Introduction to Quantum Field Theory PhL 3.26.05
by Franz Mandl 1959

There is a 1984 revision of this book by Mandl and Shaw, hard to pin down dates as there was a paperback version, etc. Mandl is credited with being a good teacher through his books, he has several of them.

My hardback Wiley Interscience copy of the 1959 edition is *ex libris* Herb Solomon dated 1969, I have no idea where I got this book or who he was.

This book contains 22 chapters of average length 7 pages, total 161 pages of text, then exercises on the entire book, then some solutions to those exercises, then an Appendix on the Dirac equation, then a little bibliography, finally an index.

Chapter 1: The Many-Particle Representation (1). We are going to play with the Klein-Gordon field equation which describes a Lorentz scalar field φ which should keep things simple for a while. We are in full special relativity right off the bat with the KG equation 1.2. Notation is that x_o = t, but in dot products we run 1 → 4 with x_4 = it, so that is how we deal with the metric tensor $g_{\mu\nu}$ which is not mentioned at all. We normalize the KG solutions to a cubical "box" of volume V which quantizes our **k** values to certain eigenvalues called $\mathbf{k}_i$. We then imagine a multi-particle state for example $|n_1\ n_2>$ which means we have n_1 particles in the $\mathbf{k}_1$ state and n_2 in the $\mathbf{k}_2$ state (ie, these being KG solutions or wavefunctions, but he does not write down the solution wavefunctions). We are doing scalar bosons here so Bose-Einstein statistics allows n>1. He calls these bosons "mesons".

We then define the a(**k**) and $a^\dagger(\mathbf{k})$ *operators* which create and destroy quanta with a appropriate multipliers so that $a^\dagger(\mathbf{k})\ a(\mathbf{k})\ |n..> = n\ |n..>$, the "occupation number operator". In 1959, this meson particle is thought to mediate strong interactions with a very short range, Yukawa is mentioned.

Comment on the KG equation: M points out that the *number* equation $E^2 = \mathbf{p}^2 + m^2$ becomes the *KG equation* when you make the usual QM operator replacements $\mathbf{p} = -i\hbar\nabla$ and $E = +i\hbar\partial_t$. So, it is hard to imagine any model of a field ψ for a free particle where ψ does NOT satisfy the KG equation.

You can of course just write $p_\mu = (\mathbf{p}, iE)$ so that $p^2 = \mathbf{p}^2 - E^2$ (Mandl metric) and then we get $p^2 = -m^2$ or $(p^2 + m^2) = 0$. But also $\partial_\mu = (\nabla, i\partial_t)$ and $\partial^2 = \nabla^2 - \partial_t^2$. Then also $p_\mu = -i\hbar\partial_\mu$ so $p^2 = -\partial^2$ and we get $(\partial^2 - m^2) = 0$ which is the KG operator equation. The plane wave solution to the KG equation $(\partial^2 - m^2)\psi = 0$ is $\psi = \exp(\pm ipx)$ because $\partial^2 \exp(\pm ipx) = -p^2 \exp(\pm ipx)$ where here p is a 4-vector c-number.

If we do this same thing in the non-rel "particle" world, we would start with $E = \mathbf{p}^2/2m$ and replace the operators and end up with the SE for a free particle, which is $+i\hbar\partial_t = -\nabla^2/2m$.

How does this compare to the traditional "wave equation" which says $(\partial_t^2 - v^2\nabla^2)f = 0$? If we assume a solution of the same form $f = \exp(\pm ipx) = \exp(\pm i[\ \mathbf{p}\bullet\mathbf{x} - Et\ ])$ we find that we have a solution as long as $E^2 = v^2\mathbf{p}^2$ or $E = v\ |\ \mathbf{p}\ |$. For a "light wave" where m=0, the wave equation and the KG equation are exactly the same and v = c = 1.

Chapter 2: Classical Fields (9) A second (upper) index α is attached to φ^α to handle things like A^μ which has more than one "component". Mandl likes fields to be real. He writes the Lagrangian action in 2.3, varies it and gets the EL equations, same as Goldstein, but in a more compact notation where

$\varphi_{,\nu} \equiv \partial\varphi/\partial x_\nu$

With the i-metric, all vector indices are lower. Our EL equations are at once in covariant form, matching Goldstein page 354 11-24. So we have

$\partial\mathcal{L}/\partial\varphi - \partial_\nu(\partial\mathcal{L}/\partial\varphi_{,\nu}) = 0$ Mandl 2.8 omitting α index

and M does not use the δ notation with L here. For the H stuff, he writes the canonical momentum calling it p in 2.11, writes H in the usual way. He then repeats G's development of discrete to continuous by letting the box get big, and we end up with H as integral of the density $\mathcal{H}$. Then we have the π momentum density and all the rest.

First example: He writes $\mathcal{L} = -(1/2)[\ (\partial_\nu\varphi)^2 + m^2\varphi^2]$ and I then used the EL equations to show that this gives the KG equation. Then using this and π, he writes $\mathcal{H}$ as in 2.20 which I verified. Notice it is a function of coordinate φ, momentum π, *and* spatial derivatives of φ, as we saw in G.

Chapter 3: Field Operators (14) Here it comes! We take our φ and π "fields" and impose the basic commutators on them similar to $[x,p] = i\hbar$ (this book sets $\hbar$=1 and c=1). The results are in 3.2 where we get a spatial delta function in the limit. Both "operators" are evaluated at the same time t.

He then does a trivial Fourier expansion of the fields into Fourier coefficients, carefully selecting a normalization constant, and then he shows that our imposed *field* commutator condition is the same as the creation operator condition, so identify those Fourier coefficients with these operators, upgrading a^* to $a^\dagger$ which makes the fields Hermitian. This is the famous "second quantization", the first being the normal idea of quantum mechanics with a simple wavefunction φ(x). M points out that in first quantized, x is the operator (and p), whereas in second quantized, x is just a c-number index, and φ is the operator, or its momentum space a(**k**) operator. I guess I never realized that distinction so simply stated. So φ is then a "field operator", the title of this chapter. So is π, the conjugate field he calls it.

Note added: Go back to the idea on page 14 that $[\varphi,\pi] \doteq i$ in a normalization sense. In our little KG model, we have $\pi = \dot{\varphi}$, so this says $[\varphi, \dot{\varphi}] \doteq i$. But we know that in a Fourier component, we are going to have $\dot{\varphi} = \pm ik_o\varphi$ so we then have $[\varphi, \pm ik_o\varphi] \doteq i$. Ignoring the sign issue, this is WHY the Fourier components in 3.3 are normalized such that $\varphi \sim 1/\sqrt{k_o}$ and then $\pi \sim \sqrt{k_o}$ as shown in 3.3. Then later on page 18 the presence of φ in H_I puts a $1/\sqrt{k_o}$ in it as in 4.4, and then $\langle n+1| H_I |n\rangle \sim 1/\sqrt{k_o}$ as shown in 4.5. This energy factor then makes its way into the ΔE calculation for pion exchange on page 21 and gets into the Yukawa potential calculation. My point in all this is that the $1/\sqrt{k_o}$ is not arbitrary, it must be there!

Chapter 4: Simple Examples (18). In this chapter M makes a "toy model" of pion-nucleon interaction. The "pion" π_0 is the scalar chargeless quantized field φ we have been dealing with. The "nucleon" is an inert fixed no-recoil point nucleon density ρ(**x**) = delta functions. The interaction Hamiltonian is gφρ could not possibly be simpler, which is why this is a great little model. The nucleon is purely classical, as he says "serving as a source and sink for meson fields". That is to say, the delta functions the nucleons "provide" removes the d^3x integral going from $\mathcal{H}$ to H and you are left with a nucleon sum over n in 4.4. The rest of the structure there comes from our expansion of the φ field, so the a and $a^\dagger$ can create and destroy pions.

It is clear that only two kinds of states will have a first-order matrix element of this H_I. In these non-vanishing matrix elements, we have a "vertex" where a pion is either created or removed, like those vertices shown on page 20.

Pause to comment: (1) I presume that $H_I = g\varphi\rho$ would be a theoretically acceptable H_I no matter how large g was. There is no assumption of "perturbation theory" at this point (I don't think), although you wonder if this might not be the linear term in an expansion of some more complex H_I that is the "true" H_I. (2) We are going to study the "interaction between nucleons" in our first example here, so we have in mind an S-matrix element of two nucleons in and then two out, and some kind of interaction in a bubble (not drawn at this point in the book). Our only interaction known is "the vertex", so we draw the four Feynman diagrams shown on page 20 which is all the diagrams with two vertices. There are no "single vertex" diagrams that can contribute to this S-matrix element! (3) Of course if we want to claim that these 2-vertex diagrams are the "important" diagrams, then we have to assume "small g" and perturbation theory.

Reminder of perturbation theory: this is well handled Schiff page 244 Chapter 8. You expand everything as in 31.2. The W is then the corrected energy of a state as you add more terms of the perturbation theory, and then 31.11 shows the usual second order result with the energy denominator and sum over intermediate states. This then is exactly what Mandl 4.7 is saying for the two Feynman diagrams (a) and (b) on p 20. The energy difference is "initial minus intermediate", but our intermediate state has an extra meson, so the difference is - ω_k as shown. The other two diagrams also have this same denominator, but they are renorm contributors so we ignore them here.

Now look at 4.5 again. Recall that the n sum is over nucleons, so we have n = 1 and 2 because we are studying the interaction of only two nucleons, and that is why we get the four terms in 4.8. I have verified each factor in this result. In 4.7 the left matrix element is the CC of the one in 4.5 so get exponent sign change for that guy.

Now look first at the self-energy terms like m=n=1 which means the two vertices are both on nucleus #1 which is drawing (c). You really need a more expanded notation to see all this detail. But yes, in the limit the sum over k is integral dk^3 and we get 4.9. The angular $d\Omega$ gives 4π and then we have a linear ∞ sitting there. So right off the bat, even our "toy theory" illustrates self-energy divergence problems.

Now define $r = |\mathbf{x}_1 - \mathbf{x}_2|$ = distance between nucleons. Then the two terms (a) + (b) in the picture are both equal and their sum contribution to ΔE is $-g^2 \exp(-mr)/(4\pi r)$, the Yukawa potential.

This is quite amazing to me. Here we have a toy theory and we find that the pion exchange graphs are lowering the energy of our two nearby nucleons! Thus, the pion exchange is causing an attractive force between the nucleons, just what we are hoping to see (hadron physics people hope to see). In the limit m=0 you get the Coulomb potential, bang! So this illustrates the often confusing idea that when two particles "exchange" a third, the energy of the two can be lowered and this exchange causes a force between the particles. And the toy theory also predicts a very short range of the pion mass where I guess he has put in the π_0 mass. Keep in mind that the "exchange of particle" is two vertices which represents a term in perturbation theory expansion.

What about higher order terms? If g is large, you would expect them to be large. In this toy theory they are in fact zero due to the no-recoil assumption, but you have to go look up why this is. You do some CT and get 4.11 which says H = T+V in some sense as shown, with no interaction, and the Yukawa just sits there, there is no perturbation theory, and you are done. [CT: another Goldstein payoff for me.]

Now let's next do "nucleon pion scattering" as shown page 22. This is just another S matrix element which has two second order graphs (and of course no first-order graphs). We add up the two terms and they cancel! Well, just from momentum conservation you would know that the result was 0 except

possibly when **k** = **k'** . Again, result is true to all orders for no-recoil nucleons. But at least we are seeing here how one might "compute" the result of a scattering experiment from quantum field theory!

Chapter 5: The Interaction Picture (24). We get a little review of "pictures" here, and we end up going with the interaction picture I.P. in which (1) states move only with H' while (2) operators move only with H_o, assuming H = H_o+ H'. So in some sense the states move "slowly" in this scheme. So operator motion can be written as 5.9 or differentially as 5.10 with a commutator, and H_o appears in each of these.

For example, we can "move" the operator $a_{\mathbf{k}}(0)$ to $a_{\mathbf{k}}(t)$ by 5.11. Doing the CH theorem, or using the trick shown, you can show that, where no argument means at t=0, and k subscript on all ω and a's

$$a_{\mathbf{k}}(t) = \exp(i\omega_{\mathbf{k}} t a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}})\, a_{\mathbf{k}} \exp(-i\omega t a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}) = \exp(-i\omega_{\mathbf{k}} t)\, a_{\mathbf{k}}$$

Now, if we write out the φ(x,t) expansion as in 5.12, we get $a_{\mathbf{k}}(t)$ in side, but we can move that back to time t=0 (where we don't show the time argument) to get

Now go back to the Fourier expansion in 3.4 page 15, which was stated for t=0. We can write for example

$$a_{\mathbf{k}}(t)\, e^{i\mathbf{k}\bullet\mathbf{x}} = a_{\mathbf{k}}\, e^{i\mathbf{k}\bullet\mathbf{x}} \exp(-i\omega_{\mathbf{k}} t) = a_{\mathbf{k}}\, e^{i\mathbf{k}\mathbf{x}}$$

Doing this makes the RHS be as shown in 5.14. This thing is almost a scalar. When you take the commutator now of φ(x) and φ(x') at arbitrary spacetime points, you get something like this:

$$[\varphi(x), \varphi(x')] \doteq \int d^3k/\sqrt{2E} \quad \int d^3k'\sqrt{2E'} \quad .....[a(\mathbf{k}), a^{\dagger}(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}') .....$$

$$\doteq \int d^3k/(2E) .... \text{scalar stuff} ....$$

And now we can use this little trick written in the Mandl metric

$$\int d^3k/(2k_o) = \int_0^{\infty} dk_o \int d^3k\, \delta(k_o - \sqrt{\mathbf{k}^2 + m^2}\,)\,/(2k_o) = \int d^4k\, \delta(k^2 + m^2) = \text{scalar!}$$

That is why, although φ is not quite a scalar, [φ(x), φ(x')]is a world scalar. I followed the computation which then shows that [φ(x), φ(x')] = i Δ(x-x') = world scalar = integral shown in 5.18. I think this Δ integral is over the positive hyperboloid surface of the function $k_o = \sqrt{\mathbf{k}^2 + m^2}$.

Now before we go off and study this fancy "invariant function" in the next chapter, we know Δ(x-x') must be zero when t=t', so that Δ(x) = 0 when t=0. But, if you start at a point (**x**,t=0) in the lightcone exterior, you can reach any other point in same by doing a boost with a suitable **x** starting point, so we know then that Δ(x) = 0 for any x^{μ} outside the light cone. And this makes sense, because φ(x) and φ(x') cannot "interfere" if x-x' is spacelike, because of speed of light. Very good. Think of non-zero commutator as interference between two things, something blocking measurement of both. If they are "far

apart" (spacelike apart), one cannot know that the other was measured, so they should both be measurable at their respect event points.

Mandl is assuming pretty much on the part of his reader: knowledge of perturbation theory in QM, knowledge of special relativity, etc.

One other detail. The two parts of 5.14 are called φ^+ and φ^-, where the first contains a(**k**) terms, and the second contains the $a^\dagger$(**k**) terms. We sort of separate the creation and destruction parts of the field.

Chapter 6: The Δ function and related functions (30). Now we are getting into the nitty gritty stuff, I remember it from long ago. First, we write Δ(x) in three different ways: 5.18 of the last chapter, then 6.1 which is at least invariant looking, but has the contour C shown (which must also be invariant somehow), and then 6.3 which has all infinite real axis integration and is manifestly invariant. Notice the appearance of the ε(k) function which detects whether the time component if k_μ is positive or negative. We are reminded that "proper" Lorentz transformations cannot change this function's value.

There are two terms visible in Δ(x) as shown in equation α page 31. We can identify each term with a contour C^+ and C^- as shown, and then we just have $\Delta = \Delta^+ + \Delta^-$ as in 6.6. Note that the C^- term in equation α *includes* the minus sign in the denominator. Note: When you change **k** to -**k** in the second term integral, you do NOT pick up a minus sign from the integral action, so the minus shown remains and appears in the equation just before 5.18, and this is repeated in 6.9.

I verified everything in this chapter, every single equation.

After talking about all these forms of Δ, M then computes the "vacuum expectation value" of the product φ(x)φ(x') as is, and the result is 6.12 and is the same as for $\varphi^+(x)\varphi^-(x')$.

He then introduces the **time ordering operator** P in 6.13, so earliest time on the right, and the VEV of that thing is called ½ Δ_F which we know is the "**Feynman propagator**" though he does not reveal this fact yet. The naive reader must wonder (as I do) why on earth you would care about time ordering those guys in the VEV. He then goes on to write this Δ_F thing in two more ways which are really the same. First, he writes it as 6.18 with the deformed real axis contour as shown. Then we push the poles a little off the axis as in Fig 8 and then we can have a real-axis k_0 integral. We then get the "famous" form 6.19 with the little -iε sitting in it. Mandl's metric is $a_\mu a_\mu = \mathbf{a}\bullet\mathbf{a} - a_0^2$ which is the opposite sign from B&D and I suppose all newer books, so we will always have to adjust equations when we compare, but that is not too hard.

The reader is given no reason why Δ_F is "interesting". And I don't think M is ever going to state the Feynman rules for this scalar field, so we won't see this "propagator" in use. But we will see the analogous QED propagators, and perhaps I have a book which does the Klein-Gordon Feynman rules. Remember that M wanted a compact book that still included the main real-world QED results, so that is why he is skipping the KG rules.

Chapter 7: Charged Mesons (36). This will be the last chapter on the "mesons" and their strong interactions wrongly modeled in perturbation theory. He will start QED in the next chapter.

Imagine combining two real φ fields into one complex one as in 7.1. Then imagine the field coordinates are φ and φ* instead of φ_1 and φ_2 so $\mathcal{L}$ is a function as shown 7.2. We then get the EL equations shown 7.3. If we use explicit $\mathcal{L}$ as in 7.6, then the EL equations gives us KG wave equations for φ and φ* which is certainly what we want. In passing, claim can write $\mathcal{L}$ in terms of φ_1 and φ_2 as shown in 7.7.

Now define a certain "current" density $j_\nu = es_\nu$ as shown in 7.8. Suppose it were true that $\partial_\nu s_\nu = 0$. Then you could conclude that $Q = \int d^3x\, \rho(x) = \int d^3x\, es_0\,(x)$ = a "conserved quantity" and we would like to associate Q with "electric charge" and associate "e" with the charge on a charged pion. Now if we impose the condition that $\mathcal{L}$ be invariant under the transformation 7.4, which is a simple phase shift of the fields in the obvious manner, then it is easy to show that $\partial_\nu s_\nu = 0$. This phase transformation is called a "gauge transformation" with no explanation given for the word "gauge".

Comments: I would associate this phase shift transformation with a generator J_3 which produces an SU(2) $R_z(\varphi)$ which does this phase shift on the field column vector $[\varphi,\varphi^*]^T$. I would expect [H, J_3] = 0 and then the quantum number "m" of J_3 would the "charge" of a particle state. We then need a little 2x2 formulation of our fields here within SU(2). What would J^2 be? This is Noether's Theorem where I think a symmetry transformation of $\mathcal{L}$ always results in a conserved current density. Mandl does not take us down this avenue, at least now.

At this point, M writes out the expansions of the $\varphi_{1,2}$ fields in the obvious way in terms of $a_{1,2}(k)$, and we find the expected commutators 7.15 among the a's. But then we define an "a" without a subscript as in 7.16 and also a $b^\dagger$, and notice that $a^\dagger \neq b$ because a sign is wrong. We then define fields φ and $\varphi^\dagger$ and use what we know to derive expansions of these as shown in 7.18. So the φ field creates a "b" particle or kills an "a" particle, and the $\varphi^\dagger$ field does the reverse. The a,b commutators are what we expect.

Now comes the major calculation in 7.23 which I did not verify: $Q = e\Sigma_k$ [$N_a(k) - N_b(k)$]. This makes the "charge" interpretation very strong. The a and b particles have "opposite charge", whatever that charge is. Whatever it is, it is "quantized" in units of the e which appeared in our current definition. We interpret this as meaning that a "particle has a discrete charge". He would like to interpret them as particle and anti-particle as well, related just by charge conjugation.

In the last section, he shows that the expansion term in φ which has $a(k)e^{ikx}$ is associated with the absorption of one "particle a", and he refers to e^{+ikx} as the "positive frequency part". This is a little confusing to me since it contains $\exp(-ik_0t)$ which one might call the negative frequency part, but I think historical convention is to write $\exp(-i\omega t)$ and call this ω "positive frequency".

Now look at 7.24 which was easy to derive using the commutator driver rule for an operator in the interaction picture. One of the two factors must be zero. If the matrix element is not zero, then the two states must differ in energy by $\omega_k = E_1 - E_2$. The initial *meson* state has thus lost energy, and we say this is because a particle with energy ω_k was absorbed -- it just went away. We don't yet have a "vertex" to do the absorbing because we are here just talking about the φ fields. At the vertex, you might expect this energy ω_k to be transferred to the other particle type.

Chapter 8: Fermions (42).