Schiff Chapters 1-4: PhL fin 12.5.08

Phil General Comments **2**

Chapter 1: The Physical Basis of Quantum Mechanics **3**
1. Experimental Background (2). 3
Inadequacy of Classical Physics (2) 3
Summary of Principle Experiments and Inferences (3) 3
2. The Old Quantum Theory (4). 4
Conceptual Difficulties (5). 5
3. Uncertainty and Complementarity (7). 5
4. Discussion of Measurement (9). 5
5. Wave packets in space and time (14). 6
Problem 5: 6

Chapter 2: The Schrodinger Wave Equation (19) **6**
6. Development of the SE. (20) 6
7. Interpretation of the Wave Function (24) 7
8. Energy Eigenfunctions (30) 7
Discrete Energy Levels 8
Continuous Energy Levels 8
9. The 1D Square Well (37) 9

Chapter 3: Eigenfunctions and Eigenvalues **9**
10. Postulates and Energy Eigenfunctions (46). 9
11. Momentum Eigenfunctions (53). 10
The Minimum Uncertainty Product. 11

Chapter 4: Bound State Problems (Discrete Eigenvalues) **12**
13. The Famous Harmonic Oscillator (66). 12
Hermite Polynomials (69) 14
Derivation of 13.11 from 13.10. _____long digression _____ 15
Comment on Generating Functions. 19
Harmonic Oscillator Wave Functions (71). 19
Correspondence with Classical Theory (73) 19
Oscillating Wave Packet (74) 20
14. 3D Spherically symmetric potentials (76) 21
15. 3D Square Well (83) 21
16. The Hydrogen Atom (88) 22
Laguerre Polynomials (92) 27
The H atom wavefunctions (93) 28
Degeneracy (94) 29
Separation in Parabolic Coordinates (95) 29
PARABOLIC COORDINATES -- a digression 29
About these coordinates from M&M p185. // using ξ^2 and η^2 , the M&M convention 29
About these coordinates from M&M p185. // using ξ and η in Schiff's (physics) convention 30
The Laplacian. 33
The Parabolic Separation and Solution 34

Phil General Comments

[No one has put **Schiff errata** on line, I have looked maybe 3 separate times, my book is 3rd edition, years were 1949, 1955 and 1968. He died in 1971 at age 55 [1915-1971] , probably the reason errata was never collected.]

Jan 23, 1971 - Dr. Leonard Schiff, a distinguished scientist and member of the **Stanford** University faculty, died apparently of a heart attack on Tuesday. He was 55 years old. A funeral service was held today. Dr. Schiff was born March 29, 1915

I downloaded a copy of the first 1949 edition as a djvu file, so I can search for things and I can see how that book has changed between that and my 1968 edition.

Anonymous Amazon review.

Legend tells that Leonard Schiff, while learning quantum mechanics from J. R. **Oppenheimer**, produced a careful set of **notes** which eventually became the first edition of this text. Soon followed a **second**, which was, for a long time, the **unanimous choice** of QM text-book in English. The **third** introduced things like group theory, a little of what was then called formal scattering theory (Lippman-Schwinger equation), and other new things. **Schiff's text was a minimalist one**, in the sense that you found everything, but in a very compact, no frills style. **A teacher was required**. **The ratio of words to formulas was quite small**. **It entirely dominated the panorama.**

I, studying by myself, tried very hard, but could not understand, for instance, his treatment of perturbation of degenerate levels. Neither could I understand the infamous "box of very large dimensions" where the systems with a continuous spectrum were trapped so that their spectra became discrete. Even less could I acquire the much desired "global view" of quantum mechanics. I was very unhappy with Schiff. Someone suggested Bohm's textbook for me, but, at that moment, it seemed too leisurely paced for my impatience. Then I hit upon the newly arrived English translation of **Landau, Lifshitz**, "and all was Light". Nowadays I think there is no more place for "ole Schiff", but I must confess some of my colleagues still swear by their second edition, with the famous greenish McGraw-Hill cover. Well, I must recognize that the Dirac equation treatment was quite good, and that you could trust the equations as they appeared in the text: **almost no typos.**

Schiff's Preface:

This is a **graduate level** text AND **reference** book, author wants to do the math, the physics, and provide examples. He won't do any kind of math super rigor. Schiff does not fess up to the history claimed above (legend), but does credit Oppenheimer who died in 1967, one year before this third edition came out (and 4 years before Schiff himself died). He says the second edition 1955 had few changes from the first in 1949 (6 year gap), but this 3rd edition (13 year gap) has some major changes. One is the addition of the complex potential section and optical theorems which I have lately been reading. More matrix, more symmetry, more approximation methods, density matrix. I suspect he also made this "his" book and not Oppie's. Wichman's criticism of the 2nd edition were considered for the third.

Book Structure: Very unusual I think. There are 14 chapters, but there are 57 numbered "sections" whose numbers just increase through the chapters, and there is a huge variation in the size of a section, ranging from perhaps 2 to 25 pages. I have put these Section titles at heading level 2 in my contents. Each of these Sections then has some number of *unnumbered* subsections with small bold capital letter headings which for me are heading level 3. Equations all bear the number of their Section, not Chapter, eg (24.3). When you start at a page, you cannot tell what Chapter you are in, but you can tell the Section from equation numbers.

Schiff wants to connect to the physical world of observation before launching off into the mathematical formalism of QM (the SE and wavefunctions).

Chapters 1 through 4 comprise 100 pages, a mere 1/5th of this book! My notes here are 36 pages.

Chapter 1: The Physical Basis of Quantum Mechanics

1. Experimental Background (2).

Inadequacy of Classical Physics (2)

Blackbody is first mentioned here. In 1900, you could measure the energy spectrum of a blackbody and people knew what the curve looked like. But how could you theoretically compute the spectrum? E&M was of course known, so you could treat the blackbody cavity using the EM wave equation with the usual 3D box wall boundary conditions of E = 0, and you found that the density of "modes" in the box was proportional to ω^2, see p 8 1 Livesey eg. Each of these modes could presumably have any energy you wanted, but in thermal equilibrium the logical thing to assume was that the energy of a mode was just $\varepsilon(\omega) = kT$, independent of the frequency of the mode. Then $f_{BB}(\omega) \sim \omega^2$ which disagrees with the known blackbody spectrum which has a peak then decays.

Each EM mode has integers n,m,ℓ to label it in 3D. Let $N = n+m+\ell$. Suppose you assumed that you could not have *any* energy in a mode, but suppose a mode could only have only discrete energies given by $E_N = N\hbar\omega$ were $\hbar$ was some constant. In this case, you think of the modes as statistical mechanical "states" and you apply Boltzmann to say that the probability of a mode in TE being occupied is $\sim \exp(-E_N/kT) = \exp(-N\hbar\omega/kT)$. This means that as ω increases, a mode is less likely to be occupied in TE, whereas every mode energy = kT says all modes are equally likely to be occupied regardless of ω. When you apply usual stat mech to this quantized energy spectrum idea, you find that instead of $\varepsilon(\omega) = kT$, you get $\varepsilon(\omega) = \hbar\omega/[e^{\hbar\omega/kT} - 1]$ and this then correctly predicts the blackbody spectrum if you select the constant $\hbar$ to have a certain value. **Planck** (1899-1901) proposed this idea, and that tiny value $\hbar$ is "his" constant. I mention all this detail because $\hbar$ is so fundamental to QM, nice to see how it came into human consciousness for the first time.

In the above description, we treat the BB as some kind of EM cavity and we say nothing about the "walls". Another description treats the walls as made of "atoms" which somehow have discrete harmonic oscillator energy levels and it is the states of these atoms that you think about, with the higher states less occupied by **Boltzman**, and then the BB radiation is controlled by the $E = \hbar\omega$ transitions between these harmonic oscillator levels. Livesey mentions these two descriptions on bottom of page 80, but the connection is not made clear in his small section on this subject. So at this point I will cease and desist on this BB subject, admitting that I only partially understand it, but historically we see its significance.

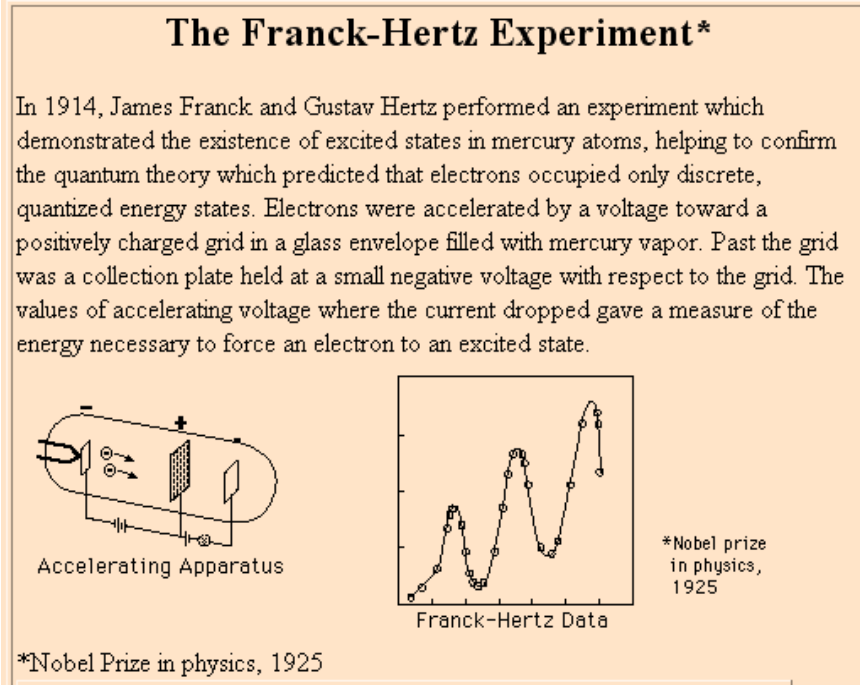
Summary of Principle Experiments and Inferences (3)

In the end, there were lots of experiments like **photoelectric** and Einstein, and **specific heats** of solids, and **Stern Gerlach**. These all suggested that certain physical parameters took only "quantized" values, hence the name QM was later used. This discreteness was one "thread" of early QM discovery. The other thread was the wave/particle duality, which involved the second major "equation" of deBroglie that

traditional particles also obey the idea that λ = h/p, not just light particles. Again, that constant h appears. People found that electrons "diffract" just like photons. In 1927, Davisson and Germer found this diffraction and confirmed **deBroglie's** idea of 1923. Table on page 4.

2. The Old Quantum Theory (4).

This is basically **Bohr's 1913 "theory of the atom"** which is that an atom has a discrete spectrum for some reason, and light "photons" of E=ħω are emitted or absorbed between these discrete levels. It is hard for me to imagine that there was a time before this basic theory existed (less than 100 years ago!). The electron and proton of an H atom could previously have had any energy you wanted, and there was no reason for the thing not to implode, no ground state, etc etc. The big hint of course was that observed atomic emission spectra had discrete lines, and the early **Ritz-Rydberg "combination principle" (1908)** which related spectral lines was consistent with the Born atom theory. The **Franck-Hertz 1914 experiment** showed that vapor atoms seemed to absorb electrons more if those electrons had certain energies (corresponding we now know to the energy gaps of the vapor atoms). Here is a little web picture of this simple experiment:



So Bohr's main idea was to say that the electron in H atoms had a quantized orbital $L_z = m\hbar$ and this led to a simple model with a simple spectrum prediction. So this was the second "quantization rule" that appeared in the human mind, the first being E= ħω. This L_z rule follows if you use the deBroglie rule for an electron orbit.

So yes, this is the "**old quantum theory**" and it provided new qualitative explanations of things like the line spectra. Sommerfeld (my friend Arnold) rephrased the angular momentum quantization rule in terms of those $\int$ pdq action objects from classical mechanics theory, but we think of that too as part of the "old" QM.

Meanwhile, the Bohr 1923 "**correspondence principle**" just meant that for large numbers of quanta, your theory had to give classical theory results, so this was a guide for quantum theory models.

Conceptual Difficulties (5).

Schiff talks first about the fact that the old Q theory was not giving perfect predictions, but now he talks about "conceptual" problems, such as the notion of a light "particle" going through one OR the other whole in a 2-slit experiment. Even if you lower intensity way down so only "one photon at a time" runs through this experiment, covering one slit removes the interference pattern. If a photon goes through one hole, how does the covering or uncovering of the "other hole" affect where that photon goes? The conceptual problem here is our picture of something "going through a hole" in a classical sense. In fact, the entire notion of "trajectory" goes out the window, and with it, the idea that "previous position exactly determines future position" also goes away, so there goes "causality". It must have been a huge mental shock to people used to the complete in-principle determinism of the mechanical 1800's. We now know that this experiment is fully explained by the notion of a wavefunction ψ and $\psi^*\psi$ probability, but for the Bohr 1913 atomic theory, there was as yet no SE and no wavefunction.

3. Uncertainty and Complementarity (7).

Various forms of the **uncertainty principle** (UP) were developed from the full "new" QM theory in **1927** by **Heisenberg**, some are listed page 8 top. Associated with this notion that you cannot measure two particle parameters in a complementary set (like p and x) with arbitrary accuracy was the 1928 Bohr **complementarily principle**, a pretty fancy name I think for what the Heisenberg idea really says. But the idea is that there is now an "in principle" claim that no experiment can ever measure both to accuracy better than the UP says.

4. Discussion of Measurement (9).

In this section Schiff considers three experiments which I will briefly comment on, but I have not studied this stuff in detail. Each experiment is a check on the UP.

Localization Experiment. On page 10 top we have photons created at point P going through a lens and focusing on a screen. In this set-up, experiment or EM theory says that the screen resolution is $\Delta x \sim \lambda/\sin\varepsilon$. On the other hand, there is uncertainty in the momentum of the photon arriving at the screen because it could have arrived from any point on the lens, so we get $\Delta p = p \sin\varepsilon$, so you get $\Delta p \Delta x \sim \lambda p$. But then deBroglie $p = h/\lambda$ makes this be the UP. You can DO this experiment, and you will find the result just quoted.

Momentum Measuring Experiment. The idea here is to use Doppler shifted frequency of an emitted photon from a moving atom to measure the momentum of that atom. If you want some precision $\Delta\nu$ on your frequency measurement, you have to allow time τ to do the measurement and $\Delta\nu = 1/\tau$ from EM or Fourier theory or whatever. This τ time causes $\Delta\nu$ which causes a $\Delta\nu$ which causes a Δp for your momentum measurement as in 4.8. Meanwhile, you don't know exactly when the atom emitted the photon, and this results in a certain Δx for its location when it emitted shown in 4.7 because you don't know when it slowed down. Again, we end up with $\Delta x \, \Delta p \sim h$,

Indicator slit experiment. The picture on page 13 shows some perhaps fluorescing indicators which will let you know through which slit the photon went on its way to the screen. I skip the details, but the conclusion is this: if you can maintain the diffraction pattern AND also determine which hold the photon

went through, then you violate the UP as shown in 4.12. So if the UP is true, then you will fail in this experiment.

5. Wave packets in space and time (14).

Here we vaguely introduce the wavefunction ψ for a "particle" with three properties: it can interfere with itself, it somehow correlates with probability of being somewhere, and it applies to an individual object (not just to an ensemble).

Next, we look at a spatial wave packet and we quote from Fourier analysis that $\Delta k \Delta x \geq 1$. When we combine this with deBroglie $p = h/\lambda$, we get the UP, which is encouraging to the idea of thinking of a classical particle perhaps as a wave packet.

Then comes a temporal wave packet spread out over some Δt and Fourier says $\Delta t \Delta \nu \geq 1$. If we combine this with Planck $E = h\nu$, we get the time UP, so more encouragement.

Basically we are going to treat everything as ψ waves in this book, so Schiff asks : does this mean we have thrown out the idea of trajectory and the particle half of the wave-particle duality? No, he says, the wave thing will give all particle predictions properly, and later in the book we will do a cloud chamber particle track as an example.

Problem 5: We are scattering a ping pong ball against a fixed ping pong ball target by doing a vertical drop of distance D. During a drop, $\Delta p_x = p \sin\theta$, if there were some skew angle θ. This results in some $\Delta x = D\sin\theta$. And $p^2 = 2m(gD)$. You might then say that $\Delta p_x \Delta x = \sqrt{2mgD}\, D \sin^2\theta$. The UP then suggests that $\sin^2\theta \geq h/\sqrt{2mgh}\, D$. So you might estimate that one bounce will result in some tiny angle θ, and then how many bounces before the bouncing stops, ie, when the falling ball completely misses the target ball. The interesting idea here is that even with a perfectly idealized situation, QM says that something will happen which is surprising.

Chapter 2: The Schrodinger Wave Equation (19)

The SE will work for non-relativistic problems describable by a potential V.

6. Development of the SE. (20)

Schiff is trying to motivate the SE. First, we imagine that plane waves will be like those from historical wave theory, so will have the forms shown in 6.3. These have definite p and fully spread out x. He first tries a traditional wave equation with its second time derivative but finds that, with plane waves, this thing will have a parameter which depends on the motion parameter p, but we don't want this because we want to be able to superpose solutions of different p into the same equation. Next he tries 6.6 with its linear time derivative, perhaps this is like a heat equation or other classical equations. If we try the expo forms of the plane waves, they solve this thing with constants not dependent on p, so we like it. So right away, we end up with the 1D free-particle SE as 6.8. It is the ODE with a linear time derivative and second space derivative that the expo wave satisfies, seems awfully reasonable. The generalization to 3D is trivial, and we then have 6.11 with ∇^2 appearing, the bane of every student.

Then, with little comment, Schiff compares 6.11 to the equation $E = p^2/2m$ and notes that this would be the same as the SE if you made the connection $E = +i\hbar\partial_t$ and $\mathbf{p} = -i\hbar\nabla$. He does not mention

Hamiltonian H=E at this point. Nor does he mention $p^\mu = i\hbar\partial^\mu$ as encompassing these both. There is no Hilbert space yet, we are just starting out.

To add forces, he does appeal to replacing $p^2/2m$ by itself + V, and there we have it, the SE in 6.16 in all it's glory, the cause of billions of hours of problem set time for students on this planet. Date is given as 1926 and paper quoted.

7. Interpretation of the Wave Function (24)

We need more than just the SE, we need to interpret **wave function** $\psi(\mathbf{r},t)$ which replaces the **trajectory** $\mathbf{r}(t)$ of classical physics. Since ψ is complex, the simplest thing that could be **probability** is $|\psi|^n$ and power n=2 is it. If this is prob, then 7.2 must be true.

On page 26 we define the **probability flux** which he calls S instead of j, and he shows that dP/dt = 0 if you take a very large outer surface, using a Gauss theorem as spelled out in A,B,C,D which appears later in the book in the collision stuff. And we see the "**continuity equation**". And we get more support for $p = \nabla$ because this looks like classical flow equations where you have a **v** in there. He is building his case.

Expectation value for r is certainly reasonable in 7.5, just weight by probability. Same for V. Now to make our little theory **consistent**, for example so 7.7 is true, we have to interpret expectation values with differential operators as shown in 7.8. This now obvious fact is less obvious to the fresh student. Again, we have no Hilbert spaces yet described, no adjoint operators, no scalar products, no nothin!

The last subsection talks about **Ehrenfest's Theorem(s)** which say this:

$\partial_t\langle\mathbf{r}\rangle = \langle\mathbf{p}\rangle/m$ like classical $\partial_t\mathbf{r} = \mathbf{p}/m$ = velocity

$\partial_t\langle\mathbf{p}\rangle = -\nabla\langle V\rangle$ like classical $\partial_t\mathbf{p} = -\nabla V$ = force

These two results are derived in full detail using nothing but the SE with a real potential V. Obviously these are two instances of "the correspondence principle" mentioned earlier, the connection to the classical world. You might think of these in terms of a wave packet particle, for example.

I try to picture myself reading this for the first time. The fancy integration by parts would be unfamiliar, and maybe Gauss as well, all integral theorems -- just pure math -- but now it is all quite clear. Recall that you always start in the middle in QM.

8. Energy Eigenfunctions (30)

In the first section, S shows that you can try separation of variables in the SE and you then find expo time dependence as in p 31 A where E is "some constant", and a TDSE with that E in 8.2. He means to say that, by looking at our earlier correspondence, we conclude that E is the energy of the particle, not just some meaningless constant. This would again follow from the correspondence idea etc etc. The terms stationary state, eigenvalue equation, eigenvalue, eigenfunction are all defined. The claim is made that for bound state problems where ψ is going to be localized, the E spectrum is always discrete. For scattering problems with non-normalized plane waves, E can be continuous.

Now, why must ψ and $\nabla\psi$ be continuous out in the middle of nowhere? I think this just follows from the ODE applied to an innocent point in space away from anything violent happening.

On page 33 S tells one way to handle an "infinite wall" in potential. Start with a finite wall, match conditions on both sides, then take the limit. You find in this way that $\psi = 0$ at the wall.

Energy Eigenvalues in One Dimension

Here we have some vintage Schiff I think. We want qualitative things to say about ψ in a bound state problem. Here are some interesting claims:

(1) Consider a wing region where E < V < 0. This would apply for example in the *wings* of the potential shown on page 35 because E is always going to be a horizontal line above the bottom of the potential well but below (drew such a line on page 35). The SE says ψ"/ψ ~ V-E. , so in the wings this is positive. Then if ψ is positive, ψ" is positive and the curve is "cupping up". But we are moving toward ψ = 0, so this cupping up must resemble expo decay as shown. Usually we normalize wavefunctions so they are positive on the right, and there will either be an even or odd number of nodes, so on the left wing ψ can have either sign, so he shows both cases there. You always have that expo decay like shape toward the axis, but on the left you can have either sign of ψ. The fact that ψ'/ψ < 0 on the right just says it is like e^{-ax} there. And on the left you get ψ'/ψ > 0 since growing like e^{+ax} as you move to the right. I don't see the importance yet of the ψ'/ψ fact, but I see why true.

(2) Consider now the middle region. In the middle where E > V, you have ψ"/ψ negative, so the curve is always cupping *toward* the axis, that is, it is sinusoidal in nature. Again, see page 73 for examples. When E>V if we write E = KE + V, we find KE > 0, so **p** = real, and this is where the particle "lives". This is the classical region where it would bounce around off the turning points at E=V. The wing tails don't exist classically. At the turning points, ψ" = 0, so points of inflection.

OK, now finally I understand all of figure 35, somehow Schiff just seems obscure till you read it several times. (a) shows the potential V being a smooth well of some sort. (b) shows how you might "integrate in" solutions from the two extremes. (d) shows a correct mating of these two solutions. Notice that we have the case (2) situation above. (c) shows incorrect mating when E is a little too large negative. (e) shows incorrect mating when I is a little too positive. So (c) must be an eigenvalue for E. So this is his way of showing why the BC force eigenvalues.

I seem to recall Saxon doing it a different way. You start at one end and integrate through, and if E is not an eigenvalue, you will blow up at the end you come out.

Discrete Energy Levels

The drawings on page 36 shows the same situation at, just above, and just below the correct eigenvalue for the next eigenstate, the one with a single node in the center. Why do more nodes mean higher E? Look at the SE in 8.2. More nodes means larger |ψ"|. To the right of the middle if ψ>0, then ψ"< 0, and LHS of 8.2 is larger at the same x (if there are more nodes), since potential is same. So get more positive E.

For a well like a HO that goes up forever, there are an infinite number of states. But for a localized well, there are only a finite number of states usually, though you can get infinite in certain cases where they compress as you reach the top.

Continuous Energy Levels

If E is larger than either side of your potential, you are in the collision type case. In the wing (or wings), you now have sin/cos instead of expo, so can linearly combine them and always match up your

wavefunctions at any E. In the other case you only had one constant in a wing, not two, because the blowing up expo was not allowed.

Final section comments on continuing the general ideas here to 3D from 1D.

9. The 1D Square Well (37)

This is the first problem for every student, the "particle in a box". Unfortunately, the math is a little messy and you don't get a nice closed form solution for the eigenvalues. The HO is nicer in this regard but is not a good example of V=0 far away. I forget what the gaussian well does, I think it gives everything analytic, but maybe not. A quick web scan gets me nothing. I think I have seen it done in some book.

S first treats the infinite wall box and we get the solutions I marked with A on page 39. We find that the energy eigenvalues are $E_n = E_1 n^2$ with n = 1,2,3... and they alternate sine and cosine with another node for each step in n. Since ψ=0 at the walls, this is similar to E field modes in a waveguide.

Next he does the finite step. Matching ψ and ψ', we again end up with two "classes" of solutions which are sine/cosine in the middle, and expo decay in the wings. He first writes out the four matching conditions in A, solves them for sin and cos in B, then divides the pairs to get the two transcendental equations involving α and β, one for each solution class (one for each parity). He then defines rescaled variables ξ and η and **graphically finds the energy levels**. He then gives a **qualitative description** of what these levels do. As V_o increases, the quarter circle get's larger, and the number of solutions increases; for any V_o there is only a finite number of solutions. The student does not see clearly the tan and cot plots and might miss the main point here because S has amplified just a section of the plots.

What he does not do is solve for the wavefunctions. That is, he does not write down coefficients A,B,C,D for the two classes of solutions, and he does not plot the results. This probably requires a little work, and his main interest here is in the energy levels. In the next chapters he will do some more analytic problems and do this extra work.

The **parity** discussion on page 42 is familiar to me. If V(x) = V(-x), then [H,P] = 0 and we can diagonalize H and P at the same time. Later we will think of V(x) as an even polynomial in X, say, and of course $PX^2P^{-1} = + X^2$ and then $PV(X)P^{-1} = V(X)$ and then PV = VP and [P,V] = 0 and so [H,P] = 0. Then P is a "symmetry" of H.

But he does not do that fancy stuff here yet. He shows that if V is even, then u(-x) = $\pm$ u(x) and so has definite parity. This simple analysis **fails if you have degenerate levels, but** then you can just "symmetrize" as he shows later on page 43. He then concludes that if you know about parity ahead of time, you can make a faster job of finding the conditions on the coefficients, you can just think about one side only and match ψ'/ψ on the + side.

By the way, he is using u(x) instead of ψ(x) in this entire chapter. I think that goes back to 8.3 p 31 where he uses ψ for the time-dependent solution, and u for the time-independent one. Perhaps he will stick with this convention in future chapters.

Chapter 3: Eigenfunctions and Eigenvalues

10. Postulates and Energy Eigenfunctions (46).

Here we have a 10 short subsections where some basics are laid out. The very first draws our attention to the idea that $\mathbf{\Omega v}_\mu = \omega_\mu \mathbf{v}_\mu$ is the EV equation for an arbitrary linear "**operator**" Ω, and that every "**dynamical variable**" relating to the motion of a particle has such an "operator". This is the **first**

"postulate". The **second** is that when you do a measurement on such an operator, you will only read *one* of the ω_μ eigenvalues. The **third** is that you can make a linear combination $\psi = \Sigma\, A_\mu \nu_\mu$ and then $|A_\mu|^2$ tells you the probability you will read this ω_μ value. So we have three sort of postulates.

This book Section deals only with the **energy EV equation** 10.2 which is the TISE. He tries to **include both bound state and collision situations** in his discussion, even though we know that things like plane waves don't normalize very well. He suggests using a **"large box"** $V=L^3$ which causes the collision continuous spectrum to become slightly discrete, and suggests using the **periodic boundary conditions** method. This has a great advantage because it means the *total* flux of far-flung regions is zero! Using this fact on page 49, he is able to show in general that eigenfunctions of different energy eigenvalues are **orthogonal**. [Much later we will learn this is property of a Hermitian operator.] The example just quoted: bottom of page 48 is formed by subtracting two pre-multiplied SE's, then the LHS of 10.4 is converted to a Gauss surface deal in 10.5 by the usual Green's identity. The periodic BC's are then invoked for both value AND derivative to argue this goes away, even in a collision situation, and so eigenfunctions are orthogonal.

He does various other things just from the SE and various **"parts integrations"**, where we end up throwing out the "parts". The need is to cast the ∇ operator back and forth inside an integral.

He does mention the issue of **degeneracy** and how we can do GSO there and get 10.7 with fully everything being **orthonormal**, even in those subspaces, but he usually won't show this detail.

He shows that the EV **"E" must be real"** on page 50, and this involves a single parts integration. [Another Hermitian operator property.]

Then he starts into the **linear combination idea** with coefficient A_E as in 10.8, derives its inversion formula, and comes up with a crude statement of the **"closure property"**, what I have always called "the **completeness condition**" of the eigenfunctions. These terms mean the same thing, and complement the "orthonormality" condition. He is **avoiding delta functions at this point**!

On page 52 he shows that **<E> is the expectation value of H**, not using that H letter yet. Again, just using the SE and here two parts integrations with that periodic BC claim again.

Finally, on page 53 he solves the **TDSE** by just adding the time phase to the coefficient A as in 10.17.

Comments: He mentions no "space" in which our dynamical operators "operate", later we will learn this is a Hilbert Space, which is on the *first* page of Shankar's book. He gets a lot of mileage out of just using the SE and parts integration and those periodic BC's. I suspect this might be a somewhat novel method as QM books go, not sure. The reader of course has to know about 3D vector calculus pretty much. He is basically saying that QM can be embodied in the coordinate space SE and to that we add our postulates concerning the wave function interpretation.

11. Momentum Eigenfunctions (53).

Another collection of short subsections. We start with 11.1 which is the EV equation for operator $\hat{\mathbf{p}}$, but this notation is not yet used. We change to **k** and the plane waves are as in 11.2. If we do box norm with periodic BC's, we get the discrete eigenvalues shown in 11.3 and $C = L^{-3/2}$ and orthonormality as in 11.5.

Then S says we can avoid the box situation using **delta functions**. The reader is assumed to know nothing about delta functions, it seems (but to know about Green's identities of vector calculus). Perhaps in 1968 these δ's were not often seen in textbooks. He takes 11.9 as a "representation" of the delta function as a limit, but he neglects to show the important unit integral, but OK, fine. He then quickly finds

the expo integral rep of the delta as in 11.10. In this normalization, we get $C = (2\pi)^{-3/2}$. When we square and integrate to show orthog, this cancels the $(2\pi)^3$ arising from the expos. This leads to that little identification $L^3 \leftrightarrow \mathbf{(2\pi)}^3$ for volume in the two systems.

On page 57 he gives a nice list of **δ function properties**, some of which are less than familiar to me, so nice to have a list. He is missing the one about δ[f(x)], however, though there are some special cases.

Then on page 58 he writes ψ(r) as a linear combination of our momentum plane waves with $A_{\mathbf{k}}$ and on page 59 he computes <p> and gets the obvious result.

12. Motion of a Free Wave Packet in 1D (60).

I see that I have never read this last Section of this chapter because there are no markings in the book. Now is the time to read it!

The Minimum Uncertainty Product. We start assuming only that ψ(x) it is a "normalized wave packet" which vanishes far away. If we define (Δx) as the usual Fourier **standard deviation** usually called σ ($(\Delta x)^2$ is the **variance**) , and we do similarly for (Δp), we can show that in fact $\Delta x\ \Delta p \geq \hbar/2$. This little proof is done in coordinate space in 1D where $p = -i\hbar\partial_{\mathbf{x}}$. As usual, we use parts integrations and a little contrived identity shown top of page 61. Since x and k are conjugate Fourier variables, this result is quite familiar to me. I did not bother doing all the steps. The SE was not used. So once you know that $\mathbf{p} = -i\hbar\nabla$, the Heisenberg Uncertainty Principle of "physics" is nothing more than the standard result of Fourier Analysis. This is perhaps a little deflating. Along the way here Schiff identifies two conditions (pencil 1 and 2 on page 61) which cause $\Delta x\ \Delta p = \hbar/2$.

Form of the Minimum Packet. Here we write down those two conditions and try to find a wave packet that *meets* these two conditions and thus will have $\Delta x\ \Delta p = \hbar/2$, and this will be called "the minimum packet". Just massaging the local information here, he comes up with the result **12.11** for this packet. It is a function of x, but it includes <x>, <p>, and Δx as constants. If we used <k>, there would be no ħ.

What is this telling us? First, it is just another Fourier result having nothing really to do with physics except as we apply it. The result is a sort of generalized Gaussian. We let <p> be the desired momentum of our wave packet, and <x> be its desired mean position (might as well have that be 0), and we select some $\sigma = \Delta x$ for the thing, and there it is. You can set the three constants any way you want [$\Delta p = \hbar/2\Delta x$], and the packet will always have the property $\Delta x\ \Delta p = \hbar/2$. Notice that this is ψ(x), a static function, there is no time here. Perhaps real particles are made of such packets, he makes no conjecture on that.

Momentum Expansion Coefficients (62) In this entire book section, Schiff is constantly using *both* box and continuum normalization I guess to show that reader that you do get the same results either way, and to give examples which might be called upon later in the book (something Schiff likes to do).

Suppose we expand this packet in terms of plane waves as in 12.15 (and 12.12 or 12.13!). We know the plane waves are also energy eigenfunctions, so we know from 10.18 that we can write 12.15. In other words, the $A_{\mathbf{k}}$ just move in time as you expect with the usual energy phase.

Change with time of this packet (63). If we regard 12.11 as a packet at t=0, how does this packet evolve in time? First, we want to compute $A_{\mathbf{k}}$ for our 12.11 packet as in 12.18 so now we know the $A_{\mathbf{k}}$. We then jam this into 12.15 to get 12.20. [Again, he has $\Sigma_{\mathbf{k}}$ in 12.19 so we are in box norm here, but then for 12.20 he has converted this to a continuous integral.] The pieces of this integral's phase are quite clear, I

have put in little pencil arrows. This is a near-gaussian integral which you would do by completing the square, or just looking it up in GR and the result is also shown in 12.20. I remember being fascinated by this at one time: here is what a free particle packet really does, we have computed it in closed exact form! I forgot to say that we have been working with a packet which is at the origin at t=0 (so $\langle x \rangle = 0$) and which is not moving so $\langle p \rangle = 0$, but surely we could have done the general case as well.

Now we are coming to a punch line. The upshot is that the packet just gets broader in time. The variance in fact is proportional to $\Delta p^2 t^2$ so width grows linearly in time. AND, the more your initial packet is constrained in space smaller Δx), we know from the UP that Δp is larger and the spread is faster! The σ is roughly increasing by the distance vt that a classical particle would move in time t with $v = \Delta p/m$. So, overall this packet maintains $\langle p \rangle = 0$, but its unruly jittery moving components are moving in a way to make the packet blur out in this way. Although $\langle p \rangle = 0$ in the aggregate, there is still a $\Delta p > 0$. I don't think the time-evolved packet stays "minimal", he does not comment on that. It does not seem to have the required 12.11 form any more.

Classical limit (64) Here comes the nice punch line. Consider the planetary model of an orbiting electron as such a time evolving packet. Is this model reasonable? If you try to spatially constrain such a packet to roughly the orbit radius, the best you can do to minimize the variance is to select $\Delta x(0) = \sqrt{\hbar t/m}$. First, here is a little proof of this claim: Consider that $(\Delta x(t))^2 = (\Delta x(0))^2 + (\Delta p(0))^2 t^2/m^2 \sim (\Delta x(0))^2 + (\Delta x(0))^{-2} \hbar^2 t^2/m^2$

$$(\Delta x(t))^2 = (\Delta x(0))^2 + (\Delta p(0))^2 t^2/m^2 \sim (\Delta x(0))^2 + (\Delta x(0))^{-2} \hbar^2 t^2/m^2$$

Write as $f(a) = a^2 + bt^2/a^2$ $\qquad$ $f'(a) = 2a - 2bt^2/a^3 = 0$ when:

$$1 = bt^2/a^4 \quad \text{or } a^4 = bt^2 = \hbar^2 t^2/m^2 \quad \text{or } a = (\Delta x(0)) = \sqrt{\hbar t/m} = \Delta x$$

Now, you would like to maintain this small packet for lots of orbit times (t >> T), but let's just try to hold it for one orbit, t = T. But you want $\Delta x << a$ so we can have our planetary model. We then find that we need $\hbar T/m << a^2$ or $ma^2/T >> \hbar$. But Bohr angular momentum is L = rp = amv and $v = 2\pi a/T$ so we then have $L = 2\pi\ m\ a^2/T$. Thus, roughly, our requirement is that $L >> \hbar$ for our little "planet" to make this work. But in fact **Bohr model** says that L in fact is $\sim \hbar$, so this condition is not met. In other words, even if you could start an orbiting electron as an optimal packet, by a single revolution it would be spread out all around the orbit! So this is a very nice conclusion of this section. It really drives a stake into the planetary model, just from the UP, which is to say, from "wave mechanics" at the most basic level. Obviously this is a crude approximate argument, since such an electron is not a "free particle" etc etc, but we get the idea.

Comments: So far, the word Hamiltonian has not been used, nor has the letter H. But hamiltonian slipped into problem 9 I notice.

Chapter 4: Bound State Problems (Discrete Eigenvalues)

13. The Famous Harmonic Oscillator (66).

The problem is set up, we go to dimensionless x variable ξ, define α and β and ω_c as in 13.3. For very large ξ, ignore λ and solution is $u = \exp(-\xi^2/2)$: $u' = -\xi u$, so that $u'' = -\xi u' - u \sim +\xi^2 u$, so solves the ODE in this limit Then let's write the ODE for H based on 13.4 and get 13.5.

We know that an ODE will have solutions, and whatever these solutions are, they can be expanded in a power series around the origin $\xi = 0$. Near $\xi=0$ we can see that the solutions of 13.2 are trig-like and therefore not-singular. Schiff follows Sommerfeld's 1929 development here which threw me for a long while, but now I think it is fine. The assumed form of 13.6 is a general power series whose lowest power is ξ^s. What is a little confusing is that ξ^s is multiplied by a "normal power" series, so that a_3 is not necessarily the coefficient of ξ^3 in the polynomial for H, but rather the coefficient of ξ^{s+3}. By writing the solution in this way, we are requiring that $a_o \neq 0$ so that "s" really is the lowest power. And since we just said that the solution is non-singular at $\xi=0$, we know that $s \geq 0$.

We then plug this power series into the ODE 13.5 and then set the coefficient of each resulting power to 0, the standard way to solve for the power series. The result are the coefficient "recursion relations" shown building up in 13.7. Let's "just do it" :

Coefficient Recursion Formulas: Let's look at how this is done: at the start, we just assume s is some integer.

$$H = \Sigma_{k=0} a_k \xi^{s+k} \qquad H' = \Sigma_{k=0} a_k (s+k) \xi^{s+k-1} \qquad H'' = \Sigma_{k=0} a_k (s+k)(s+k-1) \xi^{s+k-2}$$

Putting this into 13.5 gives [we absorb the middle term's ξ into the exponent there, and we rename the summation index to be k' in the last two terms]

$$\Sigma_{k=0} a_k (s+k)(s+k-1) \xi^{s+k-2} - 2 \Sigma_{k'=0} a_{k'} (s+k') \xi^{s+k'} + (\lambda-1) \Sigma_{k'=0} a_{k'} \xi^{s+k'} = 0$$

Now shift to k' = k-2 in last two terms so they then begin at k = +2 instead of 0:

$$\Sigma_{k=0} a_k (s+k)(s+k-1) \xi^{s+k-2} - 2 \Sigma_{k=2} a_{k-2} (s+k-2) \xi^{s+k-2} + (\lambda-1) \Sigma_{k=2} a_{k-2} \xi^{s+k-2} = 0$$

For k = 0 and k =1, only the first term exists (!) so we can say:

k=0: $(s+k)(s+k-1)\, a_0 = (s)(s-1)\, a_0 = 0$ **13.7 A**

k = 1: $(s+k)(s+k-1)\, a_1 = (s+1)(s)\, a_1 = 0$ **13.7 B**

and for k > 1 we get

$$\Sigma_{k=2} \{ a_k (s+k)(s+k-1) - 2a_{k-2}(s+k-2) + (\lambda-1)\, a_{k-2} \} \xi^{s+k-2} = 0$$
$$\Sigma_{k=2} \{ a_k (s+k)(s+k-1) - [2\,(s+k-2) - (\lambda-1)]\, a_{k-2} \} \xi^{s+k-2} = 0$$

so the recursion becomes

$$a_k (s+k)(s+k-1) - [\, 2s+2k - \lambda - 3)]\, a_{k-2} = 0$$

In this, replace with $\nu = k-2$ to get $k = \nu+2$ and so

$a_{\nu+2}(s+\nu+2)(s+\nu+1) - [\ 2s+2\nu+1-\lambda]\ a_\nu = 0$ **13.7 C**

The k=0 condition tells us there are only two possibilities, s = 0 or s = 1.

An important step now happens. We consider that the potential for the HO is even, V(x) = +V(-x) which means the solutions for each eigenvalue can be selected such that H(-ξ) = ± H(ξ). When you are talking powers, this means a solution is either all even powers or all odd powers. Since $a_o \neq 0$, we must have this situation:

s=0 => even powers = > $a_i = 0$ for i odd
s=1 => odd powers = > $a_i = 0$ for i odd

So in either case, we set ALL the odd labeled coefficients to 0, which greatly simplifies things. This is why in the recursion 13.7C Schiff says that "ν is even" -- because the odd a_ν are all zero and we don't care about the recursion for them.

The next step is to make the claim that the series must terminate if we are to get a normalizable wavefunction u(ξ) as in 13.4, and Schiff shows this top of page 69.

From 13.7C we see that termination means λ [the eigenvalue where E = λ (ħω/2)] cannot be any value, but must an integer of the form λ = 2s+2ν +1 with ν even. So here are our cases: [we use dummy label ν' in the second case so we can change it in a second]

s=0	λ = 2ν+1	ν = 0,2,4,6....
s=1	λ = 2ν'+3	ν' = 0,2,4,6....

In the second case, we want to get 2ν+1 so want 2ν+1 = 2ν'+3 , or 2ν = 2ν' + 2, or ν = ν' + 1. We then rewrite both cases, making this change in the second case:

s=0	λ = 2ν+1	ν = 0,2,4,6....	even parity
s=1	λ = 2ν+1	ν = 1,3,5,7....	odd parity

and we can then combine these into a single result

λ = 2n+1 n = 0,1,2,3,4.... parity is $(-1)^n$

and we get the very famous HO result that E = λ (ħω/2) = (2n+1) (ħω/2) . **13.8**

For a given n, the polynomial solution of 13.5 is then called $H_n(\xi)$.

This section ends with a comment that the HO **"zero point energy"** is just the ground state energy which is non zero just as it is for the square well, no big deal. It is a result of the UP, and minimizes the UP.

Hermite Polynomials (69)

At this point, Schiff's presentation is a little strange, but still good. In 13.10, we know we can create the sum shown on the RHS of 13.10. The big question is why this sum comes out being exp[$-s^2+2s\xi$] which seems an amazing fact. Schiff is now going to prove that this is true, but the fact that he is doing a proof

gets lost at first. It took me three tries to do the next section, and just for fun I will retain all the details of Plan A and Plan B which failed.

Derivation of 13.11 from 13.10. _____long digression _____

Plan A: We start assuming that 13.10 is true, then we easily verify results A and B. At this point we have

$$2s\,S = \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) \qquad A$$

$$(-2s + 2\xi)S = \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi) \qquad B$$

These sums are not equal, but here is one way to get two equal sums: Mult A by $(-2s + 2\xi)$ and B by 2s and we get

$$(-2s + 2\xi)\, \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) = 2s\; \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi)$$

Write this out as three terms

$$(-2s)\, \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) + (2\xi)\, \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) - 2s\; \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi) = 0$$

Absorb the s's into the powers [and change dummy index to n' in first term]

$$-2\, \Sigma_{n'=0}\, s^{n'+1}/n'!\; H_{n'}'(\xi) + 2\xi\, \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) - 2\Sigma_{n=1}\, s^n/(n-1)!\; H_n(\xi) = 0$$

In this first term, set n = n'+1 or n' = n-1 so in "n", sum starts at 1, not 0:

$$-2\, \Sigma_{n=1}\, s^n/(n-1)!\; H_{n-1}'(\xi) + 2\xi\, \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) - 2\Sigma_{n=1}\, s^n/(n-1)!\; H_n(\xi) = 0$$

$$-2\, \Sigma_{n=1}\, s^n/(n-1)!\; [\, H_{n-1}'(\xi) + H_n(\xi)] + 2\xi\, \Sigma_{n=0}\, s^n/n!\; H_n'(\xi) = 0$$

For n = 0, only the last term exists and we get

n=0: $\quad s^n/n!\; H_n'(\xi) = 1H_0'(\xi) = 0$

but this is no surprise since $H_0(\xi)$ = a constant. For n> 0 we find [cancel common factors]

$$-1/(n-1)!\; [\, H_{n-1}'(\xi) + H_n(\xi)] + \xi/n!\; H_n'(\xi) = 0$$

$$-1/(n-1)!\; [\, H_{n-1}'(\xi) + H_n(\xi)] + \xi/[n(n-1)!]\; H_n'(\xi) = 0$$

$$-[\, H_{n-1}'(\xi) + H_n(\xi)] + \xi/[n]\; H_n'(\xi) = 0$$

$$-nH_{n-1}'(\xi) - nH_n(\xi) + \xi\, H_n'(\xi) = 0 \qquad \text{// my final result}$$

$$-nH_{n-1}' - nH_n + \xi\, H_n' = 0$$

This relates two derivatives to one non-derivative. My first question: is this result correct or not? The two recursion formulas he gives in 13.11 *are* in fact correct says Schaum page 151, so let's apply them to the above:

$$- nH_{n-1}' - nH_n + \xi H_n' = - n(2[n-1]H_{n-2}) - nH_n + \xi (2nH_{n-1})$$

$$= -2n(n-1)H_{n-2} - nH_n + 2n\xi H_{n-1} \quad =?= 0$$

$$= -2(n-1)H_{n-2} - H_n + 2\xi H_{n-1} \quad =?= 0$$

Now in this last line replace n with n+1:

$$= -2nH_{n-1} - H_{n+1} + 2\xi H_n \quad =?= 0$$

and yes, this then replicates the second formula he gives, so this means that my result is actually correct. But how to I get from my result to his two results?

My conclusion is that I have derived a recursion formula that is correct, but which is not the one we want, and I see no obvious way to get from it to the desired targets, so lets go to

Plan B: Start over

$$A = 2s\, S = \Sigma_{n=0}\, s^n/n!\, H_n'(\xi)$$

$$B = (-2s + 2\xi)S = \Sigma_{n=1}\, s^{n-1}/(n-1)!\, H_n(\xi)$$

Notice that

$$A+B = 2\xi S$$

$$sA+sB = 2\xi sS = \xi A$$

$$(s-\xi)A + sB = 0$$

This looks more promising because everything is lower in power and we should get a simpler result? So let's do it:

$$(s-\xi)\, \Sigma_{n=0}\, s^n/n!\, H_n'(\xi) + s\, \Sigma_{n=1}\, s^{n-1}/(n-1)!\, H_n(\xi) = 0$$

$$s\Sigma_{n=0}\, s^n/n!\, H_n'(\xi) + \xi\Sigma_{n=0}\, s^n/n!\, H_n'(\xi) + s\, \Sigma_{n=1}\, s^{n-1}/(n-1)!\, H_n(\xi) = 0$$

$$\Sigma_{n'=0}\, s^{n'+1}/n'!\, H_{n'}'(\xi) + \xi\Sigma_{n=0}\, s^n/n!\, H_n'(\xi) + \Sigma_{n=1}\, s^n/(n-1)!\, H_n(\xi) = 0$$

where we change to n' dummy index in the first term. Now set n = n'+1 there so n sum then starts at 1.

$\Sigma_{n=1}\, s^n/(n-1)!\, H_{n-1}'(\xi) + \xi\Sigma_{n=0}\, s^n/n!\, H_n'(\xi) + \Sigma_{n=1}\, s^n/(n-1)!\, H_n(\xi) = 0$

For n=0 we have only the center term which says $H_0'(\xi) = 0$ and thus $H_o(\xi)$ = constant, which is true. for all other n we get

$1/(n-1)!\, H_{n-1}'(\xi) + \xi/n!\, H_n'(\xi) + 1/(n-1)!\, H_n(\xi) = 0$

$n/n!\, H_{n-1}'(\xi) + \xi/n!\, H_n'(\xi) + n/n!\, H_n(\xi) = 0$

$nH_{n-1}'(\xi) + \xi H_n'(\xi) + nH_n(\xi) = 0$

but this is the same result I got last time!!! So Plan B gives nothing new. I am stuck with this result

$- nH_{n-1}' - nH_n + \xi H_n' = 0$

Plan C: Start over again.

$A = 2s\, S = \Sigma_{n=0}\, s^n/n!\, H_n'(\xi)$

$B = (-2s + 2\xi)S = \Sigma_{n=1}\, s^{n-1}/(n-1)!\, H_n(\xi)$

This is the trick: in each equation, insert $S = \Sigma_{n=0}\, s^n/n!\, H_n(\xi)$. Then we have

A: $2s\, \Sigma_{n=0}\, s^n/n!\, H_n(\xi) = \Sigma_{n=0}\, s^n/n!\, H_n'(\xi)$

B: $(-2s + 2\xi)\, \Sigma_{n=0}\, s^n/n!\, H_n(\xi) = \Sigma_{n=1}\, s^{n-1}/(n-1)!\, H_n(\xi)$

Now indeed we have two separate equations. Deal with one at a time, A first:

$2s\, \Sigma_{n=0}\, s^n/n!\, H_n(\xi) = \Sigma_{n=0}\, s^n/n!\, H_n'(\xi)$

$2\, \Sigma_{n'=0}\, s^{n'+1}/n'!\, H_{n'}(\xi) = \Sigma_{n=0}\, s^n/n!\, H_n'(\xi)$

In the first term take n'+1 = n and n' = n-1:

$2\, \Sigma_{n=1}\, s^n/(n-1)!\, H_{n-1}(\xi) = \Sigma_{n=0}\, s^n/n!\, H_n'(\xi)$

For n=0 we get $0 = H_0'(\xi)$ which says $H_0(\xi)$ = constant, which is true. For all other n get

$2\, /(n-1)!\, H_{n-1}(\xi) = 1/n!\, H_n'(\xi)$

$2\, n/n!\, H_{n-1}(\xi) = 1/n!\, H_n'(\xi)$

$2\, nH_{n-1}(\xi) = H_n'(\xi)$

and finally we have **13.11 C.**

Now play with the B equation:

$$(-2s + 2\xi)\,\Sigma_{n=0}\, s^n/n!\; H_n(\xi) = \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi)$$

$$-2s\Sigma_{n=0}\, s^n/n!\; H_n(\xi) + 2\xi\,\Sigma_{n=0}\, s^n/n!\; H_n(\xi) = \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi)$$

$$-2\Sigma_{n''=0}\, s^{n''+1}/n''!\; H_{n''}(\xi) + 2\xi\,\Sigma_{n'=0}\, s^{n'}/n'!\; H_{n'}(\xi) = \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi)$$

Now set n"+1 = n-1 so that n" = n-2 and n = n" +2.
And set n' = n-1 so that n' = n-1 and n = n'+1. Then have

$$-2\Sigma_{n=2}\, s^n/(n-2)!\; H_{n-2}(\xi) + 2\xi\,\Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_{n-1}(\xi) = \Sigma_{n=1}\, s^{n-1}/(n-1)!\; H_n(\xi)$$

For n= 1 only the last two terms contribute and we get

$$2\xi\, H_0(\xi) = H_1(\xi)$$ // which is true, Schaum p 151

For larger n get

$$-2\ /(n-2)!\; H_{n-2}(\xi) + 2\xi\ /(n-1)!\; H_{n-1}(\xi) = 1/(n-1)!\; H_n(\xi)$$

$$-2(n-1)\ /(n-1)!\; H_{n-2}(\xi) + 2\xi\ /(n-1)!\; H_{n-1}(\xi) = 1/(n-1)!\; H_n(\xi)$$

$$-2(n-1)\, H_{n-2}(\xi) + 2\xi\, H_{n-1}(\xi) = H_n(\xi)$$

$$-2(n)\, H_{n-1}(\xi) + 2\xi\, H_n(\xi) = H_{n+1}(\xi)$$ // replaced n with n+1 in previous line

and this gives **13.11 D.**

Next question: <u>how do we know that recursions 13.11 are true by other means?</u> Schiff says somehow this is "easy to do", but I don't think it is easy at all. In Erdelyi, the three-adjacent-order recursion formula is proven (and I did it in detail in my notes) for all orthogonal polynomials at once, and the proof is quite long and uses the orthogonality of those polynomials!

So, all we can say for Schiff is that the generating function formula is *consistent with* the two recursion relations and probably you could go backwards from the two recursion relations to that generating formula, a very obscure derivation method indeed!

The recursion formula involving the three adjacent orders is what is common to all orthogonal polynomials with appropriate factors.

The other ones with derivatives are fairly specific to the poly type, and AS does not even show our simple one for the Hermites.

Schiff closes out this rather clumsy section by using the generating function formula to derive the Rodrigues' (AS p785) formula expression the H_n as derivatives of $\exp(-\xi^2)$.

Comment on Generating Functions. Erdelyi (Bateman) vol 2 has a section on orthos that I took some notes on (printed in a binder as well), but it does not mention generating functions. M&M have them, but not in general, just for specific polynomials, and the development involves integral representations and contour integration. A&S of course list them all off for all ortho polys.

These things are at a sort of intersection of many topics: differential equations (like Hermite), ortho polys, complex variables, Hilbert Spaces. Aside from Stakgold, I don't even have a single ODE book. It is not clear to me why such a simple function should even exist, but they certainly do exist as AS show.

So this is just another huge hole in my historical math knowledge. I would have to do yet another stack push on this, and I don't want to do that right now.

MM list several generating functions on page 132. But there is just no unifying comment.

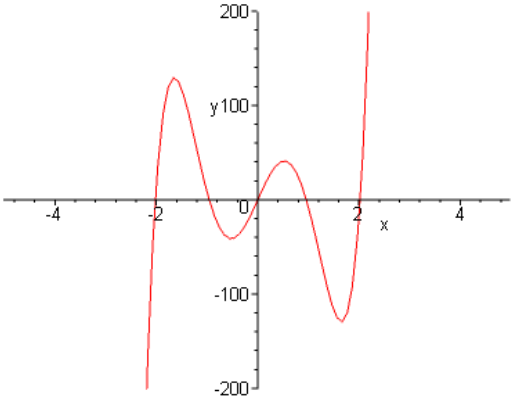
Harmonic Oscillator Wave Functions (71). Yes, our result is 13.13. Schiff uses the generating function idea to quickly compute N_n, the normalization factor. How else would you do this? You would just have to go look it up in a Hermite's section somewhere. He also obtains the orthogonality using the same quite impressive generating function method. You see the "weight" factor in 13.16, but of course when you write this in terms of $u_n(x)$ as in 13.13, there is no weight, the weight just comes from the expo part of the function. Recall the weights from Erdelyi. On page 72 S goes to compute the expectation value of x again using this tricky method.

Question: How *in general* would you compute such expectation integrals for special functions? I can't even find 13.18 in GR, for example. I know that the coordinate representation is not the simplest for the HO, but still, it is harder than I remember it being.

Correspondence with Classical Theory (73)

First off, look at the plots on page 73 of the HO wavefunctions. The ground state is just K $\exp(-\xi^2/2)$ with no multiplying polynomial. In general, when there is a polynomial, the polynomial blows up as its highest power on the left and right, but this is then tamed by the $\exp(-\xi^2/2)$ factor. Here is an example

```
> with(orthopoly);
                    [G, H, L, P, T, U]
> plot(H(5,x),x = -5..5, y= -200..200,numpoints = 100);
```



Now regarding probability function for classical motion. This notion is a little foreign. We are used to probability on QM but not in classical. The basic idea for a any periodic motion would be this:

$P(x) = k/v(x)$

where v(x) is the velocity at position x during the motion. If it is going twice as fast at some x compared to some other x, then it has half the probability of being seen there in a set of random measurements of position. Now we have:

$x(t) = A\sin(\omega t)$
$v(t) = A\omega\cos(\omega t)$

$x^2\omega^2 + v^2 = (A\omega)^2 * 1$ so $v^2 = \omega^2(A^2 - x^2)$ and this tells us

$v(x) = \omega\sqrt{A^2 - x^2}$

Therefore we have

$P(x) = \text{const} / \sqrt{A^2 - x^2}$

and this is where Schiff gets his claim of $(\xi_o^2 - \xi)^{-1/2}$ for the classical probability distribution. This is then plotted as the dotted line on page 74 (taken from the Pauling book I recently bought!) . You can see that only when n gets to a larger value to you "get" something like the classical distribution! I replicated this plot using Maple.

Now you can compute $\langle x^2 \rangle_n$ using the generating function methods and you get $\langle V \rangle = E_n/2$. This means we must have $\langle KE \rangle = E_n/2$ as well. Look at page 71 Goldstein and you see for $V \sim x^2$ we have Goldstein's n=1 and 3.29 (virial theorem) then says $\langle T \rangle = \langle V \rangle$, and we see that here in QM and of course it is also true in classical.

S then claims that one can show p73 A, so as n increases, your UP RHS increases with n. The ground state in this case has $\hbar/2$ and in fact has the exact form of our minimum packet we talked about in the last chapter.

Oscillating Wave Packet (74)

This is a fascinating little "study" and I have not attempted to verify all the math which looks substantial. The idea is credited to Schrodinger in 1926, this paper

12. E. Schrödinger, "Der stetige Übergang von der Mikro-zur Makromechanik", *Natuurwissenschaften* **14** (1926) 664–666. English translation: "The continuous transition from micro- to macromechanics", in *Collected Papers on Wave Mechanics*, ibid., pp. 41–44.

Suppose you construct as in 13.21 a gaussian packet that is offset to x = a at time t=0. You can compute the A_n coefficient to be 13.22. Jam this into 13.20 where you have a series summation. The summed phases appear on the right of E, but this is interpreted as that generating function sum and everything is then expo. The claim is that if you square this thing, it hugely simplifies giving H which says that the packet swings back and forth *without blurring out over time* just as would a classical particle!

Note: tried to get the Collected Papers on **BitTorrent**, it was .rar got to 89% and bogged down probably because it wants to use me as an uploader source for a while, not that interested, a little scary too.

Moreover, the further to the side you start it, the higher is the "n" that is the peak of the A_n distribution, and in fact this peak is at a value of n such that E_n is the classical energy!

I don't know how many such examples there are of this classical /quantum situation.

14. 3D Spherically symmetric potentials (76)

In the usual way, we separate the solution for V(r) into $R_\ell(r)Y_{\ell m}(\theta,\varphi)$. For m=0 you have the regular Legendre functions with their generating function as in 14.10 which is quite simple. You can compute the associated Legendres as in 14.12 just doing some derivatives. These have a pretty messy looking generator as shown in 14.13. The Schiff normalization for the spherical harmonics $Y_{\ell m}(\theta,\varphi)$ is shown in 14.16. This is really the same as Jackson page 64-65. There, the function P_1^1 is negative, for example. This is the Condon & Shortley and MO convention of circa 1949.

On page 81 S shows that the φ part of Y has parity of |m|, while the P_ℓ^m has parity ℓ - |m| which means that the whole $Y_{\ell m}(\theta,\varphi)$ has the parity $(-1)^\ell$ regardless of m.

The radial equation picks up the famous $\ell(\ell+1)/r^2$ term which adds to the potential. I regard this as a sort of centrifugal force that pushes outward on an orbiting electron, whereas the discussion page 81 keeps talking about it being an inward force which seems wrong to me. Large ℓ keeps the orbiter far from the center. I would have said it was analogous to the fictitious centri*fugal* force of classical dynamics and has exact the right form and dimensions.

Page 82 shows L written in Cartesian and spherical coordinates, and L^2 in sphericals. It is clearly stated that the $Y_{\ell m}(\theta,\varphi)$ are eigenfunctions of $L^2$and L_z. Schiff likes the words orbital and magnetic for ℓ and m quantum numbers. And he says that the eigenfunctions in V(r) are degenerate in energy for the different allowed m.

In general, this section is very familiar to me.

15. 3D Square Well (83)

The famous well is defined as a 3D hole (spherical) of depth $V_0 > 0$, the picture is fine.

We consider the equation for χ in the ℓ=0 case and things are the same as for the 1D well except we only use the decaying expo radially since there is only one "side" in sphericals. We end up with one of the two transcendental equations we got for the 1D square well. in the 1D case, we had solutions of both parities which is why there were two trans equations. Here the parity for a given ℓ is $(-1)^\ell$, so only one parity per ℓ, hence only one trans equations.

For $\ell > 1$, the R equation is the "**spherical Bessel equation**" which has solutions which are J and N functions but of "half integer order", and they are normally called j and n. Page 85 and 86 gives a good summary or properties. The n's blow up at ρ=0 so the solution inside the well can only be $j_\ell(\rho)$ ρ = αr and α as in 15.2. Hence, 15.11.

The exterior solutions need expo decay, and the j/n linear combination known as the "**Hankel function of the first kind**" is the one that does this, I very dimly recall this from Jackson, hence 15.14.

So for a given ℓ, we need to find A and B in the interior and exterior solutions 15.11 and 15.14 which make the solutions "match up" at r=a. This is some huge messy deal with all these Bessel functions and their derivatives. This never came up in Jackson electrostatics. No book I have seems to do this problem in detail but I think I qualitatively know what the solutions look like.

One side benefit of this section is that we get warmed up on these j and n functions which will reappear in the corresponding scattering problem.

16. The Hydrogen Atom (88)

So now we come to "the main act" of quantum mechanics. The first 2.5 pages are used up to show that you can separate the SE into CMS and "relative motion" wave functions, and then we just throw out the CMS because it is trivial -- a free particle of mass M. We care about the relative motion and we use the reduced mass μ. I think it was good to get this out of the way in a clear manner.

Next. When looking for a polynomial solution of an ODE, the first thing you always do is look at large argument and find the form of the asymptotic solution, usually some kind of decaying exponential, then you try a solution which is a polynomial times that exponential. I have no proof this works, but it does often seem to work. In the HO case 13.1, the dominant large-arg term was x^2, not E, and this led to the asymptotic form exp[$-x^2/2$] because ∂_{x^2} on this creates the needed x^2. In the H atom case 16.6 where things are 1/r-like, it is the E term that dominates for large r, but we "want" an asymptotic form that does not depend on the eigenvalue E to maximize our chances of finding a power series x asymp form solution. That, says Schiff, is the motivation for "scaling" the equation 16.6 with ρ = αr and looking for R(ρ), with the explicit α shown in 16.8. Here, α has dimensions 1/r so ρ is dimensionless.

So, how exactly does this "remove E" and turn it into ¼ ? Let Q(ρ) = R(r) and start with 16.6 which says this:

$$-a\ 1/r^2\partial_r(r^2\partial_r R) - bR/r + c/r^2R = ER$$

$\partial_r = \alpha\partial_\rho$ {dims: 1/r = 1/r } $= (\rho/r)\partial_\rho$ $r^2 = (\rho/\alpha)^2$ so we have $1/r = (\alpha/\rho)$

$-a\ (\alpha/\rho)^2\ \alpha\partial_\rho\ ((\rho/\alpha)^2\ \alpha\ \partial_\rho\ Q) - bR(\alpha/\rho) + c(\alpha/\rho)^2R = ER$ $a = \hbar^2/2\mu$ $b = Ze^2$

$-a\ \alpha^2\ (1/\rho)^2\ \partial_\rho\ (\rho^2Q) - b\ \alpha\ R/\rho + c\alpha^2(1/\rho)^2R - ER = 0$ $c = \ell(\ell+1)\hbar^2/2\mu$

$+a\ \alpha^2\ (1/\rho)^2\ \partial_\rho\ (\rho^2Q) + b\ \alpha\ R/\rho - c\alpha^2(1/\rho)^2R + ER = 0$

$+(1/\rho)^2\ \partial_\rho\ (\rho^2Q) + (b/\alpha a\ )(1/\rho)Q - (c/a)(1/\rho)^2Q - |E|\ /(a\alpha^2)Q = 0$

So now the constant term is the last term , so we want to select α to make it some constant. Schiff chooses this constant to be ¼ for reasons not clear yet, but OK. So

¼ = |E|/(aα²) $\quad\quad$ $\alpha^2 = 4/a = 8\mu|E|/\hbar^2$ $\quad\quad$ as in **16.8**

$(b/\alpha a) = 2\mu Ze^2/\alpha\hbar^2 = 2\mu Ze^2/\hbar^2 * (\hbar^2/[8\mu|E|])^{1/2} = Ze^2/\hbar * (4\mu^2\hbar^2/[8\mu|E|\hbar^2])^{1/2}$

$= Ze^2/\hbar * (\mu/[2|E|])^{1/2} = Ze^2/\hbar\sqrt{\mu/[2|E|]} \equiv \lambda$ $\quad\quad$ // define λ

$(c/a) = \boldsymbol{\ell}(\boldsymbol{\ell}+1)\hbar^2/2\mu * 2\mu/\hbar^2 = \boldsymbol{\ell}(\boldsymbol{\ell}+1)$

Notice that changing that ¼ factor would not change this last result. We then have

$+(1/\rho)^2\,\partial_\rho(\rho^2 Q) + (\lambda/\rho)Q - \boldsymbol{\ell}(\boldsymbol{\ell}+1)/\rho^2 Q - \tfrac{1}{4}Q = 0$

$+(1/\rho)^2\,\partial_\rho(\rho^2 Q) + [(\lambda/\rho) - \boldsymbol{\ell}(\boldsymbol{\ell}+1)/\rho^2 - \tfrac{1}{4}]Q = 0$

and this verifies 16.7 where again Q(ρ) = R(r) in my notation. Now we go to large ρ and equation says

$(1/\rho)^2\,\partial_\rho(\rho^2[\partial_\rho Q]) = \tfrac{1}{4}Q = (1/\rho)^2[\rho^2\partial_\rho[\partial_\rho Q] + 2\rho[\partial_\rho Q]] = \partial_\rho[\partial_\rho Q] + 2[\partial_\rho Q]/\rho \approx \partial_\rho[\partial_\rho Q]$

So try $Q = e^{-\rho/2}$ so that $\partial_\rho Q = -\tfrac{1}{2}Q$ and then $\partial_\rho[\partial_\rho Q] = +\tfrac{1}{4}Q$ and we have it.

So yes, the next step is to write

$Q(\rho) = F(\rho)\,e^{-\rho/2}$

and then we have to convert the radial equation again! Here we go. Let

$g = e^{-\rho/2}$ $\quad g' = -g/2 \quad g'' = g/4$

$Q' = (Fg)' = F'g + Fg' = (F' - F/2)g$

$Q'' = (F' - F/2)g' + (F'' - F'/2)g = -(F' - F/2)g/2 + (F'' - F'/2)g = \{-F'/2 + F/4 + F'' - F'/2\}\,g$

$= (F'' - F' + F/4)\,g$ $\quad\quad$ // notice the ¼ here !!!

$(\rho^2 Q')' = \rho^2 Q'' + 2\rho Q' = \rho^2(F'' - F' + F/4)\,g + 2\rho(F' - F/2)g$

$(1/\rho)^2\,\partial_\rho(\rho^2 Q) = (F'' - F' + F/4)\,g + 2/\rho\,(F' - F/2)g$

So our ODE becomes

$(F'' - F' + F/4)\,g + 2/\rho\,(F' - F/2)g + [(\lambda/\rho) - \boldsymbol{\ell}(\boldsymbol{\ell}+1)/\rho^2 - \tfrac{1}{4}]Fg = 0$

$(F'' - F' + F/4) + 2/\rho\,(F' - F/2) + [(\lambda/\rho) - \boldsymbol{\ell}(\boldsymbol{\ell}+1)/\rho^2 - \tfrac{1}{4}]F = 0$

$F'' + (2/\rho - 1)F' + [(\lambda - 1)/\rho - \boldsymbol{\ell}(\boldsymbol{\ell}+1)/\rho^2]F = 0$ $\quad\quad$ // **16.10** is verified

and I think we now see why we scaled to make the E term be ¼, because in the last line above we got cancellation of this ¼ and now we have no constant term sitting in the [...] multiplying F.

Now we do exactly what we did in the HO case. We assume F has the power series form 16.11 where the first term has power ρ^s and that is why $a_0 \neq 0$. And F hence Q hence R cannot blow up at r=0, so $s \geq 0$. He calls this thing L because it will become a Laguerre polynomial. So let's go again,

$F = \rho^s L$

$F' = \rho^s L' + s\rho^{s-1} L$

$F'' = \rho^s L'' + s\rho^{s-1} L' + s(s-1)\rho^{s-1} L + s\rho^{s-1} L' = \rho^s L'' + 2s\rho^{s-1} L' + s(s-1)\rho^{s-1} L$

$F'' + (2/\rho - 1) F' + [(\lambda-1)/\rho - \ell(\ell+1)/\rho^2]F = 0$ cF

$\rho^s L'' + 2s\rho^{s-1} L' + s(s-1)\rho^{s-2} L + (2/\rho - 1) (\rho^s L' + s\rho^{s-1} L) + [(\lambda-1)/\rho - \ell(\ell+1)/\rho^2] \rho^s L = 0$ $c\rho^s L$

$\rho^s L'' + [2s\rho^{s-1} L' + (2/\rho - 1) (\rho^s L')] +$

$[s(s-1)\rho^{s-2} L + (2/\rho - 1) s\rho^{s-1} L + [(\lambda-1)/\rho - \ell(\ell+1)/\rho^2] \rho^s L = 0$

$\rho^s L'' + [2s\rho^{s-1} + (2/\rho - 1) (\rho^s)] L' +$

$\{ s(s-1)\rho^{s-2} + (2/\rho - 1) s\rho^{s-1} + [(\lambda-1) \rho^{s-1} - \ell(\ell+1)\rho^{s-2}] \} L = 0$ // group L",L',L

$\rho^2 L'' + [2s\rho + (2/\rho - 1) (\rho^2)] L' +$

$\{ s(s-1) + (2/\rho - 1) s\rho + [(\lambda-1)\rho - \ell(\ell+1)] \} L = 0$ $c\rho^2 L$ // divide by ρ^{s-2}

$\rho^2 L'' + \rho[2s +2 - \rho] L' + \{ s(s-1) + (2 - \rho) s + [(\lambda-1)\rho - \ell(\ell+1)] \} L = 0$

$\rho^2 L'' + \rho[2s +2 - \rho] L' + \{ [-s + \lambda-1] \rho + s(s-1) + 2 s - \ell(\ell+1) \} L = 0$

$\rho^2 L'' + \rho[2(s+1) - \rho] L' + \{ [-s + \lambda-1] \rho + s(s+1) - \ell(\ell+1) \} L = 0$ // page 91 A

I cannot say enough for doing algebra in Word. In each step, you do just enough that it is easily trackable. Computer "paper" space is completely free (some storage cost). I always make mistakes, and then I can backtrack and easily find them, correct them, and no one is the wiser. The copy and paste ability is what makes this fly, cannot do that with "pencil and paper". And erasing ability! But to do this, I needed to get all those Greek letters and algebraic forms going for fast typing. I wonder how widespread this method is? Probably more so than I think in my isolation in the Basement.

The above is an ODE for function L(ρ) and must be true for all ρ. For ρ = 0 it must be true. At at that point, need $s(s+1) = \ell(\ell+1)$ which quadratic has $s=\ell$ but also $s = -(\ell+1)$ since $-(\ell+1)(-\ell) = \ell(\ell+1)$. We reject $s = -(\ell+1)$ because need R finite at ρ = 0, and we already said $s \geq 0$, so $s = \ell$. ! We then get

$\rho^2 L'' + \rho[2(\ell+1) - \rho]\, L' + \{[-\ell + \lambda - 1]\,\rho\}\, L = 0$

$\rho L'' + [2(\ell+1) - \rho]\, L' + [-\ell + \lambda - 1]\, L = 0$ $c\rho L$ // **16.12** is verified

Now install our power series:

$L = \Sigma_{k=0} a_k \rho^k = \Sigma_{k=0} a_k \rho^k$

$L' = \Sigma_{k=0} a_k k \rho^{k-1}$

$L'' = \Sigma_{k=0} a_k k(k-1) \rho^{k-2}$

$\rho L'' + [2(\ell+1) - \rho]\, L' + \{[-\ell + \lambda - 1]\}\, L = 0$

$\rho \Sigma_{k=0} a_k k(k-1)\rho^{k-2} + [2(\ell+1) - \rho] \Sigma_{k=0} a_k k \rho^{k-1} + \{[-\ell + \lambda - 1]\} \Sigma_{k=0} a_k \rho^k = 0$ $c \Sigma_{k=0} a_k \rho^{k+1}$

$\Sigma_{k=0} a_k k(k-1)\rho^{k-1} + [2(\ell+1) - \rho] \Sigma_{k=0} a_k k \rho^{k-1} + \{[-\ell + \lambda - 1]\} \Sigma_{k=0} a_k \rho^k = 0$

$\Sigma_{k=0} a_k k(k-1)\rho^{k-1} + 2(\ell+1) \Sigma_{k=0} a_k k \rho^{k-1} + \Sigma_{k=0} a_k \rho^k \{-k + [-\ell + \lambda - 1]\} = 0$

$\Sigma_{k=0} a_k k[(k-1) + 2(\ell+1)]\, \rho^{k-1} + \Sigma_{k=0} a_k \{-k + [-\ell + \lambda - 1]\}\, \rho^k = 0$

$\Sigma_{k=0} a_k k[k + 2\ell + 1]\, \rho^{k-1} + \Sigma_{k=0} a_k \{-k - \ell - 1 + \lambda\}\, \rho^k = 0$

$\Sigma_{k=0} a_k k[k + 2\ell + 1]\, \rho^{k-1} + \Sigma_{k'=0} a_{k'} \{-k' - \ell - 1 + \lambda\}\, \rho^{k'} = 0$ $c \Sigma_{k=0} a_k \rho^{k+1}$

Now in the second term set k' = k-1 so k = k' + 1 to get

$\Sigma_{k=0} a_k k[k + 2\ell + 1]\, \rho^{k-1} + \Sigma_{k=1} a_{k-1} \{-k + 1 - \ell - 1 + \lambda\}\, \rho^{k-1} = 0$ $c \Sigma_{k=2} a_{k-2} \rho^{k-1}$

For k=0 get contribution only from the first term which says 0 = 0, nothing new here.

For k = 1 we get contributions from the first two terms only (not the red term)

$a_k k[k + 2\ell + 1]\, \rho^{k-1} + a_{k-1} \{-k + 1 - \ell - 1 + \lambda\}\, \rho^{k-1} = 0$

$a_1[2\ell + 2] + a_0 \{-\ell - 1 + \lambda\} = 0$ => $a_1 = (\ell+1-\lambda)/(2\ell+2) * a_0$

so this one low recursion formula would be different from all the rest because it would not contain "c" .

For k>0 get [k>1]

$a_k k[k + 2\ell + 1] + a_{k-1} \{-k + 1 - \ell - 1 + \lambda\} = 0$ $c a_{k-2}$ on RHS

$a_k k[k + 2\ell + 1] + a_{k-1} \{-k - \ell + \lambda\} = 0$

Replace k = ν+1 [Since we had k > 0, this means ν ≥ 0]

$a_{\nu+1}(\nu+1)[\nu + 2\ell + 2\,] + a_\nu \{ -\nu - 1 - \ell + \lambda\} = 0$ $ca_{\nu-1}$

$a_{\nu+1} = \{ \nu + 1 + \ell - \lambda\}/\{(\nu+1)[\nu + 2\ell + 2\,]\}\, a_\nu$ **16.13** is verified +c in {}

Now for very large index we do have $a_{\nu+1}/ a_\nu \rightarrow 1/\nu$ as claimed. This means that the tail of the series is going like this:

$$C\,\nu^{100} + (C/\nu)\,\nu^{101} + (C/\nu^2)\,\nu^{102} + \ldots\ldots = C\nu^{100}\,[\,1 + 1 + 1\ldots\ldots\,] = \text{diverges!}$$

So the obvious truncation requirement is this, where $\nu_{max} \geq 0$

$\nu_{max} = \lambda - \ell - 1$ or $\lambda = \nu_{max} + \ell + 1$

Since ν_{max} and ℓ are integers, so is λ and this is going to quantize our energy E. Schiff uses these letters:

$\nu_{max} = n'$ $\nu_{max} + \ell + 1 = n' + \ell + 1 = n$ so $\lambda = n$

Since $n' = \nu_{max} \geq 0$, we have $n \geq \ell+1$ for a given ℓ. So for $\ell = 0$ we have n = 1. So thing of n as the main number, then our condition becomes $\ell + 1 \leq n$ or $\ell \leq n-1$. So when n = 1, can have only $\ell = 0$, etc.

So looking back at 16.8 we get

$\lambda^2 = Z^2e^4/\hbar^2 * (\mu/2|E|) = n^2 = \mu Z^2e^4/ 2\hbar^2 * 1/|E| \Rightarrow$

$$E = -\,\mu Z^2e^4/2n^2\hbar^2 \qquad n = 1,2,3.... \qquad \ell \leq (n-1)$$

and this duplicates the "old 1913 Bohr quantum theory" where L_z was quantized, but now we did it with the SE and we have actual wavefunction solutions. so this is the "new quantum theory" of 1923 or so.

Schiff notes that we get an infinite number of bound states as opposed to the square well where we always get a finite number, and this is one of those 1/r long range issues. The HO is long range and also has an infinite number.

What now is α , by the way? We had

$(b/\alpha a) = \lambda = n$ $a = \hbar^2/2\mu$ $b = Ze^2$

so

$\alpha = b/na = 2\mu Ze^2/n\hbar^2 = \alpha_n$ // different for each eigenfunction

Now the Bohr radius in cgs units is $a_o = \hbar^2/\mu e^2$ adjusted for our reduced mass μ. Then we have

$\alpha_n = (2Z/na_0)$ dimensions 1/r

Comments on the selection of the ¼ in 16.7. Had this been instead ¼ +c, we would have had an extra term as I show in red above throughout the calculation. This would have resulted in a *triple* recursion relation:

$$a_{\nu+1}(\nu+1)[\nu + 2\ell + 2\,] \;+\; a_\nu\,\{-\nu - 1 - \ell + \lambda\} \;= c\, a_{\nu-1}$$

This *considerably* complicates the analysis. If you build this up one step at a time, perhaps you can get a power series and it will reduce to a two component recursion relation with λ replaced by a multiple of itself. Or treat as a difference equation and solve for a_ν. Only then can you write a truncation condition for this thing. Meanwhile, our earlier definition of λ would be changed as follows, where c is now added:

$$¼ + c = \; |E|/(a\alpha^2) = (1+4c)/4 \qquad\qquad \alpha^2 = 4/a(1+4c) = 8\mu|E|/[\,\hbar^2(1+4c)]$$

$$(b/\alpha a\,) = 2\mu Ze^2/\,\alpha\hbar^2 \;= 2\mu Ze^2/\hbar^2 \;* (\hbar^2(1+4c)/[8\mu|E|])^{1/2} \;= Ze^2/\hbar * (4\mu^2\hbar^2(1+4c)/[8\mu|E|\hbar^2])^{1/2}$$

$$= Ze^2/\hbar * (\mu(1+4c)/[2|E|])^{1/2} = Ze^2/\hbar * \sqrt{\mu(1+4c)/[2|E|]} \;\equiv \lambda_{new}$$

If I did this right, then we end up with $\lambda_{new} = \sqrt{1+4c}\,\lambda$. In the footnote on page 92 Schiff suggests that if you do it this "hard way", you will probably get the following truncation condition

$$\lambda_{new}/\sqrt{1+4c} \;= (\nu_{max} + \ell + 1) = n$$

where, as he says, λ ends up being a "multiple" of an integer. Then we would end up with

$$\{\, Ze^2/\hbar * \sqrt{\mu(1+4c)/[2|E|]}\,\}^2 \;= \{\,\sqrt{1+4c}\; n\,\}^2$$

and then the (1+4c) factors would cancel and you end up with the same eigenenergies. So getting that ¼ there is quite important. Also, if we had some c in there, we would have

$$\rho L'' + [2(\ell+1) - \rho]\, L' \;+\; [-\ell + \lambda - 1 - c\rho]\;\; L \;\; = 0$$

$$\rho L'' + [(2\ell+1) + 1 - \rho]\, L' \;+\; [(\ell+\lambda) - (2\ell+1) - c\rho]\;\; L \;\; = 0$$

$$\rho L'' + [p + 1 - \rho]\, L' \;+\; [q - p - c\rho]\;\; L \;\; = 0$$

and this puts an extra ugly term in our Laguerre equation! This is there because we are using the "wrong α", meaning we are now using $\alpha^2 = 4/a(1+4c) = 8\mu|E|/[\,\hbar^2(1+4c)]$ with c≠0. No doubt, if you were to scale the argument of L in the equation above by the right factor to get the "right α", then that last term will go away.

So again, getting that ¼ into 16.7 was crucial and gives us the "right α" so that we end up with the correct equation for the associated Laguerre polynomials.

Laguerre Polynomials (92)

Schiff gives us first the generator form, then the recursion relations in two forms, one involving derivatives, and he claims these solve 16.18 where q = an integer, solutions are $L_q(\rho)$. Then the associated Laguerres are given by 16.19 and they solve 16.20 he claims, and this is what matches our equation for L with the identifications

$q = \ell + \lambda = \ell + n \qquad p = 2\ell+1$

So our solutions will be

$L_q^p(\rho) = L_{n+\ell}^{2\ell+1}(\rho)$ as shown in 16.22

Next, in 16.21 he gives the generator form for these associated Laguerres, and then in 16.22 we get an explicit expression for L_q^p which is not too complicated. He did not do this for the associated Legendres (which are part of the spherical harmonics), nor did he do it for the Hermites of HO fame.

The H atom wavefunctions (93)

We assemble the three pieces of the radial function now:

$R(r) = Q(\rho) = F(\rho)\, e^{-\rho/2} = \rho^{\ell}\, L_{n+\ell}^{2\ell+1}(\rho)\, e^{-\rho/2} * \text{normalizer}$

$\rho = \alpha_n r \qquad \alpha_n = (2Z/na_0) \qquad a_o = \hbar^2/\mu e^2$

Since ρ depends on n, we might write this as

$R(r) = \rho_n^{\ell}\, L_{n+\ell}^{2\ell+1}(\rho_n)\, \exp(-\rho_n/2) * \text{normalizer} \qquad \rho_n = \alpha_n r$

or

$R_{n\ell}(r) = (\alpha_n r)^{\ell}\, L_{n+\ell}^{2\ell+1}(\alpha_n r)\, \exp(-\alpha_n r/2) * \text{normalizer} \qquad \alpha_n = (2Z/na_0) \qquad a_o = \hbar^2/\mu e^2$

The normalization condition is $\int r^2 dr\, R_{n\ell}(r)^2 = 1$ which says $\int \rho^2 d\rho\, Q(\rho)^2 = \alpha_n^3$

$N^2 \int \rho^2 d\rho\, \rho^{2\ell}\, [L_{n+\ell}^{2\ell+1}(\rho)]^2\, e^{-\rho} = \alpha_n^3$

The integral is shown in 16.23, so we invert that result, take square root, mult by $\alpha_n^{3/2}$.

Phase: Schiff adds a -1 phase in 16.24 and then does not comment on it. It is a convention only. Here is how we can trace this. First, Schaum page 153 for regular Laguerre's gives 1 and -x+1 as the first two, where we normalize phase things so the first one is just 1. But then page 155 has L_1^1 being the first derivative and it is -1 as shown there in 29.5. This would make our most fundamental S orbital $R_{n=1,\ell=0}$ have a minus sign, we don't like that, so we add a minus in 16.24 to fix it.

Finally,

$E_n = -\ \mu Z^2e^4/2n^2\hbar^2$ $\alpha_n = (2Z/na_0)$ $a_o = \hbar^2/\mu e^2$

$E_n = -(1/a_0)\, Z^2e^2/2n^2 = -Z^2e^2/2a_on^2$ as in page **94 A**

Degeneracy (94)

Notice that $E_{n\ell m} = E_n$ for the V = -1/r problem we just solved. For each ℓ there are $(2\ell+1)$ degenerate states, but also it happens that for only this potential the different ℓ's are also degenerate. This is broken in atoms where we lose the 1/r exactness. Schiff's main point here is that when you have lots of degeneracy, you can usually find some other coordinate system where the problem separates, and those basis functions will be linear combinations of the original system basis functions due to this degeneracy. To demonstrate this, Schiff is now going to solve the H atom in parabolic coordinates! Very strange.

Separation in Parabolic Coordinates (95)

PARABOLIC COORDINATES -- a digression

About these coordinates from M&M p185. // using ξ^2 and η^2 , the M&M convention

*Note that first part of the **wiki** page uses this convention but uses σ,τ in place of ξ,η so eg $z = (\tau^2 - \sigma^2)/2$. The lower part of the wiki page does the second convention.*

*Also, **Wolfram** uses this M&M convention but has u = η and v = ξ.*

Consider the x-z plane only. A fully general parabola here is given by $z = Ax^2 + z_0$. Let ξ be some number, and pick a parabola which has $z_o = -\xi^2/2$ and $A = 1/(2\xi^2)$. then your parabola is this:

$z = 1/(2\xi^2)x^2 - \xi^2/2$ or $(z + \xi^2/2) = 1/(2\xi^2)x^2$ or $2\xi^2(z + \xi^2/2) = x^2$

As Schaum p 38 shows, for such a parabola, the distance from the tip to the focus is 1/4A. In our case, this distance is $a = \frac{1}{4}\, 2\xi^2 = \xi^2/2$. Since our parabola opens up to the + z direction, the tip is a to the left of the origin by precisely this distance (ie $|z_o|$), so we conclude that the origin is in fact at the focus. So as we vary ξ, we get a family of "confocal" parabolas all of which have the origin as their focus. Since ξ is real, we are making A>0 and getting parabolas facing to the right.

We can make another set of parabolas opening to the $-\hat{\mathbf{z}}$ direction and use η in place of ξ, and this time we select $A = -1/\eta^2$ and $z_o = +\eta^2/2$. Thus:

$z = -1/(2\eta^2)x^2 + \eta^2/2$ or $(z - \eta^2/2) = -1/(2\eta^2)x^2$ or $-(2\eta^2)\,(z - \eta^2/2) = x^2$

Thus we arrive at 5-54 of M&M page 185. He gives a picture on page 186 and you can then denote any point in the x-z plane by the two numbers ξ and η, and these are called the parabolic coordinates.

Now stare at Schaum p 38 lower right picture and imagine z going to the right and x going up (not as there labeled). Then θ would be the spherical coordinate polar angle and r would be the spherical r. The equation of this same parabola (opening to +z, one of our ξ parabolas say) is given by

$r = 2a/(1-\cos\theta)$ $a = \xi^2/2$ $r = \xi^2/(1-\cos\theta)$ $\xi^2 = r(1-\cos\theta) = r-z$

where $z = \cos\theta$. What about the -z opening η parabolas? The same formula $r = 2a/(1-\cos\theta')$ would apply, but now θ' would be the polar angle from $-\hat{\mathbf{z}}$, so $\theta' = \pi-\theta$, But $\cos\theta' = \cos(\pi-\theta) = -\cos\theta$. Thus, we have for our η parabolas

$r = 2a/(1+\cos\theta)$ $a = \eta^2/2$ $r = \eta^2/(1+\cos\theta)$ $\eta^2 = r(1+\cos\theta) = r+z$

and thus we arrive at the first two equations in Schiff page 95 **16.25**. [except we in the wrong convention]

This would be 2D parabolic coordinates. To get to 3D, you just make paraboloids of revolution and then you can pick out any point in 3D space by the triplet ξ,η and φ where φ is the usual spherical azimuth. So we have (ξ,η,φ) which are "zeye, ate-a and fie".

We can do a little more here. Notice that

$$\xi^2\,\eta^2 = r(1-\cos\theta)\,r(1+\cos\theta) = r^2\sin^2\theta \quad \Rightarrow \quad r\sin\theta = \xi\,\eta$$

and

$$\eta^2 - \xi^2 = 2r\cos\theta = 2z$$

Then we can write down

$x = r\sin\theta\cos\varphi = \xi\eta\cos\varphi$ $\xi^2 = r-z$ $r = (\eta^2+\xi^2)/2$

$y = r\sin\theta\sin\varphi = \xi\eta\sin\varphi$ $\eta^2 = r+z$ $\cos\theta = z/r = (\eta^2-\xi^2)/(\eta^2+\xi^2)$

$z = r\cos\theta = (\eta^2-\xi^2)/2$ $\tan\varphi = y/x$ $\sin\theta = 2\xi\,\eta\,/\,(\eta^2+\xi^2)$

Conventions: It seems that most people nowadays replace $\xi^2 \rightarrow \xi$ and $\eta^2 \rightarrow \eta$ with the understanding that the resulting variables take only positive values. Then we have to redo all our equations above. So here is a completely new copy of the above with these changes made:

About these coordinates from M&M p185. // using ξ and η in Schiff's (physics) convention

Consider the x-z plane only. A fully general parabola here is given by $z = Ax^2 + z_0$. Let ξ be some number, and pick a parabola which has $z_o = -\xi/2$ and $A = 1/(2\xi)$. then your parabola is this:

$z = 1/(2\xi)x^2 - \xi/2$ or $(z + \xi/2) = 1/(2\xi)x^2$ or $2\xi(z + \xi/2) = x^2$

As Schaum p 38 shows, for such a parabola, the distance from the tip to the focus is 1/4A. In our case, this distance is a = ¼ 2ξ = ξ/2. Since our parabola opens up to the + z direction, the tip is a to the left of the origin by precisely this distance, so we conclude that the origin is in fact at the focus. So as we vary ξ, we get a family of parabolas all of which have the origin as their focus. Since ξ is real, we are making A>0 and getting parabolas facing to the right.

We can make another set of parabolas opening to the $-\hat{\mathbf{z}}$ direction and use η in place of ξ, and this time we select A = -1/η and z_o = +η/2. Thus:

$z = -1/(2\eta)x^2 + \eta/2$ or $(z - \eta/2) = -1/(2\eta)x^2$ or $-(2\eta)(z - \eta/2) = x^2$

Thus we arrive at 5-54 of M&M page 185. He gives a picture on page 186 and you can then denote any point in the x-z plane by the two numbers ξ and η, and these are called the parabolic coordinates.

Now stare at Schaum p 38 lower right picture and imagine z going to the right and x going up (not as there labeled). Then θ would be the spherical coordinate polar angle and r would be the spherical r. The equation of this same parabola (opening to +z, one of our ξ parabolas say) is given by

$r = 2a/(1-\cos\theta)$ $a = \xi/2$ $r = \xi/(1-\cos\theta)$ $\xi = r(1-\cos\theta) = r-z$

where $z = \cos\theta$. What about the -z opening η parabolas? The same formula $r = 2a/(1-\cos\theta')$ would apply, but now θ' would be the polar angle from $-\hat{\mathbf{z}}$, so $\theta' = \pi-\theta$, But $\cos\theta' = \cos(\pi-\theta) = -\cos\theta$. Thus, we have for our η parabolas

$r = 2a/(1+\cos\theta)$ $a = \eta/2$ $r = \eta/(1+\cos\theta)$ $\eta = r(1+\cos\theta) = r+z$

and thus we arrive at the first two equations in Schiff page 95 **16.25**.

This would be 2D parabolic coordinates. To get to 3D, you just make paraboloids of revolution and then you can pick out any point in 3D space by the triplet ξ,η and φ where φ is the usual spherical azimuth. So we have (ξ,η,φ) which are "zeye, ate-a and fie".

We can do a little more here. Notice that

$\xi\eta = r(1-\cos\theta)\, r(1+\cos\theta) = r^2\sin^2\theta$ = $r\sin\theta = \sqrt{\xi\eta}$

and

$\eta - \xi = 2r\cos\theta = 2z$

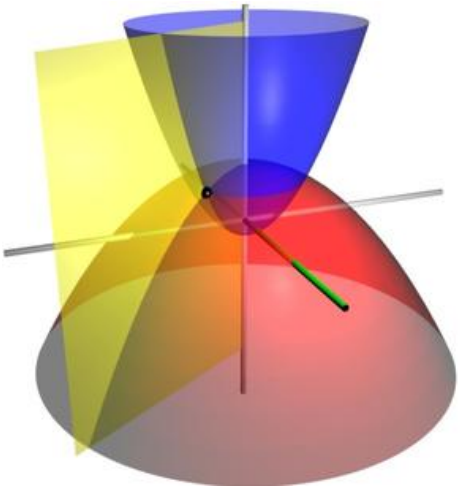
Then we can write down

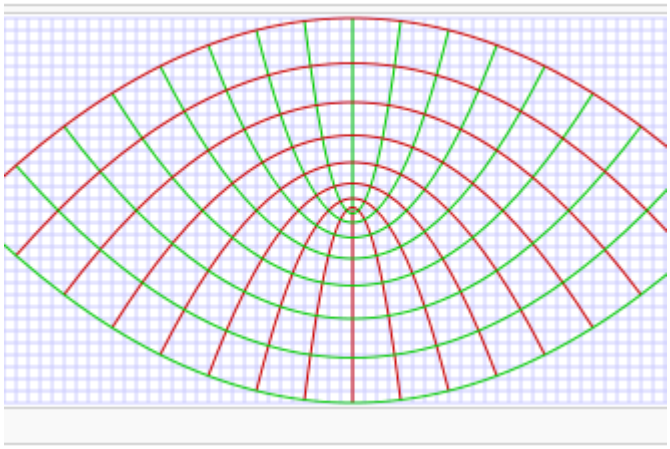
$x = r\sin\theta\cos\varphi = \sqrt{\xi\eta}\cos\varphi$ $\xi = r-z$ $r = (\eta + \xi)/2$

$y = r\sin\theta\sin\varphi = \sqrt{\xi\eta}\sin\varphi$ $\eta = r+z$ $\cos\theta = z/r = (\eta - \xi)/(\eta+\xi)$

$z = r\cos\theta = (\eta - \xi)/2$ $\tan\varphi = y/x$ $\sin\theta = 2\sqrt{\xi\eta}\ /\ (\eta+\xi)$

Here are some useful pictures: Here if z is up, blue is an ξ paraboloid and red is η and yellow is φ.





Exercise A: Since you claim to be an expert now on curvilinear coordinates, let's see you compute the metric tensor and other quantities "from first principles" for the Schiff-convention parabolic coordinates. The equations are these from above

$x = r\cos\theta\cos\varphi = \sqrt{\xi\eta}\,\cos\varphi$ $\quad\quad$ $\xi = r - z$

$y = r\cos\theta\sin\varphi = \sqrt{\xi\eta}\,\sin\varphi$ $\quad\quad$ $\eta = r + z$

$z = r\cos\theta = (\eta - \xi)/2$ $\quad\quad$ $\tan\varphi = y/x$

So my starting point is to think of the *transformation*. Once you know $T^{a}{}_{b}$ you can compute the metric tensor

$$g_{kp} = \Sigma_i \; T_k{}^i \, T_p{}^i = \Sigma_i \frac{\partial x^i}{\partial q^k}\frac{\partial x^i}{\partial q^p} \qquad\qquad g_{kk} = \Sigma_i \; T_k{}^i \, T_k{}^i = \Sigma_i \left(\frac{\partial x^i}{\partial q^k}\right)^2$$

The q^i are our new curvilinear coordinates, the x^i are the cartesians. So let's get computing:

$\partial x/\partial\xi = \eta/[2\sqrt{\xi\eta}\,]\cos\varphi$ $\quad$ $\partial x/\partial\eta = \xi/[2\sqrt{\xi\eta}\,]\cos\varphi$ $\quad$ $\partial x/\partial\varphi = -\sqrt{\xi\eta}\,\sin\varphi$

$\partial y/\partial\xi = \eta/[2\sqrt{\xi\eta}\,]\sin\varphi$ $\quad$ $\partial y/\partial\eta = \xi/[2\sqrt{\xi\eta}\,]\sin\varphi$ $\quad$ $\partial y/\partial\varphi = +\sqrt{\xi\eta}\,\cos\varphi$

$\partial z/\partial\xi = -1/2$ $\quad$ $\partial z/\partial\eta = +1/2$ $\quad$ $\partial z/\partial\varphi = 0$

I could do it all out, but I know g is diagonal, so let's just compute the diagonal elements:

$$g_{\xi\xi} = [\eta/[2\sqrt{\xi\eta}\,]\cos\varphi]^2 + [\eta/[2\sqrt{\xi\eta}\,]\sin\varphi]^2 + 1/4$$

$$= [\eta/[2\sqrt{\xi\eta}\,]^2 + \tfrac{1}{4} = \eta^2/[4\xi\eta] + \tfrac{1}{4} = \tfrac{1}{4}(\eta/\xi + 1) = 1/(4\xi)(\eta+\xi)$$

$g_{\eta\eta} = 1/(4\eta)(\eta+\xi)$ // by symmetry

$g_{\varphi\varphi} = [-\sqrt{\xi\eta}\,\sin\varphi]^2 + [+\sqrt{\xi\eta}\,\cos\varphi]^2 = \xi\eta$

Or

$g_{\xi\xi} = (\eta+\xi)/(4\xi)$ $g_{\eta\eta} = (\eta+\xi)/(4\eta)$ $g_{\varphi\varphi} = \xi\eta$

These are the quantities that M&M refer to as the Q^2 guys. So we have

$Q_\xi^2 = (\eta+\xi)/(4\xi)$ $Q_\eta^2 = (\eta+\xi)/(4\eta)$ $Q_\varphi^2 = \xi\eta$

Notice how this *differs* from the value in the M&M convention shown on page 185. You cannot just take the M&M metric tensor and replace with square root of the symbols! This is because $d(\xi^2) = 2\xi(d\xi)$, etc.

The Laplacian. This is what we need to know in our modern ξ,η coordinates.

$Q_\xi^2 = (\eta+\xi)/(4\xi)$ $g_{ij} = \text{diag}(Q_\xi, Q_\eta, Q_\varphi)$ = the diagonal metric tensor
$Q_\eta^2 = (\eta+\xi)/(4\eta)$
$Q_\varphi^2 = \xi\,\eta$

and g is diagonal because this is an orthogonal coordinate system. Recall all the things we know about this:

$\nabla\varphi = \mathbf{e}^i\,\partial_i\varphi = (\mathbf{e}_i/g_{ii})\,\partial_i\varphi = (\,1/\sqrt{g_{ii}}\,)\,\hat{\mathbf{e}}_i\,\partial_i\varphi$

$\nabla \bullet \mathbf{V} = (1/\sqrt{g}\,)\,\partial_i[\sqrt{g}\,V^i]$ // no simplification $g = g_{11}g_{22}g_{33} = \Pi_i\, g_{ii}$

$\nabla^2\varphi = (1/\sqrt{g}\,)\,\partial_i[\,g^{ij}\sqrt{g}\,\partial_j\varphi\,] = (1/\sqrt{g}\,)\,\partial_i[g^{ii}\sqrt{g}\,\partial_i\varphi\,]$

$\nabla \times \mathbf{V} = (1/\sqrt{g}\,)\,[\,V_{12}\,\mathbf{e}_3 + V_{23}\,\mathbf{e}_1 + V_{31}\,\mathbf{e}_2\,]$ // no simplification

$\sqrt{g} = Q_1Q_2Q_3$

Our interest today is going to be the Laplacian which, when written out for orthogonals, is this

$\nabla^2\psi = (1/\sqrt{g}\,)\,\partial_i[g^{ii}\sqrt{g}\,\partial_i\psi\,] = (1/Q_1Q_2Q_3)\,\partial_i[\,Q_i^{-2}\,Q_1Q_2Q_3\,\partial_i\psi]$

$= (1/Q_1Q_2Q_3)\{\,\partial_1[Q_2Q_3/Q_1 * \partial_1\psi] + \partial_2[Q_3Q_1/Q_2 * \partial_2\psi] + \partial_3[Q_1Q_2/Q_3 * \partial_3\psi]\,\}$

We have

$Q_1Q_2Q_3 = (\eta+\xi)\,(1/4)\,(1/\sqrt{\xi\,\eta}\,)\sqrt{\xi\,\eta} = (\eta+\xi)/4$

$Q_2Q_3/Q_1 = [(\eta+\xi)/(4\eta)]^{1/2}\sqrt{\xi\,\eta}\,/\,[(\eta+\xi)/(4\xi)]^{1/2} = (\xi/\eta)^{1/2}\sqrt{\xi\,\eta} = \xi$

$Q_3Q_1/Q_2 = \eta$ // by symmetry

$Q_1Q_2/Q_3 = [(\eta+\xi)/(4\xi)]^{1/2}\,[(\eta+\xi)/(4\eta)]^{1/2}\,/\sqrt{\xi\,\eta} = (\xi+\eta)/[\,4\xi\eta]$

so filling in we get

$\nabla^2\psi = 4/(\xi+\eta)\,\{\,\partial_\xi(\xi\partial_\xi\psi) + \partial_\eta(\eta\partial_\eta\psi) + \partial_\varphi[\,(\xi+\eta)/[\,4\xi\eta]\,\partial_\varphi\psi\,\}$

$= 4/(\xi+\eta)\,\{\,\partial_\xi(\xi\partial_\xi\psi) + \partial_\eta(\eta\partial_\eta\psi)\,\} + (1/\xi\eta)\,\partial_\varphi^2\psi$

which agrees with Schiff page 96 16.26 ! The result is quite simple and things are symmetric in $\xi\eta$.

_____ END DIGRESSION _____

The Parabolic Separation and Solution

We also learned from above that $r = (\eta + \xi)/2$, so

$V(r) = -Ze^2/r = -2Ze^2/(\xi+\eta)$

Now let's do the separation with $u = f(\xi)g(\eta)\Phi(\varphi)$

$-(\hbar^2/2\mu)\{\,4/(\xi+\eta)\,[\,\partial_\xi(\xi\partial_\xi u) + \partial_\eta(\eta\partial_\eta u)\,] + (1/\xi\eta)\,\partial_\varphi^2 u\,\} - 2Ze^2/(\xi+\eta)u = Eu$

$-(\hbar^2/2\mu)\{\,4/(\xi+\eta)\,[\,\partial_\xi(\xi\partial_\xi\, fg\Phi) + \partial_\eta(\eta\partial_\eta\, fg\Phi)\,] + (1/\xi\eta)\,\partial_\varphi^2\, fg\Phi\,\} - 2Ze^2/(\xi+\eta)\, fg\Phi = E\, fg\Phi$

$-(\hbar^2/2\mu)\{\,4/(\xi+\eta)\,[\,g\Phi\partial_\xi(\xi\partial_\xi f) + f\Phi\partial_\eta(\eta\partial_\eta g)\,] + (1/\xi\eta)\, fg\,\partial_\varphi^2\Phi\,\} - 2Ze^2/(\xi+\eta)\, fg\Phi = E\, fg\Phi$

$-(\hbar^2/2\mu)\{\,4/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (1/\xi\eta)\,(1/\Phi)\,\partial_\varphi^2\Phi\,\} - 2Ze^2/(\xi+\eta) = E$

$-(\hbar^2/2\mu)\{\,4\,\xi\eta/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (1/\Phi)\,\partial_\varphi^2\Phi\,\} - 2Ze^2\,\xi\eta/(\xi+\eta) = E\,\xi\eta$

$-(\hbar^2/2\mu)\{\,4\,\xi\eta/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (1/\Phi)\,\partial_\varphi^2\Phi\,\} - [\,2Ze^2\,\xi\eta/(\xi+\eta) + E\,\xi\eta\,] = 0$

$4\,\xi\eta/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (1/\Phi)\,\partial_\varphi^2\Phi - (-2\mu/\hbar^2)[\,2Ze^2\,\xi\eta/(\xi+\eta) + E\,\xi\eta\,] = 0$

$4\,\xi\eta/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (1/\Phi)\,\partial_\varphi^2\Phi + (2\mu/\hbar^2)[\,2Ze^2\,\xi\eta/(\xi+\eta) + E\,\xi\eta\,] = 0$

$4\,\xi\eta/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (2\mu/\hbar^2)[\,2Ze^2\,\xi\eta/(\xi+\eta) + E\,\xi\eta\,] = -\,(1/\Phi)\,\partial_\varphi^2\Phi$

and we have separated the Φ part and set it equal to constant m^2 and then we get the usual azimuthal functions as solutions of the Φ equation since $\partial_\varphi{}^2\Phi = -m^2\Phi$, and m = integer so single valued, same as before. Now comes the magic part: the 1/(ξ+η) from the potential matches that of the Laplacian, so

$$4\,\xi\eta/(\xi+\eta)\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (2\mu/\hbar^2)[\,2Ze^2\,\xi\eta/(\xi+\eta)+E\,\xi\eta\,] = m^2$$

$$4\,\xi\eta\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (2\mu/\hbar^2)[\,2Ze^2\,\xi\eta + E\,\xi\eta\,(\xi+\eta)\,] = m^2(\xi+\eta)$$

$$4\,[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (2\mu/\hbar^2)[\,2Ze^2 + E(\xi+\eta)\,] = m^2(\xi+\eta)/\,\xi\eta = m^2\,(1/\eta + 1/\xi)$$

$$[(1/f)\partial_\xi(\xi\partial_\xi f) + (1/g)\partial_\eta(\eta\partial_\eta g)\,] + (\mu/2\hbar^2)[\,2Ze^2 + E(\xi+\eta)\,] = m^2(\xi+\eta)/\,4\xi\eta = m^2\,(1/4\eta + 1/4\xi)$$

$$\{\;(1/f)\partial_\xi(\xi\partial_\xi f) + \mu\,E\xi\;/2\hbar^2 + (\mu/2\hbar^2)[\,2Ze^2] - m^2/4\xi\;\}$$

$$+\;\{\;(1/g)\partial_\eta(\eta\partial_\eta g) + \mu\,E\eta\;/2\hbar^2 - m^2/4\eta\;\} = 0$$

$$(1/f)\partial_\xi(\xi\partial_\xi f) + \mu\,E\xi\;/2\hbar^2 + (\mu/\hbar^2)[\,Ze^2] - m^2/4\xi = -\{\;(1/g)\partial_\eta(\eta\partial_\eta g) + \mu\,E\eta\;/2\hbar^2 - m^2/4\eta\;\} = \nu$$

where we have arbitrarily put the Z term in with the ξ side of things, and set the common constant to ν.

From here on, the solution follows exactly the same method that we used in the spherical coordinates case, and I don't do all the math now. The important part was seeing the separation! We define α^2 similarly to before (different constant), and we have λ_1 for the ξ equation and λ_2 for the η equation. We look at the asymptotic behavior and extract the expo term, we assume a power series starting at some power s, we learn that s = ± m/2, we get the truncation of the series conditions which define n_1 and n_2 exactly as before, but instead of having ℓ floating around, we have m floating around . We define n as a certain sum $n = \lambda_1 + \lambda_2$ and it is seen to be the main quantum number which makes the energy values which of course agree with the spherical method result. The functions are still associated Laguerres but have labels now involving m and n_1/n_2 instead of ℓ and n. The wavefunctions are then products of Laguerres in each parabolic variable times exp times power, and they don't bother with the normalization.

In the spherical case we have n,ℓ,m as quantum numbers for eigenstates. Here we have n_1, n_2, m. S computes the degeneracy at each n level and gets the same value n^2 that he got before. So somehow we have just shuffled the eigenfunctions around.

The fact that there is this n^2 degeneracy at each level, AND the fact that you can separate in more than one coordinate system, are claimed on the web to result from a certain O(4) symmetry that the Hamiltonian has for this H atom problem. This is in fact discussed starting page 234 in Chap 7 on Symmetry, a good place to do it, and I will wait till then.

In atomic physics, since 2 electrons can go into each orbital, we are more used to $2n^2$ degeneracy:

n=1	2	S	k shell
n=2	8	S+P	ℓ shell
n=3	18	S+P+D	m shell
n=4	32	S+P+D+F	n shell
n=5	50	S+P+D+F+G	o shell

where the numbers refer to electrons in what is called "the outer shell", see Livesey page 220.