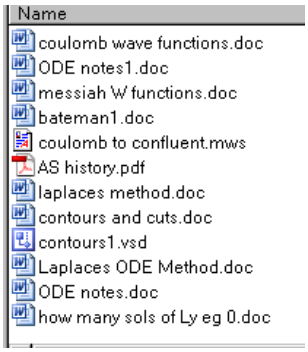
Schiff Chap 5 scattering PhL 11.24.08

Title of the chapter is : Continuous Eigenvalues: Collision Theory
Arbitrary positive energy E is specified at the start of your experiment, carried in parameter k then n.
The main difficulty in this chapter is with Coulomb scattering, hence lots of notes.

Note: I have lots of notes on EM scattering in my Optics area, but I can't find any notes on "particle scattering".

Supporting documents:
radial equation.doc conditions on R at r=0, and then comments on the Coulomb case

And from the ODE math folder



CONTENTS OF THESE NOTES

17. The 1D Positive Square Well. 2

18. Collisions in 3D 4
Scattering Cross Section (110) 4
CMS and Lab Kinematics (111) 4
Relation between cross sections (111) 6
Dependence on γ (113) 8
Asymptotic Behavior + Normalization (114) 12

19. Scattering by Spherically Symmetric Potentials 15
Long Digression. 15
Asymptotic Behavior (117) 17
Differential Cross Section. (118) 19
Total Elastic Cross Section and Optical Theorem (p 120) 23
Phase Shifts δ_ℓ (121) 24
Calculation of the Phase Shifts (121) 24
Meta Review to this Point. 25
The sign of the phase shift δ_ℓ versus sign of the potential V(r) (122) 34
Ramsauer-Townsend Effect (123). 34
Scattering by an *Infinite* Positive or Negative Infinite Spherical Square Well (124). 35
Scattering by a *Finite* Negative Spherical Square Well (126). 35
Resonance Scattering (127) 36
Angular distribution at low energies (129). 37

20. Scattering by Complex Potentials (129) 37
Conservation of Probability (130) 37
Complex Phase Shifts and Three Cross Sections (131) 39
Asymptotic Relations (133) and Reciprocity Theorem (135) 40
Generalized Optical Theorem (135) 44

21. Coulomb Scattering (138) 48

A. Coulomb Scattering in Parabolic Coordinates 48
Arguments about the assumed ansatz functional form. 49
Derivation of Equation 21.3 and finding ν // promised in paragraph above 50
Derivation of 21.5 (and in so doing, show that 21.4 is compatible with 21.3) 52
Comment on Equation 21.7. (outdated, see 6" below) 53
Comment on Equation 21.8 (outdated, see 6" below) 53
Comments on 21.7 and 21.8 added 12.17.08: 53
Derivation of 21.9 and 21.10 55
Comments on 21.9 and 21.10 added 12.17.08: 57
Review of where we stand at this point. 58
Going way back to 19.3: Using 19.3 to predict our discovered large-r form of $R_\ell(r)$ for large r. 59
Derivation of 21.11 and why $|f|^2$ is still the cross section. 60
Derivation of 21.12 and getting "unit flux" 61
Pause for Comments: 62
Notice that there is no "partial wave expansion" 62
What is the low-energy form of our solution? (We already know it !) 63
The Gamov Factor. 63

B. Coulomb Solution in Spherical Coordinates (142) 64
Derivation of 21.17 and 21.18. 64
Derivation of 21.19 (added 12.17.08) 66
Derive page 143 A. 67
Derive Equation 21.22. 71
Does optical theorem explain Coulomb form? No. 75
Direct Derivation of 21.23 as Lim $n\to 0$ $f_c(\theta) = (2/ik)\delta(1-z)$. 76
Comments on Schiff's Coulomb Scattering Approach 77
Seat of pants spherical coordinates derivation of the form 21.22 for $f_c(\theta)$. 77
Modified Coulomb Field (144) 79
Derivation of 21.27 79
Why you cannot take a limit of turning off the Coulomb force in scattering. 80
The Classical Limit of QM Coulomb Scattering (145) 80

17. The 1D Positive Square Well.

This venerable problem provides a lot. Width is a, height is V_o as page 101 shows, left edge at x=0 since no benefit to center it. The coefficients A,B,C,F,G are set up, α^2 is defined as energy over the height top page 103, and the T and R coefficients are computed as in **17.5** by matching value and derivative at both boundaries. I did not do this calculation on this reading, but I seem to have done it twice in the past since there are some check marks there. For the case that the initial energy is less than the height, we replace α

= iβ and then T is as shown in **17.7**. The figure on page 104 gives an excellent plot of T for both regimes. The regime with $E < V_0$ is the small part of the graph to the left of my vertical red line, and shows that you get "tunneling" through the barrier even if you are less high than it is. As $E \to 0$, we get $T \to 0$ due to the E in the denominator of 17.7.

For E/V ~ 1.6 you get 100% **transmission** and no **reflection**. These points of 100% transmission and 0% reflection correspond to thickness a that is an integral number of half waves, where you can tell from regular optics that the reflections from the two surfaces will cancel so you must have 100% going through. This appears in the shape of $\sin^2(\alpha a)$ in the T expression 17.5.

What happens **classically**? The curve on page 104 would be a perfect box. Think of a ball rolling toward a smooth hump. If the hump is too high, ball rolls back 100%. If the hump is too low, ball rolls over the hump 100%. I think you approach this classical limit as you make the square well more "**opaque**" which means you increase the size of the parameter grouping shown under 17.8. Thicker and higher makes it more opaque. This is why semiconductor **tunneling** effects only really appear for very thin barriers.

Now we come to the scattering "motion pictures" as he calls them. This work was done by those in the footnote on page 105 and appears in Saxon as well. You do numeric integration of the time-dependent SE. I would like to try this some day, since surely Alta can do it just fine. Let's now look at some of the results. He gives his numbers, and when I calculate the "opaqueness" it is quite large,

$mV_o a^2/\hbar^2 = (1/2)\,(70.7\pi)^2(.064)^2/1^2 = (70.7 * 3.14159 * .064)\text{^}2/2 = 101.$

page 106: Here $E = 1/2V_o$. Since the barrier is very opaque, we basically just bounce off completely. The reader is perhaps surprised by the high frequency shape of $|\psi|^2$ during the collision. How would I explain that? We must have a very large first and second ∂_x derivative to get what we are seeing. Look at the SE 6.16. Probably ∇^2 is large because the time derivative is also large, but that is not very enlightening.

Here is my explanation. You have waves reflecting off the front surface of the form sin(kx) – sin(k'x) where k and k' are in the FT gaussian. So this is roughly $\psi = 2\cos(\bar{k}x)\sin(\Delta k\, x/2)$ from Trig page 17. The cos factor is the part that is fast moving, and for it we have roughly $\bar{k} = 50\pi = \langle p \rangle$. Then $\lambda = 2\pi/\bar{k} = 1/25 = .04$. But this is ψ, and ψ^2 moves twice as fast since $\cos^2(\bar{k}x)$, so I expect the ψ^2 peaks to be about .02 apart. Now a = .064, so we might expect to see about 3.2 peaks of ψ^2 in the width of the well (we just use this width as scale in the picture). My ruler gives λ = 1 mm and a = 3.2 mm, so my explanation is perfect. The width of the spatial gaussian is irrelevant because it controls only the slower term $\sin^2(\Delta k\, x/2)$ (assuming it has some reasonable width. In these examples, the half-width is about half that of the well.

page 107: Here packet energy equals the height. Classically this would cause a rolling ball to hesitate at the top of the barrier and either go over or back based on slight energy error. In QM this fact is reflected in the fact that you have a portion of the probability bouncing back and forth for a while inside the well! It will eventually decay, we know. In this case we have some transmission but mostly reflection.

page 108: Now we have a *negative* well and packet energy half the well depth, so like page 106 but opposite sign well. The well is not really a barrier here, just something mildly in the way. We get mostly transmission as expected, but some reflection.

page 109: Same as page 107 but negative well. As we expect, we get even more transmission than in the page 108 case since more energy. The well is less of a pothole here to the larger passing vehicle. Still some very small reflection. In these cases, classically the ball just speeds up in the dip and we don't expect to see any lingering in the area of the well, so no wave appears inside the well.

18. Collisions in 3D

pp 105→109: We are reminded of work done to get equations 16.5 earlier in the book. If your potential depends only on the difference in position of two particles, then as shown on pages 89-90, you can separate the problem into motion of the CMS with mass M, and relative motion with reduced mass μ. He refers to the CMS motion of mass M energy as E', and the more interesting relative motion energy as E, and of course you add these to get the total energy. We know how a CMS moves, and our only interest in it would be to relate CMS frame calculations to LAB frame calculations, which we will be doing here very soon (and which Goldstein did). The "theory" will always be done in the CMS frame since there are only 3 degrees of positional freedom to worry about, not 6.

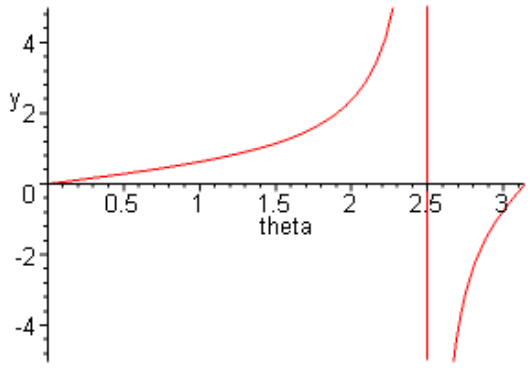
Scattering Cross Section (110)

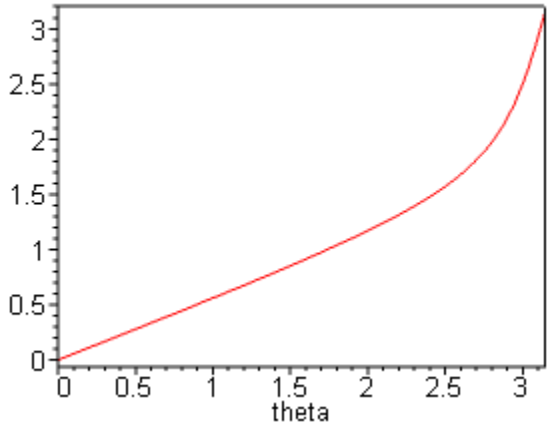
p 110: The differential and total **cross sections** are defined in the lab frame, and can also be defined in the CMS frame and will be different.

CMS and Lab Kinematics (111)

p 111-112: The elastic collision pictures on page 112 are perfect, right from the book. We have lab frame at the top, CMS frame in the middle, and the velocity vector addition at the bottom. This assumes an elastic collision where no energy is lost, and $m_3 = m_1$ and $m_4 = m_2$. You derive **18.3** easily from the bottom picture, and **you quickly arrive at 18.4** which relates the lab to the CMS angle. Here, θ_0 is the lab angle, and θ the CMS.

Note added 12.21.08. What does 18.4 say? I got this wrong on my first pass! If you want to make a plot of θ_0 versus θ, you have to use an arctan function which as a range of $(0,\pi)$ because the polar angle in either CMS or LAB of course has this range. Here are a few plots using 18.4 for $\gamma = 1/2$. The left plot shows $\tan\theta_o$ versus θ from 18.4, while the right plot shows θ_o versus θ.





But what happens if collision is inelastic: suppose Q heat is converted into final mass KE and the final masses are different?

Inelastic collision calculation: (and derivation of 18.5) (nonrelativistic!) The top figure is the same except I change final particle mass labels to m_3 and m_4 and imagine Q heat coming off. Q is something stored in the initial particles somehow that then boosts the KE of the final particles, so we have to write energy conservation like this

$$E + Q = \tfrac{1}{2} m_3 v_3^2 + \tfrac{1}{2} m_4 v_4^2$$

so that Q contributes to the final KE. Yes, Q > 0 for exothermic. So here are some equations all in the CMS frame:

$$E \equiv \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \qquad E+Q = \tfrac{1}{2} m_3 v_3^2 + \tfrac{1}{2} m_4 v_4^2 \qquad Q>0 \text{ exothermic}$$

E is the total initial energy in the CMS frame. Momentum conservation tells us that

$$m_3 v_3 = m_4 v_4$$

We can solve for v" = v_3 (see middle picture) as follows:

$$E + Q = \tfrac{1}{2} m_3 v_3^2 + \tfrac{1}{2} m_4 v_4^2 = \tfrac{1}{2} m_3 v_3^2 + \tfrac{1}{2} m_4 [m_3 v_3 / m_4]^2 = \tfrac{1}{2} m_3 v_3^2 + \tfrac{1}{2} m_3^2 / m_4 * v_3^2$$

$$= \tfrac{1}{2} (m_3/m_4) (m_4 + m_3) v_3^2$$

So we get

$$v_3^2 = 2 m_4 / m_3 * 1/(m_4 + m_3) * (E+Q)$$

The CMS speed he calls v' is my v_2 since 2 was at rest in the lab frame, and of course v" = my v_3. Since all the trig is the same, we should have 18.4 as our result, but now

$$\gamma^2 = (v'/v")^2 = \tfrac{1}{2} v_2^2 (m_4 + m_3) (E+Q) m_3/m_4$$

Now we want to relate v_2^2 to E which should be pretty simple:

$$E \equiv \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \qquad \text{and} \qquad m_1 v_1 = m_2 v_2$$

$$E = \tfrac{1}{2} m_1 (m_2/m_1)^2 v_2^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} (m_2^2/m_1 + m_2) v_2^2 = \tfrac{1}{2} (m_2/m_1) (m_2 + m_1) v_2^2$$

$$\Rightarrow \qquad v_2^2 = 2E m_1 / [m_2(m_2 + m_1)]$$

and our final answer is then

$\gamma^2 = \frac{1}{2} v_2^2 (m_4 + m_3) (E+Q) m_3/m_4 = E m_1/ [m_2(m_2 + m_1)] * (m_4 + m_3) (E+Q) m_3/m_4$

$= m_1m_3(m_4 + m_3) E / [m_2m_4 (m_2 + m_1) (E+Q)]$

which looks similar to 18.5 but is different. Rewrite as

$\gamma^2 = (m_1m_3 / m_2m_4)* M_f/M_i * E/(E+Q)$ **// verifies 18.5**

Are we both using the same E ?? I have

$E = \frac{1}{2} (m_2/m_1) (m_2 + m_1) v_2^2 = \frac{1}{2} (m_2/m_1) (m_2 + m_1) v'^2$

$= \frac{1}{2} (m_2/m_1) (m_2 + m_1) [m_1v/ (m_2 + m_1)]^2 = \frac{1}{2} (m_2/m_1)m_1^2/(m_2 + m_1) * v^2$

$= \frac{1}{2} m_1m_2/ (m_2 + m_1) * v^2$

= his E on page 111 // yes, we are both using the same E.

Comment: I suppose you could argue that $(M_i - M_f)c^2 = Q$ but we know the mass change will be very small in a fractional sense. Roughly we have

$M_f/M_i = 1 - Q/M_i$

Even in fusion, I quote from the web (just from somone's comment)

When you fuse ordinary Hydrogen into Helium, 0.7% of the mass is converted to energy. When Uranium undergoes fission only about 0.1% of the mass is converted. So in mass conversion terms, you might say that fusion is about seven times better than fission.

So perhaps $M_f/M_i = .999$ for uranium fission. So yes, Schiff is justified in omitting this factor !

Main Point: For a nonrelativistic inelastic collision, the CMS and lab angles are still related as in 18.4, and the γ factor is still given by $\gamma = v_2/v_3$ = ratio of CMS velocities of masses 2 and 3. However, $\gamma \neq m_1/m_2$ but is instead given by 18.5. If we set $Q = 0$ and $m_3 = m_1$ and $m_4 = m_2$, we recover the elastic result for γ.

Comment: When you do 1+2→3+4 kinematics relativistically, things are more complicated, and you end up using the Stanley variables s,t,u and you get that fancy triangle function. See for example pages 10-13 of Kallen 1963 book. Schiff is not interested in relativistic stuff at this point.

Relation between cross sections (111)

Derive 18.7. This is much harder than it looks. Start with

$\tan\theta_o = S_\theta / (\gamma + C_\theta)$ => $\sec^2\theta_o d\theta_o = (\gamma C_\theta+1)/(\gamma+C_\theta)^2 d\theta$ // easy to show

Now write

$\sin\theta_o d\theta_o = \{\sin\theta_o / \sec^2\theta_o\} \sec^2\theta_o d\theta_o = \{\sin\theta_o / \sec^2\theta_o\} (\gamma C_\theta + 1)/(\gamma + C_\theta)^2 d\theta$

$\Rightarrow d\theta/d\theta_o = \sin\theta_o \sec^2\theta_o (\gamma + C_\theta)^2 / \{ \sin\theta_o (\gamma C_\theta + 1) \} = \sec^2\theta_o (\gamma + C_\theta)^2 / (\gamma C_\theta + 1)$

and finally write

$\sigma_o = \sigma S_\theta / \sin\theta_0 * d\theta/d\theta_o = S_\theta / \sin\theta_0 * \sec^2\theta_o (\gamma + C_\theta)^2 / (\gamma C_\theta + 1)$

$= \sigma S_\theta (\gamma + C_\theta)^2 / (\gamma C_\theta + 1) * \{ \sec^2\theta_o / \sin\theta_0 \}$

So what remains is to convert {...} to the angle θ using 18.4. First, consider the sine:

$\sin^2\theta_0 = 1 - \cos^2\theta_o = 1 - 1/\sec^2\theta_o = (\sec^2\theta_o - 1) / \sec^2\theta_o = \tan^2\theta_o / \sec^2\theta_o$

Now looking at page 112 top, it seems impossible for θ_0 to be larger than $\pi/2$ from the billiards table argument. This tells me that all three trig functions in the above are positive. Thus we should have

$\sin\theta_0 = \tan\theta_o / \sec\theta_o$

and then our factor of interest becomes

$\{ \sec^2\theta_o / \sin\theta_0 \} = \sec^3\theta_o / \tan\theta_0 = (\sec^2\theta_o)^{3/2} / \tan\theta_0 = [1 + \tan^2\theta_0]^{3/2} / \tan\theta_0$

Now we go off and compute:

$\tan\theta_o = S_\theta / (\gamma + C_\theta) \quad \Rightarrow$

$1 + \tan^2\theta_0 = (\gamma^2 + 2\gamma C_\theta + 1)/(\gamma + C_\theta)^2 \quad$ // easy to show

Thus we have

$\{ \sec^2\theta_o / \sin\theta_0 \} = [1 + \tan^2\theta_0]^{3/2} / \tan\theta_0 = (\gamma^2 + 2\gamma C_\theta + 1)^{3/2}/(\gamma + C_\theta)^3 * (\gamma + C_\theta)/ S_\theta$

$= (\gamma^2 + 2\gamma C_\theta + 1)^{3/2} / [S_\theta (\gamma + C_\theta)^2]$

And we can now insert this {..} factor into our result above to get

$\sigma_o = \sigma S_\theta (\gamma + C_\theta)^2 / (\gamma C_\theta + 1) * \{ \sec^2\theta_o / \sin\theta_0 \}$

$= \sigma S_\theta (\gamma + C_\theta)^2 / (\gamma C_\theta + 1) * (\gamma^2 + 2\gamma C_\theta + 1)^{3/2} / [S_\theta (\gamma + C_\theta)^2]$

$= \sigma / (\gamma C_\theta + 1) * (\gamma^2 + 2\gamma C_\theta + 1)^{3/2}$

and this agrees with (18.7) except I did not get an absolute value in the denominator. Here is why that absolute value needs to be there: First, notice this fact:

$$\gamma^2 + 2\gamma C_\theta + 1 = (\gamma + C_\theta)^2 + S_\theta^2 > 0$$

so there is no question about the $(\gamma^2 + 2\gamma C_\theta + 1)^{3/2}$ factor being unambiguous. But the factor $\gamma C_\theta + 1$ could be either sign, since γ can be any positive number according to 18.4. But cross sections always have to be positive. In reality, we should have $|d\theta|$ and $|d\theta_o|$ in 18.6, because $d\theta/d\theta_o$ can be negative as our expression above shows. Thus our equation above should be

$$|d\theta/d\theta_o| = \sec^2\theta_o \, (\gamma + C_\theta)^2 / |\gamma C_\theta + 1|$$

and there we have it. So 18.7 has now been derived.

Now compare this result to Goldstein. On page G 87 we have G's version of Schiff 18.4. But G dismisses the computation above as "being quite a job for arbitrary γ" on page 88, and he does not carry it out, so I am glad I was able to do this job above.

For equal masses we get $\gamma = 1$ and things simplify:

$$\tan\theta_o = S_\theta / (\gamma + C_\theta) = S_\theta / (1 + C_\theta) = 2 \, s \, c / 2 \, c^2 = \tan \theta/2$$

where s and c refer to half angles. Thus, we get $\theta_o = \theta/2$, a famous result. This says that when your lab angle reaches 90 degrees, the CMS angle has reached 180 degrees and is exactly backwards (for particle 3, say). Our cross section relation is then this:

$$\sigma_o(\theta_o) = \sigma(2\theta_o) / (\gamma C_\theta + 1) * (\gamma^2 + 2\gamma C_\theta + 1)^{3/2} = \sigma(2\theta_o) / (C_\theta + 1) * (1 + 2C_\theta + 1)^{3/2}$$

$$= \sigma(2\theta_o) \; 2\sqrt{2} \, (1 + C_\theta)^{3/2} / (C_\theta + 1) = \sigma(2\theta_o) \; 2\sqrt{2} \, (1 + C_\theta)^{1/2}$$

$$= \sigma(2\theta_o) \; 2\sqrt{2} \, (2\cos^2\theta_0)^{1/2} = \sigma(2\theta_o) \, 4 \cos\theta_0$$

which appears as Schiff p 113 A and Goldstein page 88 3-75, another famous result.

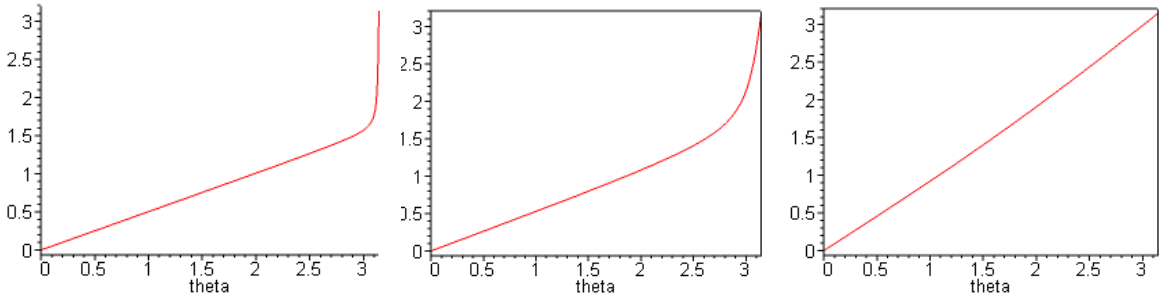
Since we are doing this stuff, let's look at the general case and see what happens:

Dependence on γ (113)

The angle θ and θ_o refer to the scattering angle of the projectile in the CMS and LAB frames. I goofed this up badly on my first pass, so these are new notes as of 12.21.08, and I went overboard putting lots of graphs below.

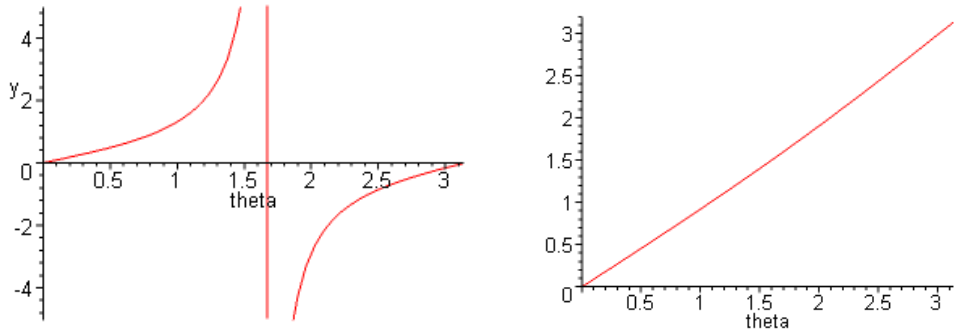
Case 1: Heavy Target. If γ is in the range (0,1), which means the target particle is heavier. Here are plots of θ_o versus θ for a few γ values in this range.

0.99 0.9 0.1



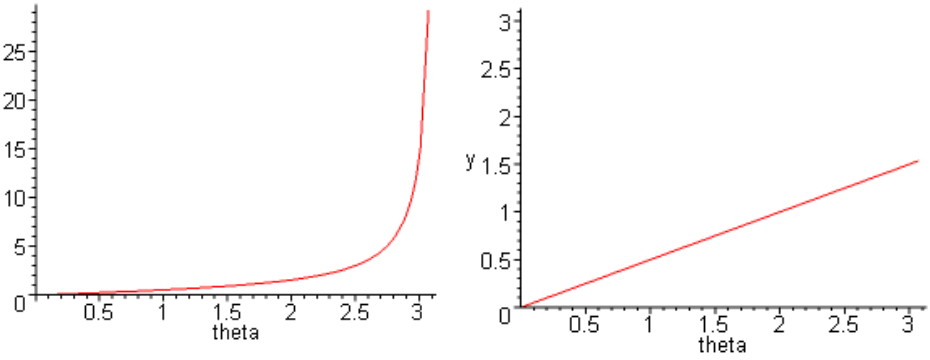
As γ gets very *small*, target is bowling ball hit by a BB and in this case CMS = LAB so the plot becomes a straight line, as we almost have with γ = 0.1. For this case, let's plot both $\tan\theta_o$ and θ_o :

$\tan\theta_o$ for γ = 0.1 θ_o for γ = 0.1



As we approach equal mass with γ = .99, notice that the line jags sharply upward at the right end, so only a tiny range near π of CMS angles can get you more than π/2 in the lab frame.

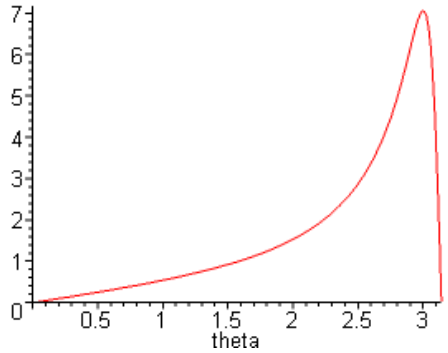
Case 2: Equal Mass. Here are the plots for $\tan\theta_0$ and θ_0 when γ = 1 exactly.



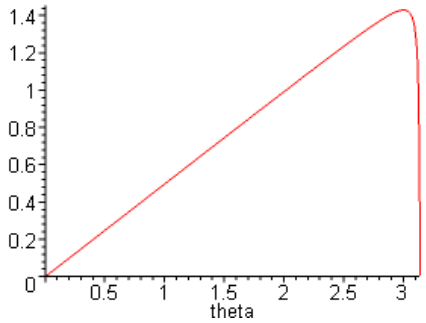
The right plot says that you cannot get to a lab angle greater than $\theta_o = \pi/2$, which is the familiar billiard ball result. So in general for γ < 1 we can get to all lab angles, but as γ → 1, the lab angle must be less than 90 degrees.

Case 3: Light Target. If γ > 1, which means the target particle is lighter. For γ = 1.01 we get

$\tan\theta_o$ for $\gamma = 1.01$

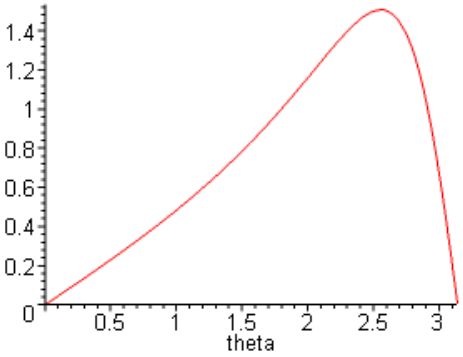


θ_o for $\gamma = 1.01$

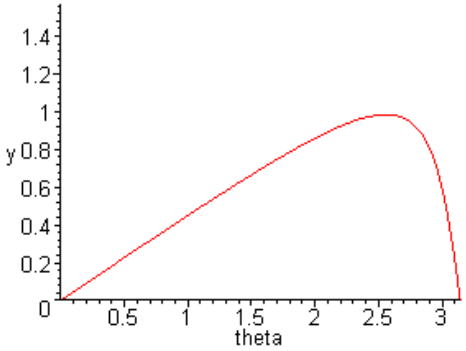


Here we see the deformation from the $\gamma = 1$ case. Now in the lab we cannot quite get to 90 degrees.

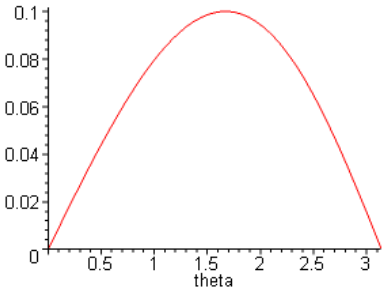
$\tan\theta_o$ for $\gamma = 1.2$



θ_o for $\gamma = 1.2$



Now as usual, all CMS angles are possible, but now the lab angle only gets up to about 1 radian or 57 degrees, and this happens at a CMS angle of 2.6 radians. So now we are not able to get anywhere near the 90 degrees. As we make γ larger, the angle gets less and less. Eventually we get to a symmetric plot for θ_0 which is this for $\gamma = 10$



Now we can only reach a lab angle of 0.1 radian or 6 degrees with our massive projectile. Finally when you shoot the bowling ball at a stationary BB, bowling ball is going to just continue on in the forward direction, the BB is too light to deflect it at all.

In all these Case 3 situations, notice that there are **two CMS angles which give the same LAB angle**, since the curve folds back down to the axis.

Resumption of original notes. In case 3, we don't get this blowup in $\tan\theta_0$. From our algebra above, we know this:

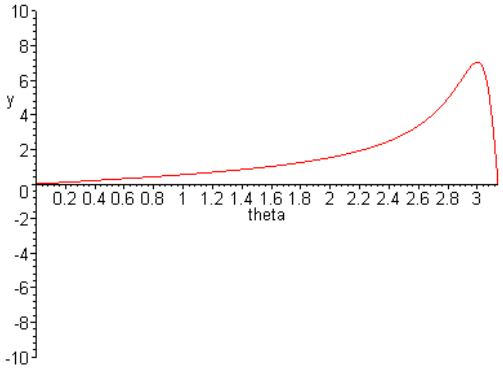
$$d(\tan\theta_o) = (\gamma C_\theta + 1)/(\gamma + C_\theta)^2 d\theta$$

This now has a zero when $(\gamma C_\theta + 1) = 0$ or $C_\theta = -1/\gamma$ which is again some angle in the range $(\pi/2, \pi)$. At this point, the angle θ_0 has reached its maximum value. If we install $C_\theta = -1/\gamma$ into our formula we find that

$$(\tan\theta_o)_{max} = S_\theta / (\gamma + C_\theta) = 1/\sqrt{\gamma^2 - 1} \qquad \text{// easy to show}$$

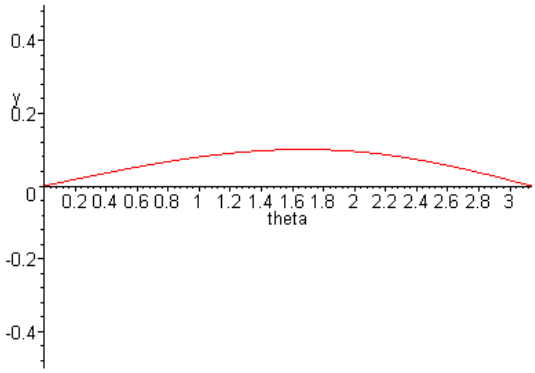
$$(\sin\theta_o)_{max} = 1/\gamma \qquad \text{// just draw the triangle}$$

So for example, suppose $\gamma = 1.01$. Then $\theta = \cos^{-1}(-1/\gamma) = 172$ degrees. At this CMS angle, the lab angle reaches its maximum value $\theta_{o,max}$. We find that $(\tan\theta_o)_{max} = 7.05$ which means $\theta_{o,max} = 82$ degrees. Here is a plot of this case



In this situation, the CMS angle can range fully from 0 to 180 degrees, but the lab angle cannot exceed 82 degrees, and this happens when CMS angle is 172 degrees. So you cannot reach the billiard ball max angle of 90 degrees.

As another example, suppose $\gamma = 10$, so our incident is now quite heavy. We find $\theta = \cos^{-1}(-1/\gamma) = 96$ degrees and $(\tan\theta_o)_{max} = 0.100 \Rightarrow \theta_{o,max} = 6$ degrees. Here is a plot:



Again, the CMS angle can be anything in 0 to π, but the lab angle cannot be more than 6 degrees. This is pretty easy to understand: the heavy incident particle just plows through the target and cannot be deviated very much no matter how it hits. In this case, in the CMS the incident particle is nearly at rest, and the target bounces off like a pea off a bowling ball and that pea can go anywhere. It is interesting that there is a certain CMS angle for the pea that maximizes the lab deflection of the bowling ball.

As Schiff points out on page 113, we have

$$\sigma_o = \sigma / (\gamma C_\theta + 1) * (\gamma^2 + 2\gamma C_\theta + 1)^{3/2}$$

having a pole at our special maximal point, the differential cross section is infinite, but integrates just fine to get the total cross section.

Another Schiff comment: as the above graphs show, there a given lab angle corresponds to two CMS angles for a scattered particle (horizontal line through the plots). He says you can determine the CMS angle for a given particle at this lab angle by measuring its energy.

Next, Schiff points out that in our case $\gamma>1$, the circle construction on page 112 demonstrates that there is a maximum angle in the lab θ_0.

So I have now read through Schiff's "dependence on γ" comments on page 113 and they all agree with what I did above, and I added his extra comments above.

Asymptotic Behavior + Normalization (114)

Schiff has many good comments to make in these short sections on page 114-116.

Review of pre-collision kinematics. Go back and look at 16.5 on page 90 where we separate CMS action from "relative motion" action for the two particle collision system. Let's just do the kinematics on our own here. We are only looking at the situation before the collision occurs, so no mention of the final particles.

$\mathbf{v}_1, \mathbf{v}_2$ = in the cms frame $\quad m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$ from P conservation

$\mathbf{v}_1 = s_1\hat{\mathbf{z}} \quad \mathbf{v}_2 = -s_2\hat{\mathbf{z}} \quad m_1 s_1 = m_2 s_2$

$\mathbf{V}_1, \mathbf{V}_2 = 0$ in the lab frame $\quad \mathbf{V}_1 = +S_1\hat{\mathbf{z}} \quad s$ = speed

$\mathbf{r}_1, \mathbf{r}_2$ = positions in the cms frame,

$\mathbf{R}_1$, $\mathbf{R}_2$ = positions in the lab frame

$\mathbf{R}_{cms} = (m_1\mathbf{R}_1 + m_2\mathbf{R}_2)/M$ position of CMS in the lab frame

$\mathbf{V}_{cms} = (m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2)/M = (m_1\mathbf{V}_1 + m_2\mathbf{V}_2)/M = (m_1/M)\,\mathbf{V}_1 = (m_1/M)\,S_1\hat{\mathbf{z}}$

$\mathbf{r}_{cms} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$ position of CMS in the CMS frame

$\mathbf{v}_{cms} = (m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2)/M = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/M = 0$

What is s_1 and s_2 ?

$\mathbf{v}_1 = \mathbf{V}_1 - \mathbf{V}_{cms} = S_1\,\hat{\mathbf{z}} - (m_1/M)\,S_1\hat{\mathbf{z}} = +(m_2/M)\,S_1\hat{\mathbf{z}}$ => $s_1 = (m_2/M)\,S_1$

$\mathbf{v}_2 = \mathbf{V}_2 - \mathbf{V}_{cms} = -\mathbf{V}_{cms} = -(m_1/M)\,S_1\hat{\mathbf{z}}$ => $s_2 = (m_1/M)\,S_1$

Notice how each s_i has the mass of the *other* particle, and how this agrees with Fig 20 (b).

Now, what is the energy associated with the motion *of* the CMS?

$E' = \frac{1}{2}\,M\,V_{cms}^2 = \frac{1}{2}\,M\,(m_1/M)^2\,S_1^2 = \frac{1}{2}\,m_1^2/M\,S_1^2$

What is the energy associated with the "relative motion" ?

$E_{rel} = \frac{1}{2}\,m_1v_1^2 + \frac{1}{2}\,m_2v_2^2 = \frac{1}{2}\,m_1(m_2/M)^2\,S_1^2 + \frac{1}{2}\,m_2\,(m_1/M)^2\,S_1$

$= \frac{1}{2}\,\mu\,S_1^2$ // just do out the algebra on the above line.

What is the lab energy of the incident particle? This is also the total initial energy.

$E_1 = \frac{1}{2}\,m_1V_1^2 = \frac{1}{2}\,m_1S_1^2$

Then we can write

$E_{rel} = \frac{1}{2}\,\mu\,S_1^2 = \mu E_1/m_1 = (m_2/M)\,E_1$ **(verifies 18.9)**

which is a result quoted for some reason in 18.9.

Now, what is the sum of the energy of the CMS motion plus the relative energy?

$E_{tot} = E' + E_{rel} = \frac{1}{2}\,m_1^2/M\,S_1^2 + \frac{1}{2}\,\mu\,S_1^2 = \frac{1}{2}\,m_1S_1^2 = E_1$ fine.

What can we say about 16.5B? It says that our effective particle has mass μ and coordinate

$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{R}_1 - \mathbf{R}_2$ and $V = V(\mathbf{r}_1 - \mathbf{r}_2) = V(\mathbf{r})$

So, we can regard the relative problem as that of a particle of mass μ in the potential of a static potential V(**r**) that is coming from a "fixed center" at **r** = 0. It is a one-body problem, not a two-body problem. This is the point Schiff makes near the bottom of page 114.

Now, what can we say about the famous general scattering form (18.10) ? Keeping in mind the lower picture and its discussion, if we go out somewhere where we don't get the incoming beam and where we are outside the potential, our SE should be this:

$$-\hbar^2/2\mu\,[\,\nabla^2 u\,] = E\,u \qquad\qquad u = f(\theta,\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}/r$$

The energy of this outgoing wave should be $p^2/2u = \hbar^2k^2/2\mu$ so we need to have

$$-\hbar^2/2\mu\,[\,\nabla^2\{\,f(\theta,\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}/r\}\,\,] = \hbar^2k^2/2\mu\,\{f(\theta,\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}/r\}$$

or

$$\nabla^2\{\,f(\theta,\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}/r\} = -k^2\,\{f(\theta,\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}/r\}$$

Here is a little theorem on top of my TK math page 8

$$\nabla^2\{\,f(\theta,\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}/r\} = f(\theta,\varphi)\,\nabla^2(e^{i\mathbf{k}\cdot\mathbf{r}}/r) + (e^{i\mathbf{k}\cdot\mathbf{r}}/r)\,\nabla^2 f(\theta,\varphi) + 2\,\boldsymbol{\nabla} f(\theta,\varphi)\bullet\boldsymbol{\nabla}(e^{i\mathbf{k}\cdot\mathbf{r}}/r)$$

The last term vanishes because $\boldsymbol{\nabla}(e^{i\mathbf{k}\cdot\mathbf{r}}/r)$ is in the $\hat{\mathbf{r}}$ direction, but $\boldsymbol{\nabla}f(\theta,\varphi)$ is only in the other two directions as sheet 6 TK shows. Also, we know from one of our data pages that $\nabla^2(e^{i\mathbf{k}\cdot\mathbf{r}}/r) = -k^2(e^{i\mathbf{k}\cdot\mathbf{r}}/r)$ exactly [staying away from r=0] so our SE above becomes

$$f(\theta,\varphi)\,\{\,-k^2(e^{i\mathbf{k}\cdot\mathbf{r}}/r)\,\} + (e^{i\mathbf{k}\cdot\mathbf{r}}/r)\,\nabla^2 f(\theta,\varphi)\,\} = -k^2\,\{f(\theta,\varphi)\,(e^{i\mathbf{k}\cdot\mathbf{r}}/r)\}$$

The first and last terms cancel, and we are left with this needing to be true

$$(e^{i\mathbf{k}\cdot\mathbf{r}}/r)\,\nabla^2 f(\theta,\varphi) = 0$$

I suppose if we go asymptotic, then whatever $\nabla^2 f(\theta,\varphi)$ is, it will be independent of r, and as we go to distant large r, then LHS = 0. This must be the meaning of Schiff's comment "it is readily verified". It does not mean that $\nabla^2 f = 0$, but for all I know that might be true as well, though he never mentions it.

The picture bottom of page 115 explains that at your detector, you don't see the incident beam due to a collimator, so the "flux" is due only to the second term in 18.10. The flux from famous 7.3 is this

$$\mathbf{j} = \hbar/(2i\mu)\,[\,\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\,]$$

Now the radial flux which we really care about will be

$$\mathbf{j}_\mathbf{r} = \hbar/(2i\mu)\,[\,\psi^*\boldsymbol{\nabla}_\mathbf{r}\psi - \psi\boldsymbol{\nabla}_\mathbf{r}\psi^*\,] \qquad\qquad \psi = f(\theta,\varphi)(e^{i\mathbf{k}\cdot\mathbf{r}}/r)$$

But

$\nabla_r\psi = f(\theta,\varphi)\,\nabla_r\,(e^{ikr}/r) = f(\theta,\varphi)\,(ik - 1/r)\,(e^{ikr}/r)$

$\psi^*\nabla_r\psi = |f|^2\,(ik - 1/r)(1/r^2)$

$\psi\nabla_r\psi^* = |f|^2\,(-ik - 1/r)(1/r^2)$

$\mathbf{j}_r = \hbar/(2i\mu)\,[\,\psi^*\nabla_r\psi - \psi\nabla_r\psi^*\,] = \hbar/(2i\mu)\,|f|^2\,[(ik - 1/r) - (-ik - 1/r)\,]/r^2$

$= \hbar/(2i\mu)\,|f|^2\;2ik/r^2 = \hbar k/\mu\;\;|f|^2/r^2 = v\;|f|^2/r^2 \qquad v = p/\mu$

The inbound flux, by the same formula, would be this

$\mathbf{j}_z = \hbar/(2i\mu)\,[\,\psi^*\partial_z\psi - \psi\partial_z\psi^*\,] \qquad \psi = e^{ikz}$

$= \hbar/(2i\mu)\,[\,ik - (-ik)] = \hbar/(2i\mu)*2ik = \hbar k/\mu = v$

The cross section is defined by flux flow as follows

$j_z\,d\sigma = j_r\,(r^2 d\Omega) \qquad$ where $r^2 d\Omega = dA$

which says

$v\,d\sigma = v\;|f|^2/r^2 * (r^2 d\Omega) = v\;|f|^2\;d\Omega$

$\Rightarrow\; \partial\sigma/\partial\Omega = |f|^2 \qquad$ // verifies **18.11**

another "famous scattering result" which appears as 18.11 on page 116.

Summary: In 18.10, we have two terms as shown. If we separately compute the incident flux only from the first term, and the outgoing flux only from the second term, then we find that $\sigma = |f|^2$. Normalization A does not matter, but relative normalization between the two terms does matter!

19. Scattering by Spherically Symmetric Potentials

Long Digression. Look back at page 77. In 14.1 we see the SE written out in spherical coordinates for a V(r). We separate as in page 77A and get 14.2 which says f(r) = g(θ,φ). As you vary r, normally you imagine f(r) would change, but it must still be equal to g(θ,φ) which has not changed. So f(r) = some constant λ. In this way, we get 14.3 and 14.4. The angular equation in 14.4 we recognize as saying

$\hat{L}^2\,Y = \lambda Y \qquad$ where $\hat{L}^2 = -\,[\,(1/S\theta)\,\partial_\theta\,[\,S\theta\,\partial_\theta\,] + (1/\sin\theta)^2\,\partial_\phi^2\,]$

and we know the eigenvalues of $\hat{L}^2$ are integer numbers of the form $\ell(\ell+1)$.

We know that we can form a set of basis functions $Y_{\ell m}(\theta,\varphi)$ upon which we can expand any SE angular solution:

$$f(\theta,\varphi) = \Sigma_{\ell m}\ f_{\ell m}\, Y_{\ell m}(\theta,\phi)$$

where see 14.14 Schiff for the $Y_{\ell m}$.

Now forgetting the SE completely, suppose we have some function $f(\theta,\varphi)$ in the abstract. We can still expand it as shown above. Jackson in 3.53 [Schiff 14.16] shows the normalization of the $Y_{\ell m}(\theta,\phi)$ and in the case that m=0 we get

$$Y_{\ell 0}(\theta,\phi) = \sqrt{(2\ell+1)/4\pi}\ \ P_\ell(\cos\theta)$$

Again forgetting the SE, if we have some function $f(\theta)$, we can expand it in the same way, but we know that the coefficients for $m\neq 0$ will all be zero. This follows from the inversion formula for the coefficients which is this

$$\int d\Omega\ f(\theta,\varphi)\, Y_{\ell m}(\theta,\phi) = f_{\ell m}$$

so that

$$\int d\Omega\ f(\theta)\, Y_{\ell m}(\theta,\varphi) = f_{\ell 0}\, \delta_{m,0}$$

so we can expand as follows:

$$f(\theta) = \Sigma_\ell\ f_\ell\, Y_{\ell 0}(\theta,\phi) = \Sigma_\ell\ \sqrt{(2\ell+1)/4\pi}\ f_\ell\, P_\ell(\cos\theta)$$

Now let's get back to that separated SE in Schiff page 77. The constant λ is of course also in the radial equation which then has some solutions $R_{E\ell}(r)$ for certain eigenenergies which here we just call E. On page 93 this leads to a set of H atom solutions of the form

$$u_{E\ell m}(r,\theta,\varphi) = R_{E\ell}(r)\, Y_{\ell m}(\theta,\varphi)$$

which we usually refer to as "orbitals". These are the "eigenstates" for the H atom. We could form some arbitrary solution of the H atom SE by taking any linear combination of the above u functions.

In the H atom in isolation, the eigenstates with different m are degenerate since no direction in space is picked out for L_z, so the listing of m values just enumerates the degeneracy. From an energy point of view, we need consider only the m=0 eigenstates. Then we could rewrite the above as

$$u_{E\ell 0}(r,\theta,\varphi) = R_{E\ell}(r)\, Y_{\ell 0}(\theta,\varphi) = R_{E\ell}(r)\,\sqrt{(2\ell+1)/4\pi}\ \ P_\ell(\cos\theta) = u_{E\ell 0}(r,\theta)$$

since there is no longer φ dependence. Then we could expand an arbitrary solution in this way

$$f(\theta,r) = \Sigma_{\mathbf{E}} \Sigma_{\ell}\ R_{\mathbf{E}\ell}(r) \sqrt{(2\ell+1)/4\pi}\ \ P_{\ell}(\cos\theta)$$

But in a scattering problem, as opposed to a bound state problem, there is a solution for any E, so I think the idea is that we replace the above with

$$f_{\mathbf{E}}(\theta,r) = \Sigma_{\ell}\ R_{\mathbf{E}\ell}(r) \sqrt{(2\ell+1)/4\pi}\ \ P_{\ell}(\cos\theta)$$

and this then is the basis of 19.1A which I have been dancing around here for 3 pages. The normalization is different and of course arbitrary. In my analysis above, $R_{\mathbf{E}\ell}(r)$ is a solution of the radial equation with some V(r) and some E sitting in it, and that is what Schiff's $R_{\ell}(r)$ must be in 19.1. Right now, it is not obvious to me why he has chosen the normalization we see there. [Remember that the scale of a radial solution like $R_{\mathbf{E}\ell}(r)$ is arbitrary.] { We shall soon see that the i^{ℓ} factor is added because it appears also in the $e^{\mathbf{ikz}}$ expansion in 19.9 }

Asymptotic Behavior (117)

The next step is to define $R_{\mathbf{E}\ell}(r) = \chi_{\ell}(r)/r$ just as we did for the H atom on page 81. We there got equation 14.17 as the radial equation for $\chi_{\ell}(r)$. If we mult through by $\hbar^2/2\mu$ we get all of **19.2.** We are no longer showing the E label, but you see E is the continuum value $E = p^2/2\mu = \hbar^2k^2/2\mu$. Recall Schiff's opening comment that "in a collision problem, E is specified in advance", page 100.

We now come to **19.3** which is completely new to me. First, we see that for very large r, $\boldsymbol{\ell}$ does not matter and yes, the solution fo 19.2 is $e^{\pm\mathbf{ikr}}$ and this is in line with 18.10. So we want to "take out" this fast k dependence by defining f as shown in 19.3. Here is some algebra (set A = 1)

$$\chi = e\cdots e^{\pm}$$

$$\partial_{\mathbf{r}}\chi = e\cdots [\ (\pm ik) + f]\ e^{\pm}$$

$$\partial_{\mathbf{r}^2}\chi = e\cdots[\ -k^2 + \partial_{\mathbf{r}}f + 2ikf + f^2]\ e^{\pm}$$

Putting this into 19.2, we get cancellation of the k^2 term as desired, and we then end up exactly with the left equality in **19.4**. So this is just a restatement of the radial equation where $f = f_{\ell}$ should be a slowly varying function. Note that U(r) is just a scaled V(r) as per 19.2.

So suppose we have this first order ODE:

$$f' + f^2 \pm 2ikf = K/r^{\mathbf{s}} \qquad \text{for large r} \qquad s = 1,2,3...$$

Our question is this: what is the asymptotic behavior of f(r)? For me, this is an unusual question, I don't have a standard method to answer it. If I assume an expo decay for f, I cannot replicate the power on the right side. Suppose we assume $f = Ar^{-\mathbf{n}}$ for large r. Then we get

$$-n\,Ar^{-n-1} + A^2r^{-2n} \pm ikAr^{-n} = K/r^s$$

In this case, yes, $\pm ikAr^{-n}$ is the leading term, and our solution is then n = s and we have

$$\pm ikAr^{-s} = Kr^{-s} \qquad \text{and so } A = K/(\pm ik)$$

In this case, consider our integral

$$\int_a^r dr'\, f(r') = A \int_a^r dr'(\, r')^{-s}$$

If s = 2,3,4..., the upper endpoint of this integral makes no contribution for large r, so we conclude that this integral is independent of r. In this case we conclude from 19.3 that:

$$\chi_\ell(r) = A * \text{constant in } r * e^{\pm ikr}$$

So there are then only two solutions for large r and we might write then as

$$\chi_\ell(r) = A\sin(kr) + B\cos(kr) \quad \text{or} \quad = C\sin(kr + D)$$

This then is the justification for 19.5 !!! That is the entire point of this digression into f. The upshot is that for large r, we can represent $\chi_\ell(r)$ in terms of an amplitude and a phase shift. In the case s = 0, we get a log divergence of the f(r') integral, so we will analyze this special case separately in the Coulomb scattering section!

Now suppose RHS 19.4 = expo decay = e^{-sr}, which implies $\ell = 0$ to get such a decay. In this case, we try an expo decay for f as well, call it e^{-nr}. Then 19.4 says this:

$$-s\,e^{-nr} + e^{-2nr} \pm ik\,e^{-nr} = e^{-sr}$$

In this case, we can through out only the f^2 term and set set n = s. The first and third terms then have to both be considered, just as he states. But again, we will have $f \sim e^{-sr}$ and the integral of f(r')dr' will still converge (even more so than before) at the upper endpoint, so we end up with exactly the same conclusion about the large-r behavior of $\chi_\ell(r)$ as before.

Comment: This 19.5 is an important and key result. In scattering, we are really talking the large r limit of things in our experiments, and in this limit, our "radial function" is very simple, just a sine. For a given partial wave ℓ, it is completely characterized by two assumed complex numbers presented here as A_ℓ' and δ_ℓ'. By comparison, the H atom radial functions are messy animals and you cannot be thinking large r there.

Now what do we know about the solution of 14.17? For the 3D square well, we have interior and exterior solutions and these involve appropriate powers. Obviously only positive powers are allowed inside to keep it from blowing up at the origin, and we get the spherical Bessel $j_\ell(r)$ there. For the H atom, we have

a similar issue, and we get Laguerre positive powers as shown on page 93 bottom. But I don't think these facts are relevant right now.

Look at 19.2. Since ℓ, k^2 and V(r) are assumed real, there will be real solutions χ . You can of course just phase such a solution to get $\chi' = e^{i\alpha}\chi$, so you can have complex solutions in that sense. In terms of 19.5, this means that δ_ℓ' = real, but we could phasor A_ℓ' to make it complex. I think we will probably be taking both of these things real. Again, we assume V(r) real at this point. (=> elastic he says)

Now comes yet another bizarre statement. If you assume δ_ℓ' = real, and if you compute your 7.3 flux based on 19.1 with 19.5, you conclude 19.6 which says there are no "particle sources" inside your large sphere. Let's skip this calculation for the moment, but let's ponder the interpretation. I guess we have to think of u(r,θ) as including both parts of 18.10, the incident and the scattered wave. In fact, we shall soon see the incoming plane wave expanded in this manner in 19.9. So the point is that u(r,θ) which is the solution of our scattering problem knows about incoming and outgoing. The large-r solution in 18.10 is appropriate to a scattering experiment.

Differential Cross Section. (118)

I claim I wrote up complete notes called "N1" for this section, but alas, I have now searched all 5 attic boxes and can find so such notes. I threw lots of notes out in 1990 and this N1 must now be gone, too bad. I see no folders on "scattering theory" or anything like that. Class was Falicov Physics 221.

Now let's go back again to the problem of scattering in a 3D well. Outside the well we have V = 0 and the "exterior solutions" of page 86-87 apply for solutions for R_ℓ for the SE radial equation. The radial equation in terms of R is shown in 14.3 page 77, and if you set $\lambda = \ell(\ell+1)$ and V = 0 you get this:

$$1/r^2 \left(r^2R'' + 2r R'\right) +\{ 2m/\hbar^2E - \ell(\ell+1)/r^2\}R = 0 \qquad \text{from 14.3}$$

$$R'' + (2/r)R' +\{ (2m/\hbar^2)E - \ell(\ell+1)/r^2\}R = 0$$

$$r^2R'' + 2rR' \{ \alpha^2r^2 - \ell(\ell+1)\} R = 0 \qquad \alpha^2 = (2m/\hbar^2)E \qquad \text{// see 15.2}$$

Now we want to scale this equation to make the r^2 coefficient one inside the {..} . Let $\rho = \alpha r$ and we have

$$R' = \partial R(r)/\partial r = \partial R(r)/\partial\rho * \partial\rho/\partial r = \alpha\, \partial_\rho R(\rho/\alpha)$$

$$R'' = \alpha^2\, \partial^2_\rho R(\rho/\alpha)$$

$$(\rho/\alpha)^2\, \alpha^2\, \partial^2_\rho R(\rho/\alpha) + 2\,(\rho/\alpha)\, \partial_\rho R(\rho/\alpha) \{ \rho^2 - \ell(\ell+1)\} R(\rho/\alpha) = 0$$

$$\rho^2\partial^2_\rho R(\rho/\alpha) +2\rho\, \partial_\rho R(\rho/\alpha) \{ \rho^2 - \ell(\ell+1)\} R(\rho/\alpha) = 0$$

$$\rho^2\partial^2_\rho f(\rho) +2\rho\, \partial_\rho f(\rho) \{ \rho^2 - \ell(\ell+1)\} f(\rho) = 0 \qquad \text{where } f(\rho) = R(\rho/\alpha)$$

$\partial^2_\rho f(\rho) + (2/\rho)\partial_\rho f(\rho) \{ 1 - \ell(\ell+1)/\rho^2 \} f(\rho) = 0$ // this is 15.4 page 84 Schiff

The second from last line above is seen to be the **"spherical Bessel differential equation"** as shown in A&S page 437. The solutions of this equation are in general

$R(\rho/\alpha) = f(\rho) = a_\ell j_\ell(\rho) + b_\ell n_\ell(\rho)$ // but AS refer to n(ρ) as y(ρ), fine

$R_\ell(r) = a_\ell j_\ell(\alpha r) + b_\ell n_\ell(\alpha r)$

Now, in our scattering case we have

$\alpha^2 = (2\mu/\hbar^2)E = (2\mu/\hbar^2)(\hbar^2 k^2/2\mu) = k^2$

So we can then write

$R_\ell(r) = a_\ell j_\ell(kr) + b_\ell n_\ell(kr) = [A_\ell \cos(\delta_\ell)] j_\ell(kr) - [A_\ell \sin(\delta_\ell)] n_\ell(kr)$

where

$a_\ell = A_\ell \cos(\delta_\ell)$ $a_\ell^2 + b_\ell^2 = A_\ell^2$ $A_\ell^2 = a_\ell^2 + b_\ell^2$

$b_\ell = - A_\ell \sin(\delta_\ell)$ $b_\ell/a_\ell = - \tan(\delta_\ell)$ $\tan(\delta_\ell) = - b_\ell/a_\ell$

so finally we arrive at 19.7 as the form R must have in a region where we set V=0. This is Schiff's "second region".

Schiff gives properties of these functions on page 85 so I will use those, he gives the books he got them from on that page.

For large ρ the **asymptotic formulas are on page 15.8** so we have

$R_\ell(r) \rightarrow [A_\ell \cos(\delta_\ell)] j_\ell(kr) - [A_\ell \sin(\delta_\ell)] n_\ell(kr)$

$= (1/kr) A_\ell \{ \cos(\delta_\ell) \cos(kr - \tfrac{1}{2}(\ell+1)\pi) - \sin(\delta_\ell)\sin(kr - \tfrac{1}{2}(\ell+1)\pi) \}$

$= (1/kr) A_\ell \cos [kr - \tfrac{1}{2}(\ell+1)\pi + \delta_\ell]$

But we know that $\cos(a - \tfrac{1}{2}\pi) = \cos(\tfrac{1}{2}\pi - a) = \sin(a)$. so we get

$R_\ell(r) \rightarrow (1/kr) A_\ell \sin [kr - \tfrac{1}{2}\ell\pi + \delta_\ell]$ // 19.8 is now verified.

We now want to message things into the form of 18.10 so we can learn what f(θ) is, then we will have an expression for our cross section in terms of the partial wave parameters.

How do I support 19.9? How do we look up something like this? I don't see it sitting in the A&S section on Bessel, but of course it involves two different kinds of functions. I don't offhand see this in Jackson anywhere. It appears in GR on page 973 with a reference to Watson's 1944 Treatise, same as Schiff's reference. I can't find it in Bateman Vol 4.

Using the orthogonality of the P_ℓ which has a factor $2/(2\ell+1)\ \delta_{\ell\ell'}$, this incoming wave series expansion is equivalent to this integral:

$$\int_{-1}^{1} P_\ell(z)\, e^{i\rho z}\, dz \ = 2\, i^\ell\, j_\ell(\rho)$$

Curiously, I cannot find this listed anywhere! Maple can't do it. It seems such a simple integral. I think the answer has to do with the **Gegenbauer** polynomials. First, from GR page 1031 we have

$$P_n(x) = C^{1/2}_n(x)$$

Then look at GR page 830 where we find that with $\nu = ½$,

$$\int_{-1}^{1} C^{1/2}_n(x)\, e^{iax}\ dx \ = \pi\sqrt{2}\, i^n \Gamma(n+1)\, a^{-1/2}\ J_{n+1/2}(a) \,/\, n!\ \Gamma(1/2)$$

$$= \pi\sqrt{2}\, i^n\, a^{-1/2}\ J_{n+1/2}(a) \,/\ \Gamma(1/2) = \pi\sqrt{2}\, i^n\, a^{-1/2}\ J_{n+1/2}(a) \,/\ \sqrt{\pi}$$

$$= \sqrt{\pi}\sqrt{2}\, i^n\, a^{-1/2}\ J_{n+1/2}(a) = 2\, i^n \{\sqrt{\pi/2a}\ J_{n+1/2}(a)\} = 2\, i^n\, j_n(a) \qquad \text{AS p 437}$$

and *finally* I have what I wanted. This took a long time! So I now consider that I have in fact derived the plane wave expansion formula. To do the integral myself, I suppose I could do a power series for both sides, etc etc. But enough of that!

Now comes some very important and somewhat mess math to carry out the program outlined above. The large r limit of 19.9 is this:

$$e^{ikz} = \Sigma_\ell\ (2\ell+1)\, i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ \sin(kr - ½\,\ell\,\pi) \qquad \text{//large r, see 15.8 and sin/cos note above}$$

Therefore, 18.10 says this: (A=1) [but let A = K, and write it in red just to trace it.]

$$u(r,\theta) = K\,\Sigma_\ell\ (2\ell+1)\, i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ \sin(kr - ½\,\ell\,\pi) + K\, f(\theta)\, e^{ikr}/r$$

But 19.1 A says (where we insert 19.8)

$$u(r,\theta) = \Sigma_\ell\ (2\ell+1)\, i^\ell\ P_\ell(\cos\theta) * R_\ell(r) \ = \Sigma_\ell\ (2\ell+1)\, i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ A_\ell * \sin(kr - ½\,\ell\,\pi + \delta_\ell)$$

Equate these two items to get

$$\Sigma_\ell\ (2\ell+1)\, i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ \sin(kr - ½\,\ell\,\pi) + f(\theta)\, e^{ikr}/r$$

$$= \Sigma_\ell\ (2\ell+1)\, i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ A_\ell / K * \sin(kr - ½\,\ell\,\pi + \delta_\ell) \qquad \textbf{// This is page 119 A}$$

Now write

$\sin(kr - \tfrac{1}{2}\ell\pi) = (1/2i) *(\exp[i(kr - \tfrac{1}{2}\ell\pi)] - \exp[-i(kr - \tfrac{1}{2}\ell\pi)] \}$

$= e^{ikr}[\, e^{-i\ell\pi/2}]/2i \; - e^{-ikr}[\, e^{+i\ell\pi/2}]/2i$

$\sin(kr - \tfrac{1}{2}\ell\pi + \delta_\ell) = e^{ikr}[\, e^{-i\ell\pi/2+i\delta}]/2i \; - e^{-ikr}[\, e^{+i\ell\pi/2-i\delta}]/2i$

Now let's write separately the two expo parts

$\Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ e^{ikr}[\, e^{-i\ell\pi/2}]/2i \;\; + \;\; f(\theta)\ e^{ikr}/r$

$= \Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ A_\ell / K\ e^{ikr}[\, e^{-i\ell\pi/2+i\delta}]/2i$

and

$\Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ \{- e^{-ikr}\ [\, e^{+i\ell\pi/2}]/2i\}$

$= \Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ (1/kr)\ A_\ell / K\ \{- e^{-ikr}[\, e^{+i\ell\pi/2-i\delta}]/2i\ \}$

Now mult each through by 2ikr and delete common expos to get

$\Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ [\, e^{-i\ell\pi/2}] \;\; + 2ik\ \ f(\theta)$

$= \Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ A_\ell / K\ \ [\, e^{-i\ell\pi/2+i\delta}]$ **// agrees with 19.10 B**

and

$\Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ [\, e^{+i\ell\pi/2}]$

$= \Sigma_\ell\ (2\ell+1)\ i^\ell\ P_\ell(\cos\theta)\ A_\ell / K\ [\, e^{+i\ell\pi/2-i\delta}]$ **// agrees with 19.10C**

From this last equation we see that

$[\, e^{+i\ell\pi/2}] = A_\ell / K\ [\, e^{+i\ell\pi/2-i\delta}] \;\; => 1 \;\; = A_\ell / K\ e^{-i\delta} \;\;\;\; => \;\; A_\ell = e^{i\delta}\ K$ **// verifies page 119 D**

where in all the above, of course, δ means δ_ℓ.

Meanwhile, consider that

$e^{-i\ell\pi/2} = (e^{-i\pi/2})^\ell = (-i)^\ell$ and $e^{-i\ell\pi/2}\ i^\ell = (-i)^\ell\ i^\ell = 1^\ell = 1$. Then the first equation 19.10B says

$\Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta) \;\; + 2ik\ \ f(\theta)$

$= \Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ A_\ell / K\ [e^{+i\delta}] \;\;\;\; = \Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ [e^{+i2\delta}]$

which tells us that

$$2ik\, f(\theta) = \Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ [e^{+i2\delta\ell} - 1] \qquad \text{// \textbf{verifies 19.11}}$$

and so finally we have $f(\theta)$ expressed in terms of the partial wave shifts δ_ℓ . Rewrite the above as

$$[e^{+i2\delta\ell} - 1] = e^{+i\delta\ell}\,[e^{+i\delta\ell} - e^{+i\delta\ell}\,] = e^{+i\delta\ell}\,[2i\,\sin(\delta_\ell)\,]$$

=>

$$2ikf(\theta) = \Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ e^{+i\delta\ell}\,[2i\,\sin(\delta_\ell)\,]$$

=>

$$f(\theta) = (1/k)\,\Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ e^{+i\delta\ell}\,\sin(\delta_\ell)$$

=>

$$\sigma = |\,f(\theta)|^2 = 1/k^2\,|\,\Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ e^{+i\delta\ell}\,\sin(\delta_\ell)|^2 \qquad \text{// \textbf{verifies 19.12}}$$

In doing all this work, we used the nonrelativistic SE in several places:

(1) We used it to verify that (18.10) is a solution of the nonrelativistic SE

(2) We used the nonrelativistic SE radial equation for large r do find solutions as j_ℓ type functions as shown in 19.7 for example.

Schiff claims, without proof, that the our three key equations 18.10, 19.11 and thus 19.12 are also true if you go through the whole thing **relativistically**. I will accept this as true for now. To show it, we need some more general argument, or we need to use the relativistic SE.

Comment: We could have blindly expanded $f(\theta)$ on the Legendres to get

$$f(\theta) = \Sigma_\ell\ f_\ell\ P_\ell(\cos\theta)$$

What we learned from all the above was the *form* of this coefficient

$$f_\ell\ =\ (2\ell+1)\,(e^{+2i\delta\ell} - 1)/(2ik)$$

Not only can we describe the $f(\theta)$ with a single real number δ_ℓ (giving f_ℓ) for each partial wave, we know something about the form of that number f_ℓ. Why does $f(\theta)$ go as $1/k$? There must be some simple general reason for that. Perhaps cross section = area, and k is the only number we have with a dimension.

Total Elastic Cross Section and Optical Theorem (p 120)

Start with our result from above

$$\partial\sigma/\partial\Omega = |\,f(\theta)|^2 = 1/k^2\,|\,\Sigma_\ell\ (2\ell+1)\ P_\ell(\cos\theta)\ e^{+i\delta\ell}\,\sin(\delta_\ell)|^2 \qquad \text{// verifies 19.12}$$

Then total cross section is

$$\sigma_{tot} = 1/k^2 \int dz\, [\Sigma_\ell\ (2\ell+1)\ P_\ell(z)\ e^{+i\delta\ell} \sin(\delta_\ell)\ ]\ [\Sigma_{\ell'}\ (2\ell'+1)\ P_{\ell'}(z)\ e^{-i\delta\ell'} \sin(\delta_{\ell'})\ ]$$

$$= 4\pi/k^2\ [\Sigma_\ell\ (2\ell+1) \sin^2(\delta_\ell)\ ]$$

where Jackson page 58 gives us $2/(2\ell+1)\ \delta_{\ell\ell'}$ and $d\varphi$ gives us 2π .

Now comes the optical theorem. Result p 120A is trivial from 19.11. Then

$$Imf(0) = Im\ (stuff/i) = Im(-i\ stuff) = -\ Re(stuff)$$

and

$$Re(stuff) = (1/2k)\ \Sigma_\ell\ (2\ell+1)\ [\ \cos(2\delta_\ell) - 1\ ] = -\ (1/2k)\ \Sigma_\ell\ (2\ell+1)\ 2 \sin^2\delta_\ell$$

so that

$$Im\ f(0) = (1/k)\ \Sigma_\ell\ (2\ell+1) \sin^2\delta_\ell = \sigma_{tot} * k/4\pi$$

or

$\sigma_{tot} = (4\pi/k)$ Im f(0) // Optical Theorem 19.14 verified

He points out that this result is much more general that our derivation here.

Comment: We have assumed that V(r) = real, and S claims this corresponds to elastic scattering, but I don't yet know how elasticity and realness of V are connected. This is a mystery at this point, I think he will clear this up when we get to complex V.

Phase Shifts δ_ℓ (121)

We should think of our "interior region" (the first region on page 118 bottom) as a place where the potential V affects the solution, so we imagine we solve the problem there somehow for $R_\ell(r)$. The claim made in this section is that you will only find large δ_ℓ phase shifts for $\ell \leq ka$. The reason is that outside this range of ℓ, $j_\ell(kr)$ is very small for $r<a$. But I guess we think of $j_\ell(kr)$ as the interior portion of $R_\ell(r)$ that is supposed to be "interacting with the potential". If $j_\ell(kr)$ is very small in this region, there is no interaction, it is as if $R_\ell(r)$ doesn't even see the potential, so you would in this case get no phase shift at all. This argument is a little hazy, but the result is going to be very true. Looking at $\ell_{max} = ka$ as our limit, we see that the partial wave method will be most useful at smaller k values, ie, low energy scattering. S points out that ℓ/k is the classical distance of closest classical approach, so if $\ell/k > a$, we expect to get no interaction, so this gives classical support to our argument. Large ℓ classically keeps the incoming particle away from the target, while large k can push it closer in. So for a given k, you classically expect there to be an ℓ beyond which you don't get much deflection, and this would be a function of the interaction range.

Calculation of the Phase Shifts (121)

We imagine the interior region ends in effect at some radius r = a, beyond which we enter region 2 where V(r) does not matter, but we still keep the ℓ term in the ODE, and this gives the solution of the form 19.7. If we can compute for the interior region that $R'_\ell(r)/ R_\ell(r) |^{r=a}$ equals some value γ_ℓ, then we can match this to this same ratio outside in region 2 at r=a. Using 19.7, we trivially obtain page **121 A.** This in turn is trivially solved for $\tan\delta_\ell$ as shown in 19.15, just write it out on 2 lines and you have it.

Now page 122 pursues this approach. Suppose we are at large enough ℓ where the phase shifts are small, so we say $\cos\delta_\ell \approx 1$ and $\sin \delta_\ell \approx \delta_\ell << 1$. Then from 121 A we can do a little quick algebra to get the general form in 19.16. We can write an exact expression from this algebra for ε_ℓ but all we really care about is that $\varepsilon_\ell <<$ the first term shown. I did not bother to do the math for 19.17 and the rest of this section (though I seem to have done it once upon a time). The main point here is that once you get into this large ℓ range, you can put an upper bound on δ_ℓ and you can thereby verify that the partial wave expansion does in fact converge! The fall off is rather strong, in fact. This seems a very technical point for Schiff to be inserting at this point.

Meta Review to this Point.

1. Classical. In a classical scattering experiment, say Rutherford, you have trajectories which are affected by the scattering potential or force. Classical equations of motion describe the trajectory. For example, in a Coulomb force the trajectory is a half hyperbola. A "theorist" can compute the trajectories from the assumed force and predict a cross section as a function of angle. The "experimentalist" can measure the cross section. There is no "wavefunction", there is no "probability" except to the extent that the incident particle beam is probably spread out over many atomic diameters so the resulting cross section is statistical in that sense. In principle, classically you could have a precision beam on one charge at a precise impact parameter and then there is nothing probabilistic about the process.

2. Quantum. In a quantum scattering experiment, there are no trajectories, there is only a wavefunction. This includes an incoming plane wave (not a particle) as shown in 18.10, and an outgoing wave (large r assumed) shown as the second term of 18.10 with scattering amplitude $f(\theta)$. The lateral extent of the incoming plane wave function is a good match to the spread-out experimental beam mentioned in 1 above. The cross section is $|f(\theta)|^2$ so this is what an experimentalist can measure. People want to know what is the potential V(r) that results in this cross section? How can you deduce V(r) from $|f(\theta)|^2$? As in the classical case, we assume some V(r) and try to predict $|f(\theta)|^2$ and then see if experiment agrees.

The connection between V(r) and $|f(\theta)|^2$ is the main point of this Schiff section; it is a quite indirect connection. A key idea is to make use of the large-r limits of various functions. The wavefunction for the scattering problem is written in two separate forms. The first form is 18.10 as just discussed, which is a large-r form. The second form is 19.1, which is a partial wave sum of solutions of the SE for this problem, which SE of course includes V(r). The reason we have this sum is as follows: for V(r), we know the SE solution is separable into radial times angular subfunctions, and in a no-φ situation those subfunctions are just the $P_\ell(\cos\theta)$. So we know that the most general form of the solution is the ℓ sum shown in 19.1 where all we know about $R_\ell(r)$ is that it solves the radial equation including V(r).

The next step is to go to some moderately large r where we think 18.10 applies, and for such r we equate our two "forms" of the solution, 18.10 and 19.1A We can do the exact expansion 19.9 of the plane wave part of 18.10 in terms of j's. As for $R_\ell(r)$, at our moderate value of r, we assume we are outside the effect of the potential so we can set V(r) = 0 in the SE. [Remember that ℓ creates an effective potential

even when the real potential is zero.] We can solve such a radial SE and find that the most general solution for a given ℓ is 19.7. This is a linear combination of the two kinds of spherical Bessel functions j and n, and we provide an unknown overall amplitude A and a j/n mixture determined by δ, so we regard then A_ℓ and δ_ℓ as the two unknown constants.

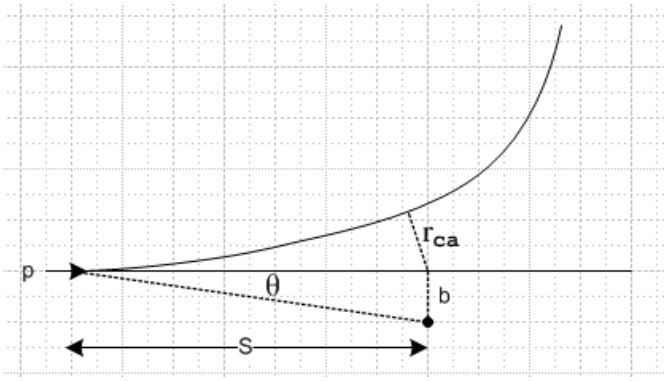
We then do the equating just mentioned, and at the same time we then go to very large r so we can use the asymptotic trig forms of the spherical Bessel functions. The "equating" then gives us our desired relation between f(θ) and the phase shifts δ_ℓ as shown in 19.11.

For low energy scattering where there are not many active partial waves [see next section!] , the experimentalist can measure the cross section at a given k, thereby measuring $|f(\theta)|^2$, and can then hopefully deduce values for the lowest several phase shifts δ_ℓ. The theorist meanwhile assumes some V(r) in the core region (small r), computes the exact $R_\ell(r)$ there, and then determines the ratio $R'_\ell(r)/ R_\ell(r)$ at the outer edge of this "internal" region and matches this to the same ratio computed from the mid-region assumed form 19.7, where by the way we also learned the $A_\ell = e^{i\delta}$ so the δ_ℓ determines *everything* in a partial wave. By doing this equating in each partial wave, the theorist can compute the phase shifts δ_ℓ.

The δ are called "phase shifts" because they measure the shift in the phase of $R_\ell(r)$ relative to that of $R_\ell(r)= j_\ell(r)$, where $j_\ell(r)$ is all there is if you have no scattering, ie, you have just the plane wave and f(θ) = 0. Thus, for low energy scattering, the small set of activated δ_ℓ values provides a **point of contact** between the experimentalist who measures $|f(\theta)|^2$ and deduces the δ_ℓ from 19.12, and the theorist who assumes some V(r) and computes what the δ_ℓ ought to be.

For high energy scattering, where perhaps hundreds of partial waves are activated, this partial wave method is probably useless. There are too many parameters. But for low energy scattering, perhaps only the S and P wave are activated, and then the entire scattering process can be described by just two real δ parameters and the theory/experiment comparison can be well made.

3. Why does low-energy scattering activate only low partial waves? Consider this picture



where the black dot is the scattering center at impact parameter b relative to the incoming particle. Since the force is central, the torque on the incoming particle is **N** = **r** x **F** = 0, so **L** is conserved, arrow is out of the plane of paper, as Goldstein confirms. At large distance S, θ is a small angle. We can say

$L = rxp = Spsin\theta \sim Sp(b/S) = pb$

$\Rightarrow \ell\hbar \approx (\hbar k)b \quad \Rightarrow \quad \ell \approx kb$

For a given b, we roughly have $\ell = kb$ so that you get more ℓ for more k. Suppose the potential has an effective range "a" such than when b > a, you get no scattering. Then the maximum ℓ you can have for a scattered particle is roughly $\ell_{max} = ka$. As you do your experiment, b takes all values $b \le a$ for your cross section measurement. We get no contribution when b > a because there is no scattering. Smaller b implies smaller ℓ for the same k since $\ell = kb$. So the upshot is that for a given k, we will get $\ell = 0,1,2...\ell_{max}$ partial waves activated, and higher ℓ waves will not be activated. This discussion assumes only that V(r) is central, we don't need to know anything about classical turning points.

This argument is a little different from Schiff's classical argument on page 121 mid page. In my picture above, it is true that b is not the closest approach distance, so I am not claiming $\ell \approx kr_{ca}$ as Schiff claims. You would think that $r_{ca} \le a$ is the condition of scattering, not $b \le a$. On the other hand, if $b = a+\varepsilon$, then the incident particle trajectory will be the straight line shown above! The incident particle never gets closer to the scattering center than $a+\varepsilon$ so there is "no scattering". So there must be scattering then whenever $b \le a$, and then my argument above is justified.

The distance of closest approach is not obviously calculable so I don't know where Schiff gets his claim, unless it is in the rough sense of b. At closest approach, we have $L = r_{ca}p_{ca}$. We know that $L = p_i b$ as well where p is the initial momentum but that is not enough to solve for r_{ca}. We really need all this stuff:

$$L = r_{ca}p_{ca} = p_i b$$

$$E_i = p_i^2/2m \quad = \hbar^2k^2/2m \quad = p_{ca}^2/2m \; + V(r_{ca})$$

so here are two equations relating p_{ca} to r_{ca} from which we could compute r_{ca} in terms of k and b. But this requires knowledge of the form of the potential. So I think my argument above is much better.

4. Diagonalization. The above partial wave expansion is an example of simplification of the problem by diagonalizing it into a set of subspaces which can be considered independently. This idea is worth reviewing at this point since rust has built up. We know in a discrete matrix problem $H\psi = E\psi$ that we might start with H being a 5x5 matrix all filled in, but if we diagonalize H fully, then the problem is completely solved, so in a sense our entire problem is one of diagonalization. Here is how this idea works in the coordinate space representation. We start with

$H\psi = E\psi$ that is $\int d^3r \langle\theta'\varphi'r'| H |\theta\varphi r\rangle\langle\theta\varphi r|\psi\rangle = E \langle\theta'\varphi'r'|\psi\rangle$

or $\Sigma_{\theta\varphi r} H_{\theta'\varphi'r',\theta\varphi r} \psi_{\theta\varphi r} = E \psi_{\theta'\varphi'r}$ // matrix form

where the number of rows in our matrix is triply infinite, there is a row for each value of r, θ and φ. So it is the same matrix problem, just with a larger matrix.

We now do a transform which is somewhat like a Fourier Transform, but it is one done "on the rotation group" whose functions are the spherical harmonics, not on the real number group whose functions are the simple exponentials we are used to in a Fourier Transform. Just as a regular Fourier Transform replaces x with k or t with ω or s, even so our transform here replaces θφ with ℓm. It just happens in this case that one side of the FT has a continuous spectrum, while the conjugate side has a

discrete spectrum, but it is still the same thing. After doing our transformation, the above matrix equation becomes

$H\psi = E\psi$ that is $\int r^2 dr \; \Sigma_{\ell m} \langle \ell' m' r' | H | \ell m r \rangle \langle \ell m r | \psi \rangle = E \langle \ell' m' r' | \psi \rangle$

or $\Sigma_{\ell m r} \; H_{\ell' m' r \, \ell m r} \, \psi_{\ell m r} = E \, \psi_{\ell' m' r}$ // matrix form

All we have done is changed from continuous variables θ,φ to discrete variables ℓ,m . In both cases, these variables represent a double infinity, so all three variables are still a triple infinity.

Note: How did we know the $r^2 dr$ factor? We could have started with the previous form, and then just deleted $\langle \theta' \varphi' r' |$ on both sides in favor of $\langle \ell' m' r' |$ as a first step toward getting to the second form. Then we could use the result that $\int d\Omega \; | \theta \varphi r \rangle \langle \theta \varphi r | = 1 = \Sigma_{\ell m} | \ell m r \rangle \langle \ell m r |$ both just treating r as some fixed parameter. This then uses up the dΩ part of $d^3 r$ leaving us with $r^2 dr$. QED.

At this point, we claim that if V = V(r), then the problem has O(3) symmetry and this means that we have the three commutator relations: $[H, \mathbf{L}] = 0$. These imply for example that $[H, L^2] = 0$ and $[H, L_z] = 0$ and $[H, L_\pm] = 0$. Now consider:

$$0 = \langle \ell' m' r' | [H, L^2] | \ell m r \rangle = \{ \ell(\ell+1) - \ell'(\ell'+1) \} \langle \ell' m' r' | H | \ell m r \rangle$$

$$0 = \langle \ell' m' r' | [H, L_z] | \ell m r \rangle = \{ m - m' \} \langle \ell' m' r' | H | \ell m r \rangle$$

This stays that if either $\ell \neq \ell'$ OR $m \neq m'$, then $\langle \ell' m' r' | H | \ell m r \rangle = 0$, so we may conclude that

$$\langle \ell' m' r' | H | \ell m r \rangle = \delta_{\ell, \ell'} \delta_{m, m'} \langle \ell m r' | H | \ell m r \rangle$$

which says that our **H matrix is partially diagonalized** in the ℓm part of the ℓmr coordinate. So this matrix is *not* fully filled in, it is zero almost everywhere. Suppose we were to draw a picture for a particular fixed value of r. Then we in theory have to make a similar picture for all other values of r, but all such pictures will look the same. And we label then the rows by ℓ, and within each ℓ, by the allowed values of m. Then our H matrix has this form

x	0	0	0	0	0	0	0	0	0	0
0	x	x	x	0	0	0	0	0	0	0
0	x	x	x	0	0	0	0	0	0	0
0	x	x	x	0	0	0	0	0	0	0
0	0	0	0	x	x	x	x	x	0	0
0	0	0	0	x	x	x	x	x	0	0
0	0	0	0	x	x	x	x	x	0	0
0	0	0	0	x	x	x	x	x	0	0
0	0	0	0	x	x	x	x	x	0	0

etc

The first red x is the non-zero matrix element (apart from r) for $\ell = 0$, then the 3x3 matrix is ℓ=1, and the 5x5 matrix is $\ell = 2$. So this is the sense in which we have achieved a partial diagonalization. We never have to think of matrix elements $< \ell'm'r' | H | \ell mr>$ outside one of these square areas, and that is why our problem is greatly simplified. We use a symmetry to partially diagonalize our Hamiltonian.

But there is still more we can learn by using the Lie Algebra. Consider

$$L_{\pm}|\ell m> = \sqrt{\ell(\ell+1)-m(m\pm1)}\ \ |\ell\ m \pm 1>$$

$$< \ell m+1 r' | H | \ell m+1 r> = [\ell(\ell+1)-m(m+1)]^{-1}\ < \ell m r' | L_+^{\dagger} H L_+ | \ell m r>$$

$$= [\ell(\ell+1)-m(m+1)]^{-1}\ < \ell m r' | L_- H L_+ | \ell m r> = [\ell(\ell+1)-m(m+1)]^{-1}\ < \ell m r' | H L_- L_+ | \ell m r>$$

but we also know that

$$L_{\mp} L_{\pm} = L^2 - L_z^2 \mp L_z$$

so we get

$$< \ell m+1 r' | H | \ell m+1 r> = [\ell(\ell+1)-m(m+1)]^{-1}\ < \ell m r' | H(L^2 - L_z^2 - L_z) | \ell m r>$$

$$= [\ell(\ell+1)-m(m+1)]^{-1}\ \{\ell(\ell+1) - m^2 - m\}\ < \ell m r' | H | \ell m r>$$

$$= < \ell m r' | H | \ell m r>$$

which says that for a given ℓ, the matrix element is *the same* for all the values of m! Thus we can write

$$< \ell m r' | H | \ell m r> = < \ell r' | H | \ell r> \qquad \text{// since not a function of m.}$$

so that within each square in our picture above, all the x's have the same value! So our simplification is very large!

Our problem now becomes

$$\int r^2 dr\ \Sigma_{\ell m}\ <\ell' m' r' | H | \ell m r><\ell m r|\psi> = E <\ell' m' r'|\psi>$$

or

$$\int r^2 dr\ \Sigma_{\ell m}\ <\ell' r' | H | \ell r>\delta_{m,m'}\delta_{\ell,\ell'}<\ell m r|\psi> = E <\ell' m' r'|\psi>$$

or

$$\int r^2 dr\ <\ell' r' | H | \ell' r><\ell' m' r|\psi> = E <\ell' m' r'|\psi> \qquad \text{// still } H\psi = E\psi\text{, the SE}$$

Because the operator $\int r^2 dr \langle \ell' r' | H | \ell' r \rangle$ is not a function of m, the wavefunction $\langle \ell' m' r | \psi \rangle$ cannot depend on m, so we write it just as $\langle \ell' r | \psi \rangle$ which is usually called $R_{\ell' E}(r)$, the "radial function". So we are now down to this:

$$\int r^2 dr \ \langle \ell' r' | H | \ell' r \rangle \langle \ell' r | \psi \rangle = E \langle \ell' r' | \psi \rangle$$

as an integral equation for the radial function which we can solve given whatever boundary conditions apply. At this point, let's replace ℓ' by ℓ to reduce clutter, so we have

$$\int r^2 dr \ \langle \ell r' | H | \ell r \rangle \langle \ell r | \psi \rangle = E \langle \ell r' | \psi \rangle \qquad *$$

and one should notice the locations of r and r' in this equation.

Digression: Showing matrix mechanics is the same as the differential equation approach.

Digression: Somehow this equation * must be completely equivalent to the usual differential equation we write for the radial function, Schiff 14.3. Let's see if we can verify that fact. Write

$$H = \hat{p}^2/2m + V(r) = -(\hbar^2/2m)\nabla^2 + V(r) = -(\hbar^2/2m) \{ \nabla^2_r + (1/r^2)(-L^2) \} + V(r)$$

$$\Rightarrow \quad \langle \ell r' | H | \ell r \rangle = \langle \ell r' | [-(\hbar^2/2m) \{ \nabla^2_r + (1/r^2)(-L^2) \} + V(r)] | \ell r \rangle$$

Replace at this point L^2 by $\ell(\ell+1)$ since we are the $|\ell m r\rangle$ basis, and we get

$$= \langle \ell r' | [-(\hbar^2/2m) \{ \nabla^2_r - (1/r^2)\ell(\ell+1) \} + V(r)] | \ell r \rangle$$

$$= [-(\hbar^2/2m) \{ \nabla^2_r - (1/r^2)\ell(\ell+1) \} + V(r)] \langle \ell r' | \ell r \rangle$$

$$= [-(\hbar^2/2m) \{ \nabla^2_r - (1/r^2)\ell(\ell+1) \} + V(r)] \delta(r'-r)/r'^2$$

Notice this very crucial fact which was messing me up earlier today: In this r-only space our measures look like this:

$$\int r^2 dr \ |\ell r\rangle\langle \ell r| = 1 \qquad \text{completeness}$$

$$\langle \ell r' | \ell r \rangle = \delta(r'-r)/r'^2 \qquad \text{orthogonality}$$

We can then write * as : (that is, we install <H> , and swing our r'^2 factor to the right side)

$$\int r^2 dr \; \langle \ell r' | \, H \, | \, \ell r \rangle \, R_{\ell E}(r) = E \, R_{\ell E}(r') \quad \Rightarrow$$

$$\int dr \;\; [- (\hbar^2/2m) \, \{ \, \nabla^2_{\mathbf{r}} - (1/r^2) \, \ell(\ell+1) \; \} + V(r) \,] \, \delta(r'-r) \, \{ r^2 R_{\ell E}(r) \} = r'^2 \; E \, R_{\ell E}(r')$$

$$\int dr \;\; [- (\hbar^2/2m) \, \{ \, \nabla^2_{\mathbf{r}} - (1/r^2) \, \ell(\ell+1) \; \} + V(r) \,] \, \delta(r'-r) \, \{ r^2 R_{\ell E}(r) \} - r'^2 \; E \, R_{\ell E}(r') = 0$$

We can now write

$$LHS = T1 + T2 = 0$$

where

$$T1 = - (\hbar^2/2m) \int dr \{ \, \nabla^2_{\mathbf{r}} \, \delta(r'-r) \} \, \{ r^2 R_{\ell E}(r) \}$$

$$T2 = \int dr \;\; [- (\hbar^2/2m) \, \{ - (1/r^2) \, \ell(\ell+1) \; \} + V(r) \,] \, \delta(r'-r) \, \{ r^2 R_{\ell E}(r) \} \quad - \quad E \, r'^2 \, R_{\ell E}(r')$$

$$= [- (\hbar^2/2m) \, \{ - (1/r'^2) \, \ell(\ell+1) \; \} + V(r') \,] \, \{ r'^2 R_{\ell E}(r') \} - r'^2 E \, R_{\ell E}(r')$$

$$= r'^2 \; \{ \; (\hbar^2/2m) \, (1/r'^2) \, \ell(\ell+1) \; + [\, V(r') - E \,] \; \} \, R_{\ell E}(r')$$

$$= r'^2 \, (\hbar^2/2m) \{ \;\; (1/r'^2) \, \ell(\ell+1) \; + (2m/\hbar^2) [\, V(r') - E \,] \;\; \} \, R_{\ell E}(r')$$

$$= - r'^2 \, (\hbar^2/2m) \;\; \{ \;\; (2m/\hbar^2) [\, E - V(r')] \;\; - \;\; \ell(\ell+1)/r'^2 \; + \; \} \, R_{\ell E}(r')$$

So at least we see here apart from the leading factor something appearing in 14.3. There is hope! If we cancel our factors $- (\hbar^2/2m)$ we have this intermediate result:

$$(1/r'^2) \int dr \{ \, \nabla^2_{\mathbf{r}} \, \delta(r'-r) \} \, \{ r^2 R_{\ell E}(r) \} \; + \{ \;\; (2m/\hbar^2) [\, E - V(r')] \;\; - \;\; \ell(\ell+1)/r'^2 \; + \; \} \, R_{\ell E}(r') = 0$$

and we are then hoping to show that the first term is what we want, namely, to show that

$$\int dr \; \{ \, \nabla^2_{\mathbf{r}} \, \delta(r'-r) \} \, \{ r^2 R_{\ell E}(r) \} = r'^2 \; \nabla^2_{\mathbf{r}'} \, R_{\ell E}(r) \qquad\qquad **$$

and if we can show this, then we have arrived at Schiff 14.3 which is the known radial SE. Notice that the hoped for result above does NOT say that you can just magically parts-over the entire radial Laplacian onto $\{ r^2 R_{\ell E}(r) \}$. In fact, that would not give the result we want.

We have to now look in detail at how parts works. So here we go:

$$\int dr \{ \, \nabla^2_{\mathbf{r}} \, \delta(r'-r) \} \, \{ \, r^2 R_{\ell E}(r) \} = \int dr \{ \, 1/r^2 \, \partial_{\mathbf{r}} \{ [\, r^2 \; \partial_{\mathbf{r}}] \, \delta(r'-r) \} \, \{ \, r^2 R_{\ell E}(r) \}$$

$$= \int dr\, \partial_r\{[\, r^2\ \partial_r]\, \delta(r'-r)\}\ \{R_{\ell E}(r)\}$$

Do a first parts integration to get

$$= -\int dr\ \ \{[\, r^2\ \partial_r]\, \delta(r'-r)\}\ \partial_r\{R_{\ell E}(r)\}\ \ = -\int dr\, \partial_r\{\delta(r'-r)\}\ [r^2\ \partial_r\ (R_{\ell E}(r))]$$

Now do a second parts integration to get

$$= +\int dr\ \{\delta(r'-r)\}\ \partial_r\ [r^2\ \partial_r\ (R_{\ell E}(r))\ ] = \partial_{r'}\ [r'^2\ \partial_{r'}\ (R_{\ell E}(r'))]$$

$$= r'^2\ \ \partial^2_{r'}\ (R_{\ell E}(r')) + 2r'\ \ \partial_{r'}\ (R_{\ell E}(r'))$$

$$= r'^2\ \ \{\ \ \partial^2_{r'}\ (R_{\ell E}(r')) + 2/r'\ \ \partial_{r'}\ (R_{\ell E}(r'))\ \}$$

But recall that

$$\nabla^2_r R\ \ =\ (1/r^2)\ \partial_r(r^2\partial_r R)\ = (1/r^2)\ [\ r^2\partial_r{}^2 R\ + 2r\ \partial_r R\ ]\ = \partial_r{}^2 R + 2/r\ \partial_r R$$

Thus we have shown that

$$\int dr\{\ \nabla^2_r\ \delta(r'-r)\}\ \{r^2 R_{\ell E}(r)\}\ \ = r'^2\ \ \{\ \ \partial^2_{r'}\ (R_{\ell E}(r')) + 2/r'\ \ \partial_{r'}\ (R_{\ell E}(r'))\ \}\ \ = r'^2\ \{\ \nabla^2_{r'} R_{\ell E}(r')\ \}$$

which was the desired result **. Thus, we have converted our integral equation,

$$\int r^2 dr\ \ \langle \ell' r'|\ H\ |\ \ell' r\rangle\langle \ell' r|\psi\rangle = E\ \langle \ell' r'|\psi\rangle \qquad\qquad *$$

into this differential equation

$$\nabla^2_{r'} R_{\ell E}(r')\ \ +\ \{\ (2m/\hbar^2)[\ E - V(r')]\ \ -\ \ \ell(\ell+1)/r'^2\ \ \}\ R_{\ell E}(r')\ = 0$$

which is precisely 14.3 of Schiff. So now perhaps we have more confidence in our "integral equation". End of digression.

So we had this result:

$$\int r^2 dr\ \ \langle \ell' r'|\ H\ |\ \ell' r\rangle\langle \ell' r|\psi\rangle = E\ \langle \ell' r'|\psi\rangle \qquad\qquad *$$

or

$$\int r^2 dr \; \langle \ell r' | H | \ell r \rangle R_{\ell E}(r) = E \, R_{\ell E}(r')$$

Suppose we solve for the radial function. We can then go back to θφ space as follows:

$$\psi(r,\theta,\varphi) = \langle r\theta\varphi|\psi\rangle = \Sigma_{\ell m} \langle r\theta\varphi|r\ell m\rangle\langle r\ell m|\psi\rangle = \Sigma_{\ell m} Y_{\ell m}(\theta,\varphi) R_{\ell E}(r)$$

Compare this to Schiff page 93 first equation in 16.24. The above ψ(r,θ,φ) is *a* solution to our SE, but it is not the most general solution. What is going on here?

We started off with a huge vector = vector equation of the form A**v** = E 1 **v** where **v** is a vector and A and 1 are matrices and E is a real number. In our application we are writing this as H **ψ** = E 1 **ψ** where **ψ** is a solution vector which spans the entire huge triply infinite Hilbert space.

Now, the implication of the block diagonal form shown above is that we could if we wanted decompose this huge space into a set of subspaces like this:

$$\mathcal{H} = \Sigma_{\ell \oplus} \mathcal{H}^{(\ell)} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} ....$$

where the meaning of the direct sum is precisely our block diagonal form. We could partition our solution vector in this same way (but I will write it as a row vector to save space)

$$\boldsymbol{\psi} = (\psi^{(0)} , \psi^{(1)} , \psi^{(2)} , \psi^{(3)} , \psi^{(4)} , ...)$$

where $\psi^{(0)}$ represents a 1-component vector, $\psi^{(1)}$ a 3-component vector, and so on. We could write for example, where we are now using ℓm subscripts,

$$\psi^{(1)} = (\psi_{11}, \psi_{10}, \psi_{1-1})$$

Now, when we write H **ψ** = E **ψ** we must realize that 0 is also a solution! And we can have mixed solutions where some of the subspace ψ's are zero and others are not. For example, here is a legal solution of our SE

$$\boldsymbol{\psi} = (\psi^{(0)}=0 , \psi^{(1)}=0 , \psi^{(2)} , \psi^{(3)}=0 , \psi^{(4)}=0 , ...)$$

so the only non-zero components are the ψ_{2m} ones. The point: this is a valid SE solution that involves only the $\ell = 2$ subspace of our problem. We have after all "diagonalized" the problem in this way.

So, let's consider separate solutions in each ℓ subspace. Then we can regard the following as possible solutions to the SE within just the ℓ subspace,

$$Y_{\ell m}(\theta,\varphi) R_{\ell E}(r)$$

The solutions are different for different m, and there are of course $(2\ell+1)$ solutions and these form a spanning basis for the ℓ subspace.

Now we write a *general solution* of our SE as a linear combination of these subspace solutions! So

$$\psi_E(r,\theta,\varphi) = \Sigma_\ell \; [\Sigma_m \; k_{\ell m} \; \{ Y_{\ell m}(\theta,\varphi) R_{\ell E}(r)\}] = \Sigma_\ell \; R_{\ell E}(r) \Sigma_m k_{\ell m} \; Y_{\ell m}(\theta,\varphi)$$

and this is the result we have been after. Recall that our problem was assumed to have SO(3) symmetry due to the V(r) assumed central potential, so the radial function does not depend on m.

If we somehow know that our solution wavefunction cannot depend on θ, then $k_{\ell m}$= 0 unless m=0 and the above becomes

$$\psi_E(r,\theta) = \Sigma_\ell \; k_{\ell 0} Y_{\ell 0}(\theta,\varphi)\, R_{\ell E}(r) = \Sigma_\ell \; [\sqrt{(2\ell+1)/4\pi} \;\; k_{\ell 0}]\, P_\ell(\cos\theta)\; R_{\ell E}(r)$$

and if we choose our constants such that

$$[\sqrt{(2\ell+1)/4\pi} \;\; k_{\ell 0}] = (2\ell+1)\, i^\ell$$

then we obtain the assumed form Schiff 19.1A.

The sign of the phase shift δ_ℓ versus sign of the potential V(r) (122)

It seems clear that if you turn on a potential U(r) > 0 and compare the radial wave function against the case U=0, the asymptotic or even finite R(r) is going to shift either to the left or the right a bit. Imagine turning a knob and watching it move. The question is: which way goes it go? Schiff argues on page 122 that turning on U>0 makes the wave "push out" relative to the dotted U=0 wave, and this is shown top left of page 123. This means that the zeros move to the right. Since zeros of the sine are at kr - ½ $\ell\pi$ + δ = nπ, moving a zero to the right (larger kr) means δ_ℓ got smaller. So the argument is that a positive repulsive potential gives you negative phase shifts and pushes curve to the right. And of course the reverse would be true for an attractive potential as shown top page 123 right side. The graphs come numeric work with a square well potential and bear out these claims.

The general argument Schiff gives, however, is I feel bogus. I agree that χ"/χ is larger for U>0 compared to U=0, but I see no reason why this implies that χ'/χ is larger. I have tried making drawings and also doing algebra but I cannot support this claim. Either I am missing some simple fact, or he needs a better argument. His argument also depends on 19.19 which I did not derive.

So OK, I could probably come up with a better argument in 8 hours of labor, but I know it is going to go one way or the other, and I am happy with the conclusion of the numeric work which he describes mid page 123.

Ramsauer-Townsend Effect (123).

Now remember that the phase shift δ_ℓ is a function of E and hence of k. Suppose you have a very low energy scattering situation where only the ℓ=0 partial wave is activated. And suppose at some energy E in the middle of some range, you have $\delta_\ell = \pi$ as shown 124 picture (and on the front cover!). This would kill off that partial wave and would kill off the cross section, so you would have a zero in dσ/dΩ at some energy, and this is the R-T effect. They found the effect in 1921 doing 0.7 eV electrons off certain atoms. The claim is that with atoms like neon and argon with closed shells, there is a sharp boundary at some r=a and you get a good cutoff on the partial waves so you can have only ℓ=0 at low energy. Schiff argues (an argument I do not follow) that this effect can only occur with an attractive potential (as found inside the

outer shell of a tight neon atom, eg). In contrast, in our 1D work, we found the notion of transmission zeros with V_o having either sign, a sine factor.

So this is an amusing effect, and one could study the radial equation in great detail to justify the various conclusions of this and the preceding sections.

Scattering by an *Infinite* Positive or Negative Infinite Spherical Square Well (124).

Think of such a thing as scattering off a super hard sphere of radius a which no wave function can penetrate, he calls it in his title a "**perfectly rigid sphere**". We have $V = \infty$ for $r < a$ and $V=0$ outside. So outside we know our solutions are 19.7, and at the surface we have $R=0$. Thus, without any work at all we have computed ALL the phase shifts exactly as shown in 19.20. For $ka << 1$, this becomes 19.21 using the small argument limits of the Bessel functions, and this shows that as ℓ increases, the phase shifts quickly go to zero. This 10.21 is a specific example of the more general crude result Schiff found earlier in 19.19 and which I did not derive on this reading.

There are now two limits of interest. In the very low energy limit, there is only $\ell=0$ and the shift there is $\delta_0 = (ka) << 1$. If you put this into 19.13, you get $4\pi a^2$ which oddly is four times the optical cross section. No reason for this factor of 4 is proposed.

The other limit is large ka and this is much harder to do. I agree with 19.23, and the claim is made that some range of ℓ dominates the sum and somehow we get a leading term as shown, but nothing here is clear to me. When the dust settles, you get twice the optical cross section of πa^2 and it is then argued that this is due to an overcounting issue: somehow one πa^2 is used to generate the non-forward scattering, and the other makes the shadow. I do not follow the argument, but probably could it I wanted.

Scattering by a *Finite* Negative Spherical Square Well (126).

In the "differential cross section" discussion above, I showed that the radial equation for no potential looks like this

$$R_\ell(r) = a_\ell\, j_\ell(\alpha r) + b_\ell\, n_\ell(\alpha r)$$

$$\alpha^2 = (2\mu/\hbar^2)E = (2\mu/\hbar^2)(\hbar^2k^2/2\mu) = k^2$$

where $E > 0$. If you have some constant V in a region, E becomes E - V. In this problem, we use $V = -V_o < 0$ for our negative finite depth well, the depth is V_o. So we get

$$\alpha^2 = (2\mu/\hbar^2)(E+V_0)$$

as in 19.25. For the interior solution that includes r=0, we can have only $j_\ell(\alpha r)$, hence 19.25. If we go to the sphere surface at r=a, our γ ratios are of course as in 19.26. If we were to stick these into page 19.15A, we have

$$\tan\delta_\ell = [\, k\, j_\ell'(ka) - \gamma_\ell\, j_\ell(ka)\,]/[\, k\, n_\ell'(ka) - \gamma_\ell\, n_\ell(ka)] \qquad \text{where } \gamma_\ell = \alpha\, j_\ell'(\alpha a)/\, j_\ell(\alpha a)$$

so this gives a closed-form result for all the phase shift tangents as a function of k, α , a and ℓ. For small k such that $ka << 1$, we can simplify the ratio using the small argument Bessel limits, and for $\ell = 0$ I just

showed that you get 19.27 A, and I see that at one time I did 19.27B so I won't repeat that here. Nothing new by doing it. In the small-k limit, again only the ℓ=0 term survives.

Then using the formula 19.13 we get

$$\sigma = (4\pi/k^2)\sin^2\delta_o \approx (4\pi/k^2)\tan\delta_o^2 = (4\pi/k^2) * [19.27A]^2$$

and if we use the value of γ_o given in text on top of page 127, I just confirmed on scratch the result 19.28. This says that we get the infinite depth well result (same either sign of V_0) times a correction (and we assume we are away from the poles!). If we take the limit of 19.28 as $V \to \infty$ we get our previous result of just $4\pi a^2$. I'm sure someone has an intuitive explanation of this 4.

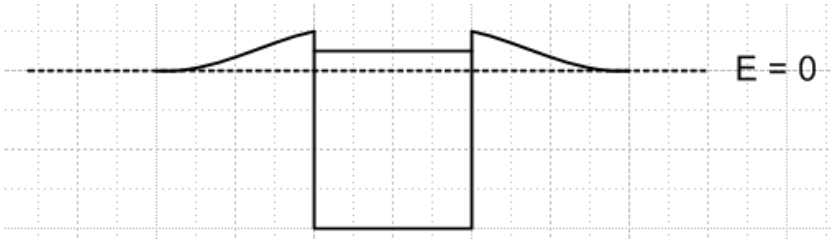
Resonance Scattering (127)

Here we continue with the "finite negative spherical square well" of the last section, and we look specifically at the differential cross sections near resonances for the ℓ=0 and ℓ=1 partial waves. Using the usual small k approximations for the Bessel functions we get the two results 19.29 and 19.30 which certainly appear to have resonance peaks in them. The math must be a bit messy because the parameters b_o and b_1 are not even stated, though said to be near unity. But we know there will be resonances because we see those poles sitting in 19.27. He is just using a little fit for α near E = 0 (low energy) where $\alpha = \alpha_0$. This is in B, and then this gets converted into a fit for the γ_ℓ which appear in the pole expressions.

He then claims that the ℓ = 1 has the sharper resonance peak.

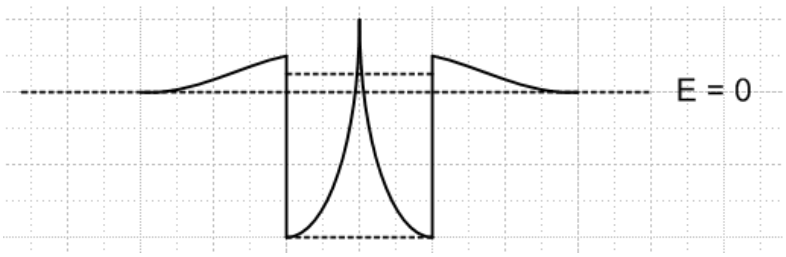
Next, he writes the total ℓ=0 cross section near its resonances. The poles occur at $\cot(\alpha a) = 0$ and so are periodic as you vary V_o. This whole resonance effect can be explained in a total new way: resonances occur at values of V_o where the well has a bound state right near the top, which is to say near E = 0. As we saw in the 1D case, the scattering builds up amplitude in the well due to the presence of this state there, and this causes a large distortion of the scattering, which appears as a scattering peak.

He then goes on to remind us of the spherical "effective potential" that we see for example in 14.17 for $\ell > 0$. So outside the r=a location we get a little "lip" in the potential (see page 128 for same picture with an arbitrary potential inside r = a , indicated by a dotted curve there).



and this allows bound states slightly above E = 0 which he calls "virtual bound states" . But I think this same situation applies for the isolated bound state problem in this 3D well.

[As you have larger ℓ, the extra potential term acts as a "centrifugal force" pushing the incoming electron away from the target. Goldberg has some pictures on page 65 on this subject. In the above picture, I suppose I should continue the $1/r^2$ curves stepped down by distance V_o, ie,



Usually one just draws the side of this picture on the right, and the conclusion that there will be bound states with E > 0 is not altered by this improved picture.]

I want to say the following, but don't know how to justify it yet, maybe need QFT. Think of the well as a sort of "atomic nucleus" and the electron is shot in with some ℓ, so to speak. If the ℓ^{th} partial wave has a resonance in the lip area at the incoming energy E, then you might say that the incident electron is annihilated, and appears temporarily as a bound state electron in the well and scattering stops. The nucleus absorbs the electron from the plane wave somehow. Then after a delay, it emits the electron again, and scattering continues. Thus we would indeed expect to see a large effect on the cross section near such a resonance. In a way, it is similar to a photon being absorbed by a normal atom and then later being reradiated. For a continuous plane wave, this process sort of happens all the time. For a packet, as we did in the 1D example on page 107, you can imagine the final movie frames as showing the electron being radiated out of the bound state in both directions. That is a 1D example and "differential cross section" means simply transmit or reflect. There is no "centrifugal lip" in 1D, so you don't see this resonance effect with a negative well. It would be interesting to see some kind of computer animation of a packet scattering from a 3D potential well.

Schiff makes some other interesting comments which I will not comment on here.

Angular distribution at low energies (129).

Suppose only $\ell = 0$ and 1 are activated. Then it is very easy to verify 19.32 A and B. For any given small ka, the $\ell = 0$ has a larger phase shift, causes more scattering, and he makes simple arguments why this is so. He then gives a nice example showing how you might barely see $\ell = 1$ in the total cross section, but it will be very dramatic in the differential cross section just because you get interference between small δ_1 and large δ_0. This interference is in the $\cos\theta$ term of A. The $\cos^2\theta$ term will be tiny because it is δ_1^2.

20. Scattering by Complex Potentials (129)

We are going to discuss an idea called "the optical model" whereby we crudely model scattering which is non-elastic by a complex potential V. This idea was first proposed late, in 1954. It would apply to scattering where there is some form of "absorption" going on which makes it inelastic. Perhaps the incoming particle is completely grabbed by the target (this would be somehow counted as part of the inelastic cross section), or perhaps the incoming particle scatters, but the target is left in an excited state, which then fits our earlier model with Q < 0. The name of this model is quite obvious, since we treat absorption in optics by making a normally real parameter be complex, such as the index, and here we are going to make the phase shift δ_ℓ be complex.

Conservation of Probability (130)

A very nicely done section. First, I (have now) derived 20.1 back in my Chapter 1-6 notes, you can see all details of the derivation there. Then you integrate to get 20.2 and here is what that equations says: " the time rate of change of the total probability inside some large volume Ω is – the net amount flowing out through the boundary of Ω, – the amount somehow being "absorbed" due to the V_I = - Im(V) > 0. So this is why this is going to model "absorption". For our real potential, we just said that the plane wave probability coming in through Ω has to **balance** that outgoing due to the spherical out-scattering. Did I ever show that this in fact does balance? First a comment: if there were no scattering, the total flux from the incoming beam would integrate to 0 over some large sphere! So the balance really says that it is the amount of flux *removed* from this plane wave (integrated over the sphere) that equals the (integral of) the outgoing scattered flux.

Well, first off, since time dependence is e^{-iEt}, we know that ψ*ψ does not depend on t, so the LHS of 20.2 is zero. For a real potential, this means that $\int \mathbf{S}.d\mathbf{A} = 0$, so the "balance" must be true. Can we show this explicitly for our canonical scattering situation? Here are some results from earlier in this notes doc:

$\psi = e^{ikz} + f(\theta)e^{ikr/r}$

$j_r = v\ |f|^2 / r^2$ $\qquad$ $v = p/\mu$
$j_z = v$

$j_z\, d\sigma = j_r\, (r^2 d\Omega)$ $\qquad$ where $r^2 d\Omega = dA$

$\partial\sigma/\partial\Omega = |f|^2$

If we integrate j_z over a large sphere, we of course get 0 (duh, I just did this). But this is not the physical situation because some of the incoming plane wave is "used up" to make the scattering. The amount used up is of course the amount that is scattered out which is this:

$\int d\mathbf{A} \bullet j_r \hat{\mathbf{r}} = j_z \sigma$ $\qquad$ // just do the integral on the LHS, where $d\mathbf{A} = r^2 d\Omega\ \hat{\mathbf{r}}$

So, you can think of $j_z\sigma$ itself as "the amount of flux that is scattered out". This is of course just what "scattering cross section" means in the first place.

Now with the complex potential, we still have e^{-iEt} for our stationary solution, so the LHS of 20.2 is still 0. Then we can define these things:

$j_z\sigma_{el} = \int d\mathbf{A} \bullet j_r \hat{\mathbf{r}}$ = total flux removed from input beam and scattered out

$j_z\sigma_{abs} = (2/\hbar) \int V_I\, \psi^*\psi\, d^3r$ = total flux removed from input beam and absorbed by scattering center
// as shown in 20.3

At this point, the total flux removed from the incoming beam is the sum of the above items

$j_z\,(\sigma_{el} + \sigma_{abs})$ = total flux removed from the incoming beam

Therefore we can compute the overall integral $\int S\bullet dA$ as follows:

$$\int S\bullet dA = \int d\mathbf{A} \bullet j_r \hat{\mathbf{r}} \; - j_z\,(\sigma_{el} + \sigma_{abs}) = j_z\sigma_{el} \; - j_z\,(\sigma_{el} + \sigma_{abs}) = - j_z\,\sigma_{abs}$$

$$= - (2/\hbar) \int V_I\, \psi^*\psi \, d^3r \qquad \text{// from 3" above}$$

and this then verifies 20.2 which now says

$$0 \; = - \int S\bullet dA - \; (2/\hbar) \int V_I\, \psi^*\psi \, d^3r$$

So now we have made a direction connection between V_I and the "absorption cross section":

$$\sigma_{abs} \; = (2/v\hbar) \int V_I\, \psi^*\psi \, d^3r \qquad // \qquad = \; - (1/v) \int S\bullet dA$$

and our "model" is starting to take shape.

Complex Phase Shifts and Three Cross Sections (131)

All of page 131 is fine, and 20.4 is just a restatement of 19.11 which we have under our belt. Going to the top of 132, if we insert 20.4 into page 132 A and use P_ℓ orthog, we get 20.5C at once. Then on scratch I multiplied out the abs val and got the form 20.5B, so all this is OK.

So at this point we can relate σ_{el} to the phase shift information. The next step is to show that we can also relate σ_{abs} to phase shift and the result will be 20.8, but this is a long calculation. As we come down through equations D,E,F,F,G, everything is easy to show except Schiff has uncharacteristically made two errors right in a row! The first is omission of $i^\ell\,(2\ell+1\,)$ in G. When you include this factor, equation 20.7 then has $(2\ell+1)$ in the numerator, and A_ℓ in H is just $e^{2i\delta}$. I think this is an editing problem where he changed normalization maybe back in 19.1, but then did not update this later part.

So I am happy down through 20.7. Let's assume for the moment that H is correct and plug it into 20.7:

$$\sigma_{abs} = 2\pi i/k \; \Sigma_\ell \; (2\ell+1) \; \{ \; -i \; |A_\ell|^2 \sin(2\beta) \; \} \; = 2\pi/k \; \Sigma_\ell \; (2\ell+1) \; |A_\ell|^2 \sin(2\beta)$$

But $|A_\ell|^2 = e^{-2\beta}$ so this becomes

$$\sigma_{abs} \; = 2\pi/k \; \Sigma_\ell \; (2\ell+1) \; e^{-2\beta} \; ½ \; [\; e^{2\beta} - e^{-2\beta}] \; = \pi/k \; \Sigma_\ell \; (2\ell+1) [\; 1 - e^{-4\beta}] \qquad \text{// verifies 20.8}$$

So now let's go back to the as-yet unverified limit shown in H

$\chi = 1/k\ A \sin(kr - \boldsymbol{\ell}\pi/2 + \delta)$
$\chi^* = 1/k\ A^* \sin(kr - \boldsymbol{\ell}\pi/2 + \delta^*)$
$\chi' = A \cos(kr - \boldsymbol{\ell}\pi/2 + \delta)$

$\chi^* \chi' = |A|^2/k \sin(kr - \boldsymbol{\ell}\pi/2 + \delta^*) \cos(kr - \boldsymbol{\ell}\pi/2 + \delta)$

Let $kr - \boldsymbol{\ell}\pi/2 = x$ so we have

$\chi^* \chi' = |A|^2/k \sin(x + \delta^*) \cos(x + \delta) = |A|^2/k \sin(x + \alpha - i\beta) \cos(x + \alpha + i\beta)$

$= |A|^2/k \sin(y - i\beta) \cos(y + i\beta)$ where $y = x + \alpha$

Adding the other part we get

$\chi^* \chi' - CC = |A|^2/k\{ \sin(y - i\beta) \cos(y + i\beta) - \sin(y + i\beta) \cos(y - i\beta) \}$

$= |A|^2/k \sin [-2i\beta]$ page 15 top Schuam

$= -i\ |A|^2/k\ sh(2\beta)$ page 31 top same

So OK, I have now verified H and then 20.8 is quite happy. We find that our partial wave expansion for the σ_{abs} involves only the β phase shift values. Now we can add the cross sections shown in 20.5 and 20.5 and we have

$| 1 - S|^2 + (1 - |S|^2) = 2 - 2\ e^{-2\beta}\cos(2\alpha)$

where I use the expansions already shown in page 132 B and page 133A. The two $e^{-4\beta}$ terms cancel and we then have verified 20.9 C. Finally,

$ReS = Re(e^{2i\alpha})(e^{-2\beta}) = (e^{-2\beta})\cos(2\alpha)$

and this gives 20.9 D.

So very nice, we can now talk about three different cross sections, and we have each expressed both in terms of complex S and in terms of the α and β .

Asymptotic Relations (133) and Reciprocity Theorem (135)

These two sections really are a single derivation of the theorem, he just puts a heading to delimit where the actual theorem is stated. He is going to partially reuse the math in the next sections on optical theorems. Warning: this is an amazingly long and messy section in terms of math fiddling. I bet there is a simpler way to get to the result.

Schiff is not quite clear in defining things, so here I am to the rescue. We understand in 20.10 that

$\mathbf{k} = k\,\hat{\mathbf{k}}$ = a potential incoming plane wave vector

$\mathbf{k_r} = k\,\hat{\mathbf{r}}$ = radially outgoing vector, same magnitude

Now suddenly he considers another potential incoming vector **k'** and he should say

$\mathbf{k'} = k\,\hat{\mathbf{k}}'$

so it too has magnitude k, it just comes in from some other direction. Then one can hardly argue with 20.11. Notice that he has *not* said that $f(\mathbf{k_r}, \mathbf{k}) = g(\mathbf{k_r} \bullet \mathbf{k}) = h(\theta)$, but maybe this would be implied if there are no other vectors available from which to form scalars. But for now we stay general. We have a general V(**r**) I presume [not momentum-dependent say] but we do not assume V(r) central.

So yes, both 20.10 and 20.11 satisfy the SE for the same $E = \hbar^2k^2/2m$, so we have

$-\hbar^2/2m\ \nabla^2 u_{\mathbf{k}} = (E-V)\ u_{\mathbf{k}}$
$-\hbar^2/2m\ \nabla^2 u_{-\mathbf{k}'} = (E-V)\ u_{-\mathbf{k}'}$

If we just divide these two equations, we get

$\nabla^2 u_{\mathbf{k}} / \nabla^2 u_{-\mathbf{k}'} = u_{\mathbf{k}} / u_{-\mathbf{k}'} \Rightarrow \quad u_{-\mathbf{k}'}\nabla^2 u_{\mathbf{k}} - u_{\mathbf{k}}\nabla^2 u_{-\mathbf{k}'} = 0$ // which is 20.12.

and my derivation shows that V could be complex, it just divides out. Our goal now is to use this fact and the previous two equations to prove the 20.16.

Digression On Green's Theorems. Thanks here to M&M page 161, they are often to the rescue. Start with the regular Gauss's Theorem

$$\int \nabla \bullet \mathbf{F}\ dV = \int \mathbf{F} \bullet d\mathbf{A}$$

Apply this to $\mathbf{F} = \varphi\nabla\psi$ and use

$\nabla \bullet \mathbf{F} = \nabla \bullet [\ \varphi\nabla\psi] = \nabla\varphi \bullet \nabla\psi\ + \varphi\nabla^2\psi$ // a standard vector identity

Then you get **"Green's First Identity"** which therefore really has nothing new to say:

$$\int \nabla\varphi \bullet \nabla\psi\ dV\ + \int\ \varphi\nabla^2\psi\ dV\ = \int\ \varphi\nabla\psi \bullet d\mathbf{A}\ = \int\ \varphi(\partial\psi/\partial n)\ dA$$

where $d\mathbf{A} = dA\ \mathbf{n}$ and n is a normal unit vector to the surface at location dA, and

$\nabla\psi \bullet \mathbf{n}\ = \partial\psi/\partial n$. // Just imagine coordinates x,y,z with $\hat{\mathbf{z}} = \hat{\mathbf{n}}$ so $\nabla\psi \bullet \mathbf{n} = \partial\psi/\partial z$

Write the first identity with the two functions swapped and subtract to get

$$\int\ (\varphi\nabla^2\psi - \psi\nabla^2\varphi)\ dV\ = \int\ [\ \varphi(\partial\psi/\partial n) - \ \psi(\partial\varphi/\partial n)\ ]\ dA$$

and this is **"Green's Second Identity"** or theorem.

So, page 134 A is just this identity applied to our two functions $u_{\mathbf{k}}(\mathbf{r})$ and $u_{-\mathbf{k}'}(\mathbf{r})$ and the surface is taken to be a sphere, so $\hat{\mathbf{n}} = \hat{\mathbf{r}}$.

Now do this:

$$u_{\mathbf{k}}(r) = e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r$$

$$\partial_{\mathbf{r}}\, u_{\mathbf{k}}(r) = ik\cos\theta\, e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})\,[\,-1/r + ik]\, e^{ikr}/r$$

$$u_{-\mathbf{k}'}(r) = e^{-ikr\cos\theta'} + f(\mathbf{k_r},-\mathbf{k}')e^{ikr}/r$$

$$\partial_{\mathbf{r}}\, u_{-\mathbf{k}'}(r) = -\,ik\cos\theta'\, e^{-ikr\cos\theta'} + f(\mathbf{k_r},-\mathbf{k}')\,[\,-1/r + ik]\, e^{ikr}/r$$

where $\cos\theta = \hat{\mathbf{r}}\bullet\hat{\mathbf{k}}$ and $\cos\theta' = \hat{\mathbf{r}}\bullet\hat{\mathbf{k}}'$. So the (...) in 134 A is this:

$$(..) = [e^{-ikr\cos\theta'} + f(\mathbf{k_r},-\mathbf{k}')e^{ikr}/r] * [ik\cos\theta\, e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})\,[\,-1/r + ik]\, e^{ikr}/r]$$

$$- [e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r] * [-\,ik\cos\theta'\, e^{-ikr\cos\theta'} + f(\mathbf{k_r},-\mathbf{k}')\,[\,-1/r + ik]\, e^{ikr}/r$$

and since large r, we neglect the $1/r^2$ terms and get this slightly simpler result.

$$(..) = [e^{-ikr\cos\theta'} + f(\mathbf{k_r},-\mathbf{k}')e^{ikr}/r] * [ik\cos\theta\, e^{ikr\cos\theta} + ik\, f(\mathbf{k_r},\mathbf{k})\, e^{ikr}/r]$$

$$- [e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r] * [-\,ik\cos\theta'\, e^{-ikr\cos\theta'} + ik\, f(\mathbf{k_r},-\mathbf{k}')\, e^{ikr}/r]$$

But now drop more $1/r^2$ terms as you multiply out, so

$$(..) = e^{-ikr\cos\theta'}\,[\; ik\cos\theta\, e^{ikr\cos\theta} + ik\, f(\mathbf{k_r},\mathbf{k})\, e^{ikr}/r] + [\; ik\cos\theta\, e^{ikr\cos\theta}\, f(\mathbf{k_r},-\mathbf{k}')e^{ikr}/r]$$

$$- e^{ikr\cos\theta}\; [\,-ik\cos\theta'\, e^{-ikr\cos\theta'} + ik\, f(\mathbf{k_r},-\mathbf{k}')\, e^{ikr}/r] - [-\,ik\cos\theta'\, e^{-ikr\cos\theta'} f(\mathbf{k_r},\mathbf{k})e^{ikr}/r]$$

The first two terms which don't involve f are

$$e^{ikr\cos\theta}\, e^{-ikr\cos\theta'}(ik\cos\theta + ik\cos\theta') = ik\,(\cos\theta + \cos\theta')\, e^{ikr(\cos\theta\, -\, \cos\theta')}$$

The two terms with $f(\mathbf{k_r},\mathbf{k})$ are these:

$$e^{-ikr\cos\theta'}\, ik\, f(\mathbf{k_r},\mathbf{k})\, e^{ikr}/r + ik\cos\theta'\, e^{-ikr\cos\theta'} f(\mathbf{k_r},\mathbf{k})e^{ikr}/r$$

$$= e^{-ikr\cos\theta'}\, ik\, f(\mathbf{k_r},\mathbf{k})\, e^{ikr}/r\; [\; 1 + \cos\theta'\,] = ik/r\,(1 + \cos\theta')\, f(\mathbf{k_r},\mathbf{k})e^{ikr(1-\cos\theta')}$$

The two terms with $f(\mathbf{k_r},-\mathbf{k}')$ are these:

$ikcos\theta\ e^{ikrcos\theta}\ f(\mathbf{k_r},\mathbf{-k'})e^{ikr}/r - e^{ikrcos\theta}\ ik\ f(\mathbf{k_r},\mathbf{-k'})\ e^{ikr}/r]$

$= e^{ikrcos\theta}\ ik\ f(\mathbf{k_r},\mathbf{-k'})\ e^{ikr}/r[\ cos\theta - 1] = -\ ik/r\ (1 - cos\theta)\ f(\mathbf{k_r},\mathbf{-k'})e^{ikr(1+cos\theta)}$

So here is my total result for (...)

$$(..) = ik\ (cos\theta + cos\theta')\ e^{ikr(cos\theta\ -\ cos\theta')} + ik/r\ (1 + cos\theta')\ e^{ikr(1-cos\theta')}\ f(\mathbf{k_r},\mathbf{k}) - ik/r\ (1 - cos\theta)\ e^{ikr(1+cos\theta)}\ f(\mathbf{k_r},\mathbf{-k'})$$

and this agrees with B except I have already thrown out those $1/r^2$ terms they show.

Now we want to show that the integral of the first term is exactly 0. To compute just the first term of the first term above, let $\theta = \theta_{\mathbf{r}}$ = the integration variable, and then θ' is just some constant angle. Then that integral is

$$\sim\ \int_{-1}^{1} dz\ z\ e^{ikrz} = \int_{-1}^{1} dz\ z\ sin(krz)$$

The second term has the same integral, but has a minus sign in the expo, so we get

$$\sim\ \int_{-1}^{1} dz'\ z\ e^{-ikrz'} = -\ \int_{-1}^{1} dz'\ z'\ sin(krz')$$

So the two terms are equal and opposite and therefore cancel exactly. Even if they didn't, you could argue that each term vanishes anyway for large r, but we don't have to assume large r here.

So here we are with

$$(..) = + ik/r\ (1 + cos\theta')\ e^{ikr(1-cos\theta')}\ f(\mathbf{k_r},\mathbf{k}) - ik/r\ (1 - cos\theta)\ e^{ikr(1+cos\theta)}\ f(\mathbf{k_r},\mathbf{-k'})$$

$$= ik/r\ \{\ (1 + cos\theta')\ e^{ikr(1-cos\theta')}\ f(\mathbf{k_r},\mathbf{k}) - (1 - cos\theta)\ e^{ikr(1+cos\theta)}\ f(\mathbf{k_r},\mathbf{-k'})\ \}$$

and so our result A now reads

$$0 = ikr \int d\Omega\ \{\ (1 + cos\theta')\ e^{ikr(1-cos\theta')}\ f(\mathbf{k_r},\mathbf{k}) - (1 - cos\theta)\ e^{ikr(1+cos\theta)}\ f(\mathbf{k_r},\mathbf{-k'})\ \}$$

and finally after a LOT of fiddling, we end up with 20.13

Now more work lies ahead. Consider the first term in 20.13 above and set $w = cos\theta'$ = integration θ. This means we are saying $\hat{\mathbf{r}} = (\theta',\varphi') = \hat{\mathbf{k}}'$ which means $\mathbf{k_r} = \mathbf{k}'$. The general form is 20.14 with $F = ik(1+w)f(\mathbf{k}',\mathbf{k})$. I can see just by looking that this integral vanishes unless $w = 1$. We do a formal treatment and claim that only the "first parts" will remain. The second term in C will go away because as we do more parts on it, we keep picking up 1/r factors, so it is effectively zero and we get only "the parts", and this is **20.15**. So the first term T1 in 20.13 evaluates to this:

$$\int d\varphi' \, (i/k)\, F(1,\varphi') = (i/k) \int d\varphi' \, F(1,\varphi') \;=\; (i/k) \int d\varphi' \, \{\, ik(1+1)\, f(\mathbf{k}', \mathbf{k}) \,\}$$

Now he claims that "the integrand" does not depend on φ' , but it sure looks like it does. Well, you have to think of $\mathbf{k}' = k\,\hat{\mathbf{z}}'$ so k' points along the z axis of the integration variables θ',φ' and so this k' does not depend at all on φ', nor of course does **k** . So our result is then - 4π f(**k**' ,**k**).

What then happens with the second term? We do everything exactly the same but this time F = ik(1-w)f and we will have 1+w in the exponent in C. In this case the version of 20.15 will be

$$-\, i/k \,[\, F(1,\varphi) e^{2ikr} - F(-1,\varphi) \,]$$

and we will then keep the second term. The leading minus comes because the parts ∂_w has opposite sign due to the exponent $e^{ikr(1+w)}$. So our result here will be

$$\int d\varphi \, (-i/k)\, (-\, F(-1,\varphi)) = -\int d\varphi \, (-i/k) \, \{-ik(1+1)\, f(\mathbf{k}_r, -\mathbf{k}')\} = +4\pi \, f(-\mathbf{k}, -\mathbf{k}')$$

In this term, with w=-1 we have k_r pointing in the $-\hat{\mathbf{z}}$ direction, so $\mathbf{k}_r = -\mathbf{k}$ etc etc.

And now all the dust has settled, and here is what pops out:

$$f(\mathbf{k}', \mathbf{k}) = f(-\mathbf{k}, -\mathbf{k}') \qquad \textbf{20.16} \qquad \text{"the reciprocity theorem"}$$

In the scattering amplitude, you can swap and negate the momenta and it stays the same.

The conclusion used the nonrelativistic SE and you were allowed to have a complex V. If I draw a little picture, it just **looks like time reversal symmetry to me**, but he says nothing about that here. But he does comment on this exact point in passing on pages 230 and 312 where he talks time reversal. He even has comments about absorption maybe *not* being reversible, due to heat generated say, so that might somehow conflict with what we have just shown above. But in our complex-V model, it seems to be completely true, so I guess our complex-V model does not model "heat" properly.

Parity Theorem. Suppose V(-**r**) = + V(**r**) which means it has even parity. This says that all results must be the same if you " reflect the entire experiment through the origin". But such reflection would cause **k** and **k**' to be replaced by their negatives. So we would say f(-**k**,-**k**') = f(**k**,**k**').

In this case, the reciprocity theory says f(**k**' ,**k**) = f(**k**,**k**'), meaning you just swap without inverting and f stays the same.

Generalized Optical Theorem (135)

We are instructed now to repeat the entire sequence above with this replacement:

$$u_{-\mathbf{k}'} \longrightarrow u_{\mathbf{k}'}^{*}$$

This is where my cut and paste machinery should sparkle! But the opener is this:

$-\hbar^2/2m\ \nabla^2 u_k = (E-V)\ u_k \qquad x \quad u_{k'}^*$

$-\hbar^2/2m\ \nabla^2 u_{k'}^* = (E-V^*)\ u_{k'}^* \qquad x \quad u_k$

Before I divided the two, but here we have to shuffle differently. Multiply each as shown above, then subtract so the E terms cancel, and get

$$-\hbar^2/2m\ [u_{k'}^*\nabla^2 u_k - u_k \nabla^2 u_{k'}^*] = -\ u_k\ u_{k'}^*\ [\ V - V^*\ ] = -\ u_k\ u_{k'}^*\ 2iIm(V) = +2i\ V_I\ u_k\ u_{k'}^*$$

$$\Rightarrow\ [u_{k'}^*\nabla^2 u_k - u_k \nabla^2 u_{k'}^*] = -\ 4mi/\hbar^2\ V_I\ u_{k'}^*\ u_k \qquad \text{// which is } \mathbf{p\ 136\ A}$$

So compared with previous, we have some * on the left, and a non-zero term on the right. What rule would convert my previous results for the LHS to the new form? I tried some "rules", but it is just too hard to track things, easier to "just do it all again" .

But first, result 136 B is just our same application of Green #2 to A. We start with a volume integral on everything, then use Green 2 to convert the first 2 (of the 3) terms. So the V_I term is just the original volume integral and it sits also in 20.19 as is.

Now do this:

$$u_k(r) = e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r$$

$$\partial_r\ u_k(r) = ik\cos\theta\ e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})\ [\ -1/r + ik]\ e^{ikr}/r$$

$$u_{k'}^*(r) = e^{-ikr\cos\theta'} + f^*(\mathbf{k_r},\mathbf{k'})e^{-ikr}/r$$

$$\partial_r\ u_{k'}^*(r) = -\ ik\cos\theta'\ e^{-ikr\cos\theta'} + f^*(\mathbf{k_r},\mathbf{k'})\ [\ -1/r - ik]\ e^{-ikr}/r$$

where $\cos\theta = \hat{\mathbf{r}}\bullet\hat{\mathbf{k}}$ and $\cos\theta' = \hat{\mathbf{r}}\bullet\hat{\mathbf{k}}'$. So the (...) in 136 B is this:

$$(..) = [e^{-ikr\cos\theta'} + f^*(\mathbf{k_r},\mathbf{k'})e^{-ikr}/r]\ *\ [ik\cos\theta\ e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})\ [\ -1/r^2 + ik/r]\ e^{ikr}]$$

$$-\ [e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r] \quad *\ [-\ ik\cos\theta'\ e^{-ikr\cos\theta'} + f^*(\mathbf{k_r},\mathbf{k'})\ [\ -1/r^2 - ik/r]\ e^{-ikr}\ ]$$

and since large r, we neglect the $1/r^2$ terms and get this slightly simpler result.

$$(..) = [e^{-ikr\cos\theta'} + f^*(\mathbf{k_r},\mathbf{k'})e^{-ikr}/r]\ *\ [\ \ ik\cos\theta\ e^{ikr\cos\theta} + ik/r\ f(\mathbf{k_r},\mathbf{k})\ e^{ikr}\ \ ]$$

$$-\ \ [e^{ikr\cos\theta} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r] \quad *\ [-\ ik\cos\theta'\ e^{-ikr\cos\theta'} - ik/r\ f^*(\mathbf{k_r},\mathbf{k'})\ e^{-ikr}\ ]$$

But now drop more $1/r^2$ terms as you multiply out, so

$$(..) = e^{-ikr\cos\theta'}[\ \ ik\cos\theta\ e^{ikr\cos\theta} + ik/r\ f(\mathbf{k_r},\mathbf{k})\ e^{ikr}\ \ ] + [\ ik\cos\theta\ e^{ikr\cos\theta}\ f^*(\mathbf{k_r},\mathbf{k'})e^{-ikr}/r]$$

$$-\ e^{ikr\cos\theta}\ \ [-\ ik\cos\theta'\ e^{-ikr\cos\theta'} - ik/r\ f^*(\mathbf{k_r},\mathbf{k'})\ e^{-ikr}\ ] - [-\ ik\cos\theta'\ e^{-ikr\cos\theta'} f(\mathbf{k_r},\mathbf{k})e^{ikr}/r]$$

The first two terms which don't involve f are [exactly the same as before !]

$$e^{ikr\cos\theta}\, e^{-ikr\cos\theta'}(ik\cos\theta + ik\cos\theta') = ik\,(\cos\theta + \cos\theta')\, e^{ikr(\cos\theta - \cos\theta')}$$

The two terms with $f^*(\mathbf{k_r},\mathbf{k'})$ are these:

$$ik\cos\theta\, e^{ikr\cos\theta}\, f^*(\mathbf{k_r},\mathbf{k'})e^{-ikr}/r + e^{ikr\cos\theta}\, ik/r\, f^*(\mathbf{k_r},\mathbf{k'})\, e^{-ikr}]$$

$$= e^{ikr\cos\theta}\, ik/r\, f^*(\mathbf{k_r}, \mathbf{k'})\, e^{-ikr}\,(\cos\theta +1) = ik/r\,(\cos\theta +1)\, f^*(\mathbf{k_r}, \mathbf{k'})\, e^{-ikr(1-\cos\theta)}$$

which agrees with the corresponding term in 20.19 apart from the $1/r^2$ I have dropped.

The two terms with $f(\mathbf{k_r},\mathbf{k})$ are these:

$$e^{-ikr\cos\theta'}\; ik/r\, f(\mathbf{k_r},\mathbf{k})\, e^{ikr} + ik\cos\theta'\, e^{-ikr\cos\theta'} f(\mathbf{k_r},\mathbf{k})e^{ikr}/r$$

$$= ik/r\,(\cos\theta' +1)\, f(\mathbf{k_r},\mathbf{k})\; e^{ikr(1-\cos\theta')}$$

which also agrees with its corresponding term in 20.19. So, I have now derived 20.19 all terms, except my derivation threw out all $1/r^2$ terms in 20.19: one in each bracket, then the last one. BUT, we need to keep this last term because it is $1/r^2$ without a fast phase! From above, the ff^* terms are

$$f^*(\mathbf{k_r},\mathbf{k'})e^{-ikr}/r * ik/r\, f(\mathbf{k_r},\mathbf{k})\, e^{ikr} + f(\mathbf{k_r},\mathbf{k})e^{ikr}/r\;\; ik/r\, f^*(\mathbf{k_r},\mathbf{k'})\, e^{-ikr}$$

$$=2ik/r^2\, f^*(\mathbf{k_r},\mathbf{k'})f(\mathbf{k_r},\mathbf{k})$$

that is to say, the two contributions are the same so we get a factor of 2. This agrees with the last term in 20.19. **This concludes our verification of 20.19.**

Since the first term is the same as before, it again integrates to 0.

Now consider the first term in 20.19 that is left, and compare with the corresponding term in 20.13. It looks to me to be exactly the same, so the result will again be $-4\pi f(\mathbf{k'},\mathbf{k})$, and that is exactly what we see in 20.20. Remember that for this term, we integrated over the $\mathbf{k'}$ angles so $\mathbf{k_r} = \mathbf{k'}$ for the first argument of f.

The second term seems very similar to the first term but there is a sign change in the exponent. This will cause an extra minus sign when we do the first parts as in the pencil work at the bottom of page 134, so the result will be $+ 4\pi f^*(\mathbf{k},\mathbf{k'})$. In this case, we are integrating over the $\mathbf{k}$ angles so $\mathbf{k_r} = \mathbf{k}$ for the first argument. So that is why the $\mathbf{k}$ and $\mathbf{k'}$ arguments of f have reverse positions in the first two terms of 20.20.

The remaining two terms in 20.20 are just copied down from 20.19. Notice that all terms are on the same footing in terms of power of 1/r. So we are reminded that we were only allowed to throw out the bracket $1/r^2$ terms because of the high speed phase multiplying those terms and I guess the stationary phase method would cause them to end up being 1/r on the footing of 20.20. In 20.13 we threw out the ff $1/r^2$ term as well because it also had a high speed phase.

So this concludes our derivation of 20.20. If I were doing this all over on my own, I would be clearer about the details, but I am here following the lead of Schiff and just checking his equations.

Now for the moment assume that V = real so the last term in 20.20 is zero. We then get

$4\pi\,[f^*(k,k') - f(k',k)] = -2ik * (\text{ff integral})$

$\Rightarrow \quad (\text{ff integral}) = 4\pi/(-2ik)\,[\,] = 2\pi i/k\,[\,]$ // which is 20.21.

This is quite different from our earlier "optical theorem" involving the forward direction f. This still allows for a general real V of the form V(**r**) which may not be azimuthally symmetric, but Schiff keeps failing to mention that fact.

Now again suppose V(-**r**) = + V(**r**). Then we can use our simplified reciprocity theorem to swap the two arguments of f with no change. Then RHS 20.21 becomes

$2\pi i/k\,[\,-f(\mathbf{k}',\mathbf{k}) + f^*(\mathbf{k}',\mathbf{k})] = -2\pi i/k\,(f - f^*) = -2\pi i/k * 2i\,\mathrm{Im}(f) = 4\pi/k\ \mathrm{Im}\,[\,f(\mathbf{k}',\mathbf{k})\,]$

and so we have verified 20.22.

Schiff wraps this little subsection with a comment that in Born Approx the above theorem shows that the imaginary part of f is second order in the perturbation theory coupling constant, whereas f itself is first order. I think this is just a standard unitarity result, nothing dramatic.

This reminds me suddenly that this Schiff book has a lot more to say about scattering beyond this chapter that I am now reading. A later chapter is called "Approximation Methods in Collision Theory", so I may have several more weeks of reading to do before I return to BD I on this stack pushdown. In any event, I do want to finish Chapter 5 here and then see what comes next.

Notice that Schiff got the entire section above in 1.5 pages. He is quite dense as the web person commented, just providing cairns for the teacher/reader.

Optical Theorem (137).

Now we set **k** = **k**' in 20.22 so we are talking "the forward direction" . In the f*f integral we can select $\mathbf{k}_r$ = **k** and the integrand $|f(\mathbf{k},\mathbf{k})|^2$ has no azi dependence, so get exactly the LHS of 132A which we know is just $\sigma_{elastic}$. So, if V = real, we would find that $\sigma_{el} = 4\pi/k$ Im(f[forward]) which is exactly what we got in page 120 19.14.

But suppose we make V be complex again. Then we have to include the V_I term in 20.20. So go back there again and redo what we did before with this term included:

$4\pi\,[f^*(k,k') - f(k',k)] = -2ik * (\text{ff integral}) - 4i\mu/\hbar^2 \int V_I\ P\ dV$

$\Rightarrow \quad (\text{ff integral}) = 4\pi/(-2ik)\,[\,] + (4i\mu/\hbar^2)/(-2ik) \int V_I\ P\ dV$

$= 2\pi i/k\,[\,] - 2\mu/(k\hbar^2) \int V_I\ P\ dV$

so this shows the extra term we have to add to the RHS of 20.21. But

$2\mu/(k\hbar^2) = (2/\hbar)\,(\mu/p) = (2/\hbar v)$

so the above becomes, according to 20.3 page 131,

(ff integral) $= 2\pi i/k\,[\,] - \sigma_{abs}$

and as before we have (see line above)

$2\pi i/k\,[\,] = 4\pi/k\ \mathrm{Im}\,[\,f(\mathbf{k},\mathbf{k})\,]$

so we then have

(ff integral) $= 4\pi/k\ \mathrm{Im}\,[\,f(\mathbf{k},\mathbf{k})\,] - \sigma_{abs}$

which says

$\sigma_{el} + \sigma_{abs} = 4\pi/k\ \mathrm{Im}\,[\,f(\mathbf{k},\mathbf{k})\,] = \sigma_{tot}$ // verifies **20.23**

I know this is just unitarity speaking somehow, so I am not surprised by Schiff's comments in the last large paragraph on page 137. He outlines a way to derive the optical theorem without the use of a potential, and comments it is also true relativistically (without proof).

The entire section top of page 138 summarizes a paper Schiff wrote in 1954, reference given. The main general point is that it is **interference** between the incoming plane wave and the scattered wave **in the shadow region** behind the scatterer that reduces the plane wave flux there, and this reduction feeds the radial scattering and the absorption, and this reduction amount is given by 4π/kIm(f(forward)), which is what the optical theorem says. He has some extra comments about the geometric shadow I think and other things which I will let pass for the moment. Yes, the region of interference is in fact a small forward cone, not a geometric parallel shadow.

21. Coulomb Scattering (138)

A. Coulomb Scattering in Parabolic Coordinates

This ought to be an important section for me because it will somehow be the non-relativistic version of QED. I did classical Coulomb scattering in Goldberg, and the entire BD Volume 1 is about the relativistic scattering using the Dirac equation and the Feynman rules. I have no idea right now why "parabolic coordinates" enter the picture, I will probably have to backtrack on that. This section 21 is the last of this long chapter. (for which I already have 46 pages of notes here!).

[At this point I stopped 11/30 and spent 12/1-6 reading and annotating *all* of Schiff up to this point. That is, I fully read and dealt with everything in Chapters 1,2,3,4. I think in retrospect that was a good idea, lots of holes got filled. In particular, I did the parabolic coordinate separation of the H atom in Chap 4 so I am familiar with those coordinates now. I also started a whole-book Meta review document and it is up to this point as well. Now noon Sat 12.6.08.]

Observation: Coulomb scattering is a very messy subject and finally I see where the notion of a "Coulomb Wave Function" is going to arise. I have *never* done this subject before, ever! Right off the bat, Schiff is going to use some W_1 and W_2 functions which have no name and no definition. so we are going to have to go off and learn what is going on mathematically. What references do I have:

Messiah treats Coulomb scattering on p 421 of vol 1 and it is pretty much the same as Schiff's, but at least Messiah provides an appendix which mentions the W_i functions on page 480, but it is pretty brief.

M&M have nothing.

Goldberger Watson have stuff on page 259 and refer to W&W for the W functions.

Bateman looks good in the special functions section.

G&R at least have some discussion as well on page 1057.

[At this point I spent 12/6→12/8 figuring out these W functions from Messiah's appendix, and I wrote three small documents! First is "contours and cuts.doc" where I cleared up my rusty memory of winding numbers and how you can deform cuts; second is "Laplace's method" which provides the basic integral representation for the confluent hypergeometric series which solves the corresponding ODE; third is "Messiah W functions", which involves contour deformations to obtained desired asymptotic forms. This was painful and took very many hours, and no motion at all occurred in Schiff during this time.]

[Eventually I learned all about the "$W_1$and W_2" functions from first principles, so the Messiah strange method of contours was no longer needed. These W_i are discussed below.]

Ansatz. The section on page 139 begins very mysteriously. Instead of using form 18.10 for e^{ikz} plane wave plus a scattered wave $f\, e^{ikr}/r$, we are going to *try* a *single form* $u_c = e^{ikz}\, f(\mathbf{r})$ for the whole thing, both terms. We will then take the asymptotic limit of this single form and we will find two terms that are similar to 18.10.

Reminder for the next section:

$x = r\sin\theta\cos\varphi = \sqrt{\xi\eta}\,\cos\varphi$	$\xi = r-z$	$r = (\eta + \xi)/2$
$y = r\sin\theta\sin\varphi = \sqrt{\xi\eta}\,\sin\varphi$	$\eta = r+z$	$\cos\theta = z/r = (\eta - \xi)/(\eta+\xi)$
$z = r\cos\theta = (\eta - \xi)/2$	$\tan\varphi = y/x$	$\sin\theta = 2\sqrt{\xi\eta}\ /\ (\eta+\xi)$

Arguments about the assumed ansatz functional form. On page 96 we wrote the SE for V= 1/r in parabolics. We then found that the equation separates if we assume a solution of the form p 96 A. Our little ansatz says this: (no φ dependence in scattering situation)

$$u_c = e^{ikz}\ H(\xi,\eta,\varphi) = e^{ik(\eta-\xi)/2}\, H(\xi,\eta) = e^{ik\eta/2}\, e^{-ik\xi/2}\, H(\xi,\eta)$$

But we know this will separate into this general form:

$$= (\, e^{-ik\xi/2}\, f(\xi)\,)(e^{ik\eta/2}\, g(\eta)\,) = F(\xi)G(\eta) = e^{ikz}\, f(\xi)\, g(\eta)$$

where F and G must then satisfy the separated equations 16.30 (written in terms of F and G). [Read carefully from here on!] We shall compute below that the resulting equation for f is 21.3 where we

replaced Z by = - ZZ' and where ν = – ik/2. We can arrive at an equation for g that is like 21.3 by setting Z = 0 (so n=0) and changing the sign of k. Thus we get

$\eta g'' + (1+ik\xi)g' - 0 = 0$ // 21.3 version for g.

But such an equation allows for the solution g(η) = 1 because it has only derivatives! Thus, one solution to our problem is then just $e^{ikz}f(\xi)$ which is what we get from above setting g(η) = 1. Now suppose f(ξ) contains a term in the large-r limit that has the form $f(\xi) \sim e^{ik\xi}$. Then in this limit we have, since ξ = r-z, we would get

$$e^{ikz}f(\xi) \sim e^{ikz}\, e^{ik\xi} = e^{ikz}\, e^{ik(r-z)} = e^{ikr}$$

and this is correct for outbound scattering. This lends "support" that the solution we happened to select above is in fact the right one to describe scattering, ie, it meets this scattering boundary condition, assuming that f(ξ) does in fact have such a term in its large-r limit (which we will soon find it does).

Derivation of Equation 21.3 and finding ν // promised in paragraph above

Our starting point then will be the SE Coulomb problem in parabolics which we know is 16.26. But we are going to try a solution of this form

$$u = e^{ikz}\, f(r-z) = e^{ik(\eta-\xi)/2}\, f(\xi)$$

I think it is easier to first write out the derivative groupings like this

$$\partial_\eta(\eta\, \partial_\eta u) = \eta\partial_\eta{}^2 u + \partial_\eta u$$

$$\partial_\eta u = ik/2\, u \qquad \partial_\eta{}^2 u = -(k/2)^2 u$$

$$\partial_\eta(\eta\, \partial_\eta u) = -\eta(k/2)^2 u + i(k/2)u = e^{ik(\eta-\xi)/2}\, [\, -\eta k^2/4 + ik/2\,]\, f$$

$$\partial_\xi(\xi\, \partial_\xi u) = \xi\partial_\xi{}^2 u + \partial_\xi u$$

$$\partial_\xi u = e^{ik(\eta-\xi)/2}\, \partial_\xi f - ik/2\, u$$

$$= e^{ik(\eta-\xi)/2}\, \partial_\xi f - ik/2\, e^{ik(\eta-\xi)/2}\, f = e^{ik(\eta-\xi)/2}\, [\partial_\xi f - ik/2\, f\,]$$

$$\partial_\xi{}^2 u = e^{ik(\eta-\xi)/2}\partial_\xi{}^2 f - ik/2\, e^{ik(\eta-\xi)/2}\partial_\xi f - ik/2\, \{\, e^{ik(\eta-\xi)/2}\, \partial_\xi f - ik/2\, u\, \}$$

$$= e^{ik(\eta-\xi)/2}\partial_\xi{}^2 f - ik/2\, e^{ik(\eta-\xi)/2}\partial_\xi f - ik/2\, e^{ik(\eta-\xi)/2}\, \partial_\xi f + (ik)^2/4\, u$$

$$= e^{ik(\eta-\xi)/2}\partial_\xi{}^2 f - ik\, e^{ik(\eta-\xi)/2}\partial_\xi f - k^2/4\, e^{ik(\eta-\xi)/2} f$$

$$= e^{ik(\eta-\xi)/2}\,[\partial_\xi^2 f - ik\,\partial_\xi f - k^2/4\, f]$$

$$\partial_\xi(\xi\,\partial_\xi u) = e^{ik(\eta-\xi)/2}\xi\,[\partial_\xi^2 f - ik\,\partial_\xi f - k^2/4\, f] + e^{ik(\eta-\xi)/2}\,[\partial_\xi f - ik/2\, f]$$

$$= e^{ik(\eta-\xi)/2}\,[\xi\,\partial_\xi^2 f - ik\,\xi\,\partial_\xi f - (\xi\, k^2/4 + ik/2)\, f + \partial_\xi f]$$

$$= e^{ik(\eta-\xi)/2}\,[\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\partial_\xi f - (\xi\, k^2/4 + ik/2)\, f]$$

Compute now the entire {..} in 16.26. We know there is no φ term. so we have

$$\{..\} = 4/(\xi+\eta)\;\left(\;[e^{ik(\eta-\xi)/2}\,[\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\partial_\xi f - (\xi\, k^2/4 + ik/2)\, f] + e^{ik(\eta-\xi)/2}\,(-\eta k^2/4 + ik/2)\, f\;\right)$$

$$= e^{ik(\eta-\xi)/2}\;4/(\xi+\eta)\;(\;\;\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\partial_\xi f - (\xi\, k^2/4 + ik/2)\, f - \eta k^2/4\, f + ik/2 f\;\;)$$

$$= e^{ik(\eta-\xi)/2}\;4/(\xi+\eta)\;(\;\;\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\partial_\xi f - k^2/4(\xi+\eta)\, f\;\;)$$

I will replace Z with -ZZ' to generalize (hold on that) and then 16.26 says this

$$(-\hbar^2/2\mu)\;\{....\} = (E - 2ZZ'e^2/(\xi+\eta)\; e^{ik(\eta-\xi)/2}\, f$$

Mult through by ξ+η and cancel the exp to get

$$2(-\hbar^2/\mu)\;(\;\;\partial_\xi^2 f - ik\,\partial_\xi f - k^2/4(1+\eta)\, f\;\;) = [\,E(\xi+\eta) - 2ZZ'e^2\,]\, f$$

$$(\;\;\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\partial_\xi f - k^2/4(\xi+\eta)\, f\;\;) = -\mu/(2\hbar^2)\,[\,E(\xi+\eta) - 2ZZ'e^2\,]\, f$$

$$= -\mu/(2\hbar^2)\;\; E(\xi+\eta) f + (\mu ZZ'e^2/\hbar^2)\, f$$

$$= -\mu/(2\hbar^2)\;\; E(\xi+\eta) f + kn)\, f \qquad\qquad n = \text{page } 138$$

I guess even though 1/r, we still set $E = \hbar^2 k^2/2\mu$ so that

$$\mu/(2\hbar^2)\; E = \mu/(2\hbar^2)\;\hbar^2 k^2/2\mu = k^2/4$$

so we then have

$$(\;\;\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\,\partial_\xi f - k^2/4(\xi+\eta)\, f\;\;) = -\,k^2/4(\xi+\eta) f + kn\, f$$

$$\xi\,\partial_\xi^2 f + (1 - ik\,\xi)\partial_\xi f - kn\, f = 0 \qquad\qquad \text{// } \textbf{verifies 21.3}$$

Notice in the line before last the astounding cancellation of two big terms such that η is then completely gone from the resulting equation. Considering all the math here, the result is amazingly simple. On the other hand, as shown in the comment above the derivation we just finished, this fact is reasonable since we already know that the SE separates in parabolics.

Does the above tell us anything about the value of ν ? Since we used 16.26 directly, there was no ν involved. But maybe we can compute it. In the above solution, we have $G(\eta) = e^{+ik\eta/2}$. Thus

$G = e^{-ik\eta/2}$ $G' = (ik/2)G$ $G'' = (ik/2)^2 G$

$[\eta G']' = \eta G'' + G' = \eta(ik/2)^2 G + (ik/2)G = -k^2\eta/4\, G + (ik/2)G$

According to 16.30b, the above is equal to this (m = 0 azimuth)

$(-k^2\eta/4 - \nu)G$

Comparing, the first terms cancel and we find that ν = – ik/2. This agrees with another time I did this calculation in my "radial equation" doc.

Derivation of 21.5 (and in so doing, show that 21.4 is compatible with 21.3)

Now write

$z = ik\xi$ $dz = ikd\xi$ $\partial_z = (1/ik)\partial_\xi$ $\partial_z^2 = (1/ik)^2\partial_\xi^2$

$F(z) = f(\xi)$

Then start from **21.4** and see what we get

$(ik\xi)\,(1/ik)^2\partial_\xi^2 f + (b - ik\xi)\,(1/ik)\partial_\xi f - a\, f = 0$

$(ik\xi)\,(1/ik)\partial_\xi^2 f + (b - ik\xi)\,\partial_\xi f - aik\, f = 0$

$\xi\partial_\xi^2 f + (b - ik\xi)\,\partial_\xi f - aikf = 0$

so this has the form of **21.3**. Compare this to 21.3 and we find that

$b = 1$ $aik = nk \Rightarrow$ $ai = n \Rightarrow$ $a = -in$

so we expect our solution to be

$f(\xi) = F(-in, 1, ik\xi)$ // **verifies 21.5** // F = Φ or Bateman

All I know so far is this: We tried u of the form $u = e^{ikz} f(r-z) = e^{ik(\eta-\xi)/2} f(\xi)$, we stuffed this into the SE in parabolic coordinates with the 1/r potential, equation 16.24, we ended up with an ODE for f(ξ), and we solved that ODE and got the result above as a viable solution.

Comment added 12.20.08: What I just showed above was this: If you consider the Coulomb SE in parabolic coordinates ξ and η, as shown in equation 16.26 page 96, you might look for a separable solution in the form $u(\xi,\eta) = F(\xi)G(\eta) = (e^{-ik\xi/2} f(\xi))(e^{ik\eta/2} g(\eta))$. Separated equations 16.30 for F,G

become 21.3 for f, and its partner [which has n=0 and k→-k] for g, and it happens that ν = -ik/2. A possible solution for the partner equation is g = 1 which gives the entire solution in the form $e^{ik(\xi-\eta)/2}$ f(ξ) = e^{ikz} f(ξ). Equation 21.3 for f is equivalent to the confluent HG equation 21.4 with a = -in b = 1 and x = ikξ and we then have f(ξ) = C Φ(-in, 1, ikξ) where I used Bateman's Φ in place of F.

We might argue that this is the unique HG solution because ξ = r-z and we must have a solution which is well behaved at r = z which means cosθ = 1, that is, we don't want to get a diverging power of $\sin^2(\theta/2)$ as we approach this limit along the 3D entire line θ = 0 degrees from r=0 to r=∞. The nice function Φ = 1 along this line. Perhaps this argument is weak since it turns out that with the analytic continuation of the Φ function, we do in fact get diverging powers of $\sin^2(\theta/2)$ and these yield the correct Rutherford cross section.

I think a better argument for selecting just Φ is that its asymptotic limit gives exactly the right "setup for the scattering problem" with an incoming Coulomb Plane Wave. This is discussed in more detail in my "radial equation" doc.

Comment on Equation 21.7. (outdated, see 6" below)

These results are developed in the Messiah appendix and I have written this up in three separate documents because the topic was so complicated. One was "Laplaces method" which shows how you can create an integral representation for the solution of the confluent hypergeometric equation 21.4. The second is called "contours and cuts" which clarifies what the contour is that Messiah uses. The third is "Messiah W functions" which shows how that contour can be written as the sum of two contours, and each of these contours then defines a function W_r such that $W_1 + W_2 = F$ (HG series). Messiah then goes on to find the large argument behavior of these two functions, and THAT is what is then quoted in 21.7. It does seem odd that Schiff gives no references (well, see page 139, W&W as usual).

Comment on Equation 21.8 (outdated, see 6" below)

Now we shift gears and jump to 21.8 on page 140, having already done all the W_1 stuff elsewhere, see doc "messiah W functions.doc" and related docs. We know that W_1 and W_2 are both solutions of the ODE because each has a closed contour, so each will have no parts. We know that $W_1 + W_2 = F$, the nice power series solution analytic around z = 0. Therefore, we know that $W_1 - W_2$ will also be a solution, but it will involve the "second kind" solution that will likely be singular at r-z=0. On this grounds, I regard 21.8 as a reasonable definition of an "irregular solution".

Comments on 21.7 and 21.8 added 12.17.08: I *now know* these facts (see Messiah notes doc or Bateman notes doc)

$$W_1(a,c,x) = e^{+i\pi a}\,\Gamma(c)/\Gamma(c-a)\,\Psi(a,c,x)$$

$$W_2(a,c,x) = \Gamma(c)/\Gamma(a) * e^{i\pi(a-c)}\; e^{x}\,\Psi(c-a,c,-x)$$

I know from Bateman page 278 (1) that we can write Ψ(a,c,x) in a Frobenius-like series in 1/x which is just this (I quote Bateman)

$$\Psi(a,c,x) = x^{-a}\;\Sigma_{n=0}\,(a)_n(a-c+1)_n\,(-x)^{-n}/n!$$

If we define

$$g(A,B,z) = \Sigma_{n=0} (A)_n (B)_n z^{-n}/n! = 1 + AB/z + A(A-1)B(B-1)/2z^2 + \ldots = g(B,A,z)$$

then we have the following results, *which agree with* Schiff **21.7**,

$$\Psi(a,c,x) = x^{-a} \Sigma_{n=0} (a)_n (a-c+1)_n (-x)^{-n}/n! = x^{-a}\, g(a,a-c+1,-x)$$

$$W_1(a,c,x) = e^{+i\pi a}\, \Gamma(c)/\Gamma(c-a)\, \Psi(a,c,x) = \Gamma(c)/\Gamma(c-a) * (-x)^{-a}\, g(a,a-c+1,-x)$$

$$\Psi(c-a,c,-x) = (-x)^{-c+a} \Sigma_{n=0} (c-a)_n (-a+1)_n (x)^{-n}/n! = (-x)^{-c+a}\, g(c-a,-a+1,x)$$

$$W_2(a,c,x) = \Gamma(c)/\Gamma(a) * e^{i\pi(a-c)}\, e^x\, \Psi(c-a,c,-x) = \Gamma(c)/\Gamma(a) * (x)^{a-c}\, e^x\, g(c-a,-a+1,x)$$

I also know that these W_1 and W_2 are exactly the terms in the Bateman equation $\Phi \sim \Psi + \Psi'$ which is my shorthand for Bateman p259 (7) , so we have $\Phi = W_1 + W_2 = F$, as stated Schiff page 140 D. It is this Bateman form $\Phi \sim \Psi + \Psi'$ which provides the analytic continuation of the solution to a point where you have convergent forms for the asymptotic behavior.

Now in my "coulomb wave function " doc at the end I talk about GW stuff, and there I note that

$$Y \sim W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2}\, \Psi(1/2 + \mu-\kappa;\, 1+2\mu;\, z) \sim W_1$$
$$Y^* \sim W_{-\kappa,\mu}(-z) = e^{+z/2} z^{\mu+1/2}\, \Psi(1/2 + \mu+\kappa;\, 1+2\mu;\, -z) \sim W_2$$

There, we found two solutions called "coulomb wave functions" which are

$$F_L(AS) = 1/2\,[Y + Y^*] \rightarrow \sin(\theta_L) \sim W_1 + W_2$$
$$G_L(AS) = 1/(2i)\,[Y - Y^*] \rightarrow \cos(\theta_L) \sim -i\,[W_1 - W_2]$$

so in Schiff's **21.8**, he is just in passing mentioning this other G solution.

BUT, in my GW notes and in all my study of the coulomb wave functions, we were always dealing with these parameters (because there we were doing spherical coordinates, not parabolic)

$$a = L+1-i\eta \qquad c = 2L+2 \qquad \mu = L+1/2 \qquad \kappa = L+1 - (L+1-i\eta) = i\eta$$

where L was the index of "the partial wave", and where here η = means n. But in our solution here, although we can formally state all the same relationships for functions Φ, Ψ, W_1, W_2 F and G, we are using instead

$$a = -i\eta \qquad c = 1 \qquad \mu = 0 \qquad \kappa = 1/2 + i\eta$$

There is no L because in parabolic coordinates there are no partial wave sum, the solution is given as a single function instead of a sum over L.

Derivation of 21.9 and 21.10

Now, in parabolics we have r = (η + ξ)/2. I think we can regard η as some fixed value, and then identify large r with large ξ. So we now need the large ξ limit of 21.5. Let's then do it this way

$f(\xi) = F(a=-in, b=1, z=ik\xi)$ $\quad a-b+1 = -in \quad a = -in \quad b = 1 \quad z = ik\xi \quad a-b = -in - 1 \quad b-a = 1+in$
$1-a = 1+in$

$\alpha = a = -in \qquad \beta = a-b+1 = -in \qquad arg3 = -z = -ik\xi$
$W_1(-in, 1, ik\xi) = \Gamma(1)/\Gamma(1+in)\,(-ik\xi)^{in}\,[\ 1 + (-in)(-in)/(-ik\xi) + O(1/\xi^2)\ ] \qquad \Gamma(1)= 1$

$W_2(-in, 1, ik\xi) = \Gamma(1)/\Gamma(-in)\,e^{ik\xi}(ik\xi)^{-in-1}\,[\ 1 + (1+in)(1+in)/(ik\xi) + O(1/\xi^2)\ ]$

We can add these then to find that

$f(\xi) \rightarrow 1/\Gamma(1+in)\,(-ik\xi)^{in}[\ 1 + (-in)(-in)/(-ik\xi) + O(1/\xi^2)\ ]$

$+ 1/\Gamma(-in)\,e^{ik\xi}(ik\xi)^{-in-1}\,[\ 1 + (1+in)(1+in)/(ik\xi) + O(1/\xi^2)\ ]$

If we throw out the $1/\varepsilon^2$ terms, we get

$f(\xi) \rightarrow 1/\Gamma(1+in)\,(-ik\xi)^{in}[\ 1 + n^2/(ik\xi)\ ]$ $\qquad W_1$

$+ 1/\Gamma(-in)\,e^{ik\xi}(ik\xi)^{-in-1}\,[\ 1 + (1+in)^2/(ik\xi)\ ]$ $\qquad W_2$

Now set $\xi = (r-z)$ to see if we are getting anywhere

$f(\xi) \rightarrow 1/\Gamma(1+in)\,(-ik\,(r-z))^{in}[\ 1 + n^2/(ik\,(r-z))\ ]$ $\qquad W_1$

$+ 1/\Gamma(-in)\,e^{ik(r-z)}\,(ik\,(r-z))^{-in-1}\,[\ 1 + (1+in)^2/(ik\,(r-z))\ ]$ $\qquad W_2$

$u = e^{ikz}f(\xi) \rightarrow 1/\Gamma(1+in)\,\{e^{ikz}\,(-ik\,(r-z))^{in}\}[\ 1 + n^2/(ik\,(r-z))\ ]$ $\qquad W_1$

$+ 1/\Gamma(-in)\,e^{ikr}\,(ik\,(r-z))^{-in-1}\,[\ 1 + (1+in)^2/(ik\,(r-z))\ ]$ $\qquad W_2$

At least we get a desired e^{ikr} factor sitting somewhere in the result. Looking at 21.9 and 21.10, we have to ask: how does something called θ get in there, and what is θ ? Well $z = r\cos\theta$ where θ is our scattering angle in the CMS system. So as $r\rightarrow\infty$, we also have z getting large certainly in the forward direction.

Now let's take a few hacks at things in the W_1 term:

$(-ik\,(r-z))^{in} = (-i)^{in}\,[\,k(r-z)]^{in} = (-i)^{in}\exp\{\,in \ln [\,k(r-z)]\,\}$

Then we have

$$\{e^{ikz}\ (-ik\ (r-z))^{in}\} = (-i)^{in} \exp\{ikz + in \ln [\ k(r-z)]\ \} = (-i)^{in} \exp\{\ i[kz + n \ln(k(r-z))\ \}$$

and we can write

$$(-i)^{in} = (e^{-i\pi/2})^{in} = e^{+n\pi/2}$$

so our W_1 term becomes

$$1/\ \Gamma(1+in)\ \{e^{ikz}\ (-ik\ (r-z))^{in}\}[\ \ 1 + n^2/(ik\ (r-z))\ ]$$

$$= 1/\ \Gamma(1+in) * e^{+n\pi/2}\ \exp\{\ i[kz + n \ln(k(r-z))\ \}\ [\ \ 1 + n^2/(ik\ (r-z))\ ]$$

and this **exactly matches the first term in 21.9 for W_1**.

Now for the W_2 term, which so far is this:

$$1/\Gamma(-in)\ e^{ikr}\ (ik\ (r-z))^{-in-1}\ [\ \ 1 + (1+in)^2/(ik\ (r-z))\ ]$$

So we write

$$(ik\ (r-z))^{-in-1} = (i)^{-in-1}\ \ [\ k(r-z)]^{-in-1} = e^{(i\pi/2)(-in-1)} * e^{(-in-1)\ln[k(r-z)]}$$

$$= e^{+i\pi n/2}\ (-i)\ \ expi\ \{(-n+i)\ln[k(r-z)]\}$$

so we then have for the W_2 term

$$1/\Gamma(-in)\ e^{ikr}\ \ (-i\ e^{+n\pi/2})\ e^{-in\ln[k(r-z)]}\ /\ k(r-z)\ [\ \ 1 + (1+in)^2/(ik\ (r-z))\ ]$$

$$= \ 1/\Gamma(-in)\ e^{i\{kr-\ n\ln[k(r-z)]\}}\ \ (-i\ e^{+n\pi/2})\ \ /\ k(r-z)\ [\ \ 1 + (1+in)^2/(ik\ (r-z))\ ]$$

$$= 1/\Gamma(-in)\ expi\ \{\ kr-\ n\ln[k(r-z)]\ \}\ \ (-i\ e^{+n\pi/2})\ \ /\ k(r-z)\ [\ \ 1 + (1+in)^2/(ik\ (r-z))\ ]$$

Now finally since this will be the scattering term, set $z = r\cos\theta$

$r-z = r(1-\cos\theta) = 2r\sin^2(\theta/2)$ and then W_2 term becomes

$$= 1/\Gamma(-in)\ expi\ \{\ kr-\ n\ln[k\ 2r\sin^2(\theta/2)]\ \}\ (-i\ e^{+n\pi/2})\ \ /\ k\ 2r\sin^2(\theta/2)\ [\ \ 1 + (1+in)^2/(ik\ 2r\sin^2(\theta/2)\ \ \ )\ ]$$

Now write

$$n\ln[k\ 2r\sin^2(\theta/2)] = \ n\ln[2kr] + n\ln[\sin^2(\theta/2)]\ \ \ \text{to get}$$

$= 1/\Gamma(-in)\ \text{expi}\ \{\ kr - \text{nln}(2kr)\ \}/r\ \ \text{expi}\ \{-\text{nln}[\sin^2(\theta/2)]\ \}\ (-i\ e^{+n\pi/2})\ \ /\ k\ 2\sin^2(\theta/2)\ [\ 1 + ...]$

and now finally we have the correct radial factor showing in 21.9 which I call $e\cdots/r$. Looking then at 21.9, the following factors should comprise $f(\theta)$:

$1/\ \Gamma(1+in) * e^{n\pi/2} * f(\theta)$

$= \ 1/\Gamma(-in)\ \text{expi}\ \{-\text{nln}[\sin^2(\theta/2)]\ \}\ (-i\ e^{+n\pi/2})\ /\ k\ 2\sin^2(\theta/2)\ [\ \ 1 + (1+in)^2/(ik\ 2r\sin^2(\theta/2)\ \ \ )\ ]$

I think we are going to throw out the 1/r term in the [...] since we already have our 1/r for scattering. So this then gives us

$f(\theta) = \Gamma(1+in)/\ i\Gamma(-in) * \text{expi}\ \{-\text{nln}[\sin^2(\theta/2)]\ \}\ /\ k\ 2\sin^2(\theta/2)$

which **exactly agrees with 21.10 A**. Now write

$\Gamma(1-in)\ = -in\ \Gamma(-in)$ Schaum p 101 => $1/\ \Gamma(-in)\ = -in/\ \Gamma(1-in)$

so that

$\Gamma(1+in)/\ i\Gamma(-in)\ = \Gamma(1+in)/\ i\ \ * (-in)/\ \Gamma(1-in)\ = -\ n\ \Gamma(1+in)/\ \Gamma(1-in)$

Now $\Gamma(z)$ is real analytic, so we know that $\Gamma(1-in) = \Gamma(1+in)^*$. So write

$\Gamma(1+in)\ = Re^{i\eta}$ $\Gamma(1-in)\ = Re^{-i\eta}$

Then get

$\Gamma(1+in)/\ i\Gamma(-in) = \ -\ n\ \Gamma(1+in)/\ \Gamma(1-in)\ = -n\ e^{+2i\eta}\ \ = n\ e^{i\pi\ +2i\eta}$

and our result then becomes

$f(\theta) = n\ e^{i\pi\ +2i\eta} * \text{expi}\ \{-\text{nln}[\sin^2(\theta/2)]\ \}\ /\ k\ 2\sin^2(\theta/2)$

$= n\ \exp\{-in\ln[\sin^2(\theta/2)] + i\pi + 2i\eta_o\ \}\ /\ k\ 2\sin^2(\theta/2)$

which **verifies 21.10 B** which came from the W_2 term.

Comments on 21.9 and 21.10 added 12.17.08: What have we done here? We took $u = e^{ikz}f(\xi)$ where $f(\xi)$ is the confluent function $\Phi(-in, 1, ik\xi)$ which we can regard as $W_1 + W_2$ which are just the two Ψ terms in $\Phi = \Psi + \Psi'$. The variable here is $\xi = r-z = r-\cos\theta$. We then installed the large-r limit for the Ψ functions and we ended up with 21.9 + 21.10 from which we extract our Rutherford cross section formula.

The formula shown in 21.12 is the "unit flux" (see details below) normalized version of $e^{ikz}f(\xi)$ and it is referred to as a "coulomb wave function". But since this is in parabolic coordinates, this wave function has nothing to do with the "coulomb wave functions" F and G discussed in A&S Chapter 14.

This fact caused me much confusion at first. The A&S chapter relates to solving this same problem in spherical coordinates.

Review of where we stand at this point.

We are now at the bottom of page 140. What exactly have we shown here?

We are trying to do a quantum mechanics treatment of Coulomb scattering with its potential 1/r. We have the Schrodinger equation for the Coulomb case in 16.26 in parabolic coordinates. We make a hypothesis that our solution wavefunction u which solves this SE will have the form $u = e^{ikz}f(\xi)$ where $\xi = r-z$ is one of the parabolic coordinates. When this assumed form is plugged into parabolic SE 16.28, we get the new ODE 21.3 which is fairly simple, where symbol n = (parameters)/k and (parameters) = $\mu ZZ'e^2/\hbar^2$. We are somewhat amazed how the other parabolic variable η which appears in 16.28 magically cancels away in the result 21.3 which is of course an equation in the single variable ξ.

Pause for dimensions: Bohr radius in cgs is $\hbar^2/me^2$ so the grouping (parameters) has dimensions 1/r. Then **n is dimensionless** as expected, a dimensionless version of 1/k . The second term in 21.3 shows that the variable ξ **has dimensions of R**, which we know anyway since $\xi = r-z$. Each term in 21.3 is then dimensionless and **f is dimensionless**. This is the right way to solve a problem. All the ugly stuff is packed inside that dimensionless n. [In other of my documents n shows up as η]

Now, we look at our new SE for f which is 21.3, and we recognize it as the confluent hypergeometric equation and we know that a solution that is well behaved at $\xi = 0$ ($\cos\theta=1$) will be $f = F(-in, 1, ik\xi)$. We then use the work done in Messiah's appendix to compute the large-r limit of $u = e^{ikz}f = e^{ikz}F$. In computing this limit, we make use of the two separate functions W_1 and W_2 whose large argument behavior is given by the expansions in 21.7, which appear on page 482 of Messiah. These expansions were obtained in a complex sequence of operations which involved Laplace's Method applied to the Kummer ODE and manipulation of the integration contour to get convergence for large argument, and this is what led us to require those two components W_1 and W_2. [All later clarified with the Bateman connection.]

The large-r limit of our u of the assumed form $e^{ikz}f(r-z)$ is then displayed in 21.9 where we have kept all expansion terms of order 1/r. This meant keeping two terms in the W_1 expansion, and only the leading term in the W_2 expansion. We want large r because we are doing a scattering experiment. We can then compare the result for u to our older form which was this:

$$u = e^{ikz} + f(\theta)e^{ikr} / r$$

We then see that in the Coulomb case this does not apply, and instead we get this form

$$u = e^{ikz} * \varphi(r-z, k, n) + f(\theta)\, e^{i[kr-n\ln(2kr)]}/r$$

where we can think of the function φ as some kind of correction factor on the plane wave, and we pick up a ln(2kr) extra factor in our e^{ikr} phase. [see radial equations.doc for more on this]

Going way back to 19.3: Using 19.3 to predict our discovered large-r form of $R_\ell(r)$ for large r.

Now let's pause again to go back to page 117 and 19.3 and review what was going on there. We have an ODE for the radial function $\chi_\ell(r)$. We assume the form 19.3 and jam that assumed form into the ODE, and we get a little non-linear DE for function f as shown in 19.4, which I have verified. In this equation, we have $U(r) = (2\mu/\hbar^2)V(r) = (2\mu/\hbar^2)[\ ZZ'e^2/r\ ]$. But recall from a few paragraphs above that

$n = (\text{parameters})/k$ and $(\text{parameters}) = \mu ZZ'e^2/\hbar^2 = nk$

$\Rightarrow U(r) = (2\mu/\hbar^2)[\ ZZ'e^2/r\ ] = 2nk/r$

So our little DE for function f is this

$f' + f^2 + 2ikf = 2nk/r + \ell(\ell+1)/r^2$ // the + sign goes with e^{+ikr} in 19.3

Since we care about large r, in the Coulomb case we throw out the centrifugal term and we get

$f'(r) + f^2(r) + 2ikf(r) = 2nk/r$

Let's take as a trial solution $f(r) = A/r$. Then the first two terms will be $1/r^2$ and we discard them for large r, and the last term is $2ikA/r$ and we get

$2ikA/r = 2nk/r \Rightarrow \quad iA = n$ or $A = -in$ // look familiar?

Then our solution is

$f(r) = -in/r$

Then in 19.3 we integrate this to get

$$\exp\left[\ \int f(r)dr\ \right] = \exp[\ -in\ln(r) + K)] = e^K\ e^{-in\ln(r)}$$

and our solution for R becomes

$$R \sim e^{-in\ln(r)} * e^{ikr}/r = e^{i[kr\ -\ n\ln(r)]}/r$$

and **this gives us an early prediction** that for the Coulomb potential we are going to have a scattered wave that has this form. Our detailed calculation gives 21.9 which shows exactly this scattering phase in the exponential!

How did this little f equation work out for a faster-decaying potential? If you let $V = 1/r^2$ or any higher power, you still have the third f equation term dominating and you get $f(r) = -in/r^2$. But the difference now is this:

$$\exp\left[\ \int f(r)dr\ \right] = \exp[\ +in/2r + K)] = e^K \quad \text{as } r\to\infty$$

and then

$$R = \chi/r = A\, e^{K}\ e^{ikr}/r$$

So any solution f(r) whose integral $I[f(r)] \to 0$ as $r \to \infty$ will result in the standard e^{ikr}/r form. Thus, the Coulomb potential is a "special case" where this is not true, and we have to do all this work.

So here we are with our solution for Coulomb scattering which is this

$$u = e^{ikz} * \varphi(r\text{-}z, k, n) \quad + f(\theta)\, e^{i[kr - n\ln(2kr)]}$$

with $f(\theta)$ as given in 21.10.

Derivation of 21.11 and why $|f|^2$ is still the cross section.

Now, can we still argue that $|f|^2$ is the differential cross section. Let's go back and reexamine that argument. A book scan takes us equation 18.10 + 18.11 circa, which takes us to page 12 roughly in our current notes where we had:

$$j_z\, d\sigma = j_r\, (r^2 d\Omega)$$

where the two fluxes were given by

$$\mathbf{j_z} = \hbar/(2i\mu)\, [\, \psi^* \partial_z \psi - \psi \partial_z \psi^*\,] \qquad \psi = e^{ikz}$$

$$= \hbar/(2i\mu)\, [\, ik - (-ik)] = \hbar/(2i\mu) * 2ik = \hbar k/\mu = v$$

and

$$\mathbf{j_r} = \hbar/(2i\mu)\, [\, \psi^* \nabla_r \psi - \psi \nabla_r \psi^*\,] \qquad \psi = f(\theta,\varphi)(e^{ikr}/r)$$

$$= v\ |f|^2/r^2$$

So we really need to recompute both these fluxes to make sure that our $f(\theta)$ has the same meaning. We now have

$$\mathbf{j_z} = \hbar/(2i\mu)\, [\, \psi^* \partial_z \psi - \psi \partial_z \psi^*\,] \qquad \psi = e^{ikz} * \varphi(r\text{-}z, k, n) = e^{ikz} * e^{in\ln[k(r-z)]}$$

$$\mathbf{j_r} = \hbar/(2i\mu)\, [\, \psi^* \nabla_r \psi - \psi \nabla_r \psi^*\,] \qquad \psi = f(\theta,\varphi)(e^{ikr}/r) * e^{in\ln(2kr)}$$

Then we will have

$$d\sigma/d\Omega = (j_r/j_z)\, r^2$$

It is not obvious to me that the result will be the same, though it seems reasonable. We can see that the flux is proportional to derivatives wrt z or r. In either case, derivatives of the correction factors give 1/r terms which I think can be *dropped*. For example, consider

$$\psi = e^{ikz} * e^{inln[k(r-z)]}$$
$$\partial_z\psi = \partial_z(e^{ikz})\, e^{inln[k(r-z)]} + e^{ikz}\, e^{inln[k(r-z)]} * in\,(-k)/\{\, k(r-z)\,\}$$

so at large r, we can ignore the second term and we just get the original term times a phase which will cancel against the ψ^*. The exact same idea happens for the $\partial_r\psi$ flux.

In this case, we can then treat the correction factor just as a multiplicative phase $e^{i\delta}$ in the calculation. Then, looking at our computations earlier, we will get for example

$$\nabla_r\psi = e^{i\delta}\, f(\theta,\varphi)\, \nabla_r\, (e^{ikr}/r) = e^{i\delta}\, f(\theta,\varphi)\, (ik - 1/r)\, (e^{ikr}/r)$$

$$\psi^*\nabla_r\psi = |f|^2\, (ik - 1/r)(1/r^2)$$

and this phase has disappeared! The result of the calculation is the same and we then get the same result as before for j_r. The exact same argument applies to j_z with its new $e^{i\delta}$ phase. The fact that the δ phase depends on r does not prevent it from cancelling away when we apply ψ^*.

So now I am happy with the conclusion that $|f_c(\theta)|^2$ is still the differential cross section!

When we square 21.10, the messy phase goes away and form A is at once obvious. Now we can write

$$n/k = [\mu ZZ'e^2/\hbar^2 k]/k = \mu ZZ'e^2/[\hbar^2 k^2] = \mu ZZ'e^2/p^2 = \mu ZZ'e^2/[\mu v]^2 = ZZ'e^2/[\mu v^2]$$

Thus

$$(n/2k)^2 = \{\, ZZ'e^2/[2\mu v^2]\,\}^2 \qquad \text{// which verifies 21.11 B.}$$

So amazingly after all this very messy math, we end up with the Rutherford formula which agrees with Goldstein page 84. We have now done a QM derivation of this formula for the first time! Later in BD1 we see the relativistic version of this done in QED, but here we are happy with non-relativistic QM based on the SE. Here we used parabolic coordinates, but perhaps that was not necessary.

Derivation of 21.12 and getting "unit flux"

Now, how do we **normalize our "incident beam" to unit flux?** I think the "incident beam" is the first term in 21.9 where we ignore the little correction term say for large negative z. If my comments above are correct about how the messy expo just passes through the flux calculation, then:

$$\psi = e^{ikz} \qquad \Rightarrow$$

$$\mathbf{j}_z = \hbar/(2i\mu)\, [\, \psi^*\partial_z\psi - \psi\partial_z\,\psi^*\,] = \hbar/(2i\mu)\, [\, ik - (-ik)] = \hbar/(2i\mu)*2ik = \hbar k/\mu = v$$

Then, just making up a factor with C as yet known, we have

$$\psi = \{C/\Gamma(1+in) * e^{+n\pi/2}\} \; e^{ikz} * e^{inln[2krsin^2(\theta/2)]}$$

=>

$$j_z = |\; C/\Gamma(1+in)\;|^2 * e^{+n\pi}\} \; v$$

because recall that the fancy phase factor cancels away with the ψ^* application. If we want to have $j_z = 1$, "unit flux", then we must have

$$|\; C/\Gamma(1+in)\;| \;=\; e^{-n\pi/2}/\sqrt{v}$$

Suppose we choose $C = \Gamma(1+in)\, e^{-n\pi/2}/\sqrt{v}$ which is presumably a complex number. Then the above will be true, **and so we obtain the result C on page 141**. Our complete solution with this normalization is then

$$u_c = (1/\sqrt{v}\,)\,\Gamma(1+in)\, e^{-n\pi/2}\, e^{ikz}\, F(-in,1,ik\xi) \qquad \text{// which is } \mathbf{21.12\ D}$$

Recall from above that

$$\xi = r-z = r(1-\cos\theta) = 2r\sin^2(\theta/2) \qquad z = r\cos\theta$$

so we can express our result in spherical coordinates as

$$u_c(r,\theta) = (1/\sqrt{v}\,)\,\Gamma(1+in)\, e^{-n\pi/2}\, e^{ikr\cos\theta}\, F(-in,1,2ikr\sin^2(\theta/2)) \qquad n = (\mu ZZ' e^2/\hbar^2)/k \quad E = (\hbar k)^2/2\mu$$

which is result **21.12 E**

Pause for Comments: Equation 21.12 is THE solution to our Coulomb scattering problem as a function of r,θ, normalized to "unit flux". Only when we look at its asymptotic form 21.9 is the scattering nature of this solution revealed, and we can identify the amplitude $f_c(\theta)$. This is *the* solution of the SE for this problem that meets the required boundary conditions of a beam coming in "from the left" and then scattering outbound. Since this is a scattering problem, the energy E, best represented as k, takes continuous values.

Notice that there is no "partial wave expansion", there is no ℓ . Why is this? We really solved the problem in parabolic coordinates where we had ξ,η,φ instead of r,θ,φ . Recall how $\ell(\ell+1)$ appears as a "separation constant" when you solve in r,θ,φ and you end up with $R_\ell(r)\, Y_{\ell m}(\theta,\varphi)$ where Y diagonalizes L^2 and L_z. And in the no-φ case we get $R_\ell(r)\, P_\ell(\cos\theta)$.

When we solved the H atom in parabolics, we got a separation constant called ν on page 96, you see it in the separated equations 16.30. Thus, the solutions of the H atom are going to be labeled by ν and m, since φ is present in that problem. On page 97, this ν is converted to $\lambda_1 \sim n_1$ for the ξ equation, and $\lambda_2 \sim n_2$ for the η equation, then see 98 which shows the dual-Laguerre form of the H atom solution in

parabolics. So in effect this ν is a quantum number like ℓ for the eigenstates for the H atom. You make a general solution by doing a "partial wave" sum over different values of ν and m. [Thus, the parabolic scattering solution can be thought of as being a single "parabolic partial wave" with ν = -ik/2]

So why in the scattering problem is there no "partial wave" sum over ν ? Well, we can think of the parabolic coulomb scattering solution can be thought of as being *a single* "parabolic partial wave" with ν = -ik/2.

What is the low-energy form of our solution? (We already know it !)

$$u_c(r,\theta) = (1/\sqrt{\nu})\,\Gamma(1+in)\, e^{-n\pi/2}\, e^{ikr\cos\theta}\, F(-in,1,2ikr\sin^2(\theta/2)) \qquad n = (\mu ZZ' e^2/\hbar^2)/k \quad E = (\hbar k)^2/2\mu$$

I need to go off and compute something first.

$$\Gamma(1+in)\,\Gamma(1-in) = (in)\Gamma(in)\,(-in)\Gamma(-in) = n^2\,\Gamma(in)\,\Gamma(-in) = n^2\,\pi/[-in\sin(in\pi)] \qquad \text{Schaum 16.8}$$

$$= in\pi/\sin(in\pi) = in\pi/[ish(\pi n)] = n\pi/sh(\pi n)$$

Well, we can say that

$$|\, e^{-n\pi/2}\Gamma(1+in)\,|^2 = n\pi\, e^{-n\pi} / sh(\pi n) = 2\pi\, n\, e^{-n\pi} / (e^{n\pi} - e^{-n\pi}) = 2\pi n / (e^{2n\pi} - 1)$$

which **agrees with 21.13** see below.

Thus, we can say

$$|u_c(r,\theta)|^2 = (1/v)\;\; 2\pi n / (e^{2n\pi} - 1)\;|\, F(-in,1,2ikr\sin^2(\theta/2))\,|^2$$

$$|u_c(0,\theta)|^2 = (1/v)\;\; 2\pi n / (e^{2n\pi} - 1)$$

OK, back to my question on the low-k limit. You cannot get this result trivially from our F formula above, because we have the combination kr appearing in the third argument. We want k→0 for low energy, but we want r→∞ for "scattering", so you cannot blindly take limit kr→0 and expand F in its normal power series.

Instead, I think you could use 21.10 for large n. But we just get 21.11 for the cross section, and yes, it blows up as $1/v^4 \sim 1/k^4$. The bottom line: we have an exact result in 21.11 for all (non-rel) energies k, so we don't have to think about a special "low energy limit". Certainly low k justifies the non-rel SE.

The Gamov Factor. When you collide charged nuclei, the amount of hadronic nuclear reaction depends on how much nuclear overlap you get, which is roughly proportional to $|u_c(0,\theta)|^2$ shown above, if we imagine the nuclei are very small. If nucleus is size r = a, then we need ka <<1 . For such "low energies", the overlap would be roughly $|u_c(0,\theta)|^2$ x (volume of the smaller nucleus), very crude. But

$$n = (\mu ZZ' e^2/\hbar^2)/k \quad E = (\hbar k)^2/2\mu$$

We have not until this moment talked about the sign of n! If we have repulsion, n>>0 for "low energy" and we can then say

$$|u_c(0,\theta)|^2 = (1/v)\ 2\pi n / (e^{2\pi n} -1) \approx (1/v)\ 2\pi n\, e^{-2\pi n} \sim 1/k * 1/k * e^{-K/k} \sim e^{-K/k}/k^2$$

Thus, you would expect an exponential drop-off in your nuclear reaction rate on the order of $e^{-K/k}$ as you go to low energies. This expo barrier is the "Gamov Factor" of nuclear physics.

B. Coulomb Solution in Spherical Coordinates (142)

Oddly, this is the first time in this book that we see the radial equation written in this form 21.16. To derive it, let's go back to page 77 14.3 where we had

$$1/r^2\partial_r(r^2\partial_r R) + \{2\mu/\hbar^2[\ E-V(r)] - \ell(\ell+1)/r^2\ \}\ R = 0$$

which is the radial equation for a central potential V(r). Now install $V = -e^2/r$ as the potential energy for th H atom, but adjust to be $V = +ZZ'e^2/r$ for general collision case. Then look at the grouping

$$2\mu/\hbar^2[\ E-V(r)] = 2\mu/\hbar^2[\ \hbar^2k^2/2\mu \quad - \quad +ZZ'e^2/r] = k^2 - 2\mu/\hbar^2 * ZZ'e^2/r$$

Recall $n = (\mu ZZ'e^2/\hbar^2)/k$, so this second term is 2nk/r so we get

$$2\mu/\hbar^2[\ E-V(r)] = k^2 - 2nk/r$$

and our radial equation is then

$$1/r^2\partial_r(r^2\partial_r R) + \{\ k^2 - 2nk/r\ - \ell(\ell+1)/r^2\ \}\ R = 0 \qquad // 21.16$$

Derivation of 21.17 and 21.18. Rather than grind through this in my usual way, I thought I would have Maple do it for me. See filed Schiff 21_17.mws. Here is what is in that file:

```
> restart;
> R := r^L*exp(I*k*r)* f(r);    // this is our assumed form for R(r)
```

$$R := r^{L}\, e^{(I k r)}\, f(r)$$

This is the LHS of equation 21.16:

```
> X := simplify(r^(-2)* diff((r^2*diff(R,r)),r) + (k^2 - 2*n*k/r - L*(L+1)/r^2)*R);
```

$$X := \frac{e^{(I k r)}\left(2 I r^{L} k f(r) + 2 r^{L}\left(\frac{\partial}{\partial r} f(r)\right) + 2 I r^{L} L k f(r) + 2 r^{L} L\left(\frac{\partial}{\partial r} f(r)\right) + 2 I r^{(L+1)} k\left(\frac{\partial}{\partial r} f(r)\right) + r^{(L+1)}\left(\frac{\partial^2}{\partial r^2} f(r)\right) - 2 r^{L} f(r) n k\right)}{r}$$

Now divide out the leading factors seen above, but not the i

```
Y := simplify(r^(-L+1)* exp(-I*k*r)*X);
```

$$Y := 2 I k f(r) + 2\left(\frac{\partial}{\partial r} f(r)\right) + 2 I L k f(r) + 2 L\left(\frac{\partial}{\partial r} f(r)\right) + 2 I r k\left(\frac{\partial}{\partial r} f(r)\right) + r\left(\frac{\partial^2}{\partial r^2} f(r)\right) - 2 f(r) n k$$

Now want to see this as separate terms, luckily expand does this for us:

```
expand(Y);
```

$$2 I k f(r) + 2\left(\frac{\partial}{\partial r} f(r)\right) + 2 I L k f(r) + 2 L\left(\frac{\partial}{\partial r} f(r)\right) + 2 I r k\left(\frac{\partial}{\partial r} f(r)\right) + r\left(\frac{\partial^2}{\partial r^2} f(r)\right) - 2 f(r) n k$$

Then

```
collect(% , f);
collect(%,diff(f(r),r));
```

$$(2 I L k + 2 I k - 2 n k) f(r) + 2 I r k\left(\frac{\partial}{\partial r} f(r)\right) + 2\left(\frac{\partial}{\partial r} f(r)\right) + 2 L\left(\frac{\partial}{\partial r} f(r)\right) + r\left(\frac{\partial^2}{\partial r^2} f(r)\right)$$

$$(2 I k r + 2 + 2 L)\left(\frac{\partial}{\partial r} f(r)\right) + (2 I L k + 2 I k - 2 n k) f(r) + r\left(\frac{\partial^2}{\partial r^2} f(r)\right)$$

where we have computed the LHS of 21.16 multiplied by some factors, so this thing = 0. I can then rewrite the last line as follows:

$$r\,\partial_r^2 f + [\,2ikr + 2(\ell+1)\,]\,\partial_r f + [\,2ik(\ell+1) - 2nk]f = 0$$

which agrees with 21.17, so we know there are no typos!

Now compare this with CHG equation 21.4 which says

$$r\,\partial_r^2 F + (b-r)\,\partial_r F - aF = 0$$

To get into this form, as usual we need to rescale things. So try

$$f(r) = F(\alpha r) \qquad f'(r) = \alpha\, F'(\alpha r) \qquad f''(r) = \alpha^2\, F''(\alpha r)$$

$$r\,f'' + [\,2ikr + 2(\ell+1)\,]\,f' + [\,2ik(\ell+1) - 2nk]f = 0$$

$$r\,\alpha^2\, F''(\alpha r) + [\,2ikr + 2(\ell+1)\,]\,\alpha\, F'(\alpha r) + [\,2ik(\ell+1) - 2nk]\, F(\alpha r) = 0$$

$$r\,\alpha^2\, F''(\alpha r) + [\alpha\, 2ikr + \alpha\, 2(\ell+1)\,]\, F'(\alpha r) + [\,2ik(\ell+1) - 2nk]\, F(\alpha r) = 0$$

$$(\alpha r)\ F''(\alpha r) + [2ik(\alpha r) + \alpha\, 2(\boldsymbol{\ell}+1)\ ]/\alpha\ F'(\alpha r) + [\ 2ik(\boldsymbol{\ell}+1) - 2nk]\ /\alpha F(\alpha r) = 0$$

$$\rho F''(\rho) + [2ik\rho + \alpha\, 2(\boldsymbol{\ell}+1)\ ]/\alpha\ F'(\rho) + [\ 2ik(\boldsymbol{\ell}+1) - 2nk]\ /\alpha F(\rho) = 0 \qquad \rho \equiv \alpha r$$

$$\rho F''(\rho) + [2ik\rho/\alpha + 2(\boldsymbol{\ell}+1)\ ]\ F'(\rho) + [\ 2ik(\boldsymbol{\ell}+1) - 2nk]\ /\alpha\ F(\rho) = 0 \qquad \rho \equiv \alpha r$$

Set $\alpha = -2ik$ in order for the $-\rho$ in the F' term to match our desired form, so have

$$\rho F''(\rho) + [-\rho + 2(\boldsymbol{\ell}+1)\ ]\ F'(\rho) + [\ 2ik(\boldsymbol{\ell}+1) - 2nk]\ /[-2ik]\ F(\rho) = 0 \qquad \rho \equiv \alpha r$$

$$\rho F''(\rho) + [-\rho + 2(\boldsymbol{\ell}+1)\ ]\ F'(\rho) + [\ 2ik(\boldsymbol{\ell}+1)\ /[-2ik] - 2nk/[-2ik]]\ F(\rho) = 0 \qquad \rho \equiv \alpha r$$

$$\rho F''(\rho) + [-\rho + 2(\boldsymbol{\ell}+1)\ ]\ F'(\rho) + [\ -(\boldsymbol{\ell}+1) - in]\ F(\rho) = 0 \qquad \rho \equiv \alpha r$$

Comparison then says

$$b = 2(\boldsymbol{\ell}+1)\quad a = \boldsymbol{\ell} + 1 + in$$

So our solution is then

$$f(r) = F(\rho) = F(\boldsymbol{\ell} + 1 + in,\ 2(\boldsymbol{\ell}+1);\ \rho) = F(\boldsymbol{\ell} + 1 - in,\ 2(\boldsymbol{\ell}+1);\ -2ikr)$$

and **we have verified 21.18.**

Now let's skip 21.19 for the moment.
Result 21.20 is trivial.

Derivation of 21.19 (added 12.17.08) From our confluent doc summary, recall that we did these changes to get from the SE radial equation for R(r) to the confluent HG equation for y(x):

$$R = \chi/r \qquad \chi(r) = w(\rho) = \rho^{\boldsymbol{\ell}+1} e^{-x/2}\ y(x) \qquad \Rightarrow \qquad R(r) = \rho^{\boldsymbol{\ell}} e^{-x/2}\ y(x) = r^{\boldsymbol{L}}\ e^{-x/2}\ f(x)$$

This is the functional change Schiff is using in **page 142 A** where he uses $x = -2i\rho$. As noted in my doc, this just gives a c.c. solution which is still a solution. He then writes $R(r) = r^{\boldsymbol{L}}\ e^{+ikr}\ f(x)$ where he is using $f(x) = c_{\boldsymbol{L}}\Phi(L+1+i\eta,\ 2L+2,-2i\rho)$. Now, A&S on page 538 define [I write $c_{\boldsymbol{L}}$ for his $C_{\boldsymbol{\ell}}$ to avoid confusion with the $C_{\boldsymbol{L}}$ of AS that I use below]

$$F_{\boldsymbol{L}} = C_{\boldsymbol{L}}\ \rho^{\boldsymbol{L+1}}\ e^{-i\rho}\ \Phi(L+1-i\eta,\ 2L+2,+2i\rho) \qquad \rho = kr$$

so

$$F_{\boldsymbol{L}}^{*} = C_{\boldsymbol{L}}\ \rho^{\boldsymbol{L+1}}\ e^{+i\rho}\ \Phi(L+1+i\eta,\ 2L+2,-2i\rho)$$

and we know that (and indeed showed in our confluent doc notes that)

$$F_{\boldsymbol{L}}^{*} \rightarrow \sin\theta_{\boldsymbol{L}}$$

Now we know from A&S and our own derivation (coulomb wave function.doc) that

$$C_L = 2^L e^{-\pi\eta/2} |\Gamma(L+1+i\eta)|/ \Gamma(2L+2) \quad = 2^L e^{-\pi\eta/2} \Gamma(L+1+i\eta)e^{-i\sigma}/ \Gamma(2L+2)$$

So we may conclude that

$$f(x) = c_L\Phi(L+1+i\eta, 2L+2,-2i\rho) \rightarrow c_L\ \rho^{-L-1} e^{-i\rho}(1/C_L) \sin\theta_L$$

and therefore

$$R(r) = r^L e^{+i\rho} f(x) \ \rightarrow\ r^L e^{+i\rho} c_L\ \rho^{-L-1} e^{-i\rho}(1/C_L) \sin\theta_L$$

$$= \rho^L k^{-L}\ c_L\ \rho^{-L-1}(1/C_L) \sin\theta_L\ = k^{-L} (1/kr) (c_L/ C_L) \sin\theta_L$$

$$= c_L (1/kr) (2k)^{-L} e^{+\pi\eta/2} e^{+i\sigma}/ \Gamma(2L+2)/ \Gamma(L+1+i\eta) \sin\theta_L$$

and so now I have in effect **derived 21.19** and right now it gets a red check. Schiff used n for η and he uses η_L for the coulomb phase shift σ_L.

Derive page 143 A. // this is a *long* section, had lots of trouble on earlier attempts

We want to compare the small r limit of our spherical and parabolic solutions. The limit of the spherical solution is quite simple. From page 142 it is just this:

$$R_\ell(r) \approx e^{ikr}\ r^\ell\ C_\ell \quad \approx C_\ell\ r^\ell$$

where we set F = 1 and $e^{ikr} = 1$.

Getting the corresponding thing for the parabolic solution is *much harder*. You insert 21.12 E into 21.20 and get this [recall that u here is normalized to "unit flux"]

$$R_\ell(r) = (2\ell+1)/2 * \{ v^{-1/2} \Gamma(1+in)e^{-n\pi/2}\} \ \{\ \int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}\ F(-in, 1, ikr(1-z)) \ \}$$

Schiff suggests that a "complete evaluation of this integral can be avoided". I think he is wrong about that. Every term in the F series contributes, not just the first 1 or 2 terms, due to z the integration, even in the small r limit. So we need to find a way to do this integral correctly. I know this because I did just the first series term and got the wrong answer, and then I did the second term and saw that it contributed to the answer, and then I saw that every term in the series was going to contribute. I threw these computation notes out now, no reason to further clutter this doc.

One idea is to use the integral representation for the CHF as in Messiah page 480

$$F(a,b,\xi) \ = \ (1-e^{-2i\pi a})^{-1} \Gamma(b) / [\ \Gamma(a)\Gamma(b-a)\ ] \int_C dt\ e^{\xi t}\ t^{a-1}\ (1-t)^{b-a-1}$$

over a certain contour C = Γ_c and b must be an integer. We can apply this using

a = -in b = 1 ξ = ikr(1-z) a-1 = -in-1 b-a = 1+in b-a-1 = in

so our expansion would be

$$F(-in, 1, ikr(1-z)) = (1-e^{-2i\pi(-in)})^{-1}\ \Gamma(1)/[\Gamma(-in)\Gamma(1+in)] \int_C dt\ e^{ikr(1-z)t}\ t^{-in-1}\ (1-t)^{in}$$

Before continuing, we can clean up the leading factors a bit. Recall that

$$\Gamma(-in)\Gamma(1+in) = -\pi/\sin(in\pi) = -\pi/(i\ \mathrm{sh}(n\pi)) = i\pi / \mathrm{sh}(n\pi)$$

Meanwhile,

$$(1-e^{-2i\pi(-in)}) = (1-e^{-2\pi n}) = e^{-\pi n}(e^{\pi n} - e^{-\pi n}) = 2\ e^{-\pi n}\ \mathrm{sh}(\pi n)$$

So the following combination of terms appears *inverted* in the above F expansion:

$$(1-e^{-2i\pi(-in)})\ [\Gamma(-in)\Gamma(1+in)] = 2\ e^{-\pi n}\ \mathrm{sh}(\pi n) * i\pi/[\ \mathrm{sh}(\pi n)\ ] = 2\pi\ i\ e^{-\pi n}$$

So our simplified expansion, including $\Gamma(1) = 1$, becomes

$$F(-in, 1, ikr(1-z)) = (e^{\pi n}/2\pi\ i)\ e^{ikr} \int_C dt\ e^{-ikrzt}\ t^{-in-1}\ (1-t)^{in}$$

So let's insert this now into our expansion above for $R_\ell(r)$

$$R_\ell(r) = (2\ell+1)/2 * \{ v^{-1/2}\ \Gamma(1+in)e^{-n\pi/2}\} \{ \int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}\ F(-in, 1, ikr(1-z)) \}$$

$$= (2\ell+1)/2 * \{ v^{-1/2}\ \Gamma(1+in)e^{-n\pi/2}\}\ (e^{\pi n}/2\pi)\ e^{ikr} \int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz} \int_C dt\ e^{-ikrzt}\ t^{-in-1}\ (1-t)^{in}$$

$$= (2\ell+1)/2 * \{ v^{-1/2}\ \Gamma(1+in)e^{+n\pi/2}\}\ (1/2\pi i)\ e^{ikr} \int_C dt\ t^{-in-1}\ (1-t)^{in} \{ \int_{-1}^{1} dz\ P_\ell(z)\ e^{ikr(1-t)z} \}$$

But now recall an integral we used in expanding e^{ikz} in partial waves, it appears above in these notes:

$$\int_{-1}^{1} P_\ell(x)\ e^{iax}\ dx = 2\ i^\ell\ j_\ell(a) \qquad P_\ell(z) = C_\ell^{1/2}(z) \quad \text{Gegenbauer} \qquad \text{GR page 830}$$

which we now apply with a = kr(1-t). We then have

$$\int_{-1}^{1} dz\ P_\ell(z)\ e^{ikr(1-t)z} = 2\ i^\ell\ j_\ell[kr(1-t)]$$

and the above then becomes

$$R_\ell(r) = (2\ell+1)/2 * \{ v^{-1/2}\, \Gamma(1+in) e^{+n\pi/2}\} \; (1/2\pi i)\, e^{ikr} \int_C dt\, t^{-in-1} (1-t)^{in} \{2\, i^\ell\, j_\ell[kr(1-t)]\}$$

NOW (having done all the z integration) we should be able to take the small r limit

$$j^\ell(\rho) = \rho^\ell / (2\ell+1)!! \qquad \text{Schiff page 85}$$

which then gives us

$$j_\ell[kr(1-t)] = (kr(1-t))^\ell / (2\ell+1)!! = (kr)^\ell \; (1-t)^\ell/(2\ell+1)!!$$

and we can them combine our two 1-t factors to get

$$R_\ell(r) = (ikr)^\ell / (2\ell+1)!!\, (2\ell+1) \{ v^{-1/2}\, \Gamma(1+in) e^{+n\pi/2}\} (1/2\pi i)\, e^{ikr} \left[\int_C dt\, t^{-in-1} (1-t)^{in+\ell} \right]$$

The integral appearing in the [..] is exactly that beta function integral which appears in Messiah p 481 B.5 *with the correction* I have made (!) where we identify

$$x = -in \qquad y = \ell + in + 1 \qquad x-1 = -in - 1 \qquad y-1 = in + \ell \qquad x + y = \ell + 1$$

$$\int_C dt\, t^{-in-1} (1-t)^{\ell+in} = (1 - e^{-2\pi i(-in)})\, \Gamma(-in)\Gamma(\ell + in + 1)/\Gamma(\ell+1) = (1 - e^{-2\pi n})\, \Gamma(-in)\Gamma(\ell + in + 1)/\Gamma(\ell+1)$$

So our ongoing and ever-changing R now becomes

$$R_\ell(r) = (ikr)^\ell / (2\ell+1)!!\, (2\ell+1) \{ v^{-1/2}\, \Gamma(1+in) e^{+n\pi/2}\} (1/2\pi i)\, e^{ikr}\, (1 - e^{-2\pi n})\, \Gamma(-in)\Gamma(\ell + in + 1)/\Gamma(\ell+1)$$

Now it's time for more cleanup before continuing.

$$\Gamma(1+in)\, \Gamma(-in) = -\pi/\sin(in\pi) = -\pi/ i\, sh(n\pi) = i\pi / sh(n\pi) \qquad \text{// Schaum p 102}$$

$$e^{+n\pi/2}(1 - e^{-2\pi n}) = e^{-n\pi/2}\, e^{+n\pi}\, (1 - e^{-2\pi n}) = e^{-n\pi/2}\, (e^{n\pi} - e^{-n\pi}) = +2\, e^{-n\pi/2}\, sh(n\pi)$$

So we may replace this combination of factors,

$$\Gamma(1+in)\, \Gamma(-in)\, e^{+n\pi/2}(1 - e^{-2\pi n}) = +2\, e^{-n\pi/2}\, sh(n\pi) * i\pi / sh(n\pi) = 2i\pi\, e^{-n\pi/2}$$

and then we have

$$R_\ell(r) = +2i\pi\, e^{-n\pi/2}\; (ikr)^\ell / (2\ell+1)!!\, (2\ell+1) \{ v^{-1/2} \}\; (1/2\pi i)\, e^{ikr}\, \Gamma(\ell + in + 1)/\Gamma(\ell+1)$$

$$= e^{-n\pi/2}\; (ikr)^\ell / (2\ell+1)!!\, (2\ell+1) v^{-1/2}\, e^{ikr}\, \Gamma(\ell + in + 1)/\Gamma(\ell+1)$$

Now I guess it is time to compare this to the other form which was [I kept e^{ikr} in both though really 1]

$$R_\ell(r) \approx e^{ikr}\ r^\ell\ C_\ell$$

The conclusion of this match is that

$$C_\ell = e^{-n\pi/2}\,(ik)^\ell/(2\ell+1)!!\ (2\ell+1)v^{-1/2}\Gamma(\ell+in+1)/\Gamma(\ell+1)$$

$$= e^{-n\pi/2}\,(2ik)^\ell\,2^{-\ell}/(2\ell+1)!!\ (2\ell+1)v^{-1/2}\Gamma(\ell+in+1)/\Gamma(\ell+1)$$

$$= (2ik)^\ell\,e^{-n\pi/2}\,\Gamma(\ell+in+1)\,v^{-1/2}/(2\ell)!$$

$$* \{\ 2^{-\ell}/(2\ell+1)!! * (2\ell+1)/\Gamma(\ell+1)\ (2\ell)!\ \}$$

where the bracket {..} is supposed to come out being 1 to agree with Schiff p 143 A.

Now we invoke two results.

1. First, from GP p 938 we have

$$\Gamma(n+1/2) = \sqrt{\pi}\ 2^{-n}\ (2n-1)!!$$

But I want to see 2n-1 = 2ℓ+1 so need 2n = 2ℓ + 2 or n = ℓ+1. Then the above says:

$$\Gamma(\ell+3/2) = \sqrt{\pi}\ 2^{-\ell-1}\ (2\ell+1)!!$$

But

$$\Gamma(\ell+3/2) = \Gamma(\ell+1/2+1) = (\ell+1/2)\,\Gamma(\ell+1/2)$$

so we have

$$(\ell+1/2)\,\Gamma(\ell+1/2) = \sqrt{\pi}\ 2^{-\ell-1}\ (2\ell+1)!!$$

or

$$(2\ell+1)\,\Gamma(\ell+1/2) = \sqrt{\pi}\ 2^{-\ell}\ (2\ell+1)!!$$

so

$$1/(2\ell+1)!! = \sqrt{\pi}\ 2^{-\ell}\ /\ [\ \Gamma(\ell+1/2)\ (2\ell+1)\ )]$$

2. Next, look at GR page 938 which has this doubling formula:

$$\Gamma(2x) = 2^{2x-1}/\sqrt{\pi}\ \ * \Gamma(x)\,\Gamma(x+1/2)\ \ = \Gamma(2x+1)/2x$$

which says

$$\Gamma(2x+1) = (2x)! = 2x\, 2^{2x-1}/\sqrt{\pi} * \Gamma(x)\,\Gamma(x+1/2) = x\ 2^{2x}\,\Gamma(x)\,\Gamma(x+1/2)/\sqrt{\pi}$$

or

$$(2\ell)! = \ell\ 2^{2\ell}\,\Gamma(\ell)\,\Gamma(\ell+1/2)/\sqrt{\pi}$$

The bracket above is then

$$\{\ 2^{-\ell}/(2\ell+1)!! * (2\ell+1)/\Gamma(\ell+1)\ (2\ell)!\ \}$$

$$= \{\ 2^{-\ell}\sqrt{\pi}\ 2^{-\ell} / [\Gamma(\ell+1/2)(2\ell+1)] * (2\ell+1)/\Gamma(\ell+1)\ \ell\ 2^{2\ell}\,\Gamma(\ell)\,\Gamma(\ell+1/2)/\sqrt{\pi}\ \}$$
$$= \{\ \sqrt{\pi}\ / \Gamma(\ell+1/2) * 1/\Gamma(\ell+1)\ \ell\,\Gamma(\ell)\,\Gamma(\ell+1/2)/\sqrt{\pi}\ \}$$
$$= \{\ 1 / \Gamma(\ell+1/2) * 1/\Gamma(\ell+1)\ \ell\,\Gamma(\ell)\,\Gamma(\ell+1/2)\}$$
$$= \{\ 1/\Gamma(\ell+1)\ \ell\,\Gamma(\ell)\}$$
$$= \{\ 1/[\Gamma(\ell+1)]*\ \ell\,\Gamma(\ell)\}$$
$$= \{\ 1/[\ell\,\Gamma(\ell)]*\ \ell\,\Gamma(\ell)\}$$
$$= \{\ 1\}$$

And FINALLY (after about 10 hours) I have **verified Schiff page 143 A**. In doing so, I found that the typo in the Messiah appendix was -x and not +x as I first thought, so that got fixed too. This was a pretty stiff exercise.

Equation 21.21 then trivially results from stuffing 142 A with 21.18 for f and 143 A for C_ℓ into 21.15.

I have reviewed to this point , 12.20.08

Derive Equation 21.22.

Review: Back on page 119 we did that "comparison of two forms" trick and ended up with 19.11 as the partial wave expansion of f(θ), where the partial wave "amplitude" was $(e^{2i\delta\ell} - 1)$. The two things compared there were (1) the general form 18.10 for u(r,θ) which defined f(θ), and (2) the partial wave expansion for u(r,θ) given in 19.1, where R was a solution of the radial SE. The form 18.10 was already in asymptotic form, so we had to take the large R limit of our solution of the SE in order to do the comparison. This mid-range solution for R was a linear combination of spherical Bessel functions since we set V= 0 there (see 19.7), and then we went even farther as $r\to\infty$ to use asymptotic forms of the Bessels.

So, we could do a similar thing here, and that is why we need result 21.19 which gives the large r form of the radial function in the Coulomb case. Let's assume this result for the moment and try to derive 21.22. (1) our "version" of 18.10 which defines $f_c(\theta)$ is 21.9 which as before has two terms. We will likely need both terms to get our final result here. (2) our "version" of 19.1 is 21.15 with 21.19 as the large r form of R. [I carried out this program in "radial equation.doc", and here below I do a different method. The radial doc method makes it clear that everything goes about the same as the non-Coulomb case except the plane wave has no outgoing piece so the "-1" is missing in Coulomb that is present in non-Coulomb.]

But wait, maybe there is a faster and simpler method. Unlike the general case treated earlier, here we have an explicit form for f(θ) which is 21.10. Here it is

$$f(\theta) = (n/k) (1/(1-z)) \exp i(-n \ln [(1-z)/2] + \pi + 2\eta_o)$$

$$= (n/k) e^{i(\pi+2\eta)} \exp(-i n \ln [(1-z)/2]) /(1-z)$$

$$= (n/k) e^{i(\pi+2\eta)} [(1-z)/2]^{-in} /(1-z) = (n/k) e^{i(\pi+2\eta)} 2^{+in} (1-z)^{-in-1}$$

which is a simple power of (1-z). So try an expansion of the form 21.22

$$f(\theta) = \Sigma_{\ell'} B_{\ell'} P_{\ell'}(z)$$

We can invert in analogy to 21.15 and 21.20 to find that

$$B_\ell = (2\ell+1)/2 * \int dz P_\ell(z) f(\theta) = (2\ell+1)/2 * (n/k) e^{i(\pi+2\eta)} 2^{+in} * \int dz P_\ell(z) (1-z)^{-in-1}$$

A promising integral is GR page 797 which shows this:

$$\int_{-1}^{1} dx P_\ell(x)(1+x)^\alpha = \int_{-1}^{1} dx P_\ell(-x)(1-x)^\alpha = (-1)^\ell \int dz P_\ell(z) (1-z)^\alpha$$

Let z = -x, dz = - dz, reverse end points and the second result above. Then use Schaum p 147 which we know well that $P_\ell(-x) = (-1)^\ell P_\ell(x)$ to get the third result. So we then have

$$\int dz P_\ell(z) (1-z)^\sigma = (-1)^\ell 2^{\sigma+1} [\Gamma(\sigma+1)]^2 / \Gamma(\sigma+\ell+2)\Gamma(1+\sigma - \ell)$$

and now set $\sigma = - in-1$ so $\sigma+1 = -in$ and we get

$$\int dz P_\ell(z) (1-z)^{-in-1} = (-1)^\ell 2^{-in} [\Gamma(-in)]^2 / \{\Gamma(-in+\ell+1)\Gamma(-in - \ell)\}$$

So our computed coefficient is therefore

$$B_\ell = (2\ell+1)/2 * (n/k) e^{i(\pi+2\eta)} 2^{+in} (-1)^\ell 2^{-in} [\Gamma(-in)]^2 / \{\Gamma(-in+\ell+1)\Gamma(-in - \ell)\}$$

<u>which amazingly is supposed go come out being this</u>

$$B_\ell = (2\ell+1) e^{2iq} /(2ik) \qquad \text{where q = phase of } \Gamma(in+\ell+1)$$

which then implies that we should be able to show that

$$e^{2iq} = - i n e^{i2\eta} (-1)^\ell [\Gamma(-in)]^2 / \{\Gamma(-in+\ell+1)\Gamma(-in - \ell)\} \qquad (*)$$

η = phase of $\Gamma(in+1)$ q = phase of $\Gamma(in+\ell+1)$

Looking for q, lets process the gammas as follows

$$\text{gammas} = [\Gamma(-in)]^2 / \{\Gamma(-in+\ell+1)\Gamma(-in-\ell)\} = [\Gamma(-in)]^2 / \{\Gamma(-in+\ell+1)\Gamma(-in-\ell)\} * \{\Gamma(in+\ell+1)/\Gamma(in+\ell+1)\}$$

$$= [\Gamma(-in)]^2 \{\Gamma(in+\ell+1)/\Gamma(-in+\ell+1)\} / \{\Gamma(-in-\ell)\Gamma(in+\ell+1)\}$$

But then we can identify

$$\{\Gamma(in+\ell+1)/\Gamma(-in+\ell+1)\} = e^{2iq}$$

So we get

$$\text{gammas} = e^{+2iq}\Gamma(-in)]^2/\{\Gamma(-in-\ell)\Gamma(in+\ell+1)\}$$

Now we can do something with the denominator

$\Gamma(p)\Gamma(1-p) = \pi/\sin(p\pi)$ $p = [-in-\ell]$

$\Gamma(-in-\ell)\Gamma(1+in+\ell) = \pi/\sin([-in-\ell]\pi)$ $= -(-1)^\ell\pi / ish(n\pi)$

Then we have

$$\text{gammas} = -e^{2iq}[\Gamma(-in)]^2 ish(n\pi)(-1)^\ell/\pi$$

What to do with $[\Gamma(-in)]^2$?

$$[\Gamma(-in)]^2 = [\Gamma-(in)]^2\Gamma(in)/\Gamma(in) = |\Gamma(-in)|^2\{\Gamma(-in)/\Gamma(in)\} = |\Gamma(-in)|^2 e^{-2ic}$$

where c = phase of $\Gamma(in)$. And Schaum page 102 bottom says

$$|\Gamma(in)|^2 = \pi/[n\,sh(\pi n)] = |\Gamma(-in)|^2$$

so we then have

$$\text{gammas} = -e^{2iq}[\Gamma(-in)]^2 ish(n\pi)(-1)^\ell/\pi$$

$$= -e^{2iq}e^{-2ic}\pi/[n\,sh(\pi n)]\,ish(n\pi)(-1)^\ell/\pi$$

$$= -e^{2iq}e^{-2ic}\,1/[n]\,i(-1)^\ell = -i(-1)^\ell(1/n)e^{2iq}e^{-2ic}$$

Then we have

$$\text{RHS}(*) = -\, i\, n\, e^{i2\eta}\, (-1)^{\ell}\, [\Gamma(-in)]^2 \,/\, \{\Gamma(-in+\ell+1)\Gamma(-in-\ell)\}$$

$$= -\, i\, n\, e^{i2\eta}\, (-1)^{\ell} * \text{gammas}$$

$$= +\, i\, n\, e^{i2\eta}\, (-1)^{\ell} * i\, (-1)^{\ell}\, (1/n)\, e^{2iq}\, e^{-2ic}$$

$$= -\, e^{i2\eta}\, e^{2iq}\, e^{-2ic}$$

so at least we are getting a phase as the result, which Schiff did claim. Now we have

$\eta = \arg \Gamma(1+in)$
$q = \arg \Gamma(in+\ell+1)$
$c = \arg \Gamma(in)$

One simplification here is to say $\Gamma(1+in) = in\, \Gamma(in)$ so that

$$\arg \Gamma(1+in) = \arg (in) + \arg \Gamma(in) = \pi/2 + \arg \Gamma(in)$$

Then we know that $\eta = \pi/2 + c$ so we have

$$\text{RHS}(*) = -\, e^{i2\eta}\, e^{2iq}\, e^{-2ic} = -\, e^{2i(\eta - c + q)} = e^{2i([\pi/2 + c] - c + q)} = +\, e^{2iq}$$

Thus, we have exactly **verified Schiff's result 21.22**. [This method does not make it very revealing why the result comes out so simple!]

Now as we go to high energy, $n \to 0$ and $\eta_\ell \to 0$ and we get result B. We can prove 21.23 by doing what he says, and we get

$$2 = \int_{1-\varepsilon}^{1} dz\, P_\ell(z)\, 4\, \delta(1-z) = 2$$

because we only integrate half the delta function at z=1, and of course $P_\ell(z) = 1$ at z=1. Now here is how you get 21.24. It is really this

$$\int_{1-\varepsilon}^{1} dz * B = (1/2ik) \int_{1-\varepsilon}^{1} dz\, 4\delta(1-z) = (1/2ik) * 4 * 1/2 = 1/(ik) = -i/k$$

Note added: How do we know that $\Sigma(2\ell+1)P_\ell(z) = 0$ for $z \neq 1$? Schiff does not even comment on this fact which I might seem feasible but is not obvious. Here is the answer, which also gives the delta function at the same time. Define

$$\varphi_\ell(z) = \sqrt{(2\ell+1)/2}\, P_\ell(z)$$

On the interval -1 to 2, these form a complete set of real orthonormal polynomials with integration measure 1. For such a set, we know that

$$\int_{-1}^{1} dz\, \varphi_\ell(z)\, \varphi_{\ell'}(z) = \delta_{\ell,\ell'} \qquad \text{orthonormal}$$

$$\Sigma_{\ell=0}\, \varphi_\ell(z)\, \varphi_\ell(z') = \delta(z-z') \qquad \text{complete}$$

In bra-ket notation, both these things are very obvious. So write out the second one

$$\Sigma_{\ell=0}\, (2\ell+1)^*\, P_\ell(z)\, P_\ell(z') = 2\, \delta(z-z')$$

Now set z' = 1 so that $P_\ell(z') = 1$, and we get

$$\Sigma_{\ell=0}\, (2\ell+1)^*\, P_\ell(z) = 2\, \delta(z-1)$$

which agrees with 21.23 apart from a factor of 2 which I think is a problem with one of our methods. Since this is right at the end of the range, we have to sort of redefine things a bit and we get his result. But more importantly, this shows that for $z \neq 0$ we get

$$\Sigma_{\ell=0}\, (2\ell+1)^*\, P_\ell(z) = 0 \qquad z \neq 1$$

which is the fact I was wondering about.

Does optical theorem explain Coulomb form? No. The above derivation is fine, but there must be some *basic reason* that the partial wave amplitude comes out having such a super simple form. I am thinking maybe the optical theorem has something to do with this. Suppose we write

$$f(\theta) = \Sigma_\ell\, B_\ell\, P_\ell(z)$$

and install this in three places in page 137 B with no absorption. We get something like this

$$2\pi \int dz\, |[\Sigma_\ell\, B_\ell\, P_\ell(z)]|^2 = 4\pi/k\ \ \mathrm{Im}\,[\Sigma_\ell\, B_\ell\, P_\ell(z)] = 4\pi/k\, \Sigma_\ell\, P_\ell(1)\mathrm{Im}(B_\ell)$$

$$\int dz\, [\Sigma_\ell\, B_\ell\, P_\ell(z)]\, [\Sigma_{\ell'}\, B_{\ell'}^*\, P_{\ell'}(z)] = \Sigma_\ell\, P_\ell(1)\, \{2/k^*\, \mathrm{Im}(B_\ell)\}$$

$$[\Sigma_\ell\, B_\ell\, \Sigma_{\ell'}\, B_{\ell'}^*\,] \int dz\, P_\ell(z)\, P_{\ell'}(z) = \Sigma_\ell\, P_\ell(1)\, \{2/k^*\, \mathrm{Im}(B_\ell)\}$$

$$[\Sigma_\ell\, B_\ell\, \Sigma_{\ell'}\, B_{\ell'}^*\,]\, (2/2\ell+1)\delta_{\ell\ell'} = \Sigma_\ell\, P_\ell(1)\, \{2/k^*\, \mathrm{Im}(B_\ell)\}$$

$$[\Sigma_\ell\, |B_\ell|^2\,]\, (2/2\ell+1) = \Sigma_\ell\, P_\ell(1)\, \{2/k^*\, \mathrm{Im}(B_\ell)\} = \Sigma_\ell\, \{2/k^*\, \mathrm{Im}(B_\ell)\}$$

$\Rightarrow |B_\ell|^2 = (2\ell+1)\,\mathrm{Im}(B_\ell)/k$

Let $B_\ell = |B_\ell| e^{i\varphi}$, then we have $\mathrm{Im}(B_\ell) = |B_\ell| \sin\varphi$ so

$|B_\ell|^2 = (2\ell+1)\,|B_\ell| \sin\varphi /k \qquad \Rightarrow \qquad |B_\ell| = (2\ell+1) \sin\varphi /k$

$\Rightarrow B_\ell = (2\ell+1)\, e^{i\varphi} \sin\varphi /k = (2\ell+1)(\, e^{2i\varphi} - 1)/2ik$

which does not seem to be the result we want. But of course this is just the usual result for elastic scattering, and Coulomb scattering has "complications". In elastic, if $\varphi = 0$ for a phase shift, then $B_\ell = 0$, since we have $(\, e^{2i\varphi} - 1) = 0$. But in Coulomb, if all phase shifts are 0, $B_\ell = (2\ell+1)$ and you get a forward delta spike for your amplitude $f(\theta)$. This is NOT the first e^{ikz}-like term in 21.9. But maybe this is not so strange, maybe we just get this kind of thing when we take the $n\to 0$ limit of the Rutherford amplitude.

Direct Derivation of 21.23 as Lim $n\to 0$ $f_c(\theta) = (2/ik)\delta(1-z)$. This took me quite a while, so I want to write it down. We want the $n\to 0$ limit of (21.10). In this limit, $\eta_0 = 0$ with no singular problems. If we write $\sin^2\theta/2 = (1-z)/2 = \varepsilon$, then we have

$$2ikf_c(\theta) = -ne^{-in\ln\varepsilon}/\varepsilon = -n\,\varepsilon^{-in-1}$$

We are going to take $n\to 0$ while holding k constant, so it is the "parameters" going to 0, perhaps μ. Now looking at the above, if $\varepsilon>0$, the limit is going to be $= -n * 1/\varepsilon \to 0$ as $n\to 0$ so for any *finite* ε, the limit is 0. At $\varepsilon=0$ we have a singular mess of some sort. For finite n, we have a single pole in ε at $\varepsilon=0$ times a phase, so our result looks infinite there. That is, $g(\varepsilon) = \lim_{n\to 0} [-n\,\varepsilon^{-in-1}]$ seems infinite at $\varepsilon=0$. We might then model this thing as $A\,\delta(\varepsilon) + B\,\delta'(\varepsilon) + \ldots$ So then let's consider

$$g(\varepsilon) = \lim_{n\to 0} [-n\,\varepsilon^{-in-1}] = A\,\delta(\varepsilon) + B\,\delta'(\varepsilon)$$

Then multiply each side by an analytic function $h(\varepsilon)$ and integrate from 0 to b. We know $\delta(\varepsilon)$ will integrate to 1/2, and the integral of $\delta'(\varepsilon) = -\delta(0)$ from the parts part. So RHS = $A/2 - B\delta(0)$. Now compute the LHS.

$$\int_0^b d\varepsilon\, g(\varepsilon) = \lim_{n\to 0} [-n \int_0^b d\varepsilon\, \varepsilon^{-in-1}\ ] = \lim_{n\to 0} [-n\,(1/-in)\,\varepsilon^{-in}\Big|_0^b\ ]$$

$$= \lim_{n\to 0} [-n\,(1/-in)\,(b^{-in} - 0)\ ] = (1/i)\lim_{n\to 0} [b^{-in}] = (1/i) \qquad \text{since } x^0 = 1 \ \ \ln 1 = 0$$

Thus it must be true, comparing LHS = RHS, that $A/2 = 1/i$ and $B = 0$. We would show higher order derivatives of δ also zero in similar fashion, since half integral of any of them is singular. So we have shown that

$$\lim_{n\to 0}[2ikf_c(\theta)] = (2/i)\,\delta(\varepsilon) = (2/i)\,\delta([1-z]/2) = (4/i)\,\delta(1-z)$$

and thus

$\lim_{n\to 0} f_c(\theta) = (2/ik)\,\delta(1-z)$

which agrees with the combination of page 143 B and 21.23. This then supports the general form for f which is shown in 21.22. That is, the partial wave amplitude must be something that → 1 for each partial wave. Of course above, I computed the result 21.22 exactly.

Comments on Schiff's Coulomb Scattering Approach

OK, time to back up yet again and see what has happened. We are looking for a solution of the SE which has something like the traditional asymptotic form $e^{ikz} + f(\theta)e^{ikr}/r$. But it turns that no solution of the SE with V = 1/r in fact *has* this form for large r. For a *conventional* V(r) we were able to get this form for the solution, and further, we got an expression for f(θ) in partial waves with $B_\ell = (2\ell+1)(e^{i\delta\ell} - 1)$ and we could then compute the phase shifts δ_ℓ indirectly from V(r) by matching BC's at some radius r=a etc etc.

So in the Coulomb problem we want to find a solution of the SE that looks at least something like the traditional form. There is exactly one solution and it's asymptotic form is given in 21.9 with C set as in page 141 C if we want unit incoming flux. We have mysterious *extra phase factors* (in red, parens)

$$\text{solution} = e^{ikz}\ (e^{in\ln[k(r-z)]}) + f(\theta)\, e^{ikr}/r * (e^{-in\ln[2kr]})$$

While perhaps disturbing, these phase factors don't affect flux calculations for large r, and we still have $|f|^2$ as $d\sigma/d\Omega$ and we get the exactly correct Rutherford formula.

Now, let's look back at how Schiff "came up with" the correct solution above. The correct solution which gives the above as the large-r limit is just the normalized 21.5 which is F(-in,1,ik(r-z)). This is an amazingly compact result! All we did to get this result was examine the SE in parabolic coordinates and *try out* a solution of the form $e^{ikz}f(r-z)$. And we also *tried out* a first-kind solution F with third argument +ikξ instead of -ikξ which would have given e^{-ikr}/r in the large r limit, so we would have rejected that.

After doing this and getting the very simple non-partial-wave closed form result 21.5, we were able to cast f(θ) into a partial wave expansion, and it came out having a very simple result 21.22 which seemed rather amazingly simple. We then did a few calculations which "lent support" to this simple result.

Seat of pants spherical coordinates derivation of the form 21.22 for $f_c(\theta)$.

Now I want to think about, as history has thought about, how you would solve this problem without using parabolic coordinates. We pretend that we only know about spherical coordinates. I expect to be using those famous A&S Coulomb Wave functions somehow.

We realized that the radial SE [for χ where R = χ/r] yields the p 538 A&S Coulomb Wave Function Equation and we read there about the normalized solutions F_ℓ and G_ℓ which have the interesting (but as yet unused by us) large r limits that $F_\ell \to \sin(\theta_\ell)$ and $G_\ell \to \cos(\theta_\ell)$ and therefore $G_\ell + i\,F_\ell \to e^{i\theta\ell}$, just a phase. This means then that $G_\ell - i\,F_\ell = e^{-i\theta\ell}$. Here, θ is the special phase

$$\theta_\ell = \theta_L = kr - n\ln(2kr) - \ell\pi/2 + \sigma_L = kr - n\ln(2kr) + \delta_\ell \qquad \delta_\ell \equiv -\ell\pi/2 + \sigma_L$$

Meanwhile, we know that the exact solution to our whole problem must have this general form:

$$\psi = \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, F_\ell(r)\, /kr$$

because the G_ℓ are badly behaved at r=0 (see details in radial equation.doc) . We don't yet know what the B_ℓ are. We know the large r limit of the above

$$\psi \;\rightarrow\; \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell \sin\theta_L/kr$$

Now suppose we break down this known solution into two terms by trivially writing

$$F_\ell = (F_\ell + i\, G_\ell)/2 \; + (F_\ell - i\, G_\ell)/2 \;=\; (i/2)\, \{\, (G_\ell \; - iF_\ell) \;-\; (G_\ell \; + iF_\ell)\, \}$$

so we get

$$\psi = (i/2)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell \;\; \{\, (G_\ell \; - iF_\ell) \;-\; (G_\ell \; + iF_\ell)\, \}\, /kr$$

$$= \; (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell \;\; (G_\ell \; - iF_\ell) \quad - \quad (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, (G_\ell \; + iF_\ell)$$

Now we take the large r limit in both terms to get

$$\psi \;=\; (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, e^{-i\theta\ell} - \quad (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, e^{i\theta\ell}$$

Let's now extract the non-ℓ portion of the phases out of the sums to get

$$\psi \;= e^{-i\{kr-n\ln(2kr)}\; (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, e^{-i\delta\ell} \qquad\qquad \delta_\ell \equiv -\,\ell\pi/2 + \sigma_L$$

$$-\quad e^{+i\{kr-n\ln(2kr)}\; (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, e^{i\delta\ell}$$

which is basically just a division into incoming + outgoing in terms of e^{ikr}/r. We have really done "not very much", but we have used asymptotic information from A&S.

But now I call upon the discussion in "radial equation.doc" of the so-called (by me only) Coulomb Plane Wave which has the following large-r partial wave expansion (derived there in Appendix A)

$$e^{ikz+i\ln[k(r-z)]} \;\approx\; e^{-i\{kr-n\ln(2kr)}\; (i/2kr)\, [\, \Sigma_\ell\; (2\ell+1)\, P_\ell(\cos\theta)\, (-1)^\ell\,]$$

which we notice is entirely "incoming", unlike the traditional e^{ikz}. We want to make the first term in ψ *be* this plane wave function, so that will require that

$$(-1)^\ell \;=\; i^\ell\, B_\ell\, e^{-i\delta\ell} \qquad \Rightarrow \qquad i^\ell\, B_\ell\, e^{+i\delta\ell} = (-1)^\ell\, e^{+2i\delta\ell} \;=\; (-1)^\ell\, (-1)^\ell\, e^{+2i\sigma\ell} \;= e^{+2i\sigma\ell}$$

Therefore, we have the following for ψ

$$\psi \;= e^{ikz+i\ln[k(r-z)]} - e^{+i\{kr-n\ln(2kr)}\; (i/2kr)\, \Sigma_\ell\, (2\ell+1)\, P_\ell(z)\;\; i^\ell\, B_\ell\, e^{i\delta\ell}$$

$$= e^{ikz+inln[k(r-z)]} + [\, e^{+i\{kr-nln(2kr)}/r] \;\; \{ \, (1/2ik) \, \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \, e^{+2i\sigma\ell} \, \}$$

and the object in {...} is defined as f(θ) and thus we have produced 21.22. QED.

Modified Coulomb Field (144)

I have a problem with Schiff's derivation here of 21.27, so I have provided my own, see a few paragraphs below.

What is my Schiff Problem here? His equation page 144 A is trivially true based on his definitions of F and G being related to W_1 and W_2. If we replace F in 21.12 with his expression A, we do in fact get 21.25, no argument there. The problem is getting to 21.26. I suppose we know that asymptotic forms for the F in u_c and for the new term added here, so I could imagine the result comes out right. But here is my big problem: he refers to W_2 as the ingoing term and he wants this term not changed, which motivates his form for equation A. I like the idea of not changing the incoming term, because this is our Coulomb Plane Wave and we want it to stay put. My problem is that the W_2 term is really the *outgoing* term as shown in 21.9,so something is amiss. There is just something loose here, and probably I would find that his asymptotic form does NOT follow if you do all the algebra.

So I will roll my own.

Derivation of 21.27

Suppose you have just the Coulomb potential active outside some radius r = a. In the full Coulomb case, we knew that in the Coulomb region, the solution had to be just F_ℓ due to the r=0 requirement. But if something else is going on near the center, as we enter the Coulomb region at r = a + ε, we will have an unknown mixture of F_ℓ and G_ℓ since in general a lincom of these is the Coulomb solution. In the large r limit, this means instead of getting just sine, we will get a mixture of sine and cosine which we can of course represent as just the sine with an added phase shift due to the activities inside the r=a region. We might call this the hadron phase shift or something like that, call it φ_ℓ . So our asymptotic behavior which we write in the ψ equation will be this:

$$\psi \;\; = \rightarrow \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \;\; i^\ell \, B_\ell \sin \, [kr - \eta ln(2kr) - \ell\pi/2 \; + \sigma_\ell + \varphi_\ell \,]/kr$$

If we then carry through the same analysis as before, everything is the same but we have this extra phase shift, so we would find that

$$f(\theta) \;\; = (1/2ik) \, \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \;\; e^{+2i(\sigma\ell+\varphi\ell)}$$

$$= \; (1/2ik) \, \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \, 2 \, e^{+2i\sigma\ell} \, e^{+2i\varphi\ell}$$

$$= \; (1/2ik) \, \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \, 2 \, e^{+2i\sigma\ell} \, (e^{+2i\varphi\ell} \; - 1 + 1)$$

$$= \; (1/2ik) \, \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \, 2 \, e^{+2i\sigma\ell} \, (e^{+2i\varphi\ell} \; - 1) \; + \; (1/2ik) \, \Sigma_\ell \, (2\ell+1) \, P_\ell(z) \;\; e^{+2i\sigma\ell}$$

$$= (1/k)\,\Sigma_\ell\,(2\ell+1)\,P_\ell(z)\,2\,e^{+2i\sigma\ell}\,e^{i\varphi\ell}\sin\varphi_\ell \quad + f_c(\theta) \qquad \text{// verifying Schiff 21.27}$$

where we have written the result in several ways, the last giving something you would add to the pure coulomb result. So if there were no "nuclear" shift, the first term would be zero.

Note: Schiff does not here mention the idea of **shielding** of the Coulomb force, but this is treated in Messiah. I think I could roll my own for this too, and may do so at some point. You have three regions: a tiny nuclear region with some $V_n(r)$, a relatively large Coulomb region inside the electron shells with $V_c(r)$, then finally a very large outside region where V(r) = 0. You match up all the solutions at the two boundaries.

Why you cannot take a limit of turning off the Coulomb force in scattering.

On the other hand, if there were no Coulomb activity at all, the answer would be this:

$$f(\theta) = (1/2ik)\,\Sigma_\ell\,(2\ell+1)\,P_\ell(z) f_\ell = (1/2ik)\,\Sigma_\ell\,(2\ell+1)\,P_\ell(z)\,(e^{+2i\delta\ell} - 1)$$

In this case, notice that the f_ℓ [as we might call the last factor] is not unimodular and could be zero. If we were to take the $\sigma_\ell = 0$ limit of the previous result, we get $f_\ell = e^{+2i\varphi\ell}$ which is unimodular, so we know that this limit cannot be working right. The reason is the long-range nature of Coulomb, and the fact that if we gradually turn off the Coulomb effect by taking limit n→0, the Coulomb Plane Wave

$$\psi = e^{i\{kz\ +n\ \ln[k(r-z)]\}}$$

does not approach e^{ikz} uniformly over r-z. No matter how small you make n, you can make r-z small enough to make the log term still be present. Another way is to say that you can always make θ small enough that the log term is still there. This relates to the δ(1 - cosθ) discussion on Schiff page 143 where we find that $f_c(\theta)$ does not → 0 as the Coulomb phase shifts all go to zero, but → δ(1 - cosθ).

We know from above that all of the scattered flux comes from the 'second term' in ψ_1 in the Coulomb case which is why there is no "-1" in f_ℓ. If you *could* take the limit just discussed, then suddenly the e^{ikz} term would appear in the limit, and it WOULD drive something into the outgoing flux, so this confirms the idea that this limit does not work.

Schiff's point in this discussion is the following: in Coulomb scattering, everything you measure comes only from the scattering center, none comes from the Coulomb plane wave "passing through". But in the forward direction, we get f(θ) ~ δ(1 - cosθ) which blows up at θ=0. It is blowing up due to scattering, not due to forward transmission of some plane wave e^{ikz}. Regardless of your "impact parameter" coordinate, an incoming particle is scattered very slightly, perhaps 10^{-10} of a degree, and this shows up as forward scattering. When you integrate over all impact parameters out to infinity, you get the delta peak in the forward direction.

The Classical Limit of QM Coulomb Scattering (145)

Schiff does lots of rough details to get a condition, but what it amounts to in the end is n >> 1, where n was our dimensionless parameter n = constants/k. Thus, the classical limit is the LOW energy limit! For the non-coulomb case the classical limit is the HIGH energy limit, and he comments on this fact.

The way you see quantum effects in Coulomb scattering involves identical particles, but large n makes this wash out. We have not yet done this identical particle problem.

Done: 12.20.08. My notes for this tough chapter are longer than the chapter!