Magnetic Moment of the Electron PhL 11.2.08

One identifies this "moment" by looking for a **B•S** term "somewhere".

1. How spin appears out of nowhere in the Dirac equation non-rel limit

In BD page 11, they show in 1.26 the Hamiltonian with the usual $\boldsymbol{\pi} = \mathbf{p} - e/c\mathbf{A}$ term in place of **p** to "bring in" the EM field, along with φ. The **α** matrix consists of off-diagonal **σ** matrices, so when we break things down into 2-spinors, 1.26 looks like this:

$$H = \boldsymbol{\alpha}\bullet\boldsymbol{\pi} + \beta m + e\varphi = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \bullet \pi + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Phi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m+\Phi & \boldsymbol{\sigma}\bullet\boldsymbol{\pi} \\ \boldsymbol{\sigma}\bullet\boldsymbol{\pi} & -m+\Phi \end{pmatrix}$$

Using two components, we then have

$$H\begin{pmatrix} f \\ g \end{pmatrix} = \left\{ \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \bullet \pi + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Phi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\pi}\bullet\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\pi}\bullet\boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} + \begin{pmatrix} m+\Phi & 0 \\ 0 & -m+\Phi \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$= \boldsymbol{\pi}\bullet\boldsymbol{\sigma}\begin{pmatrix} g \\ h \end{pmatrix} + \begin{pmatrix} m+\Phi & 0 \\ 0 & -m+\Phi \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{//= flip + nonflip}$$

Now we rescale by saying $f = \varphi\, e^{-imt}$ and $g = \chi\, e^{-imt}$ and the time-dependent SE becomes

$$i\partial_t\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \boldsymbol{\pi}\bullet\boldsymbol{\sigma}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} + \begin{pmatrix} \Phi & 0 \\ 0 & -2m+\Phi \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad \text{//= flip + nonflip}$$

which we just write out as two separate equations:

$$i\partial_t\varphi = c\boldsymbol{\pi}\bullet\boldsymbol{\sigma}\,\chi + \Phi\,\varphi$$
$$i\partial_t\chi = c\boldsymbol{\pi}\bullet\boldsymbol{\sigma}\,\varphi + (-2mc^2+\Phi)\,\chi$$

where I have added back the c's. In the *non-rel limit* in the second equation we ignore terms that don't contain c to get

$$0 \approx c\boldsymbol{\pi}\bullet\boldsymbol{\sigma}\,\varphi - 2mc^2\chi \qquad \Rightarrow \qquad \chi \approx (\boldsymbol{\pi}\bullet\boldsymbol{\sigma}/2mc)\,\varphi \quad = \text{the small spinor}$$

Now sub this in for χ in the first equation to get

$$i\partial_t\varphi = c\boldsymbol{\pi}\bullet\boldsymbol{\sigma}\ (\boldsymbol{\pi}\bullet\boldsymbol{\sigma}/2mc)\,\varphi + \Phi\,\varphi = [\,\boldsymbol{\pi}\bullet\boldsymbol{\sigma}\ \boldsymbol{\pi}\bullet\boldsymbol{\sigma}\,/2m + \Phi]\ \varphi$$

We can now interpret [] as the non-rel Ham for 2-spinor φ. We then go off and do some math to find that

$\boldsymbol{\pi}\bullet\boldsymbol{\sigma}\ \ \boldsymbol{\pi}\bullet\boldsymbol{\sigma}\ = \boldsymbol{\pi}^2 - e\hbar/c\ \boldsymbol{\sigma}\bullet\mathbf{B}$ where $\boldsymbol{\pi} = \mathbf{p} - e/c\mathbf{A}$ $\mathbf{B} = \nabla x \mathbf{A}$

This last piece of math is not very obvious, but I did the details in the raw notes, there is nothing unusual about it. So we end up with this:

$$i\partial_t\varphi = [\ \boldsymbol{\pi}\bullet\boldsymbol{\sigma}\ \ \boldsymbol{\pi}\bullet\boldsymbol{\sigma}\ /2m + \Phi]\ \varphi$$

$$= (\mathbf{p} - e/c\mathbf{A})^2/2m - e\hbar/2mc\ \boldsymbol{\sigma}\bullet\mathbf{B}$$
$$= (\mathbf{p} - e/c\mathbf{A})^2/2m - \mu_B\ \boldsymbol{\sigma}\bullet\mathbf{B}$$
$$= (\mathbf{p} - e/c\mathbf{A})^2/2m - 2\ \mu_B\ \mathbf{S}\bullet\mathbf{B}$$

where $\mathbf{S} = 1/2\boldsymbol{\sigma}$ and carries no $\hbar$ dimensions. So here is the famous **Bohr magneton** falling out in our lap with the **gyro factor of 2**. So this is how "spin" falls out of the Dirac theory. The gyro factor is 2.0000.. !

2. Now, what does this have to do with Feynman graphs?

Remember that Feynman said his approach is about the *solution* of the problem, not the Hamiltonian. Our scattering matrix S_{fi} is all about scattering processes, not looking at terms in Hamiltonians to find what the magnetic moment is. S_{fi} is a scattering amplitude. So I am a bit puzzled here.

Well, BD has not stressed it very much, but we are really doing perturbation theory. We know the exact solution to the Dirac equation with no EM fields, we know how to add in the EM, but we don't know how to solve the resulting theory exactly so we do perturbation theory.

Perturbation theory works in terms of an interaction Hamiltonian H'. If we look at 1.26, we should regard this as

$H' = \boldsymbol{\alpha}\bullet(-e\mathbf{A}) + e\varphi = \gamma^0\boldsymbol{\gamma}\bullet(-e\mathbf{A}) + \gamma^0\gamma^0 e\varphi = e\gamma^0\gamma^\mu A_\mu = e\gamma^0 \not{A}$ since $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$

Then let's compute our first order perturbation theory matrix element:

$$\int d^3x\ \psi^\dagger H'\ \psi$$
$$= \int d^3x\ \psi^\dagger [e\gamma^0\gamma^\mu A_\mu]\ \psi$$
$$= \int d^3x\ e\ \bar{\psi}\gamma^\mu\psi\ A_\mu$$
$$= \int d^3x\ J^\mu A_\mu$$

Now, in this first order perturbation theory evaluation of the energy, where do we find our $\boldsymbol{\sigma}\bullet\mathbf{B}$ thing so we can identify our basic magnetic moment? From page 13 we can arrange for a uniform **B** field by doing this:

$\mathbf{A} = \frac{1}{2}\ \mathbf{B}\ x\ \mathbf{r}$ $\varphi = 0$

Then we have from above

$$\int d^3x\, \psi^T H' \psi = -\int d^3x\, \mathbf{J} \bullet \mathbf{A} = -\tfrac{1}{2} e \int d^3x\, \bar{\psi} \gamma \psi \bullet (\mathbf{B} \times \mathbf{r}) = -\tfrac{1}{2} e\, \mathbf{B} \bullet \int d^3x\, (\mathbf{r} \times \bar{\psi} \gamma \psi)$$

I just perused my own ancient notes and see myself there following an approach like this. Suppose I just identify from the above the magnetic moment to be

$$\mathbf{m} = \tfrac{1}{2} e \int d^3x\ \mathbf{r} \times \bar{\psi} \gamma \psi = \tfrac{1}{2} \int d^3x\ \mathbf{r} \times \mathbf{J}$$

and this is just the classical expression for a magnetic moment, good. See Jackson p 146.

So here is one way you could get a perturbation theory expansion for **m**: expand ψ inside the integral! Imagine that we replace ψ here with the exactly correct Ψ to all orders, then we say

$$\mathbf{m} = \tfrac{1}{2} e \int d^3x\ \mathbf{r} \times (\bar{\Psi} \gamma \Psi)$$

But then we do our expansion from earlier notes which says

$$\Psi = \psi + S_F\, e\not{A}\, \Psi$$
$$S_F' = S_F + S_F\, e\not{A}\, S_F'$$

where recall that S_F' is the "full" propagator. So this does imply an expansion for m, which is what I have been looking for. I think to next order we might say

$$\Psi = \psi + S_F\, e\not{A}\, \psi$$
$$\bar{\Psi} = \bar{\psi} + \bar{\psi}\, e\not{A}\, \gamma^0 S_F^\dagger \qquad \text{I think}$$

Then we get something like this for m

$$\mathbf{m} = \tfrac{1}{2} e \int d^3x\ \mathbf{r} \times ([\bar{\psi} + \bar{\psi}\, e\not{A}\, \gamma^0 S_F^\dagger]\, \gamma\, [\psi + S_F\, e\not{A}\, \psi\ ])$$

$$\tfrac{1}{2} e \int d^3x\ \mathbf{r} \times \bar{\psi}\ \{ \gamma + e\not{A}\, \gamma^0 S_F^\dagger\, \gamma + \gamma\, S_F\, e\not{A} + e\not{A}\, \gamma^0 S_F^\dagger\, \gamma\, S_F\, e\not{A}\, \psi \}\, \psi$$

where the last term looks like our "vertex correction" somewhat. So at least this suggests how you might have a Feynman graph expansion which alters the magnetic moment in some way.

3. Where does the Gordon Decomposition fit into all this stuff?

On page 36 we see this written out,

$$j^\mu = \bar\psi_2 \gamma^\mu \psi_1 = (1/2m)\,[\,\bar\psi_2 (i\partial^\mu \psi_1) - (\,i\partial^\mu \bar\psi_2)\,\psi_1\,] - (i/2m)\, i\partial_\nu(\bar\psi_2 \sigma^{\mu\nu} \psi_1)$$

and we recognize the [...] term as the Klein-Gordon type current, and we associate the last term somehow with the Dirac particle spin, and this is sometimes called "the spin current". Now suppose we have

$$\psi_1 = e^{-ip.x}\ u(p,s)\sqrt{m/E}$$
$$\psi_2 = e^{-ip'.x}\, u(p',s)\sqrt{m/E'}$$

Then we get

$$(i\partial^\mu \psi_1) = i\,{*}{-ip^\mu}\,\psi_1 = p^\mu\psi_1$$
$$(i\partial^\mu \psi_2) = i\,{*}{-ip'^\mu}\,\psi_2 = p'^\mu\psi_2$$

$$(-i\partial^\mu \psi_2^{*}) = p'^\mu\psi_2{*}$$
$$(-i\partial^\mu \bar\psi_2) = p'^\mu\bar\psi_2$$
$$(i\partial^\mu \bar\psi_2) = -\,p'^\mu\bar\psi_2$$

$$i\partial_\nu(\bar\psi_2 \sigma^{\mu\nu} \psi_1) = [-\,p'_\nu\bar\psi_2 \sigma^{\mu\nu} \psi_1 + \bar\psi_2 \sigma^{\mu\nu} p_\nu\psi_1\,] = (p-p')_\nu\ \bar\psi_2 \sigma^{\mu\nu} \psi_1$$

So insert these pieces to get

$$j^\mu = \bar\psi_2 \gamma^\mu \psi_1 = (1/2m)\,[\,\bar\psi_2 (i\partial^\mu \psi_1) - (\,i\partial^\mu \bar\psi_2)\,\psi_1\,] - (i/2m)\, i\partial_\nu(\bar\psi_2 \sigma^{\mu\nu} \psi_1)$$

$$= (m/\sqrt{EE'})\ (1/2m)\{\,(p^\mu + p'^\mu)\,\bar u(p',s)\,u(p,s) - i(p-p')_\nu\ \bar u(p',s)\,\sigma^{\mu\nu}\,u(p,s)\}$$

So this must be Gordon in momentum space. In particular, it says [see also BD 3.28 p 37]

$$j^i = (m/\sqrt{EE'})\ (1/2m)\{\,(p^i + p'^i)\,\bar u(p',s)\,u(p,s) - (i/2m)\,(p-p')_\nu\ \bar u(p',s)\,\sigma^{i\nu}\,u(p,s)$$

$$\text{second term} = (m/\sqrt{EE'})\ *\ (-i/2m)\ q_\nu\ \bar u(p',s)\,\sigma^{i\nu}\,u(p,s)$$

and we could put this into our formula for the magnetic moment

$$\mathbf{m} = \tfrac{1}{2}\int d^3x\ \ \mathbf{r}\times\mathbf{j}(\mathbf{r})\ \ =?=\ \tfrac{1}{2}\,(1/2\pi)^3\int d^3p\ i\nabla_p \times \mathbf{j}(p)$$

In some manner I don't understand, I think the first term gives no contribution here and everything comes from the second term. Therefore, if you do some Feynman graph calculation and you find that you have made an alteration to the coefficient of $q_\nu\ \bar u(p',s)\,\sigma^{i\nu}\,u(p,s)$, then you must also be altering the magnetic moment. Sakurai makes some comments to this effect on page 109. This then is the way it works in BD page 172 equation 8.64 where we find the alteration $1 \rightarrow 1 + \alpha/2\pi$.

Well here is a better comment I think. You can see that the first terms are associated with **r** x **p** action after the limit q=0 is taken, so probably the first terms are associated with currents created by the motion of the electrons along some path, that is to say, the electron current has some *orbital angular momentum* and creates a current in that sense. Thus a Bohr H electron has such a current component (but of course we need a packet and we need to integrate over d^3p and all that stuff). So the first term does not necessarily give 0 to the computation of **m**. It is the "orbital" part. The second term houses the intrinsic spin effect, and this is what we care about when we talk about an electron's intrinsic magnetic moment.

Suppose we take an electron at rest. Then you would set $p^i = 0$ and the entire first term stuff in the current would go away. Somehow the last term must NOT go away in some sense because we still have a spin current.

Let's try the form shown above

$$[\nabla_{\mathbf{p}} \times \mathbf{j}(p)]_i = \varepsilon_{ijk}(\partial/\partial p_j)\, j^k = \varepsilon_{ijk}\,(\partial/\partial p_j)\,\left[-(i/2m)\,(p-p')_\nu\ \bar{u}(p',s)\,\sigma^{k\nu}\,u(p,s)\right]$$

$$\approx -(i/2m)\,\varepsilon_{ijk}\,g^{j}{}_{\nu}\,\bar{u}(p',s)\,\sigma^{k\nu}\,u(p,s) = -(i/2m)\,\varepsilon_{ijk}\,g^{j}{}_{\ell}\,\bar{u}(p',s)\,\sigma^{k\ell}\,u(p,s)$$

$$= -(i/2m)\,\varepsilon_{ijk}\ \bar{u}(p',s)\,\sigma^{kj}\,u(p,s) \qquad \text{// product of two antisymmetric tensors}$$

Now we know that (from our pencil TK sheet)

$$\sigma^{kj} = \varepsilon_{kji}\Sigma^i$$

so

$$\varepsilon_{ijk}\,\sigma^{kj} = \varepsilon_{ijk}\,\varepsilon_{kjm}\,\Sigma^m = -\varepsilon_{ijk}\,\varepsilon_{mjk}\,\Sigma^m = 2\,\delta_{im}\,\Sigma^m = 2\,\Sigma^i$$

so we have now shown that

$$[\nabla_{\mathbf{p}} \times \mathbf{j}(p)]_i = -(i/m)\ \bar{u}(p',s)\,\Sigma^i\,u(p,s)$$

and now this really looks "spin like" since $\Sigma = \mathrm{diag}(\sigma,\sigma)$. So we have

$$\mathbf{m} = \tfrac{1}{2}\int d^3x\ \mathbf{r} \times \mathbf{j}(\mathbf{r}) \ \ =?= \tfrac{1}{2}\,(1/2\pi)^3 \int d^3p\ i\nabla_{\mathbf{p}} \times \mathbf{j}(p)$$

$$= \tfrac{1}{2}\,(1/2\pi)^3 \int d^3p\ (-i/m)\,\bar{u}(p,s)\,\boldsymbol{\Sigma}\,u(p,s)$$

$$= \tfrac{1}{2}\,(-i/m) < \boldsymbol{\Sigma} >$$

and in a non-rel limit this will become

$$= \tfrac{1}{2}\,(-i/m) < \boldsymbol{\sigma} > \ = (-i/m) < \mathbf{S} > \qquad S = \text{spin}$$

and I suppose the normalization is wrong, but recall that we usually write

$$\mathbf{m} = g\,\mathbf{S}$$

4. Summary of the last section

The Gordon thing says this, more or less

$$j^\mu = \bar\psi_2 \gamma^\mu \psi_1 = (1/2m)\,[\,\bar\psi_2\,(i\partial^\mu \psi_1) - (\,i\partial^\mu\bar\psi_2)\,\psi_1\,] - (i/2m)\,i\partial_\nu(\bar\psi_2\,\sigma^{\mu\nu}\,\psi_1)$$
$$= (m/\sqrt{EE'})\;(1/2m)\{\,(p^\mu + p'^\mu)\,\bar u(p',s)\,u(p,s)\,\} - (i/2m)\,(p-p')_\nu\;\bar u(p',s)\,\sigma^{\mu\nu}\,u(p,s)$$

$$= \text{const} * \bar u(p',s)\,\{\,(p^\mu + p'^\mu)/2m + i\,q_\nu\sigma^{\mu\nu}/2m\,\}\,u(p,s) \qquad q = p' - p$$

which breaks the electron momentum space current into a convection flow piece (traditional) plus a spin piece.

Now, we have a general formula for magnetic moment of a current, namely

$$\mathbf{m} = \tfrac{1}{2}\int d^3x\;\; \mathbf{r}\times\mathbf{j}(\mathbf{r}) \;\; = \tfrac{1}{2}\,(1/2\pi)^3\int d^3p\; i\nabla_{\mathbf{p}}\times\mathbf{j}(p)$$

and if we insert the $\sigma^{\mu\nu}$ term of j^μ into this we get (math done above)

$$\mathbf{m} = \tfrac{1}{2}\,(1/2\pi)^3\int d^3p\;(-i/m)\;\bar u(p,s)\,\mathbf{\Sigma}\,u(p,s) = \text{constant times } \langle\mathbf{\Sigma}\rangle$$

and in the non-rel limit this becomes

$$\mathbf{m} = \text{constant times } \langle\boldsymbol{\sigma}\rangle = g\,\langle\mathbf{S}\rangle \qquad\qquad \mathbf{S} = \tfrac{1}{2}\,\boldsymbol{\sigma}$$

Therefore, if a Feynman diagram causes an alteration of "the current" which looks like this:

$$j^\mu{}_{\mathtt{altered}} = \text{const} * \bar u(p',s)\,\{\,(p^\mu + p'^\mu)/2m + i\,(1+f)\,q_\nu\sigma^{\mu\nu}/2m\,\}\,u(p,s)$$

then you must conclude that the gyromagnetic ratio has been altered as well

$$g_{\mathtt{altered}} = g(1+f).$$

The "vertex part" diagrams cause just this kind of alteration, where the γ^μ inside the current sees this alteration

$$\gamma^\mu \;\rightarrow\; \gamma^\mu + \Lambda_c{}^\mu$$

as for example in BD1 page 171-2.

5. Group theory comments

Remember that in the Dirac representation we have

$$J^{\mu\nu} = \tfrac{1}{2}\,\sigma^{\mu\nu} = i/4\,[\gamma^{\mu}, \gamma^{\nu}] \qquad \sigma^{\mu\nu} = i/2\,[\gamma^{\mu}, \gamma^{\nu}]$$

so the σ^{ij} are basically the *angular momentum generators* in the Dirac space, and this is what we associate with "spin". So if we associate magnetic moment with "spin", it seems reasonable that it is this $\sigma_{\mu\nu}$ term that is involved.