ_R, A_R

In order to do this, we must first **regularize** the divergences in our integrals.

Regularization by a Momentum cut-off

The most obvious regularization is to simply forbid any momenta above an scale Λ . Then, the integral becomes

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4} \sim \int_0^\Lambda dk \frac{1}{k} = \log 0 - \log \Lambda$$

The UV divergence has been **regularized**. This isn't very satisfactory though, since this breaks gauge invariance.

Dimensional Regularization

The most usual way to regulate the integrals is to work in $d = 4 - 2\epsilon$ dimensions rather than 4 dimensions.

$$\int \frac{d^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}} \frac{1}{k^4} \sim \int_0^\infty k^{3-2\epsilon} dk \frac{1}{k^4} = \int_0^\infty dk \frac{1}{k^{1+2\epsilon}} = \left[-\frac{k^{-2\epsilon}}{2\epsilon} \right]_0^\infty$$

we have increased the power of k in the denominator, making the integral finite

More precisely, our original integral (ignoring masses for simplicity) gives:

$$\int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{1}{k^2} \frac{1}{(k+p)^2} = \underbrace{i(4\pi)^{\epsilon-2} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}}_{\text{finite}} \underbrace{\frac{(-p^2)^{-\epsilon}}{\epsilon(1-2\epsilon)}}_{\text{divergent as } \epsilon \rightarrow 0}$$

Notice that it is rather arbitrary which bit one wants to absorb or subtract off.

One could subtract off only the pole in epsilon, i.e. $\frac{i}{(4\pi)^2} \frac{1}{\epsilon}$ for the above integral.

This is known as the **Minimal Subtraction**, denoted **MS**.

Alternatively we could have removed some of the finite terms too,

$$\text{e.g. } i(4\pi)^{\epsilon-2} \Gamma(1+\epsilon) \frac{(p^2/Q^2)^{-\epsilon}}{\epsilon} = \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + \gamma_E - \log(p^2/Q^2) \right)$$

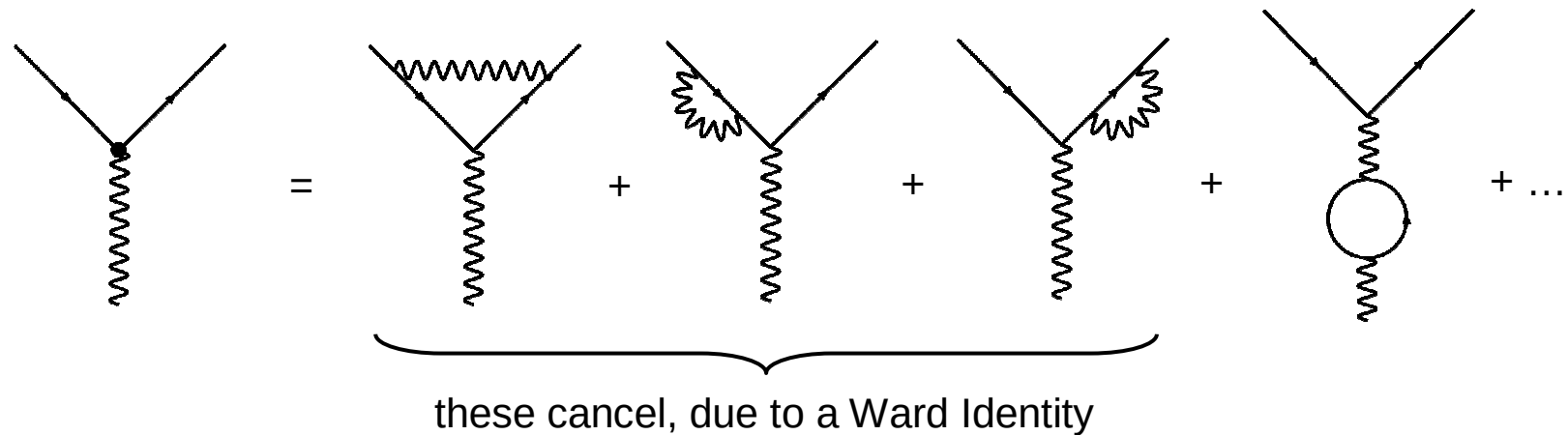
 [Euler-Mascheroni Constant]

This choice is known as **$\overline{\text{MS}}$**

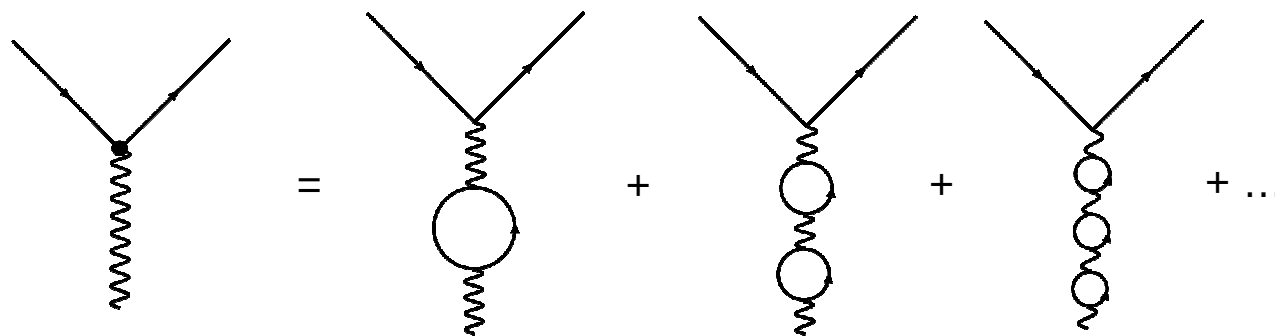
Also notice the **renormalization scale Q**.

Running couplings

How does the QED coupling e change with quantum corrections?



I can include some extra loops by....



$$e(Q) = e_0 \left(\frac{1}{1 + I(Q)} \right)$$

$$\left[I(Q) = \text{diagram of a fermion loop with two external photon lines} \right]$$

In terms of $\alpha \equiv \frac{e^2}{4\pi}$, we find

$$\alpha(Q) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log(Q^2/\Lambda^2)}$$


but since this was general, I could have chosen to evaluate my coupling at a different scale

e.g.

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log(\mu^2/\Lambda^2)}$$

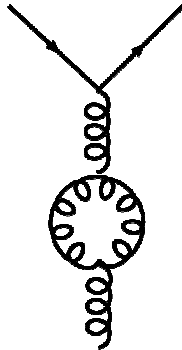
I can use this second equation to eliminate α_0 (which is infinite) from my first equation.

$$\alpha(Q) = \frac{\alpha(\mu)}{1 - \frac{\alpha(\mu)}{3\pi} \log(Q^2/\mu^2)}$$

The QED coupling changes with energy.

We can do the same thing for QCD, except we have some extra diagrams

e.g.



We find,

$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{1 + \frac{\alpha_s(\mu)}{4\pi} \beta_0 \log(Q^2/\mu^2)}$$

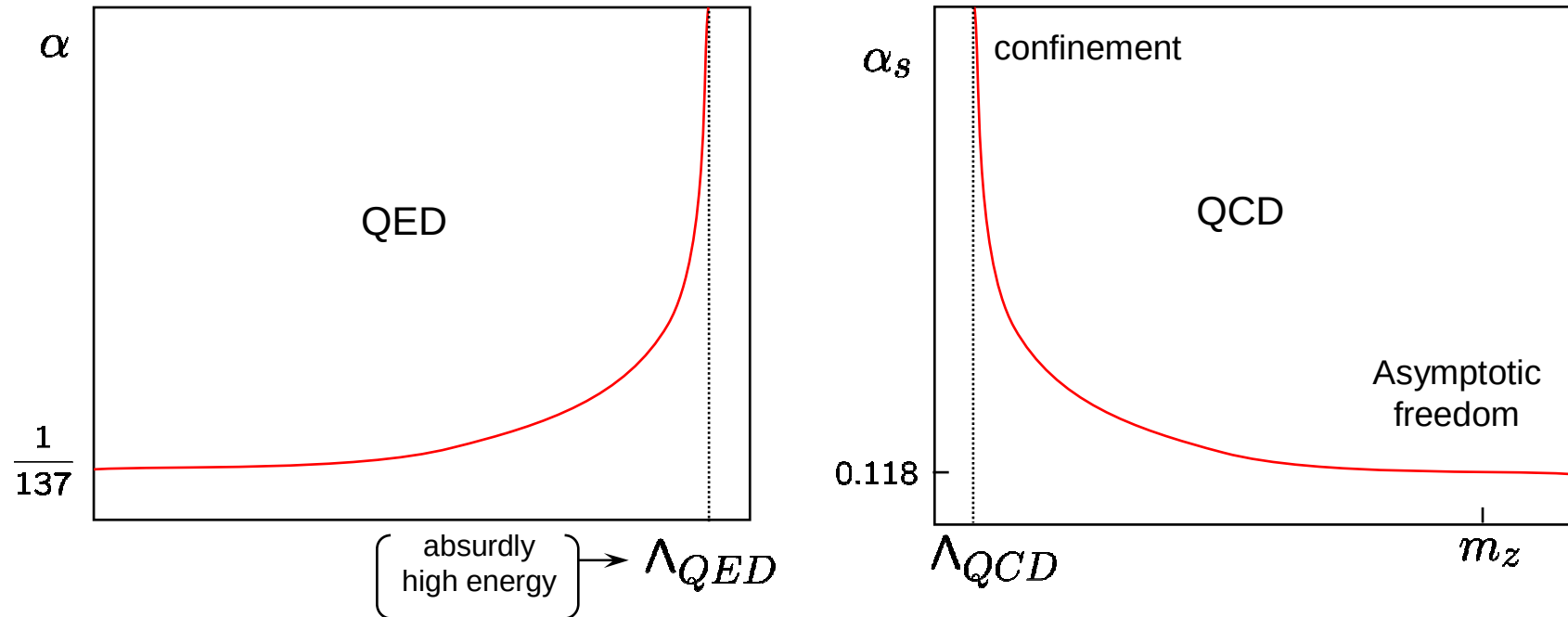
where $\beta_0 = \frac{11N_c - 2N_f}{3}$ $\left(\begin{array}{l} N_c = \# \text{ of colors} = 3 \\ N_f = \# \text{ of active flavors} \end{array} \right)$

At higher orders in perturbation theory we will have more contributions. The complete evolution of the coupling is described by the **beta function**

$$\beta(\alpha_s(Q^2)) = Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2}$$

$$\beta(\alpha_s) = -\beta_0 \alpha^2 + \mathcal{O}(\alpha^3)$$

For $N_f \leq 16$, the QCD and QED couplings run in the opposite direction.



At low energies QCD becomes strong enough to **confine** quarks inside hadrons.
(The β function is not proof of this!)

At high energies QCD is **asymptotically free**, so we can use perturbation theory.