Relativistic Particle Momentum (BD) PhL 8.25.08

For the moment, let's go with Bjorken Drell's metric which uses g = diag(1,-1,-1,-1).

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \; p.p = g_{\mu\nu}p^{\mu}p^{\nu} = (p^0)^2 - \mathbf{p}^2$$

where $p^{\mu} = (p^0, \mathbf{p}) = (p^0, p^j) = (p^0, -p_j)$. So **p** means contravariant index (upper).
[**Note**: Weinberg use negative of this metric tensor, he uses $\eta_{\mu\nu} = -g_{\mu\nu}$.] Now, we know that

$x^{\mu} = (x^0, \mathbf{r}) = (ct, \mathbf{r})$ // dim = L => $x.x = (ct)^2 - \mathbf{r}^2$ $x^0 = ct$

$p^{\mu} \equiv m dx^{\mu}/d\tau = \gamma m dx^{\mu}/dt = \gamma m v^{\mu}$ where $\gamma = 1/\sqrt{1 - (v/c)^2}$

$$\Rightarrow \quad p^0 = \gamma m dx^0/dt = \gamma mc \qquad \text{// dim = ML/T = momentum}$$
$$\mathbf{p} = \gamma m \, d\mathbf{r}/dt = \gamma m\mathbf{v} = \gamma\boldsymbol{\beta}mc = \boldsymbol{\beta}p^0 \qquad \text{// dim = ML/T = momentum}$$
$$\mathbf{p}/p^0 = \boldsymbol{\beta}$$

Then have

$$p.p = (p^0)^2 - \mathbf{p}^2 = (\gamma mc)^2 - (\gamma m\mathbf{v})^2 = (\gamma mc)^2 [\, 1 - (v/c)^2] = (\gamma mc)^2 \gamma^{-2} = m^2c^2$$

Claim: $E = \gamma mc^2 = cp^0$ => $p^0 = E/c$ // See Goldstein p 202

Then: $m^2c^2 = p.p = (p^0)^2 - \mathbf{p}^2 = (E/c)^2 - \mathbf{p}^2$

$$\Rightarrow \quad (E/c)^2 = \mathbf{p}^2 + m^2c^2$$

$$\Rightarrow \quad E^2 = (c\mathbf{p})^2 + (mc^2)^2 \qquad \text{// all terms are dim = } E^2$$

There are no $\hbar$'s in special relativity stuff, that is QM.

Non relativistic limit:

$$E = \sqrt{(c\mathbf{p})^2 + (mc^2)^2} = mc^2\sqrt{(c\mathbf{p}/mc^2)^2 + 1} = mc^2\sqrt{(\mathbf{p}/mc)^2 + 1}$$

$$= mc^2 [\; 1 + \tfrac{1}{2} (\mathbf{p}/mc)^2 \,] = mc^2 + \mathbf{p}^2/2m$$

Now if we set c=1 the above results become:

$p.p = m^2$ $p^0 = \gamma m = E$ $\gamma = 1/\sqrt{1 - v^2}$
$p^0 = E$ $E^2 = \mathbf{p}^2 + m^2$

Warning about ∇: We know that $\partial_\mu = \nabla_\mu = \partial/\partial x^\mu$ is a *covariant* vector operator, as shown in physics questions. When we write the **∇** bolded 3D symbol, we mean it like this: $\mathbf{\nabla \bullet f(x)} = \partial_j f^j(x)$. Therefore, we have this unusual situation:

$A_\mu f^\mu = A^0 f^0 - \mathbf{A \bullet f}$ where $\mathbf{A \bullet f} = \mathbf{A}^i f^i$

but

$\partial_\mu f^\mu = \partial_0 f^0 + \mathbf{\nabla \bullet f}$ where $\mathbf{\nabla \bullet f} = \partial_j f^j$

Hyperbolic boost parameters:

$\gamma = ch(u)$	$E = \gamma\, mc^2 = ch(u)\, mc^2$	$E/c = ch(u)\, mc$
$\gamma\beta = sh(u)$	$p = \gamma\beta mc = sh(u)\, mc$	
$\beta = th(u)$	$v = \beta\, c = th(u)\, c$	

$p.p = (E/c)^2 - p^2 = \cosh^2(u)\,(mc)^2 - \sinh^2(u)\,(mc)^2 = (mc)^2$

$2sh^2(u/2) = [\, sh(u) - 1\,] \Rightarrow \quad 2mc^2\, sh^2(u/2) = mc^2\,[\, sh(u) - 1\,] = (E - mc^2)$

$2ch^2(u/2) = [\, ch(u) + 1\,] \Rightarrow \quad 2mc^2\, ch^2(u/2) = mc^2\,[\, ch(u) + 1\,] = (E + mc^2)$

$$sh(u/2) = \sqrt{\frac{E - mc^2}{2mc^2}} \qquad ch(u/2) = \sqrt{\frac{E + mc^2}{2mc^2}} \qquad th(u/2) = \sqrt{\frac{E - mc^2}{E + mc^2}}$$

$sh(u) = p/mc = \beta\gamma$ $\qquad ch(u) = E/mc^2 = \gamma$ $\qquad th(u) = v/c = \beta = cp/E$

And this useful fact

$$th(u/2) = \sqrt{\frac{E - mc^2}{E + mc^2}} = \mathrm{sqrt}(\{\, (E^2 - (mc^2)^2)/(\, (E + (mc^2))^2\,\}) = (cp)/(\, E + (mc^2))$$