An explanation of the rotation of a vector operator PhL 9.25.08

Overview. This document started out attempting to explain the rule R **V** R^{-1} = R^{-1} **V** where **V** and R are operators in a QM Hilbert Space, and R is the usual active 3x3 rotation matrix for E^3. **V** is a "Cartesian vector operator", and R is "a rotation operator" in the QM space. On my Rotation Matrices TK page I show this equation where R acts in the j=1/2 spinor space, but this is just an example. Here, we examine the equation more generally, and take a look at what happens if the Hilbert Space is E^3 which is similar to the j=1 space but not the same. Here we were looking for a non-QM example. This Section 7 is quite long and does not contribute to the main flow, I recommend skipping it unless you are interested.

Along the way, many general ideas are brought in which are relevant. The notion of "going forward or backward" separately on states and operators in a QM Hilbert Space; the "forward-forward theorem"; moving states and not the operators can be interpreted as observer moving to a new "reference frame" in a general sense. The 0 commutator implies a "symmetry". We examine two kinds of ket notation (priming internally or externally), and interpret these as "active" and "passive" viewpoints. Running states forward is equivalent to running the operators backward, and vice versa. We are able to prove our rule for the 3-momentum operator **P** by working with the momentum basis where the states |**p**> are eigenkets of **P**.

We then go on to generalize the equation from the rotation group to the Lorentz and Poincare groups, and we generalize from a "vector" operator to a general operator which transforms according to some LG representation. Finally, we generalize from such operators to their corresponding *field* operators.

Subsidiary ideas appear in a supporting document "bras and kets ... .doc". In particular, a stumbling block was overcome when it was realized that a boost, although non-unitary in the 4-dimensional vector representation, *is* unitary in the infinite dimensional QM Hilbert Space providing the kets which span this space are normalized covariantly.

Along the way we looked at Stakgold, Messiah, Schiff, Tinkham, and B&D 1 and 2.

The final results are summarized in the last subsection called Conclusions.

Here are some general questions I wrote down when starting this document:

"What does this mean?"

"Where does it come from?" "How can you prove it?"

"Is it meaningful only in the context of quantum mechanics?"

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Here is the TOC of the supported "bras and kets" doc:

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1. Unitary transformations in Quantum Mechanics: the "forward-forward rule". As a preliminary, suppose you have a *unitary* operator U which you apply to states and operators (in the Quantum Mechanical Hilbert space) as shown here

$|\varphi\rangle \rightarrow U|\varphi\rangle$ (forward) $\quad O \rightarrow UOU^{\dagger}$ (forward)

Then the following is true:

$$\langle\varphi|\, O \,|\psi\rangle \rightarrow \langle\varphi|\, U^{\dagger}\,(UOU^{\dagger})\, U\,|\psi\rangle = \langle\varphi|\,(U^{\dagger}U)O(U^{\dagger}U)\,|\psi\rangle = \langle\varphi|\, O\,|\psi\rangle$$

This is true for ANY operator O (and for ANY states), not just for a scalar operator. It would be true for the operator P_k which is the k^{th} component of the momentum operator, which we know is not a scalar operator. If we just do states with no operator, then in order to get

$$\langle\varphi \mid \psi\rangle \rightarrow \langle\varphi \mid \psi\rangle$$

the U in $|\varphi\rangle \rightarrow U|\varphi\rangle$ must be unitary. This would correspond to O = 1, the unit operator.

The underlying structure here is that we have a Hilbert Space with a complex inner product which has the property that $\langle a|b\rangle = \langle b|a\rangle^*$. The adjoint operator comes from $\langle A\psi|\varphi\rangle = \langle\psi|A^{\dagger}\varphi\rangle$. It can really all be understood in matrix terms with a complex inner product. [Antilinear operators ignored here.] Note that a Hilbert Space is defined by the choice of basis elements which span the space.

Now we might use the following odd language for the above: we might say that our operator U "rotates" the states "forward" and $UOU^\dagger$ "rotates" the operator O "forward", and if we "rotate" both things "forward" at the same time, we get no change. Quantum mechanics is invariant under such a transformation because ALL matrix elements are unaffected by such a transformation.

The above would be true, for example, if U were a rotation operator R.

Note on boosts: In the 4x4 vector representation of the LG, boosts are non-unitary operators and in fact we have $B^\dagger = B$. However, the question of whether the boost operator acting in the infinite-dimensional QM Hilbert Space is unitary or not is a different issue. It really depends on how you define your HS. If it is one that is spanned by 3D kets $|\mathbf{p}\rangle$, then the boost is not unitary and this can be seen by the fact that the normalization of states is boost dependent, roughly we have $\langle \mathbf{p}|\mathbf{p}'\rangle = (E/m)\,\delta^3(\mathbf{p}-\mathbf{p}')$. For example, in the rest frame E/m = 1, but in a boosted frame it is some other number. I think there is a similar situation where we might have $\langle \mathbf{x}|\mathbf{x}'\rangle = (t/\tau)\,\delta^3(\mathbf{x}-\mathbf{x}')$ where τ is proper time. If you are only doing rotations, the HS spanned by these states is OK for unitarity of rotations. If you are also going to be doing boosts, you need to use a different HS which is spanned by 4-vector kets $|p\rangle$ or $|x\rangle$ where $\langle p|p'\rangle = \delta^4(p-p')$ etc. I don't think I have ever seen this subject commented on in one of my books, but I will keep my eyes peeled. When dealing with the Dirac spinors, you must do things covariantly like this.

Messiah on page 644 talks about this "forward-forward rule" for unitary operators, but seems to give the idea not special name. His U is called T, for transformation. Unitarity is required (or anti-unitarity, fine), and you see everything I have talked about above on this page. Messiah then goes on at length talking about symmetries of various kinds. In my book, a symmetry is an operator which commutes with the Hamiltonian, so eigenstates can then be taken as simultaneous energy and symmetry eigenstates, think nonrel hydrogen atom.

Forward-forward Rule restated with "primed ket labels". Here, for example, $|\varphi'\rangle$ is a ket in the same HS which contains the ket $|\varphi\rangle$, and U maps $|\varphi\rangle$ into $|\varphi'\rangle$. Notice that earlier we used $\rightarrow$ notation and avoided the use of primed labels, but I think they are actually helpful.

$$|\varphi\rangle \rightarrow U|\varphi\rangle \equiv |\varphi'\rangle \ \text{(forward)} \qquad O \rightarrow UOU^\dagger \equiv O' \ \text{(forward)}$$

Then the following is true (for any operator, but only for unitary U).

$$\langle\varphi'|\, O' \,|\,\psi'\rangle = \langle\varphi|\, U^\dagger\,(UOU^\dagger)\, U\,|\psi\rangle = \langle\varphi|\,(U^\dagger U)O(U^\dagger U)\,|\psi\rangle = \langle\varphi|\, O\,|\psi\rangle$$

$$\langle\varphi'|\,\psi'\rangle = \langle\varphi|\, U^\dagger U\,|\psi\rangle = \langle\varphi|\,\psi\rangle$$

Note on "primed kets" : active and passive viewpoints . The above "primed ket label" notation is the "active" viewpoint where $|\varphi\rangle$ and $|\varphi'\rangle$ are different kets in the same HS. A primed observer sees ket $|\varphi'\rangle$, the unprimed observer sees $|\varphi\rangle$. In what I call the "passive" viewpoint, you could regard there being only a single ket marked by φ to describe a QM state, and the primed and unprimed observers have primed and unprimed Hilbert Spaces, so that the primed observer sees $|\varphi\rangle'$ and the unprimed observer sees $|\varphi\rangle$. Notice that the "quantum numbers" are the same φ in both cases, and that the prime is on the outside of the ket. We can also of course do the bras this way, such as $'\langle\psi|$. I used to think this notation might be helpful, but I now believe there is nothing to be gained by it, and one should just take the

"passive" view and use $\langle\varphi'|$ and $|\varphi'\rangle$ as the primed observers states. This primed observer is "rotated" in the single HS relative to the unprimed one. I have never seen a book or paper which uses the $|\varphi\rangle'$notation, but I will be on the lookout and will report back right here if I find an example.

Here is an example, which leads to the words active and passive used above. Consider the HS which is E^2. We have a vector **V** in this space. We can imagine a second HS occupied by a primed observer which has the axes rotated slightly negatively. There is only one vector **V**, and its components are different in the two spaces, so we have for example $\hat{\mathbf{x}}\bullet\mathbf{V} \neq \hat{\mathbf{x}}'\bullet\mathbf{V}$. This would be the $|\mathbf{V}\rangle \neq |\mathbf{V}\rangle'$ notation and we have basis vectors $|\hat{\mathbf{x}}\rangle \neq |\hat{\mathbf{x}}\rangle'$. This is what I call the "passive" view.

Another viewpoint (the "active" one) is that we have only one HS, and the primed observer sees vector **V**' and we have $\hat{\mathbf{x}}\bullet\mathbf{V} \neq \hat{\mathbf{x}}\bullet\mathbf{V}'$. This would be the $|\mathbf{V}\rangle \neq |\mathbf{V}'\rangle$ notation. The vector **V**' is rotated positively some small amount relative to **V**. When I write down "rotation matrices", I always present them such that $\theta>0$ implies a positive RHR rotation.

Comment on the above notations: When I talked about a primed and unprimed observer, I did not mean to imply that these two observers must be in different Lorentz reference frames, though that would be one possibility. I just meant that we have SOME transformation $U|\varphi\rangle \equiv |\varphi'\rangle$ and think of $|\varphi'\rangle$ as being the ket seen by a "primed observer". For example, U might be an isospin rotation, and then we might think of a primed observer who is rotated relative to an unprimed observer in isospin space, or is rotated about an axis in charge space, or who is "inverted" from another observer via parity. We associate kets with observers since kets must be used by an observer to measure an expectation value of an operator, which is the only connection we have between the QM space and that occupied by observers.

2. Symmetry in Quantum Mechanics. Now, suppose we only do ONE of the two operations shown above. For example, suppose we define our "transformation" by moving the states, but not the operator. Then we have (U is assumed unitary)

$|\varphi\rangle \rightarrow U|\varphi\rangle \equiv |\varphi'\rangle$ $\qquad O \rightarrow O$ $\qquad$ " states forward"

Then the following is true:

$$\langle\varphi|O|\psi\rangle \rightarrow \langle\varphi'|O|\psi'\rangle = \langle\varphi|U^{\dagger}OU|\psi\rangle$$

but we cannot continue without more information. If it happens that [O,U] = 0 (commutator), then we *can* in fact continue in a particular way and say

$$\langle\varphi|O|\psi\rangle \rightarrow \langle\varphi'|O|\psi'\rangle = \langle\varphi|U^{\dagger}OU|\psi\rangle = \langle\varphi|U^{\dagger}UO|\psi\rangle = \langle\varphi|O|\psi\rangle$$

In this case, we can regard the operator U as a "symmetry" of the operator O. Usually in this context we have O = H, the Hamiltonian, and the states $|\psi\rangle$ are some H eigenstates. Operator U might be a rotation, and then [H,R] = 0 says that H is a rotational scalar, so rotation is a symmetry of the Hamiltonian. This would be the case for the non-rel free particle H for example, having $p^2/2m$ in it. Rotating the energy eigenstates on both sides causes no change to the matrix element.

Similarly, we could define a "transformation" where the states stay put and the operator moves:

$|\varphi\rangle \rightarrow |\varphi\rangle$ $\qquad O \rightarrow UOU^{\dagger} \equiv O'$ $\qquad$ "operator forward"

Then the following is true:

$$\langle\varphi|O|\psi\rangle \rightarrow \langle\varphi|O'|\psi\rangle = \langle\varphi|UOU^{\dagger}|\psi\rangle$$

Again, if it happens that [O,U] = 0, we can finish and say

$$\langle\varphi|O|\psi\rangle \rightarrow \langle\varphi|O'|\psi\rangle = \langle\varphi|UOU^{\dagger}|\psi\rangle = \langle\varphi|O|\psi\rangle$$

and we would say that transforming the operator O has no effect on matrix elements of O, and so U is a symmetry of O.

My comments here about "symmetry" are not consistent with those of Moore in his Appendix 1 notes, where he gives what appears to me to be my "forward-forward" theorem which I don't think has anything to do with a "symmetry" in one's theory.

In either case above (only move states, or only move operators), we could "continue" if we knew the effect on the operator O of a sandwich with U. We considered the simplest case of commutation above, which we called a symmetry, but even without such a symmetry, we might know something about how O transforms. Cases will occur below.

3. The forward/backward equivalence theorem.

Here is the idea that taking the states forward is the same as taking the operator backward (and vice versa), where we simply use our results from above, interchanging U with $U^{\dagger}$ in the operator backward situation:

states forward: $|\varphi\rangle \rightarrow U|\varphi\rangle \equiv |\varphi'\rangle \qquad O \rightarrow O$

$$\langle\varphi|O|\psi\rangle \rightarrow \langle\varphi'|O|\psi'\rangle = \langle\varphi|U^{\dagger}OU|\psi\rangle$$

operator *backward*: $|\varphi\rangle \rightarrow |\varphi\rangle \qquad O \rightarrow U^{\dagger}OU \equiv O'$

$$\langle\varphi|O|\psi\rangle \rightarrow \langle\varphi|O'|\psi\rangle = \langle\varphi|U^{\dagger}OU|\psi\rangle$$

To reach the conclusion that doing the above either way "gives the same result" does not require U to be unitary, nor is anything assumed about the operator O, it need not be a scalar for example.

In all of the above, the (forward) transformation of the operator might do something like this:

$UOU^{\dagger}$ = linear combination of operators perhaps including O itself

Our "equation of interest" concerns vector operator components and we do get exactly such a linear combination as shown above, and it involves in fact only the set of operators acted upon:

$$RO_kR^{-1} = (R^{-1})_{kj}\,O_j \qquad k = 1,2,3 \qquad (R^{-1})_{kj} = \text{some numbers}$$

In this case, we deal with rotations and we know that $R^{\dagger} = R^{-1}$ meaning R is unitary. We have not of course yet "proved" this claimed transformation of vector operators, we are just using it as an example of some transformation U acting on some operator O.

In Levitt notes, I talked about rotating the experimental apparatus while not rotating the observer "states", and also the reverse, rotating the observation states and not rotating the experimental apparatus. Rotating the states forward was equivalent to rotating the experiment backwards. I identified the states with observation, and the operator with the apparatus. So that is just what the above concerns. In Levitt, the operator would be the nuclear spin operator **I**, and the states would be up and down relative to some direction like z.

4. Transformations that involve the spacetime coordinate x^μ .

I wrote this section before realizing that the $|\alpha'\rangle$ and $|\alpha\rangle'$ notations were essentially the same, as outlined above. So let's just regard this as an exercise in notation. Things do work out correctly. That is, we think here about the two passive viewpoints, the two Hilbert Spaces alluded to earlier.

If the transformation U has an effect on x^μ (perhaps a rotation or a Lorentz transformation), then we can talk about relative "frames of reference" in the traditional sense (Messiah calls them referentials) , and we can introduce the notion of a ket and an operator as being in one or another frame of reference. For notational convenience, we often talk about a "starting frame" where things have no primes, and then "some other frame" where things are primed. In these frames we refer to the spacetime coordinate of a point as x^μ and x'^μ, for example. In this situation (as opposed to an internal symmetry situation) , we have a new notation which does not appear above, namely a ket like this: $|\alpha\rangle'$ which means a state "in the primed reference frame", as opposed to $|\alpha'\rangle$ which would be some state in the unprimed starting reference frame having quantum number α'.

Our "forward-forward" rule in terms of frames would be this:

$$\langle\alpha \mid O \mid \beta\rangle = {}'\langle\alpha \mid O' \mid \beta\rangle'$$

Everything on the left is in the unprimed frame, everything on the right is in the primed frame. If we measure a matrix element, it must be the same number independent of our frame of reference. This is the situation where we "rotate" both the experiment AND the observer ("forward-forward" situation). Think of rotating the camera that is looking at the experiment and observer -- moving the camera cannot change what happens in the experiment. We would accompany the above equation with the following:

$\|\alpha\rangle \rightarrow U\|\alpha\rangle = \|\alpha\rangle'$	$O \rightarrow UOU^\dagger = O'$
forward	forward

Then we have

$$'\langle\alpha \mid O' \mid \beta\rangle' = \langle\alpha \mid U^\dagger UOU^\dagger U \mid \beta\rangle = \langle\alpha \mid O \mid \beta\rangle$$

but this DOES require that U be unitary. Somehow there must be a modification of our "forward-forward" rule that makes the non-unitary boosts come out in some reasonable manner. [Well, if the forward "rotation" is done in a covariant manner for a boost, then U is unitary for the boost, see elsewhere.]

Now, in a situation where the ket eigenvalues are Cartesian 3-vectors, such as in the momentum or coordinate space representations, we can make the following identification:

$|\mathbf{p}\rangle'$ = the ket $|\mathbf{p}\rangle$ viewed from a backward-rotated frame = $R\,|\mathbf{p}\rangle = |R\,\mathbf{p}\rangle = |\mathbf{p}'\rangle$

Clarification: What the above means is this: We have a ket $|\mathbf{p}\rangle'$ in the primed Hilbert Space H' that belongs to the primed observer O' in the "passive" viewpoint. Yes, this Hilbert Space H' is "backward rotated" relative to the Hilbert Space H of the unprimed observer O. But in the end, we want to work only in the Hilbert Space H, and in that space, we know that $|\mathbf{p}\rangle'$ *corresponds to* $R|\mathbf{p}\rangle = |\mathbf{p}'\rangle$ which is a *forward* rotated ket in H. Rather than use some excess words like $|\mathbf{p}\rangle' \rightarrow$ corresponds$\rightarrow |\mathbf{p}'\rangle$, we just say that $|\mathbf{p}\rangle' = |\mathbf{p}'\rangle$. Technically, saying $|\mathbf{p}\rangle' = |\mathbf{p}'\rangle$ is illegal because the kets are in different Hilbert Spaces altogether. We have used this equality only in small sections below where the results are not too important. In our "proof" of $R\ \mathbf{P}\ R^{-1} = R^{-1}\mathbf{P}$, we do not use this external-to-ket prime notation.

So in this case, we may simply move the apostrophe inside the ket. When ket eigenvalues are spherical vectors as opposed to Cartesian vectors, you cannot make such a simple replacement. In fact

$$|jm\rangle' = R|jm\rangle = R^{j}_{mn}\,|jn\rangle$$

For J=1, recall that this is not the same as R though it is 3x3.

Question: do I have any books that use this kind of "external to ket prime" notation? I do not see it in Messiah. Nor in Schiff. Still, I think it is an OK notation.

4.1 Transformation in the coordinate representation of a QM Hilbert Space

The coordinate representation of course gets us involved in "transformations that affect space/time", so I put this little subsection here. The coordinate representation implies *wavefunctions*. Consider the following, where ψ_r is a generic *component* of a wave function (eg, a Dirac spinor)

$$\psi_r(\mathbf{x}) = \langle \mathbf{x}|\psi_r\rangle$$
$$\psi'_r(\mathbf{x}') = \langle \mathbf{x}'|\psi'_r\rangle = \langle \mathbf{x}|R^{\dagger}|\psi'_r\rangle \qquad \text{where } \mathbf{x}' = R\mathbf{x}$$

If ψ_r transforms according to some representation S of the rotation group, then we know that

$$R^{\dagger}|\psi'_r\rangle = S(R)_{rs}\,|\,\psi_s(p)\rangle$$

and then we get this rule for the transformation of the *wavefunction* components

$$\psi'_r(\mathbf{x}') = S(R)_{rs}\,\psi_r(\mathbf{x}) \qquad \text{where } \mathbf{x}' = R\mathbf{x}$$

One should note that $\mathbf{x}'$ and $\mathbf{x}$ are the same point in 3-space which have different coordinates for our two observers -- primed and unprimed. In the above we assumed that $|\mathbf{x}'\rangle = R|\mathbf{x}\rangle$ is the coordinate ket seen by the primed observer in the single HS, which is our "active viewpoint". It is natural that the primed observer deals with $\psi'_r(\mathbf{x}')$ and not $\psi'_r(\mathbf{x})$. This subject is further explored in document "bras and kets and coordinate transformations". And we come back to this subject later in this document.

5. Linear momentum as an example of our equation of interest.

In this section, we "prove" that $R\ \mathbf{V}\ R^{-1} = R^{-1}\mathbf{V}$ must be true for $\mathbf{V} = \mathbf{P}$, the momentum operator. We do this in the momentum basis where the kets which are eigenkets of **P**.

Our (so far only *claimed* to be true) equation of interest for the vector operator **P** is this: (operator forward)

$$\mathbf{P}' \equiv R\ \mathbf{P}\ R^{-1} = R^{-1}\mathbf{P} \qquad \text{write out as} \qquad R\ P_k\ R^{-1} = [R^{-1}\mathbf{P}]_k = (R^{-1})_{kj}P_j \qquad 5.1$$

Of course this would be true for any rotation so we can also write

$$\mathbf{P}'' \equiv R^{-1}\ \mathbf{P}\ R = R\ \mathbf{P} \qquad \text{// operator backward} \qquad 5.2$$

Suppose we have some momentum *eigenstates* $|\mathbf{p}\rangle$, where **P** is the momentum *operator*

$$\mathbf{P}\ |\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

Consider a new momentum eigenstate $|\mathbf{p}'\rangle$, where R is a rotation *operator* (in the QM HS)

$$|\mathbf{p}'\rangle = R\ |\mathbf{p}\rangle = |\ R\ \mathbf{p}\rangle \qquad \text{// states forward} \qquad 5.3$$

That is to say, $|\mathbf{p}'\rangle$ is a state whose momentum eigenvalue is given by

$$\mathbf{p}' = R\ \mathbf{p}. \qquad 5.4$$

I would say that we are rotating the momentum eigenstates "forward" (if we used R^{-1} I would call it a "backward" rotation).

We certainly expect to have the following be true: (assuming box normalization, etc etc)

$$\langle \mathbf{p}' |\ \mathbf{P}\ |\ \mathbf{p}' \rangle = \mathbf{p}' \qquad \text{since } \mathbf{P}\ |\mathbf{p}'\rangle = \mathbf{p}'|\mathbf{p}'\rangle \qquad 5.5$$

From the above we have

$$\mathbf{p}' = \langle \mathbf{p}' |\ \mathbf{P}\ |\ \mathbf{p}' \rangle \text{ (by 5.5)} = \langle \mathbf{p} |\ R^{-1}\mathbf{P}\ R\ |\ \mathbf{p} \rangle \text{ (by 5.3 used twice and } R^{\dagger} = R^{-1})$$

If we now assume result 5.2 above is valid (operator backward), then we can continue

$$\mathbf{p}' = \langle \mathbf{p}' |\ \mathbf{P}\ |\ \mathbf{p}' \rangle = \langle \mathbf{p} |\ R^{-1}\mathbf{P}\ R\ |\ \mathbf{p} \rangle = \langle \mathbf{p} |\ R\ \mathbf{P}\ |\ \mathbf{p} \rangle = R\ \langle \mathbf{p} |\ \mathbf{P}\ |\ \mathbf{p} \rangle = R\ \mathbf{p} = \mathbf{p}'$$

and everything "comes out correctly". If $R^{-1}\mathbf{P}\ R = R\ \mathbf{P}$ were NOT true, then I guess you would have to assume that $R^{-1}\mathbf{P}\ R = \mathbf{Q}$, some "other operator", and you would need $\langle \mathbf{p} |\ \mathbf{Q}\ |\ \mathbf{p} \rangle = R\ \mathbf{p}$ and this would have to be true for all **p**. Probably there is some uniqueness theorem one could conjure up to say there is only one operator that does the right thing in your Hilbert space, and it must be therefore that $\mathbf{Q} = R\ \mathbf{P}$. [

See Messiah page 633 for such a uniqueness theorem, but he is quoting his own page 250 where this is not quite proven. I accept it as true.]

Notice in the above algebra we have used $|\mathbf{p}'\rangle = R\,|\mathbf{p}\rangle$ as a forward-rotated state, and $\mathbf{P}'' \equiv R^{-1}\mathbf{P}\,R$ as a backward-rotated operator.

We could make a similar argument for the position operator **X** (another vector operator) using the position eigenstates $|\mathbf{x}\rangle$.

Question: [here we use some external prime notation.] What is the *meaning* of the backward-rotated operator $\mathbf{P}'' \equiv R^{-1}\mathbf{P}\,R$? From the above we certainly have

$$\langle \mathbf{p} \mid \mathbf{P}'' \mid \mathbf{p}\rangle = \langle \mathbf{p}' \mid \mathbf{P} \mid \mathbf{p}'\rangle = \mathbf{p}' \qquad (0)$$

As a passing comment, we point out that the equality $\langle \mathbf{p} \mid \mathbf{P}'' \mid \mathbf{p}\rangle = \langle \mathbf{p}' \mid \mathbf{P} \mid \mathbf{p}'\rangle$ is an example of our "forward- forward theorem" for matrix elements. As we go from the LHS to the RHS, we are making these changes:

$|\mathbf{p}\rangle \rightarrow |\mathbf{p}'\rangle = \mathbf{R}\,|\mathbf{p}'\rangle$ (which is "forward on the states") and
$\mathbf{P}'' \rightarrow \mathbf{P} = R\,\mathbf{P}''\,R^{-1}$ (which is "forward on the operator").

Now consider the following manipulations,

$$\mathbf{P}\,|\mathbf{p}\rangle = \mathbf{p}\,|\mathbf{p}\rangle \Rightarrow \quad R\mathbf{P}\,|\mathbf{p}\rangle = \mathbf{p}R\,|\mathbf{p}\rangle \Rightarrow \quad R\mathbf{P}R^{-1}\,R\,|\mathbf{p}\rangle = \mathbf{p}\,R\,|\mathbf{p}\rangle$$

$$\Rightarrow (R\mathbf{P}R^{-1})\,|\mathbf{p}\rangle' = \mathbf{p}\,|\mathbf{p}\rangle' \Rightarrow \quad \mathbf{P}'\,|\mathbf{p}\rangle' = \mathbf{p}\,|\mathbf{p}\rangle' \text{ where } \mathbf{P}' \equiv R\mathbf{P}R^{-1} = \text{forward}$$

Thus, if we go to a primed system where both the operator and states are forward rotated, we get the eigenvalue equation

$$\mathbf{P}'\,|\mathbf{p}\rangle' = \mathbf{p}\,|\mathbf{p}\rangle' \qquad \mathbf{P}' \equiv R\mathbf{P}R^{-1} = \text{forward} \qquad |\mathbf{p}\rangle' = R\,|\mathbf{p}\rangle = |\,R\,\mathbf{p}\rangle = \text{forward}$$

Similarly, we could make the double-primed system be one with states and operators backward:

$$\mathbf{P}''\,|\mathbf{p}\rangle'' = \mathbf{p}\,|\mathbf{p}\rangle'' \qquad \mathbf{P}'' \equiv R^{-1}\mathbf{P}\,R = \text{backward} \qquad |\mathbf{p}\rangle'' = R^{-1}\,|\mathbf{p}\rangle = |\,R^{-1}\,\mathbf{p}\rangle = \text{backward}$$

The point here is that the "meaning" of the primed operators is that they are the corresponding operators in other reference frames than the unprimed frame we normally work in. [This *does* seem to show some utility to using the notation with external ket priming notation. The last result seems less clear if we write it in the form. $\mathbf{P}''\,|\mathbf{p}''\rangle = \mathbf{p}\,|\mathbf{p}''\rangle$. Notice that $\mathbf{P}\,|\mathbf{p}''\rangle = \mathbf{p}''\,|\mathbf{p}''\rangle$.]

Comment: So, in some sense we have proven that $R^{-1}\mathbf{P}\,R = R\,\mathbf{P}$ is valid for the momentum operator. I think the idea is then to *define* a "vector operator" as one which transforms in this same way. Then we don't have to "prove" that $R^{-1}\mathbf{V}\,R = R\,\mathbf{V}$ is true for a vector operator.

Comment: If a vector operator **V** transforms in this way, that is consistent with the idea that a bra-ket *matrix element* of **V** transforms as a regular vector. Consider this example with some rotated observing states:

$$\mathbf{A'} = '\langle\alpha \mid \mathbf{V} \mid \beta\rangle' = \langle\alpha \mid R^{-1}\mathbf{V}\, R \mid \beta\rangle = \langle\alpha \mid R\ \mathbf{V} \mid \beta\rangle = R\langle\alpha \mid \mathbf{V} \mid \beta\rangle = R\ \mathbf{A}$$

If $R^{-1}\mathbf{V}\, R = R\ \mathbf{V}$ were not true, then we would not have $\mathbf{A'} = R\ \mathbf{A}$ and it would not be true that the expectation value of a vector operator transforms as a vector, and the entire theory would be inconsistent. So in some sense, this is another "proof" that $R^{-1}\mathbf{V}\, R = R\ \mathbf{V}$ must be true in any QM Hilbert space and for any vector operator.

Based on the above, we can guess the HS transformation rule for a rotational tensor:

$$\mathbf{V'} = R^{-1}\mathbf{V}\, R = R\ \mathbf{V} \quad \text{meaning} \quad V'_k = R^{-1}\ V_k\, R = [R\ V]_k = R_{ka}V_a$$

$$T' = R^{-1}T\, R = R\ R\ T \quad \text{meaning} \quad T'_{jk} = R^{-1}T_{jk}\, R = R_{ja}\ R_{kb}\ T_{ab}$$

Again, this would cause a matrix element of the operator T to have the correct tensor transformation.

6. Spin states and the angular momentum as example of vector operator

[More external ket priming notation is used here. Here it is hard to put the prime inside the ket. You cannot say for example that |jm'> = |jm>' or even that |jm>'→ |jm'> by "correspondence" the way we could for |**p**>' → |**p**'>. Here we really want to use that external notation and | jm>' → R | jm>.

Our equation of interest is now $R^{-1}\ \mathbf{J}\, R = R\ \mathbf{J}$, and we would say

$$'\langle jm \mid \mathbf{J} \mid jm\rangle' = \langle jm \mid R^{-1}\ \mathbf{J}\, R \mid jm\rangle = \langle jm \mid R\ \mathbf{J} \mid jm\rangle = R\ \langle jm \mid \mathbf{J} \mid jm\rangle$$

$$'\langle jm \mid J_z \mid jm\rangle' = \langle jm \mid R^{-1}\ J_z\, R \mid jm\rangle = \langle jm \mid R\ J_z \mid jm\rangle = R_{zk}\ \langle jm \mid J_k \mid jm\rangle = m\ R_{zz}$$

In particular, suppose j = ½ then we might write

$$|1/2,m\rangle = |00;m\rangle.$$

And we would say

$$|1/2,m\rangle' = R(\theta,\varphi)\ |1/2,m\rangle = |\theta\varphi;m\rangle$$

Then the above says

$$\langle \theta\varphi;m \mid J_z \mid \theta\varphi;m\rangle = m\ R_{zz}(\theta,\varphi) = m\cos\theta$$

which looks right to me. If θ = π, you flip and get -m and if θ = π/2, you get 0.

7. Relation to non-QM concepts. Can I relate this **P** stuff to the ancient discussion of rotating vectors actively forward versus rotating the basis vectors passively backward -- a discussion has nothing to do with quantum mechanics?

rotate a vector forward: $\mathbf{V}' = \mathsf{R}\ \mathbf{V}$

rotate a basis vector backward: $\hat{\mathbf{x}}' = \mathsf{R}^{-1}\hat{\mathbf{x}}$

=> $V'_{\mathbf{x}} \equiv \mathbf{V}'\bullet\hat{\mathbf{x}} = \mathsf{R}\ \mathbf{V}\bullet\hat{\mathbf{x}} = \mathbf{V}\bullet\mathsf{R}^{-1}\hat{\mathbf{x}}$ (since dot product is a scalar) $= \mathbf{V}\bullet\hat{\mathbf{x}}' \equiv V_{\mathbf{x}'}$

To emphasize the inner product, rewrite this as

$(\mathbf{V}', \hat{\mathbf{x}}) = (\mathbf{V}, \hat{\mathbf{x}}')$ => $(\mathsf{R}\mathbf{V}, \hat{\mathbf{x}}) = (\mathbf{V}, \mathsf{R}^{-1}\hat{\mathbf{x}})$

where in *this* Hilbert Space, our rotating operator is R = R, it just happens. Our language here would be the following: " if we rotate the vector **V** forward and keep our basis functions, it is the same as if we keep our vector, but rotate the basis functions backward. " We are seeing an application of this statement in the context of an inner product, as in QM. What happens then if we "rotate the basis vectors forward and at the same time rotate the vector forward" ?

$(\mathbf{V}', \hat{\mathbf{x}}') = (\mathsf{R}\mathbf{V}, \mathsf{R}\,\hat{\mathbf{x}}) = (\mathbf{V}, \mathsf{R}^{-1}\mathsf{R}\,\hat{\mathbf{x}}) = (\mathbf{V}, \hat{\mathbf{x}})$

There is in this case "no change" in the scalar product value as a number. So this would be our "forward-forward theorem". In all of the above, the notation $\hat{\mathbf{x}}$ refers explicitly to "the x unit vector" and not other unit vectors.

Now let's use some "healthy notation" for our unit vectors, to stay out of trouble:

$\mathbf{e}^{(i)}$ = unit vectors, for example $\mathbf{e}^{(1)} = \hat{\mathbf{x}}$

$\mathbf{e}^{(i)}{}_j$ = jth component of the unit vector $\mathbf{e}^{(i)}$, for example $\mathbf{e}^{(i)}{}_j = \delta_{ij}$

rotate a vector forward: $\mathbf{V}' = \mathsf{R}\ \mathbf{V}$

rotate basis vectors backward: $\mathbf{e}'^{(i)} = \mathsf{R}^{-1}\,\mathbf{e}^{(i)}$ eg $\hat{\mathbf{x}}' = \mathsf{R}^{-1}\hat{\mathbf{x}}$

=> $\mathbf{e}'^{(i)}{}_j = (\mathsf{R}^{-1})_{jk}\,\mathbf{e}^{(i)}{}_k = (\mathsf{R}^{-1})_{ji} = \mathsf{R}_{ij}$

A distinction between two similar equations. Now right at this point, let us "explore" in more detail a certain equation above:

$\mathbf{e}'^{(i)} = \mathsf{R}^{-1}\,\mathbf{e}^{(i)}$

The RHS here must be a linear combination of basis vectors, so we have

$$\mathbf{e'^{(i)}} = R^{-1}\,\mathbf{e^{(i)}} = a^{(i)}{}_j\;\mathbf{e^{(j)}}$$

Now dot the right equation above with $\mathbf{e^{(k)}}$ on the left and we get

$$(\mathbf{e^{(k)}}, R^{-1}\,\mathbf{e^{(i)}}) = a^{(i)}{}_k = (R^{-1})_{ki}$$

Thus we have

$$\mathbf{e'^{(i)}} = R^{-1}\,\mathbf{e^{(i)}} = a^{(i)}{}_j\;\mathbf{e^{(j)}} = (R^{-1})_{ji}\,\mathbf{e^{(j)}} = R_{ij}\,\mathbf{e^{(j)}}$$

So lets pick two equations above and put them one right after the other:

$$\mathbf{e'^{(i)}} = R^{-1}\,\mathbf{e^{(i)}} \quad \Rightarrow \quad \mathbf{e'^{(i)}}_a = (R^{-1})_{ab}\,\mathbf{e^{(i)}}_b = (R^{-1})_{ai} \qquad (1)$$

$$\mathbf{e'^{(i)}} = R_{ij}\,\mathbf{e^{(j)}} \quad \Rightarrow \quad \mathbf{e'^{(i)}}_a = R_{ij}\,\mathbf{e^{(j)}}_a = R_{ia} \qquad (2)$$

What do each of these equations "say"? The first says that $\mathbf{e'^{(i)}}$ is a "backwards rotated" version of $\mathbf{e^{(i)}}$. The second equation says that $\mathbf{e'^{(i)}}$ is a linear combination of the $\mathbf{e^{(j)}}$ with weights R_{ij} . If we set the two RHS's equal, we get this interesting fact

$$R^{-1}\,\mathbf{e^{(i)}} = R_{ij}\,\mathbf{e^{(j)}} \Rightarrow \quad (R^{-1})_{ab}\,\mathbf{e^{(i)}}_b = R_{ij}\,\mathbf{e^{(j)}}_a \qquad (3)$$

That is, when we back-rotate $\mathbf{e^{(i)}}$, we get a linear combination of $\mathbf{e^{(i)}}$ as shown with coefficients R_{ij}.

Today's Big Lesson: by having clean notation, you don't confuse equation (1) with equation (2) above.

Operators in E3. Now back to our trying to draw a parallel with QM. So far we are lacking any kind of "operator" object in our E^3 Hilbert space. So imagine a simple matrix T_{ij} which we will call by the fancy name: rank 2 tensor. We can then *define* an operator T as follows: (a "linear operator in E^3"), looking for guidance at Stakgold page 147 equation 2.34:

$$T\{\mathbf{e^{(j)}}\} = \mathbf{e^{(k)}}T_{kj} = T_{kj}\;\mathbf{e^{(k)}} = T^{T}_{jk}\,\mathbf{e^{(k)}}$$

and then we get *the desired result*:

$$(\mathbf{e^{(i)}} \mid T \mid \mathbf{e^{(j)}}) = (\mathbf{e^{(i)}} \mid T_{kj}\;\mathbf{e^{(k)}}) = T_{kj}\;(\mathbf{e^{(i)}} \mid \mathbf{e^{(k)}}) = T_{kj}\;\delta_{ik} = T_{ij}$$

Notice that $T\{\mathbf{e^{(j)}}\} \neq T_{jk}\;\mathbf{e^{(k)}}$ as you might at first think (normal matrix ordering) ! If you define T in *this* way, then the matrix indices come up backwards in the above "matrix element" evaluation.

Now, suppose T were "the rotation operator". Then the above would say

$$R\{\mathbf{e^{(j)}}\} = R^{T}_{jk}\,\mathbf{e^{(k)}} = R^{-1}{}_{jk}\,\mathbf{e^{(k)}} = R\;\mathbf{e^{(j)}} = \text{a forward rotation of the } j^{th} \text{ basis vector}$$

where the last equality comes from (3) above with $R \to R^{-1}$. We can look at a component of this equation if we like

$$R\{\mathbf{e}^{(j)}\}_a = R^{T}_{jk}\,\mathbf{e}^{(k)}{}_a = R^{-1}{}_{jk}\,\mathbf{e}^{(k)}{}_a = [R\ \mathbf{e}^{(j)}]_a$$

Let's see what R{..} does to an arbitrary vector **V**:

$$R\ \{\ \mathbf{V}\ \} = R\{V_i\,\mathbf{e}^{(i)}\} = V_i\,R\{\mathbf{e}^{(i)}\} = V_i\,R^{-1}{}_{ik}\,\mathbf{e}^{(k)} = R_{\ ki}V_i\,\mathbf{e}^{(k)} = (R\ V)_k\,\mathbf{e}^{(k)} = V'_k\,\mathbf{e}^{(k)}$$

and this is just an forward "active" rotation of the vector **V** into the vector **V'** = R **V**. So of course our operator R {...} rotates all vectors forward, be they unit vectors or arbitrary vectors.

Comment: At this point 4 PM Sat 9.26.08 I have just finished extracted myself from a large puddle of quicksand which was caused by imprecise notation. I will now continue on as if there were never a problem.

I am hunting for an animal with the following approximate appearance in the E^3 world:

$$R^{-1}\mathbf{A}\,R = R\,\mathbf{A}$$

where **A** is a vector operator of some sort. In the discussion just above, the vector **V** has been "like a state" in QM, not "like an operator". I think we know that an operator in E^3 must be a matrix. As noted above, we have

$$(\mathbf{e}^{(i)}\mid T\mid \mathbf{e}^{(j)}) = T_{ij}$$

The thing T is the operator in the E^3 space, while T_{ij} is a matrix element of this operator. We can certainly support the idea of non-commuting operators in E^3 as follows:

$$(\mathbf{e}^{(i)}\mid AB\mid \mathbf{e}^{(j)}) = (AB)_{ij} = A_{ik}B_{kj} \neq (\mathbf{e}^{(i)}\mid BA\mid \mathbf{e}^{(j)})$$

For the rotation operator we would say

$$(\mathbf{e}^{(i)}\mid R\mid \mathbf{e}^{(j)}) = R_{ij} = R_{ij}$$

where I like to use R_{ij} to indicate the 3x3 matrix. Now consider our equation of interest which is the main point of this entire document, which has grown to 12 pages and has resolved nothing so far:

$$R^{-1}\mathbf{A}\,R = R\,\mathbf{A}$$

There are really two distinct "things" going on in this equation that I think can be made clearer by having better notation. First, we can expose the vector component as follows

$$R^{-1}A_k\,R = [R\ \mathbf{A}]_k = R_{ki}\,A_i$$

The next thing we need to expose is the "matrix sense" of this equation. It really says this

$$(R^{-1})_{ab}\,(A_k)_{bc}\,(R)_{cd} = R_{ki}\,(A_i)_{ad}$$

and in our particular space E^3, we can write this as

$$(R^{-1})_{ab}\,(A_k)_{bc}\,(R)_{cd} = R_{ki}\,(A_i)_{ad} = ([R\,\mathbf{A}]_k)_{ad} = (A'_k)_{ad}$$

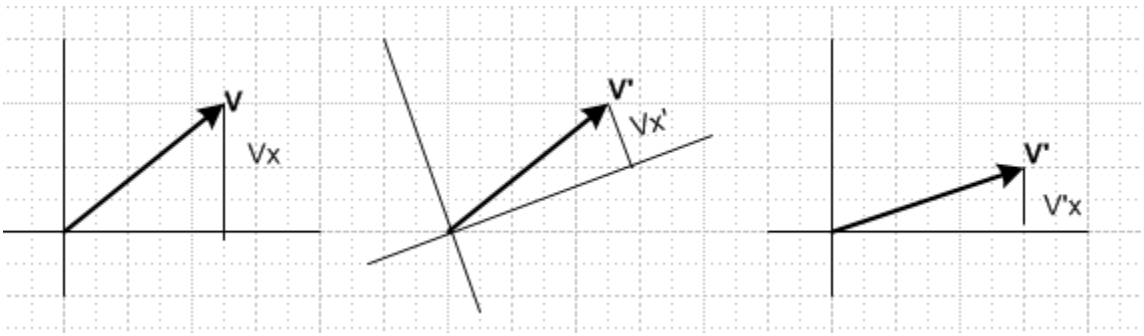
Can such an equation "be true" ? Rewrite again as

$$(A'_k)_{ad} = ([R\,\mathbf{A}]_k)_{ad} = (R^{-1})_{ab}\,(R^{-1})_{dc}\,(A_k)_{bc}$$

This sure looks like the way a tensor is supposed to transform! Here is how vectors and tensors transform

$$V'_a = R_{ab}V_b$$
$$T'_{ab} = R_{ai}R_{bj}T_{ij}$$

In these two equations, V' and T' are objects observed in a reference frame that has been rotated by R from the unprimed frame. That is to say, in the primed frame, we have $\mathbf{e}'^{(i)} = R\,\mathbf{e}^{(i)}$. The left two pictures show the situation, left is unprimed frame, middle is primed frame.



When I normally write $V'_a = R_{ab}V_b$ or V' = RV, I mean by "R" the active forward rotation of the vector V in the unprimed frame and I don't think about any unprimed frame. But as you can see, the rotation R is rotating vector V to V' backwards (relative to usual RHR). Therefore, the R's which appear in the above two equations are the inverses of my usual R matrices which are written in forward active form. So in terms of these matrix, I would write the above "transformation rules" as

$V'_a = (R^{-1})_{ab}V_b$	transformation rules for tensors rank1
$T'_{ab} = (R^{-1})_{ai}\,(R^{-1})_{bj}T_{ij}$	rank 2

But at the same time, I would indicate an active rotation of a vector V by

$V'_a = (R)_{ab}V_b$	active rotation of V into V'

Conclusion: If we work in E^3 space with its usual inner product, we can say

$$A'_k = [R\,\mathbf{A}]_k = (R^{-1})_{ab}\,(R^{-1})_{dc}\,(A_k)_{bc}$$

$$(A'_k)_{ad} = ([R\,\mathbf{A}]_k)_{ad} = (R^{-1})_{ab}\,(R^{-1})_{dc}\,(A_k)_{bc}$$

Start again with better notation on this last topic.

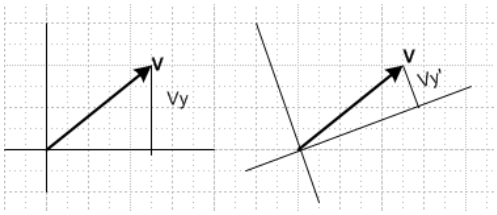
1. The transformation rules. Above, I wrote these as

$$V_a' = R_{ab}V_b$$
$$T'_{ab} = R_{ai}R_{bj}T_{ij}$$

but this notation is way too ambiguous for what I am now attempting. Let's consider the first one and write it instead like this

$$V_{a'} = e'^{(a)} \bullet \mathbf{V} = {}^{p}R_{ab}(\theta)\,V_b \qquad \text{where } \mathbf{e}'^{(a)} = {}^{a}R(\theta)\,\mathbf{e}^{(a)}$$

First, the new notation $V_{a'}$ indicates the component of vector **V** obtained by projecting **V** on the a' axis. So we put the prime down with the a. Putting it up with the V makes you think of some other vector **V**' and we don't have any other vectors at this point. Secondly, the rotation is a passive one by amount θ, and we indicate that by left superscript "p". Let's roll out our little picture above again:



In our "usual situation" with vector **V** in the first quadrant, we can *see* that $V_{y'}$ is smaller than V_y. Let's make sure this is compatible with our claimed passive rotation:

$${}^{p}R_{ab}(\theta)\,V_b = \begin{pmatrix} C\theta & S\theta \\ -S\theta & C\theta \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} V_xC\theta + V_yS\theta \\ -V_xS\theta + V_yC\theta \end{pmatrix} = \begin{pmatrix} V_{x'} \\ V_{y'} \end{pmatrix}$$

and yes, you see that $V_{y'}$ is definitely smaller than Vy in the sense I mean because we have the – sign in that term. If **V** were at 45 degrees, then if θ were 45 degrees as well, we would have $V_{y'} = 0$.

Now what about the T_{ij} rule above. What notation can we use for this? The dyadic notation would be sort of interesting here, because we could say:

$$T_{ij} = \mathbf{e}^{(i)} \bullet T \bullet \mathbf{e}^{(j)} = (\mathbf{e}^{(i)},\, T\,\mathbf{e}^{(j)}) = (\mathbf{e}^{(i)},\, T\,\mathbf{e}^{(j)}) \qquad [T\,\mathbf{e}^{(j)}]_a = T_{ab}\,\mathbf{e}^{(j)}{}_b$$

And then the thing on the left ought to be

$T_{i'j'} = \mathbf{e}'^{(i)} \bullet T \bullet \mathbf{e}'^{(j)}$

so I guess I will use the double notation $T_{i'j'}$. So here then are our cleaned up rules:

$V_{a'} = {}^{p}R_{ab}(\theta)\, V_b$ $\qquad\qquad$ $\mathbf{e}'^{(a)} = {}^{a}R(\theta)\, \mathbf{e}^{(a)}$

$T_{a'b'} = {}^{p}R_{ai}(\theta)\ {}^{p}R_{bj}(\theta)\ T_{ij}$

In contrast, my "usual" active rotation of a vector would be written like this:

$V'_a = {}^{a}R_{ab}(\theta)\, V_b$ $\qquad$ or $\qquad$ $\mathbf{V}' = {}^{a}R(\theta)\, \mathbf{V}$

Now we really do have a new vector and it is called **V'**.

2. Consideration of various operators in our E^3 space. My conjectured "rule" of interest has this form:"

$R^{-1}\mathbf{A}\, R = R\, \mathbf{A}$ $\qquad$ meaning: $\qquad$ $R^{-1}A_k\, R = [R\, \mathbf{A}]_k = R_{ki}\, A_i$

where both R's and R are all active, or all are passive. In the 3x3 space E^3 I imagine this rule becomes

$R^{-1}\mathbf{A}\, R = R\, \mathbf{A}$ $\qquad$ meaning: $\qquad$ $R^{-1}A_k\, R = [R\, \mathbf{A}]_k = R_{ki}\, A_i$

We can write this out showing matrix indices to get:

$(R^{-1})_{ab}(A_k)_{bc}\, R_{cd} = R_{ki}\, (A_i)_{ad}$

where R and A_k are 3x3 matrices, and I just indicate R by the funny font for historical reasons. We can rewrite the above "conjectured equation" as

$R_{ki}\, (A_i)_{ad} = (R^{-1})_{ab}\, (R^{-1})_{dc}\, (A_k)_{bc}$

Now let's nail things down by saying that R is an active rotation, so the above becomes

${}^{a}R_{ki}\, (A_i)_{ad} = ({}^{a}R^{-1})_{ab}\, ({}^{a}R^{-1})_{dc}\, (A_k)_{bc} = ({}^{p}R)_{ab}\, ({}^{p}R)_{dc}\, (A_k)_{bc}$

where all rotations are by positive angle θ. We now recognize the RHS as the RHS of our tensor transformation equation which was this:

$T_{a'b'} = {}^{p}R_{ai}(\theta)\ {}^{p}R_{bj}(\theta)\ T_{ij}$ $\qquad$ or $\qquad$ $(A_k)_{a'd'} = {}^{p}R_{ab}(\theta)\ {}^{p}R_{dc}(\theta)\ (A_k)_{bc}$

Thus, our conjectured "rule" implies the following:

$(A_k)_{a'd'} = {}^{a}R_{ki}\, (A_i)_{ad}$

This says that the ad matrix element of the tensor A_k observed in a primed frame (which is rotated in a positive sense relative to the unprimed frame) can be expressed as a linear combination of the same matrix elements of the various components of A_i weighted by the numbers ${}^aR_{ki}$. This could be written out as

$$(\mathbf{e}'^{(a)}, A_k \mathbf{e}'^{(d)}) = {}^aR_{ki} (\mathbf{e}^{(a)}, A_i \mathbf{e}^{(d)})$$

Process this a little bit,

$$({}^aR \mathbf{e}^{(i)}, A_k {}^aR \mathbf{e}^{(d)}) = (\mathbf{e}^{(a)}, {}^aR_{ki} A_i \mathbf{e}^{(d)})$$

$$(\mathbf{e}^{(i)}, {}^aR^{-1} A_k {}^aR \mathbf{e}^{(d)}) = (\mathbf{e}^{(a)}, {}^aR_{ki} A_i \mathbf{e}^{(d)})$$

and at least we have recovered our conjectured operator equation ${}^aR^{-1} A_k {}^aR = {}^aR_{ki} A_i$ which we started with, just an error check.

I sure wish I could think of some classical operator in E^3 space. Could we say that R_k is a vector operator with matrix elements $(R_k)_{ab}$? I don't thinks so. Or how about the rotation generators J_k with $(J_k)_{ab}$. That seems more viable. But these generators are not physical E^3 things I can picture.

If I cannot imagine even a single example of an operator A_k in E^3, it is pretty hard to say that E^3 is a useful place to try to gain incite on our fancy equation. I can think of scalar operators like the polarization tensor.

If I jump to quantum mechanics then I can think of lots of vector operators. But then I have to say what the basis states are. If they are momentum states, then we can say, setting a = d,

$$(\mathbf{e}'^{(a)}, A_k \mathbf{e}'^{(d)}) = {}^aR_{ki} (\mathbf{e}^{(a)}, A_i \mathbf{e}^{(d)}) \rightarrow \langle p' | P_k | p' \rangle = {}^aR_{ki} \langle p | P_i | p \rangle$$

and then this says $p'_k = {}^aR_{ki}\, p_i$ which says $\mathbf{p}' = {}^aR\, \mathbf{p}$ which makes me very happy.

Another example: imagine we have some kind of basis states, and we have a vector operator. Then our equation above says:

$$\langle a' | P_k | a' \rangle = {}^aR_{ki} \langle a | P_i | a \rangle \quad \text{or} \quad \langle \mathbf{P} \rangle' = {}^aR \langle \mathbf{P} \rangle$$

and now state $|a\rangle$ can be any QM base state, just necessarily a momentum eigenstate. Perhaps it is an energy eigenstate. Our rule is saying that the *expectation value* of a vector operator as viewed from rotated base states is just R times the same expectation value viewed from the unrotated states.

8. What does Tinkham have to say on this subject?

His book title is "Group theory and Quantum Mechanics", so we are inside QM all the time here. On page 125 we see our equation of interest in this form

$$\mathbf{V}' = P_R \mathbf{V} P_R^{-1} = R^{-1} \mathbf{V}$$

which of course matches our equation as stated at the top of this document. This is a *definition* of a "vector operator", and he writes out the component equation just as I have done.

Tinkham's notation is unfamiliar to me (right now), so let's back up a little to grok it better. On page 32 we have

$$P_{\mathbf{R}}f(\mathbf{x}) = f(R^{-1}\mathbf{x})$$

which I would write as

$$P_{\mathbf{R}}f(\mathbf{x}) \equiv \langle \mathbf{x}|R|f\rangle = \langle R^{-1}\mathbf{x}\,|f\rangle = \langle\, R^{-1}\mathbf{x}\,|f\rangle = f(R^{-1}\mathbf{x}) =$$

Tinkham works exclusively in the coordinate representation, and you don't see bras and kets anywhere in his book. Fine. So in the notation above, R is my Hilbert Space operator, R is a 3x3 matrix, $f(\mathbf{x})$ is a scalar function of space, and $P_{\mathbf{R}}$ is something I don't use much but which I might call $\hat{R}$, the coordinate space representation of the rotation operator.

Now armed with the above, let's try to decipher page 125 equation A which says

$$\hat{O}'\psi' = (\hat{O}\psi)' \quad \text{where } \psi' = \hat{P}_{\mathbf{R}}\psi$$

where I now insist on putting a hat on everything I think is a coordinate-space operator (as opposed to an abstract QM Hilbert Space operator). Like $P_{\mathbf{R}}$, I think O and O' are coordinate-space operators. My guess is that we can regard $O\psi$ as some new function φ, say, and then I guess that we have

$$(\hat{O}\psi)' = \varphi' = \hat{P}_{\mathbf{R}}\varphi = \hat{P}_{\mathbf{R}}(\hat{O}\psi) \qquad \text{// ie, this is the meaning of } (O\psi)'$$

Then his equation A is saying this:

$$\hat{O}'\,\psi'(x) = \hat{O}'\,\hat{P}_{\mathbf{R}}\,\psi(x) = \hat{P}_{\mathbf{R}}(\hat{O}\psi) = \hat{P}_{\mathbf{R}}\hat{O}\,\psi(x)$$

which I write again as

$$\hat{O}'\,\hat{P}_{\mathbf{R}}\,\psi(x) = \hat{P}_{\mathbf{R}}\hat{O}\,\psi(x) \qquad \text{// which is his equation B}$$

from which we can conclude that

$$\hat{O}' = \hat{P}_{\mathbf{R}}\hat{O}\hat{P}_{\mathbf{R}}^{-1} \qquad \text{// which is his equation 5.54 = C}$$

So fine, he is just doing what I usually do, but he works in the space of functions of space, instead of the space of abstract kets where we don't select a basis.

The operator O appearing above need not be a scalar. This just defines an operator O' which is in a new reference frame, though he does not say that right here. Let's now drop the hats. I think he should say at this point.

$$O' = ROR^{-1} = O \qquad \text{defines a scalar operator}$$

$\mathbf{V}' = R\mathbf{V}R^{-1} = R^{-1}\mathbf{V}$ defines a vector operator, and so on.

Now he makes an odd comment. He says that $P_R f(\mathbf{x}) = f(R^{-1}\mathbf{x})$ is a "convention". Well, I suppose someone else might have said $Q_R f(\mathbf{x}) = f(R\mathbf{x})$ and then $Q_R = P_{Rinv}$ which would be pretty strange. You would have

$$Q_R f(x) \equiv \langle \mathbf{x} | R^{-1} | f \rangle = \langle R\,\mathbf{x} | f \rangle$$

which I suppose would be an OK thing to do if you stay in coordinate space all the time. So OK, I accept it as a convention, and I approve of the convention he has selected!

So Tinkham basically DEFINES a vector operator by the transformation rule. He then goes on to compare such a Cartesian vector to a spherical vector and here his work has been valuable to me, hard to find this written up anywhere. He goes on to talk about Cartesian and spherical tensors, same comment. But as far as "our equation" goes, he agrees with us, and has no derivation insights to add. His opening "requirement" that $O'\psi' = (O\psi)'$ does nothing for me at all. Why should this be true, what does it mean?

Comment: Tinkham got me more interested in the coordinate-space representation, which is of course what we know as wave mechanics, as opposed to matrix mechanics. This got me off on another tangent, trying to interpret certain wave-mechanics statements in matrix mechanics.

9. Lorentz transformations of (1) vector operators and (2) vector field operators.

Let's make three jumps at once:

(1) from 3-vector operators to 4-vector operators;
(2) these operators are fields so functions of spacetime;
(3) from rotations to Lorentz transformations.

We want to know "What is the transformation rule similar to $R\ \mathbf{V}\ R^{-1} = R^{-1}\ \mathbf{V}$, but for a 4-vector operator+field under a Lorentz transformation?

A. Collection of irrelevant information. (skip this subsection)

Because we have operators which are functions of space, the coordinate representation plays a special role, so let's use it where possible.

Let's consider the Lorentz Transformation Λ using Tinkham's coordinate representation to say:

$$\hat{\Lambda}\,\psi(x) = \langle x | \Lambda | \psi \rangle = \langle \Lambda^{\dagger} x | \psi \rangle = \psi(\Lambda^{\dagger} x) = \psi(\Lambda^{\dagger} x)$$

and this would be true for rotations, boosts, or general combinations. If the states $\langle x|$ are normalized covariantly, then we know $\Lambda^{\dagger} = \Lambda^{-1}$ (unitary).

Aside: Now here is one useful fact which I noticed somewhere in Messiah:

$$\langle \mathbf{x} | O(\mathbf{X}) | \psi \rangle = O(\mathbf{x}) \langle \mathbf{x} | \psi \rangle = O(\mathbf{x})\, \psi(\mathbf{x})$$

Notice that O(**X**) is an operator in the HS, whereas O(**x**) is just a function of space. You justify the above fact by thinking of O(**X**) , say, as a polynomial in **X**, and you move **X** off to the left where it hits the <**x** | and turns into just "**x**" outside. This works because **X** is a Hermitian operator so that $\mathbf{X}^\dagger = \mathbf{X}$. I put this comment here, but don't think I will be using it for the time being.

B. Lorentz transformations of a 4-vector field.

Now how does a regular (ie, not an operator) 4-vector transform under a LT? What is the meaning of "active rotation" of an object when that object which is a function of x? I conjecture that this is the correct rule for the transformation of a vector field (not a field operator, just a field) under Lorentz transformations:

$$(V')^\mu(x') = \mathbf{\Lambda}^\mu{}_\nu \; V^\nu(x) \quad \text{or} \quad V'(x') = \mathbf{\Lambda}\; V(x) \qquad \text{where } x' = \mathbf{\Lambda}x \qquad (1)$$

$$(V'')^\mu(x'') = (\mathbf{\Lambda}^{-1})^\mu{}_\nu \; V^\nu(x) \quad \text{or} \quad V''(x'') = \mathbf{\Lambda}^{-1}V(x) \qquad \text{where } x'' = \mathbf{\Lambda}^{-1}x \qquad (2)$$

We are observing the vector V in two transformed frames of reference. The points x and x' and x" are the same point in this "passive" point of view, but this same point has different coordinates to the three observers. Notice, by the way, the similarity between the first equation above and the Dirac idea from the bra-kets document

$$\psi'_i(x') = S(a)_{ij}\, \psi_j(x) \qquad (3)$$

In the vector case, μ is a component index, and that is what i is in the spinor case. We briefly discussed the transformation of "wavefunctions" in Section 4.1 above. Here we can think of $V^\mu(x)$ as a wavefunction that transforms according to the vector representation of the Lorentz Group.

C. Lorentz transformations of a 4-vector field operator

I shall attempt to follow BD2 page 21 as carefully as possible because the result is tricky. They start off with a non-operator transformation like (3) above which appears as equation A embedded in their text. They write this as follows, where they are being "generic": [see also section 4.1 above]

$$\varphi'_r(x') = S(a)_{rs}\, \varphi_s(x) \qquad (4)$$

where r and s indicate "components" of some unspecified field type and S(a) is the appropriate matrix which is some representation of the Lorentz Group. To understand where the above equation comes from in the Dirac Theory at least, see "bras and kets" doc. BD2 refer to $\varphi_s(x)$ as a "classical field amplitude". For a vector field, I have stated in subsection B above that we would write

$$(V')^\mu(x') = \mathbf{\Lambda}^\mu{}_\nu \; V^\nu(x) \qquad (4) \text{ vector field}$$

Then they try to re-express equation (4) in "quantum field theory" where φ is no longer a wavefunction (a field) but becomes a field operator. So they write this for their re-expression of (4)

$$\langle\alpha'|\,\hat{\varphi}_{\mathbf{r}}(x')\,|\beta'\rangle = S(a)_{\mathbf{rs}}\,\langle\alpha|\,\hat{\varphi}_{\mathbf{s}}(x)\,|\beta\rangle \qquad (5)$$

where I put a hat on φ to indicate it is a field operator. The identification they are making therefore is this:

$$\varphi_{\mathbf{s}}(x) = \langle\alpha|\,\hat{\varphi}_{\mathbf{s}}(x)\,|\beta\rangle \qquad (6a)$$

$$\varphi'_{\mathbf{s}}(x') = \langle\alpha'|\,\hat{\varphi}_{\mathbf{s}}(x')\,|\beta'\rangle \qquad (6b)$$

This (6b) is the crucial item here and BD2 do comment on it prior to 11.64. Here, we "move the kets forward" and we *don't* move the operator forward, it does not acquire a prime. However, the argument of the field operator is expressed in the primed frame coordinates because that is what we have in the classical field amplitude. Remember that x and x' are really the same point, just written in the different observer's coordinates. By the way, in Section 2 above concerning symmetry, we talked about this idea of doing a transformation where the states move and the operator does not move. In the last paragraph there, we talked about how we could "continue" if we knew how the U sandwich acted on operator O.

In the single electron case, I guess $\hat{\varphi}$ would create an electron from the vacuum, so |β> could be the vacuum and <α| could be a single electron state. In that case, the above correspondence seems very reasonable to me, and therefore so too does the transformation rule (5).

Now let's do the usual idea of transforming kets: (forward)

$$|\beta'\rangle = \Lambda\,|\beta\rangle$$

where Λ is my unitary Hilbert Space Lorentz transformation. Then rule (5) becomes

$$S(a)_{\mathbf{rs}}\,\langle\alpha|\,\hat{\varphi}_{\mathbf{s}}(x)\,|\beta\rangle = \langle\alpha'|\,\hat{\varphi}_{\mathbf{r}}(x')\,|\beta'\rangle = \langle\alpha|\,\Lambda^{-1}\,\hat{\varphi}_{\mathbf{r}}(x')\,\Lambda\,|\beta\rangle$$

and we then have this implied result

$$\Lambda^{-1}\,\hat{\varphi}_{\mathbf{r}}(x')\,\Lambda = S(\Lambda)_{\mathbf{rs}}\;\hat{\varphi}_{\mathbf{s}}(x) \qquad x' = \mathbf{\Lambda}x \qquad (7a)$$

Now in the above suppose we take $\Lambda \rightarrow \Lambda^{-1}$ (including $x' = \mathbf{\Lambda}^{-1}x$). Then we have

$$\Lambda\,\hat{\varphi}_{\mathbf{r}}(x')\,\Lambda^{-1} = S(\Lambda^{-1})_{\mathbf{rs}}\;\hat{\varphi}_{\mathbf{s}}(x) \qquad x' = \mathbf{\Lambda}^{-1}x \qquad \Rightarrow \qquad x = \mathbf{\Lambda}x'$$

Now swap the dummy symbols x and x' and rewrite the above to get

$$\Lambda\,\hat{\varphi}_{\mathbf{r}}(x)\,\Lambda^{-1} = S(\Lambda^{-1})_{\mathbf{rs}}\;\hat{\varphi}_{\mathbf{s}}(x') \qquad x' = \mathbf{\Lambda}x \qquad (7b)$$

and this is **The Big Result** I have been after. BD2 in fact extend this to $x' = \mathbf{\Lambda}x + b$ which is an arbitrary Poincare group transformation, and then they replace operator Λ with operator U(a,b) which represents an arbitrary and unitary Poincare transformation. We then get their 11.67.

The above agrees with Moore's result

$$U(\omega)\varphi_a(x)U^*(\omega) = D^{-1}_{ab}(\omega)\varphi_b(\Lambda x)\,, \qquad (3.3.1)$$

where Moore uses * to indicate † for the Hilbert Space operator.

Now let's specialize our result above to the case of a 4-vector field operator and write

$$\Lambda\, A^{\mu}(x)\, \Lambda^{-1} \;=\; (\mathbf{\Lambda}^{-1})^{\mu}{}_{\nu}\, A^{\nu}(x') \qquad\qquad x' = \mathbf{\Lambda} x \qquad\qquad (8)$$

and again we invert this first, and then write it again with the swap $x \leftrightarrow x'$:

$$\begin{aligned} \Lambda^{-1}\, A^{\mu}(x)\, \Lambda \;&=\; (\mathbf{\Lambda})^{\mu}{}_{\nu}\, A^{\nu}(x') \qquad\qquad x' = \mathbf{\Lambda}^{-1} x \quad , x = \mathbf{\Lambda} x' \\ \Lambda^{-1}\, A^{\mu}(x')\, \Lambda \;&=\; (\mathbf{\Lambda})^{\mu}{}_{\nu}\, A^{\nu}(x) \qquad\qquad x = \mathbf{\Lambda}^{-1} x' \quad , x' = \mathbf{\Lambda} x \end{aligned} \qquad (9)$$

Now let's interrupt to go back to our 3-vector case where we wrote these vector transformation rules (see equations 5.1 and 5.2 in Section 5 above, where there I used **P**)

$$\mathbf{V}' \;\equiv\; R\, \mathbf{V}\, R^{-1} \;= R^{-1}\mathbf{V} \qquad \text{// operator forward} \qquad (10a)$$

$$\mathbf{V}'' \;\equiv\; R^{-1}\mathbf{V}\, R \;= R\;\; \mathbf{V} \qquad \text{// operator backward} \qquad (10b)$$

and we showed how this was consistent with the required transformation of a QM matrix element:

$$\mathbf{E}' = \text{'}\langle \alpha \,|\, \mathbf{V} \,|\, \beta \rangle\text{'} \;= \langle \alpha \,|\, R^{-1}\mathbf{V}\, R \,|\, \beta \rangle \;= \langle \alpha \,|\, \mathbf{V}'' \,|\, \beta \rangle$$

$$= \langle \alpha \,|\, R\;\; \mathbf{V} \,|\, \beta \rangle \;= R \langle \alpha \,|\, \mathbf{V} \,|\, \beta \rangle = R\, \mathbf{E} \qquad\qquad (10c)$$

In our 4-vector and LG case, we have the following analogous transformations:

$$(A')^{\mu}(x') \;\equiv\; \Lambda\, A^{\mu}(x)\, \Lambda^{-1} \;=\; (\mathbf{\Lambda}^{-1})^{\mu}{}_{\nu}\, A^{\nu}(x') \qquad x' = \mathbf{\Lambda} x \qquad \text{// operator forward} \qquad (11a)$$

$$(A'')^{\mu}(x') \;\equiv \Lambda^{-1}\, A^{\mu}(x')\, \Lambda \;=\; (\mathbf{\Lambda})^{\mu}{}_{\nu}\, A^{\nu}(x) \qquad x' = \mathbf{\Lambda} x \qquad \text{// operator backward} \qquad (11b)$$

where as in the 3-vector case, I just *define* some A' and A" operators to the left of the transformation equations. Now we can write down the corresponding transformation of a QM matrix element:

$$(\mathbf{E}')^{\mu}(x') = \text{'}\langle \alpha \,|\, A^{\mu}(x') \,|\, \beta \rangle\text{'} = \langle \alpha \,|\, \Lambda^{-1} A^{\mu}(x')\, \Lambda \,|\, \beta \rangle = \langle \alpha \,|\, (A'')^{\mu}(x') \,|\, \beta \rangle$$

$$= \langle \alpha \,|\; (\mathbf{\Lambda})^{\mu}{}_{\nu}\; A^{\nu}(x) \,|\, \beta \rangle \;\;=\;\; (\mathbf{\Lambda})^{\mu}{}_{\nu}\; \langle \alpha \,|\, A^{\nu}(x) \,|\, \beta \rangle \;= (\mathbf{\Lambda})^{\mu}{}_{\nu}\, E^{\nu}(x) \qquad (11c)$$

and I suppose if we don't show the 4-vector indices we can write this again as (where $x' = \mathbf{\Lambda} x$)

$$(\mathbf{E}')(x') = \text{'}\langle \alpha \,|\, \mathbf{A}(x') \,|\, \beta \rangle\text{'} = \langle \alpha \,|\, \Lambda^{-1}\mathbf{A}(x')\, \Lambda \,|\, \beta \rangle = \langle \alpha \,|\, (\mathbf{A}'')(x') \,|\, \beta \rangle$$

$$= \langle \alpha \,|\; (\mathbf{\Lambda})\mathbf{A}(x) \,|\, \beta \rangle \;\;=\;\; (\mathbf{\Lambda}) \langle \alpha \,|\, \mathbf{A}(x) \,|\, \beta \rangle \;= (\mathbf{\Lambda})\mathbf{E}(x) \qquad (11c')$$

where I use bold to indicate a 4-vector (something I don't normally do). The important new item here is where to you put x, and where do you put x', in the various fields. I think I could argue that the requirement for 11.c to be true is enough for me to derive 11.b even without peeking at BD2 (but I was having trouble getting it right until I looked at BD2).

D. Conclusions: We started with these rules for how 3-vector operators transform under rotations:

$$\mathbf{V}' \equiv R\,\mathbf{V}\,R^{-1} = R^{-1}\mathbf{V} \quad \text{// operator forward} \quad (10a)$$

$$\mathbf{V}'' \equiv R^{-1}\mathbf{V}\,R = R\,\mathbf{V} \quad \text{// operator backward} \quad (10b)$$

where R is a unitary rotation operator in the QM Hilbert space of interest. We then should have stated the corresponding results for a 4-vector operator in a covariantly normalized Hilbert space in which we know that the Lorentz transformation Λ is unitary for boosts and rotations or any combination,

$$(A')^{\mu} \equiv \Lambda\, A^{\mu}\, \Lambda^{-1} = (\mathbf{\Lambda}^{-1})^{\mu}{}_{\nu}\, A^{\nu} \quad \text{// operator forward} \quad (11a)$$

$$(A'')^{\mu} \equiv \Lambda^{-1}\, A^{\mu}\, \Lambda = (\mathbf{\Lambda})^{\mu}{}_{\nu}\, A^{\nu} \quad \text{// operator backward} \quad (11b)$$

And then finally we generalized the results to 4-vector field operators where we had

$$(A')^{\mu}(x') \equiv \Lambda\, A^{\mu}(x)\, \Lambda^{-1} = (\mathbf{\Lambda}^{-1})^{\mu}{}_{\nu}\, A^{\nu}(x') \quad x' = \mathbf{\Lambda}x \quad \text{// operator forward} \quad (11a)$$

$$(A'')^{\mu}(x') \equiv \Lambda^{-1}\, A^{\mu}(x')\, \Lambda = (\mathbf{\Lambda})^{\mu}{}_{\nu}\, A^{\nu}(x) \quad x' = \mathbf{\Lambda}x \quad \text{// operator backward} \quad (11b)$$

And going one more step with help of BD2, we see how an arbitrary field which belongs to some representation of the Lorentz Group transforms. First, here would be the pure operator transformation

$$(A')_{\mathbf{r}} \equiv \Lambda\, A_{\mathbf{r}}\, \Lambda^{-1} = S(\mathbf{\Lambda}^{-1})_{\mathbf{rs}}\, A_{\mathbf{s}} \quad \text{// operator forward}$$

$$(A'')_{\mathbf{r}} \equiv \Lambda^{-1}\, A_{\mathbf{r}}\, \Lambda = S\,(\mathbf{\Lambda})_{\mathbf{rs}}\, A_{\mathbf{s}} \quad \text{// operator backward}$$

and we then regard our rule $R\,\mathbf{V}\,R^{-1} = R^{-1}\mathbf{V}$ as a special case of this where $\Lambda = R$, where $S\,(\mathbf{\Lambda})_{\mathbf{rs}} = R$ and where dim = 3. For the Poincare group, there is no change to the above.

For *field* operators we just add x and x' in the right places

$$(A')_{\mathbf{r}}(x') \equiv \Lambda\, A_{\mathbf{r}}(x)\, \Lambda^{-1} = S(\mathbf{\Lambda}^{-1})_{\mathbf{rs}}\, A_{\mathbf{s}}(x') \quad x' = \mathbf{\Lambda}x \quad \text{// operator forward}$$

$$(A'')_{\mathbf{r}}(x') \equiv \Lambda^{-1}\, A_{\mathbf{r}}(x)\, \Lambda = S\,(\mathbf{\Lambda})_{\mathbf{rs}}\, A_{\mathbf{s}}(x) \quad x' = \mathbf{\Lambda}x \quad \text{// operator backward}$$

and for Poincare we replace $\Lambda \rightarrow U(\Lambda,b)$ and set $x' = \mathbf{\Lambda}x + b$.