Schiff Chap 7 Symmetry PhL 12.27.08

Title of the chapter is : Symmetry in Quantum Mechanics

Intro: This chapter talks separately about geometrical symmetry and dynamical symmetry. Only the last section 30 is about dynamical symmetry.

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26. Space and Time Displacements (188).

The issue of active vs passive comes up at once. Schiff likes active where we move the wavefunction or the ket, rather than move the coordinate axes. This entire section 26 (7 pages) consists of very short subsections and I have carefully maintains Schiff's structure in the notes below.

Unitary Displacement Operator. Using a Taylor expansion for $\psi(x+\rho)$, S concludes that the unitary translation operator must be $T(\mathbf{a}) = e^{-i\mathbf{a}\bullet\mathbf{p}}$ where **p** is the Hermitian operator we know and love.

Equations of Motion. If you operate on a ket with this T, the resulting ket will solve the SE only if [T,H] = 0, as he shows. If this is true, we have a situation of translation invariance.

Symmetry and Degeneracy. Think of $\langle\mathbf{r}|\mathbf{p}\rangle = e^{i\mathbf{p}\bullet\mathbf{r}}$, the momentum eigenfunctions. Suppose [H,R] = 0 for rotations. Then $0 = \langle\mathbf{r}|[H,R]|\mathbf{p}\rangle = \langle\mathbf{r}|HR\mathbf{p}\rangle - \langle\mathbf{r}|RH\mathbf{p}\rangle = \langle\mathbf{r}|H|\mathbf{p}'\rangle - E_{\mathbf{p}}\langle\mathbf{r}|\mathbf{p}'\rangle$ for all r, so it must be true that $H|\mathbf{p}'\rangle = E_{\mathbf{p}}|\mathbf{p}'\rangle$ so state $|\mathbf{p}'\rangle$ is energy-degenerate with state $|\mathbf{p}\rangle$. Whenever the transformed state like $|\mathbf{p}'\rangle$ is linearly independent of the original state $|\mathbf{p}\rangle$, symmetry causes degeneracy. In the above argument, we could have omitted $\langle\mathbf{r}|$ altogether. With it, we conclude that $e^{i\mathbf{p}\bullet\mathbf{r}}$ and $e^{i\mathbf{p}'\bullet\mathbf{r}}$ are degenerate eigenfunctions of H. Schiff talks about this with general $[H,\Omega] = 0$. In the case of the H atom, we have $[H,L_{\pm}] = 0$, so the different m-labeled states are degenerate.

Matrix elements of displaced states. This just gets reader used to the idea of the sandwich of U around an operator. On the left of A, it is the states that "have moved" and have primes. On the right we have the unmoved states, but the operator has moved. Then 26.6 is just an example of how an operator can be moved by sandwich of a unitary operator, in this case translation. Notice: $|\beta'\rangle = U|\beta\rangle$

The Group Concept. A fast outline of what a group is, and description of various words used in connection with groups: compact, abelian, continuous, connected, isomorphism, matrix representation. In a 2-page footnote, Schiff gives 9 group theory references of which I have 4. Schiff's 3rd 1968 edition probably added this stuff, but preface does not say so. Groups were getting to be (and still are) a hot topic.

Time Displacement. Notice the lack of symmetry between time and space here. We use **r** in a wavefunction or on a bra to be a basis element of a representation, but we use t as a parameter of a state. We write $\langle\mathbf{r}|\alpha\rangle$ but we don't write $\langle t|\alpha\rangle$. Instead, we write $|\alpha(t)\rangle$. Time and space are NOT on an equal footing in non-relativistic QM (as the SE shows!). But, the operator U which does time translation has

the same form as the space translation operator, but the sign is opposite, and this was done by definition of the time translated state. We want to get a + sign in our $e^{+iHt/\hbar}$ and this moves a state forwards in time. Look at 26.9 and imagine that $\alpha(t)$ somehow "peaks" at t=0, Then the transformed state $\alpha(t-\tau)$ peaks at $t = \tau$, so we have moved this little "time wave" into the future by amount τ. H must have no explicit t dependence for this to work. In this case, since [H,H] = 0, we know that the transformed state satisfies the SE, and of course we know from earlier that this is how states move in the Schrodinger picture. Oddly, Schiff does not mention this fact that we just read about in the last chapter. He gives a counter example showing that if H = H(t), the time-translated state no longer satisfies the SE.

Comment: We certainly know this SE fact (Schrodinger picture)

$$i\hbar\partial_t |\beta_S(t)\rangle = H\,|\beta_S(t)\rangle \qquad \Rightarrow \qquad |\beta_S(t)\rangle = e^{-iHt/\hbar}\,|\beta_S(0)\rangle$$

But in this section, U is defined in 29.9 to push something *backward* in time

$$U(\tau)\,|\alpha(0)\rangle = |\alpha(-\tau)\rangle \qquad \text{which is why} \qquad U(\tau) = e^{+iHt/\hbar}$$

You can of course define U however you want. It does seem a little odd he did it this way. I think he is doing some kind of "active" transform in analogy with what he did for momentum and space translation.

27. Rotation and Unitary Groups (194).

Proper Rotation Group. He describes SO(3). Groups with smooth parameters are Lie Groups, and this one is compact and non-abelian. There are of course 3 parameters. Matrices R are unitary.

Geometrical Isomorphism. This group is isomorphic to the set of points in a sphere of radius π. A vector in this sphere I would call $\mathbf{n} = \theta\hat{\mathbf{n}}$ and the rotation I would call $e^{-i\mathbf{n}\bullet\mathbf{J}}$ but we have not gotten there yet. He claims that points on opposites sides of this sphere are the same, and this are externally connected outside the sphere somehow, so the group is doubly-connected, something a little vague to me.

Infinitesimal Rotations. First we can accept I think that $\mathbf{r}' = \mathbf{r} + \mathbf{n} \times \mathbf{r}$ for a small rotation using the right hand rule. where a point r's distance from the rotation axis is $\sin\theta$ where θ is the polar angle of point r relative to n. Now look at equation A on page 197. Here, we are doing "active" so we move the coordinates backwards, ket notation makes this clear. Then go to small rotation, do Taylor again, use usual vector identity and you find that the rotation generators are $-\mathbf{L} = -\mathbf{r} \times \mathbf{p}$. Very good.

Spin of a Vector Particle. The hypothesis here is that a "vector" particle has a 3 component column vector as its wavefunction which he calls $\boldsymbol{\psi}$ (bolded for vector). In equation A page 198 we see the same reverse motion of the coordinates as before, but now we also get forward motion of the state vector by the 3x3 rotation matrix R, so now R appears twice. This results in the generator J = L+S where L is the same **r**x**p** that it was before, and **S** is an internal 3x3 matrix job. Here our basis states are $|\,\mathbf{r}\,;\ s,m_{\mathbf{s}}\rangle$ and in this basis, L is as stated. We are not yet in the basis $|\,\boldsymbol{\ell},m_{\boldsymbol{\ell}}\ ;\ s,m_{\mathbf{s}}\rangle$ where L would ALSO be a matrix.

Comrels for the generators. For r x p, we can compute the comrels from the $[x,p_{\mathbf{x}}]$ type comrels and we get the usual results, where $\hbar$ is included on the RHS. The S matrices on page 198 satisfy the same, and

the general rule he likes to write as **J** x **J** = i$\hbar$**J** . He uses the phrase **Lie Algebra** here for the first time, and shows how you can "integrate" the small angle result to get the usual exponential result with the usual minus sign in the exponent. He always uses $\boldsymbol{\varphi}$ as my vector **n**. It seems to me that φ looks too much like a wavefunction or an azimuthal angle, but at least it does suggest an angle which n does not. Fine.

Choice of a representation. Schiff is going to "solve" the O(3) Lie Algebra to find the eigenstates. He assumes states j,m which are eigenstates of J^2 and J_3. The J^2 eigenvalue is called f(j) at this point. Equation 27.20 says that, unless you want all states to be 0 which gives no useful result, you look for matrix elements of $J_\pm$ that might not be zero. This leads to a definition of λ_m as in 27.22. So in this section, he has just "set up" the problem.

Values of m, f(j) and λm . In this section he will solve the problem set up in the last section. The reader can easily verify that the equation $h_{m-1} - h_m = 2m$ has the only solution $h_m = C - m(m+1)$ [I just did this in pencil] . To kep the RHS from going negative in 27.24, m has to truncate at both high and low ends leading to max values m_1 and m_2 at the extremes, and things just be integrally spaced to get truncation. He then solves 27.24 for these two values and finds that $m_2 = -m_1 - 1$, so $m_1 - m_2 = 2m_1 + 1$ but must be integer, so m = integer or half integer, no other options. Then he shows that $\langle J^2 \rangle = m_1(m_1+1)$. Then rename m_1 to get j. Then the whole problem is solved with results in 27.25. It is good to do this in the open and not just quote the results from somewhere!

Angular momentum matrices (203) He quotes the generator matrices on page 203 for j = 1/2, 1 and 3/2. you could compute these yourself by using the given matrix elements of $J_\pm$ in 27.25 since these are just lincoms of J_x and J_y. Only non-zero values will be on the first off diagonals. He sort of fails to mention this fact that I suppose is obvious to the reader.

Connection with Spherical Harmonics. Using the known $L_\pm$ angular coordinate reps in 27.27, he claims you can show (27.28) and conclude 27.29 which is my usual way to write things. In words he says that the spherical harmonics can be thought of as $W_{\ell m,\ \theta\varphi}$ being a unitary transformation between the angular coordinate representation and the angular momentum representation, he uses those words.

Spin Angular Momentum. I think he is just claiming that a free particle must have some spin S. We saw an example earlier of spin 1.

Covering Group. Make that sphere have larger radius 2π, then surface of this larger sphere is all of the same value, either +1 or -1 for half-odd-integrals, as he calls them. For the π sphere, you might have +i at one end of a diameter and -i at the other end for the half-odds. So this new sphere is isomorphic to the covering group of SO(3) which is then singly connected, but this is all hazy to me.

About U(2) and SU(2). The j=1/2 representation of SO(3) is exactly SU(2). But he claims in fact that SU(2) is isomorphic to that larger covering group since you can do things like 3π unique rotations.

The Groups U(n) and SU(n). The first has n^2 real parameters, the second has n^2 -1.

Why is this? For U(n) there are at first $2n^2$ reals, but unitary means $U = e^{iH}$ with H hermitian. So the question is: how many parameters does such an H have? Call it $2n^2$ real parameters to start with, 2n on

the diagonal and $2(n^2-n)$ in each triangle. The diagonals have to be real, so we lose n real parameters there, and the triangles are CC, so lose $(n^2 - n)$ reals there, so

$$2n^2 - n - (n^2-n) = n^2$$

Then SU(n) puts one condition on these elements to get det = 1, so n^2-1 for SU(n).

Claim: you can represent any g in U(n) as $e^{i\theta/n}$ h where h is in SU(n). Here θ is the extra real parameter. Why is this? It seems to me that U(1) element could just be $e^{i\theta}$ without that /n item. In any event, you have added an extra parameter and the result is still closed.

Generators of U(n) and SU(n). A lot of good basic stuff here. We know SU(n) has n^2 -1 real parameters. The rank is the max number of *commuting* generators and is of course 1 for SU(2). Claim is that SU(3) has rank 2 so you can pick (or maybe find) two of the λ_i generators. Racah shows that rank = number of combinations of the generators you can make which commute with all generators. Casimir showed that one of these Racahs was a bilinear thing like J^2 for SU(2).

The SU(3) Group. For SU(3) there is one of these bilinear Casimir things, and the other is a trilinear messy combination says Schiff. On page 209 Schiff uses ε_{ijk} for the first time in this book I think, but intends it only for n=3. For n = 8 he calls this thing f_{ijk} . Very strange.

But then he clouds the picture by saying on page 210 that *all* the commuting operators are Casimirs, not just the bilinear ones. Thus, he says SU(3) has two Casimirs. I think this is now the standard usage. One is the sum of the squares of the 8 generators.

So he gives a little information related to SU(3) and has "The Eightfold Way" Gell-Mann reference from their 1964 book. I think this SU(3) idea for particles first arose around 1961 and quarks were proposed in 1964. Those years directly preceded this 1968 book, and also directly preceded my 1970 appearance at Berkeley. Chew's S matrix was going down, and the QCD stock was going up. My awareness of history was zero at that time!

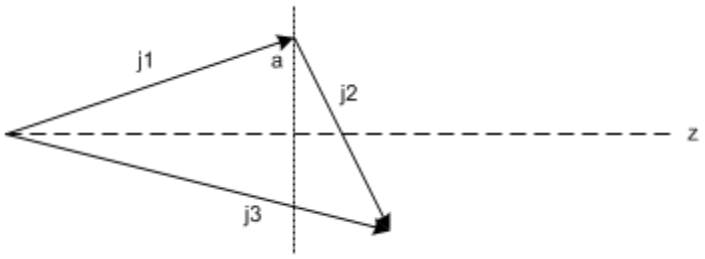
Representation of the SU(3) generators in terms of x and p. I had no idea this was even possible! He wants to identify 3 of the generators with the usual L_i combinations of x and p, then 5 more as an $\ell = 2$ tensor of some sort. The tensor Q_{ij} is shown in a sort of mixed format in 27.44 which in itself seems very strange to me. But then each of these is associated with one of the eight λ_i on page 212, where the values of parameters α and β are kept free. The total sum Casimir is then as shown in 27.47. For the right choice of α and β, this can surely match the 3D isotropic HO Hamiltonian, so that HO's solutions can be taken to be manifolds of the same value of Casimir C call it c, similar to j. I wonder how many states there are for a given c? We will find out soon!

Would this be a "geometrical symmetry" for the HO? What kind of "rotation" preserves $xy + p_x p_y$? Well, what kind of rotation preserves $xp_y - yp_x$? The answer to the second question is R_z in 3-space, but the answer to the first would have to be some rotation in the full 6-space that is phase space. Enough.

Why can you even *find* combinations of the three x and p that form a representation of the SU(3) Lie Algebra? The [x,p] commutator somehow induces this fancier SU(3) Lie algebra through these combinations. I can just put this question on the list for now.

28. Combination of Angular Momentum States; Tensor Operators (212)

The "triangle rule" seems a little vague. If you add two real 2D vectors **A** and **B** to get **C**, they do form a triangle. In this case, we know that the max length that **C** can have is when **A** and **B** are aligned, where C = A + B. Similarly, the least length that **C** can have is when **A** and **B** are opposed, where C = |A-B|. So the rule regarding the length of C is that |A-B| < C < A+B. For angular momenta combining, we know that $|j_1 - j_2| < j_3 < j_1 + j_2$ and this would apply if we were to draw a triangle of these three vectors having normal lengths. That is, $\mathbf{j_1}$ would have length j_1 , not $\sqrt{j_1(j_1+1)}$. So this is a triangle construction, yes, but the angle a has to be chosen so that j_3 has one of the allowed lengths. The starting j_1 and j_2 can be assumed to have legal lengths.



In this picture we imagine that we have translated the triangle so the left end lies on the z-axis and we have then rotated about the z axis to get the triangle into the plane of paper. So this triangle does give the triangle rule inequality sandwich. It also gives the right result that you add up the z components in the obvious way. So I guess this is a good name after all.

Eigenvalues of the Total Angular Momentum (213). This is really a direct product space exercise, but Schiff does not cast it in the that language, but I'm sure Messiah does. So we are going to have for example

$$1 \otimes 1 = 0 \oplus 1 \oplus 2$$

The states on the left are enumerated as $|1m_1\rangle \otimes |1m_2\rangle$ and on the right as |jm>. Each J_i does nothing to the "other" space, the two spaces are assumed fully decoupled. Thus

J_1 really means $J_1 \otimes 1$

and acts as unity in the second subspace. The idea is that the matrix $T_{mm',JM} = \langle j_1 m j_2 m' | JM \rangle$ is just a unitary basis change from one representation to another, and these ARE the C-G coefficients. Schiff shows a direct method of computing the coefficients by just "fiddling" with the raising and lower operators. For example,

$J_+ = J_{1+} \otimes 1 + 1 \otimes J_{2+} = J_{1+} * J_{2+}$ // in my own notation somewhere.

C-G coefficients, recursion relations, construction procedure, some particular values.

Thus for example

$J_+ |jm\rangle = f(j,m) |jm+1\rangle$ where f is the usual raiser coefficient.

But we also have

$$J_+ |jm\rangle = J_+ \{ |m_1m_2\rangle\langle m_1m_2|jm\rangle \} = \langle m_1m_2|jm\rangle [J_{1+}{}^* J_{2+}] |m_1m_2\rangle$$

$$= \langle m_1m_2|jm\rangle \{ J_{1+} \otimes 1 + 1 \otimes J_{2+} \} \{ |j_1m_1\rangle \otimes |j_2m_2\rangle \}$$

$$= \langle m_1m_2|jm\rangle \{ f(j_1,m_1) |j_1m_1+1\rangle \otimes |j_2m_2\rangle + f(j_2,m_2) |j_1m_1\rangle \otimes |j_2m_2+1\rangle \}$$

$$= \langle m_1m_2|jm\rangle \{ f(j_1,m_1) |m_1+1,m_2\rangle + f(j_2,m_3) |m_1,m_2+1\rangle$$

and then comparing the above, we can conclude that

$$f(j,m) |jm+1\rangle = \langle m_1m_2|jm\rangle \{ f(j_1,m_1) |m_1+1,m_2\rangle + f(j_2,m_3) |m_1,m_2+1\rangle$$

where we now have a mixture of known functions f with some states and with some C-G coefficients which we want to discover. If we were to close this with $\langle jm+1|$ we would get

$$f(j,m) = \langle m_1m_2|jm\rangle \{ f(j_1,m_1) \langle jm+1|m_1+1,m_2\rangle + f(j_2,m_3) \langle jm+1|m_1,m_2+1\rangle$$

and now we have an equation relating three different C-G coefficients. By messing around in this way, and forcing that maximum case phase to be +1, you can solve for the coefficients. I don't really care about the details right now, I can imagine how it all goes.

So look now at the bottom case on page 218. The $|m_1m_2\rangle$ states are listed off on the left side in an order of decreasing m_1+m_2. Thus, the first state is $|11\rangle$ and the last is $|-1-1\rangle$. In each grouping where the sum is equal, states are listed with the leftmost index going from largest to smallest. I have added my own column in pencil which shows the total $m_1 + m_2$, and of course this is decreasing as claimed.

Along the top states are listed off again, according to decreasing possible m_J. The upper number is J, the lower is m_J for each column heading. You see that the M_J's decrease. For each group of equal m_J, the J's are listed largest to smallest.

An entry in the table must be zero if $m_J \neq m_1 + m_2$ so this causes entries only in boxes on the diagonal as shown. The dimension of a box is given by the number of states in either basis that have a certain value of m_J. I think this is easiest understood in the JM basis. You see that ALL J values can have M = 0, and there are three possible J values in this case, 1,2,3, so the middle box is 3x3 and the outer boxes decrease in size by 1. In the general integer case, there will be (I think) $(j1+j2)-(j1-j2)+1 = 2j_2+1$ values of J for the middle box (and in the direct product expansion), where j_2 is the smaller one. So in the 1x1 case, this gives 3 which is correct. In a 2x2 case we would have 5.

The matrix $\langle j_1mj_2m'|JM\rangle = T_{mm',M}$ is in block diagonal form. Obviously this is all related to how the group representation functions transform, and I suppose that comes up soon when we talk tensor operators. Or maybe we just close the sum rule with some Euler angles $\langle\varphi\theta\psi|$.

Matrix elements for rotated states (219). This is a strange little section. It asks what happens if you take a general operator Ω and sandwich it between rotations. S points out that what happens to Ω is determined by the infinitesimal rotation case which is controlled by [Ω,J]. But I know that. And in general I know this for a Cartesian vector operator

$$R\mathbf{V}R^{-1} = R^{-1}\,\mathbf{V}$$

and more generally I know that

$$R\mathbf{V}_{jm}R^{-1} = [D^{(j)}{}_{mm'}]^{-1}\mathbf{V}_{jm'}$$

for a tensor operator.

Irreducible Tensor Operators (219). These are defined in analogy with the rules for spherical harmonics, but tradition uses k and q in place of j and m for the operator. So 28.18 is like 28.17.

Products of Tensor Operators, or of one Operator and One State; Wigner Eckart. I think a give Schiff demerits for a lack of clarity here, although he comments on this on page 222. When he writes

$$T(j_1m_1)|\,j_2m_2\rangle = \Sigma_{jm}\ |\,jm\rangle\langle jm\,|\,j_1m_1;\,j_2m_2\rangle$$

you want to ask "How can this be true for an arbitrary operator T? The RHS seems to show no effect at all of this operator. It might be a vector operator, and there are lots of DIFFERENT vector operators, so how can it make no difference which one you use? " The answer is that |jm> DOES depend very much both on the nature of the operator T, and the nature of the state $|\,j_2m_2\rangle$. So it would be clearer were we to write it as Messiah does, but even he is a little hazy. I might write:

$$T(j_1m_1)|\,j_2m_2;\,\sigma_2\rangle = \Sigma_{jm}\,|\,jm;\,T,\,\sigma_2,\,j_1,\,j_2\rangle\langle jm\,|\,j_1m_1;\,j_2m_2\rangle$$

Let's assume that, from the definition of a tensor operator, we could show that the above is true, where the states $|\,jm;\,T,\,\sigma_2,\,j_1,\,j_2\rangle$ have the angular momentum properties |jm>, and the other things in the state are just parameters or labels. The sum is over the usual triangle rule. Now, suppose we close with a bra of this type: $\langle\,j_3m_3;\,\sigma_3\,|$. Then we get

$$\langle\,j_3m_3;\,\sigma_3\,|T(j_1m_1)|\,j_2m_2;\,\sigma_2\rangle = \Sigma_{jm}\langle\,j_3m_3;\,\sigma_3\,|\,jm;\,T,\,\sigma_2,\,j_1,\,j_2\rangle\langle jm\,|\,j_1m_1;\,j_2m_2\rangle$$

Now our next claim is that the left term on the RHS kills off the jm sum and we have

$$\langle\,j_3m_3;\,\sigma_3\,|T(j_1m_1)|\,j_2m_2;\,\sigma_2\rangle = \langle\,j_3m_3;\,\sigma_3\,|\,j_3m_3;\,T,\,\sigma_2,\,j_1,\,j_2\rangle\langle\,j_3m_3\,|\,j_1m_1;\,j_2m_2\rangle$$

$$= f(j_3,\,\sigma_3 \quad T,\,j_1 \quad j_2\,\sigma_2 \quad m_3)\langle\,j_3m_3\,|\,j_1m_1;\,j_2m_2\rangle$$

A final thing we then have to show is that the function f does not depend on m_3. I think you can show this by considering this equality

$$\langle\,j_3m_3;\,\sigma_3\,|\,J_-\ J_+\,|\,j_3m_3;\,T,\,\sigma_2,\,j_1,\,j_2\rangle = \langle\,j_3m_3;\,\sigma_3\,|\,J^2 - J_z^2 - J_z\,|\ j_3m_3;\,T,\,\sigma_2,\,j_1,\,j_2\rangle$$

I think the LHS will then give the same factor as the RHS, but will have m_3+1, and this will then show that the object is the same for m_3 as it is for $m_3 + 1$, and just therefore be independent of m_3. So we end up with

$$< j_3 m_3; \sigma_3 |T(j_1 m_1)| j_2 m_2; \sigma_2> = < j_3 m_3; \sigma_3 | j_3 m_3; T, \sigma_2, j_1, j_2> < j_3 m_3 | j_1 m_1; j_2 m_2>$$

$$= < j_3 \ \sigma_3 \,||||\, T(j_1) \,||||\, j_2 \ \sigma_2 > \ < j_3 m_3 | j_1 m_1; j_2 m_2> \delta_{m3,\, m1+m2}$$

where I expose on the RHS the obvious property of the C-G coefficient. I show a quad bar inside the reduced matrix element to make my result look like Tinkham page 132 5-67. People then seem to define the double bar version differently. Messiah page 573 shows things pulling out $1/\sqrt{2j_3+1}$ where this is the left bra's j value. Schiff page 223 in 28.24 shows this same factor pulled out, but they he adds an extra phase!

The **Tinkham** approach to this situation is much better I think, though he falls back on coordinate space wavefunctions when that is not necessary. Here is my version of Tinkham. Suppose we start with our Wigner theorem matrix element of interest and insert $R^{-1}R$ twice and then fiddle:

$$<j_3 m_3 \sigma_3| T(j_1 m_1)| j_2 m_2 \sigma_2> \ = \ <j_3 m_3 \sigma_3| R^{-1}R \ T(j_1 m_1) \ R^{-1}R| j_2 m_2 \sigma_2>$$

$$= \ < R \ \ j_3 m_3 \sigma_3 | R \ T(j_1 m_1) \ R^{-1} | R \ j_2 m_2 \sigma_2>$$

$$= D^{(j3)}_{m3',m3}(\text{angles}) \ D^{(j1)}_{m1',m1}(\text{angles}) \ D^{(j2)}_{m2',m2}(\text{angles}) <j_3 m_3' \sigma_3| \ T(j_1 m_1')| j_2 m_2' \sigma_2>$$

where I may have some * missing and some indices in the wrong order. Tinkham then combines the rightmost two of the three D's with the C-G coefficients where they are defined as the coefficients in the reduction of the direct product representation. The next step is to apply $\int$ d(angles) to both sides. On the LHS nothing happens if norm right, and on the right side we get D function orthogonality, which Tinkham unfortunately never writes down anywhere, and this then gives the Wigner Eckart result.

Now let's ponder Tinkham versus Schiff a bit. Tinkham has used the finite rotations, Schiff is trying to use the generators only. In Tinkham, we never actually try to *apply* T to the state $|j_2 m_2 \sigma_2>$ so we don't have to wonder about how to label such a state. T just sits there in the middle all the time. We rotate each state in the QM HS, and we claim this rotation does not affect σ_2 nor does it add more quantum numbers as state labels.

Schiff's approach is to DEFINE his state |jm> as in 28.11, and to DEFINE the operators T(jm) as in 28.19, and work from there.

I suppose it is all OK, but somehow I just don't feel convinced by Schiff's presentation. He had to say something about all this stuff and he gave it his best shot, with lots of labor on his part I admit.

29. Space and Time Inversion (224)

Classical equations usually involve second time derivatives so have time reversal invariance. Remember that the SE only has a first time derivative.

Space Inversion. The 3x3 matrix that does this to a Cartesian vector is obvious.

I am always unhappy with Schiff's starting points, and here we go again. I think the symbol I is just that 3x3 matrix on page 224, so the conjecture of 29.1 is this:

$\psi(-r) = \omega\psi(r)$ $\langle -r|\psi\rangle = \omega\langle r|\psi\rangle$

and I see no need for an α and α' label.

Unitary Inversion Operator. Now I acted in R^3 , but U_I which I will call U acts in the QM HS. So 29.2 says this:

$\langle r|U|\alpha\rangle = \langle r|\alpha'\rangle = \hat{U}\langle r|\alpha\rangle$ or $\psi_{\alpha'}(r) = \hat{U}\,\psi_\alpha(r)$ // 29.2

so he uses the same symbol for two different animals. My U is the HS operator, and my $\hat{U}$ is the corresponding operator that acts on wavefunctions in the coordinate representation.

Let's try to derive 29.3 directly and ignore this junk discussion so far:

LHS $= \hat{U}\,\psi_\alpha(r) = \langle r|U|\alpha\rangle = \langle U^\dagger r\,|\,\alpha\rangle = \omega\langle -r|\alpha\rangle$ = RHS

If this is to be true, he must be assuming that

$\langle U^\dagger r\,| = \omega\langle -r|$ or $U\,|r\rangle = \omega^*\,|-r\rangle = \omega^*\,|Ir\rangle$

Then we would have $U^2\,|r\rangle = (\omega^*)^2\,|r\rangle$

I guess $|r\rangle$ means a certain kind of particle located at r, and certainly then $(\omega^*)^2$ must be unimodular and therefore so must be ω. If we somehow equate this squared parity operator with a rotation by 2π (but why on earth would we do that?) then we might argue that $\omega^2 = 1$ for an integer spin particle, and $\omega^4 = 1$ for a half integer one. This is all complete BS ! There is more going on here that Schiff does not know or is not saying.

Intrinsic Parity. I think Schiff was not really up on this and had to put something into his book on the subject. He then talks about particles having "intrinsic parity" and that pions are pseudoscalar in this regard, which nucleons are assumed to have + parity. I notice that Messiah says not much on this subject.

Inverted States and Operators. What does U do to various operators? He tries it out on these operators: **r**, **p**, **L**, **S**, **J** . The claim is that each does what you think classically. **r** and **p** are polar vectors, the ang moms are axial vectors.

Time Reversal. In place of U for parity, we now have T. An argument on page 228 shows why T cannot be a normal linear operator.

Antilinear operators. If we make T be antilinear, then the contradiction of the last section goes away. Also, he shows that if [T,H] = 0, then $T\psi$ satisfies the same SE as ψ but with t replaced with -t.

Antiunitary operators. Try writing T = UK where U is normal unitary. This T is then called antiunitary. This K itself is antiunitary.

Time reversal operator for a Spin 0 particle. We expect r not to change sign under t, but p should changes sign and so should L. These facts are shown on page 230. In this case, just using T = K does the job.

Time reversal operator for other Spin particles. If we go with the usual "ang mom" spin matrices on page 203, we see that for j = 1/2 and j = 1 and j = 3/2, it is S_y that is purely imaginary and S_x and S_z are purely real. I presume this will be true for any spin in the normal convention for these matrices. But we want to have the operator notion that **S**T = -T**S** since **S** is an angmom. Call it T^{-1}**S**T = -**S**. If we try our spin 0 operator T = K, this rule only works for the S_y matrices since they are imaginary. I presume by the way that $T^{-1} = (UK)^{-1} = K\,U^{-1}$ and we have to stop there. So I agree, K does not work.

Suppose we try $T = \exp(-i\pi S_y)K$. Then:

$$T^{-1}S_iT = K\exp(+i\pi S_y)\,S_i\exp(-i\pi S_y)K = K\exp(-i\pi S_y^*)\,S_i^*\exp(i\pi S_y^*)KK$$

$$= \exp(+i\pi S_y)\,S_i^*\exp(-i\pi S_y)$$

For i = y, this gives just $S_y^* = -S_y$ which is correct. For other cases, I have to use my sandwich rules which I quote are these

$$\exp(-i\theta J_y)\;J_z\exp(+i\theta J_y) = J_z\cos\theta + J_x\sin\theta$$
$$\exp(-i\theta J_y)\;J_x\exp(+i\theta J_y) = J_x\cos\theta - J_z\sin\theta$$

and if I set $\theta = -\pi$ these become

$$\exp(i\pi J_y)\;J_z\exp(-i\pi J_y) = J_z\cos\pi - J_x\sin\pi = -J_z$$
$$\exp(i\pi J_y)\;J_x\exp(-i\pi J_y) = J_x\cos\pi + J_z\sin\pi = -J_x$$

and sure enough, since in our case J_z and J_x are real, we can add a * on the left to get

$$\exp(i\pi J_y)\;J_z^*\exp(-i\pi J_y) = -J_z$$
$$\exp(i\pi J_y)\;J_x^*\exp(-i\pi J_y) = -J_x$$

and now we have all three S_i doing the right thing! For spin-1/2 we know that

$$\exp(-i\pi S_y) = = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\,\sigma_2$$

so in this case we have $T = UK = [-i\sigma_2]\,K$.

So, in the usual spin matrix basis, I now know what the T operator is. It is a matrix of the same dimension as the spin matrices, ie, of dimension (2j+1). This is more than I know for the P operator due to the lack of clarity of the previous section. Perhaps $P|\alpha\rangle$ = phase x $|\alpha\rangle$.

Wigner 1932 is given credit for the first person to write about time reversal in QM. The W-E theorem was 1930. I think Clebsch and Gordon did their coefficients around 1866 in a field of math they called Abelian Functions at that time. I suspect this relates directly to group representation theory which maybe came a little later.

Systems of Several Particles. Consider a set of n particles of the same spin, write T as in 29.22. Conclude that T^2 is then as shown, all real. Then break result up this way:

integer spin particles: $T^2 = +1$

half integer spin particles $T^2 = (-1)^n$

Next topic: Question is this: is Tu_k degenerate with u_k , an energy eigenfunction? Cannot be if $T^2 = -1$! Suppose we have a crystal made of atoms each of which has an odd number of electrons, so we have this case. Then the claim is that any energy eigenstate of such an atom's electrons must be degenerate with degeneracy at least two from this time reversal fact. This is called **Kramer's degeneracy** (1930) in a crystal. You can break this by applying a B field and thus split these levels. What about in an isolated atom with an odd number of electrons? Maybe the degeneracy there is obvious anyway. So in typical fashion, Schiff brings up a subject and gives only the haziest of comments about it. This degeneracy is associated with time reversal of a system with an odd number of half-integral spin particles.

Reality of Eigenfunctions. Claim is that all non-degenerate energy eigenfunctions are real in a system that has time-reversal invariance. If you have a system with only m= 0 angular momentum states (as in spinless scattering theory), then again, time reversal invariance implies real eigenfunctions. We then might associate complex eigenfunctions with a lack of time reversal invariance, and this is born out in the scattering theory which we modeled with a complex potential, and there the phase shifts came out complex, so the wavefunctions were indeed complex. So the point is that there is a connection between the reality of eigenfunctions in certain situations with time reversal invariance of the Hamiltonian.

30. Dynamical Symmetry (234)

We are reminded that symmetry often implies degeneracy. The comment about problems being solvable in multiple coordinate systems implies extra degeneracy, but reason is not made clear, I could not write down my version of the reason very well. Here we deal with symmetries that S claims have no geometric basis. But the two examples considered here do have *classical* dynamical symmetries. This may not be the case for other QM problems with dynamic symmetry.

I never heard of the Laplace-Runge-Lenz vector (LRL vector) as a Coulomb force conserved vector, but here is the proof from wiki:

This *is* the RLR vector: $$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

Direct Proof of Conservation

A central force $\mathbf{F}$ acting on the particle is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = f(r)\frac{\mathbf{r}}{r} = f(r)\hat{\mathbf{r}}$$

for some function $f(r)$ of the radius r. Since the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved under central forces, $\frac{d}{dt}\mathbf{L} = 0$ and

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d\mathbf{p}}{dt} \times \mathbf{L} = f(r)\hat{\mathbf{r}} \times \left(\mathbf{r} \times m\frac{d\mathbf{r}}{dt}\right) = f(r)\frac{m}{r}\left[\mathbf{r}\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) - r^2\frac{d\mathbf{r}}{dt}\right]$$

where the momentum $\mathbf{p} = m\frac{d\mathbf{r}}{dt}$ and where the triple cross product has been simplified using Lagrange's formula

$$\mathbf{r} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \mathbf{r}\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) - r^2\frac{d\mathbf{r}}{dt}$$

The identity

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r^2) = 2r\frac{dr}{dt}$$

yields the equation

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = -mf(r)r^2\left[\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right] = -mf(r)r^2\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

For the special case of an inverse-square central force $f(r) = \frac{-k}{r^2}$, this equals

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = mk\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{d}{dt}(mk\hat{\mathbf{r}})$$

Therefore, **A** is conserved for inverse-square central forces

$$\frac{d}{dt}\mathbf{A} = \frac{d}{dt}(\mathbf{p} \times \mathbf{L}) - \frac{d}{dt}(mk\hat{\mathbf{r}}) = \mathbf{0}$$

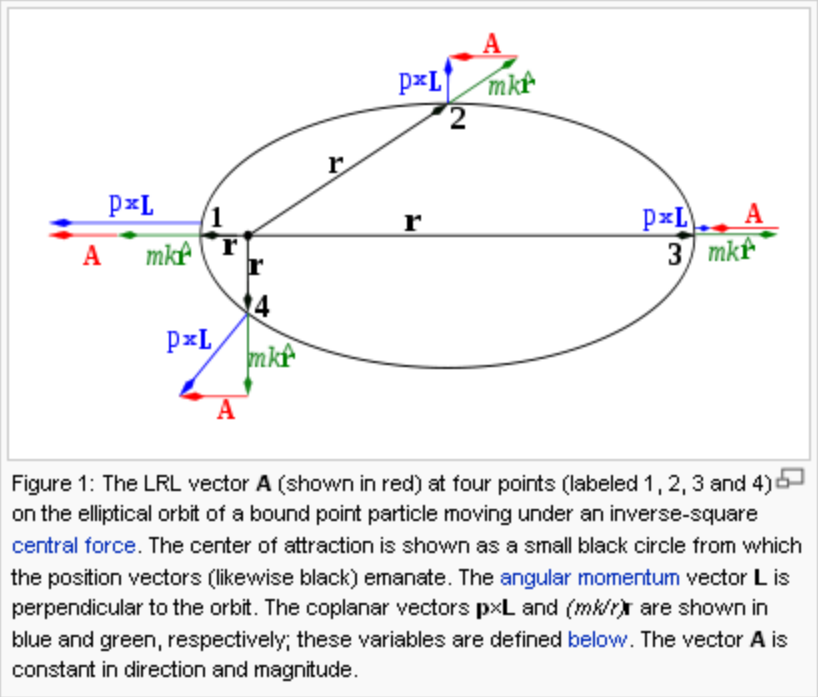


Figure 1: The LRL vector **A** (shown in red) at four points (labeled 1, 2, 3 and 4) on the elliptical orbit of a bound point particle moving under an inverse-square central force. The center of attraction is shown as a small black circle from which the position vectors (likewise black) emanate. The angular momentum vector **L** is perpendicular to the orbit. The coplanar vectors **p**×**L** and $(mk/r)\mathbf{r}$ are shown in blue and green, respectively; these variables are defined below. The vector **A** is constant in direction and magnitude.

So there it is, and **L•A** = 0 so **A** lies in the orbital plane and the above figure shows the vector **A** at four points on an orbit. If you change the orbiting direction, **A** = **p** x **L** = $mk\hat{\mathbf{r}}$ does not change because both **p** and **L** change sign.

Schiff's vector is **M** = **A**/μ , just rescaled.

Now so much for classical mechanics. Here is an interesting comment:

• A **generalized** conserved LRL vector **A** can be defined for all central forces, but this generalized vector is a complicated function of position, and usually not expressible in closed form.

• "An essential property of the LRL vector, which makes this conserved quantity unusual, is that for the three-dimensional Lagrangian of the system **there does *not* exist a so-called cyclic coordinate** corresponding to it, whereas for other conserved quantities there is such a cyclic coordinate. Thus the conservation of the LRL vector must be derived directly, e.g., by the method of Poisson brackets, as described below. Conserved quantities of this kind are called "**dynamic**", in contrast to the usual "geometric" conservation laws, e.g., that of the angular momentum."

• This vector has a great history outlined in wiki. In fact a guy **Hermann** in 1710 first noticed this thing, **Laplace** rediscovered it in 1799, then **Hamilton** in 1847 found it again, as his "eccentricity vector" involved in the fact that **p** moves on a circle, see below. Then **Gibbs** in 1901 did roughly the above vector analysis derivation. **Runge** wrote a German text on vectors (1919) and mentioned this Gibbs thing, and then that example was quoted in another book by **Lenz** on QM of the H atom (1924) and *their* names stuck.

Here is the fact I never knew, that while a planet orbits, the tip of its **p** vector moves in a perfect circle. When I do the algebra below, I get the circle equation but with -A replaced by +A.

$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$

$\mathbf{L} \times \mathbf{A} = \mathbf{L} \times (\mathbf{p} \times \mathbf{L}) - mk\,(\mathbf{L} \times \hat{\mathbf{r}})$ note $\mathbf{p} \bullet \mathbf{L} = 0$

$= L^2\mathbf{p} + mk\,\hat{\mathbf{r}} \times \mathbf{L}$

$\Rightarrow \quad L^2\mathbf{p} = \mathbf{L} \times \mathbf{A} - mk\,\hat{\mathbf{r}} \times \mathbf{L}$

so the starting point below is correct. Now use picture above to conclude that (I define my θ as being θ=0 at the most distant orbital point)

$\mathbf{L} \times \mathbf{A} = LA\,(-\hat{\mathbf{y}})$

$\hat{\mathbf{r}} = \hat{\mathbf{x}} C_\theta + \hat{\mathbf{y}}\, S_\theta$

$\mathbf{p} = p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}}$

$\Rightarrow \quad L^2\mathbf{p} = \mathbf{L} \times \mathbf{A} - mk\,\hat{\mathbf{r}} \times \mathbf{L}$

$L^2(p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}}) = LA\,(-\hat{\mathbf{y}}) - mk\,(\hat{\mathbf{x}} C_\theta + \hat{\mathbf{y}}\, S_\theta) \times L\hat{\mathbf{z}}$

$L(p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}}) = A\,(-\hat{\mathbf{y}}) - mk\,(\hat{\mathbf{x}} C_\theta + \hat{\mathbf{y}}\, S_\theta) \times \hat{\mathbf{z}}$

$L(p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}}) = A\,(-\hat{\mathbf{y}}) - mk\,(-\hat{\mathbf{y}}\, C_\theta + \hat{\mathbf{x}}\, S_\theta)$

$\Rightarrow$

$Lp_x = -\,mk\,S_\theta$
$Lp_y + A = +\,mk\,C_\theta$

$p_x = -\,(mk/L)\,S_\theta$
$p_y + A/L = +\,(mk/L)\,C_\theta$

$p_x^2 + (p_y + A/L)^2 = (mk/L)^2$

so I have +A in there, not -A. But it will be -A if I flip all three axes, which is the same as putting the origin at the right focus instead of the left focus, in which case $\mathbf{A} = A\hat{\mathbf{x}}$. In my derivation above, the only thing that changes if we put force at right focus is $\mathbf{L} \times \mathbf{A} = LA\,(+\hat{\mathbf{y}})$ which is of course exactly the term we want to change sign.

The picture below then makes sense. The angle η is just the fixed angle shown in the picture.

Circular momentum hodographs

The conservation of the LRL vector **A** and angular momentum vector **L** is useful in showing that the momentum vector **p** moves on a circle under an inverse-square central force.[6][7] Taking the cross product of **A** and **L** yields an equation for **p**

$$L^2\mathbf{p} = \mathbf{L} \times \mathbf{A} - mk\hat{\mathbf{r}} \times \mathbf{L}$$

Taking **L** along the *z*-axis and the major semiaxis as the *x*-axis yields the equation

$$p_x^2 + (p_y - A/L)^2 = (mk/L)^2$$

In other words, the momentum vector **p** is confined to a circle of radius *mk*/*L* centered on (0, *A*/*L*). The eccentricity *e* corresponds to the cosine of the angle η shown in Figure 3. For brevity, it is also useful to introduce the variable $p_0 = \sqrt{2m\,|E|}$. This circular hodograph is useful in illustrating the symmetry of the Kepler problem.

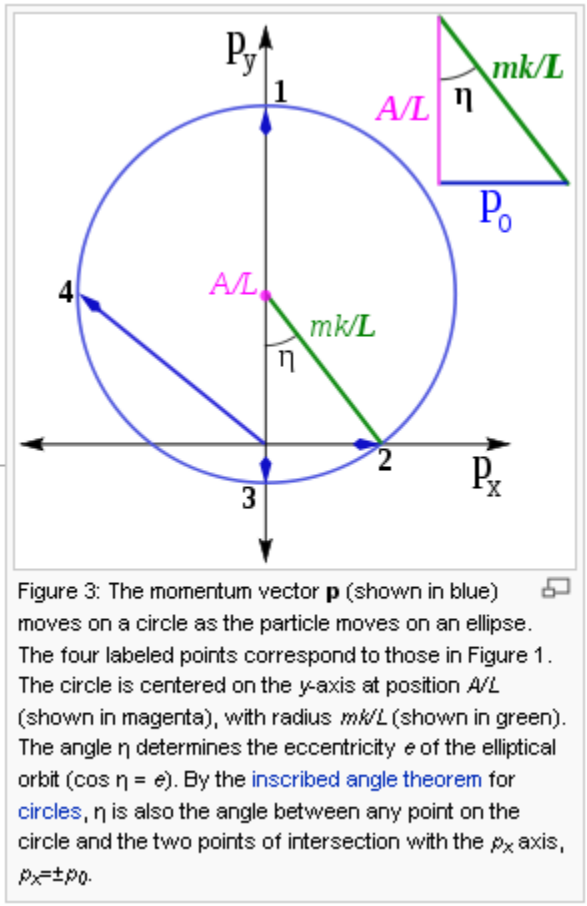


Figure 3: The momentum vector **p** (shown in blue) moves on a circle as the particle moves on an ellipse. The four labeled points correspond to those in Figure 1. The circle is centered on the *y*-axis at position *A*/*L* (shown in magenta), with radius *mk*/*L* (shown in green). The angle η determines the eccentricity *e* of the elliptical orbit (cos η = *e*). By the inscribed angle theorem for circles, η is also the angle between any point on the circle and the two points of intersection with the p_x axis, $p_x = \pm p_0$.

The claim above then is that cosη = e, the eccentricity.

The wiki site then describes Fock's construction that the **A** vector components are the generators of rotations in 4-space which causes orbit changes which do not change energy, and in QM we have [**A**,H] = 0 so we have three new commuting generators. Wiki then claims that the comrels of the **L** and **A** are those for SO(4), so that is the symmetry of this problem.

Comment: Assuming the above to be true, then just as we make eigenstates of **L** , so to speak, and find that states have degeneracy 2ℓ+1, even so we might make simultaneous eigenstates of **L** and **A** , so to

speak again, and then we will find some new degeneracy rule that will explain the "accidental" degeneracy seen in the H atom problem. As we know, that degeneracy will be n^2 and this is worked out in detail below.

Hydrogen Atom (236). Now back to Schiff. He uses a symmetrized form of **A** (or **M**) as shown in 30.5, and then he claims that with much fiddling, you find that [**A**,H] = 0 as in 30.6. He also shows that **A•L** = **L•A** = 0 as operators.

Now he starts into the commutators. The L with L ones are the usuals and he has these so far

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k \qquad // 30.7$$
$$[L_i, M_j] = i\hbar\varepsilon_{ijk}M_k \qquad // 30.8$$

and I know these two just say that **L** and **M** are rotational vectors. The last ones are these

$$[M_i, M_j] = -(2i\hbar/\mu)\, \varepsilon_{ijk}H\, L_k$$

which is weird since we have the Hamiltonian *operator* H sitting in there. We are now told to restrict our interest to a subspace of the QM HS which has H = E. For our bound state problem here, we have E < 0, so we can then rescale the M's as shown in 30.10 and this then cleans up our algebra so it is then this:

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k \qquad // 30.7$$
$$[L_i, M'_j] = i\hbar\varepsilon_{ijk}M'_k \qquad // 30.8$$
$$[M'_i, M'_j] = i\hbar\varepsilon_{ijk}\, L_k \qquad // 30.11$$

The claim is that these 6 generators are those for the SO(4) group. Before continuing, suppose we had E>0. Then we would have an extra i in the relation between M' and M from 30.10 and then the sign in the last commutator above on the RHS would be - instead of plus. This then would give exactly the comrels for the J and K generators of SO(3,1) = the Lorentz group, as I outline in my "meandering proof" doc.

But here we have a + sign there and it is SO(4).

The O(4) group. Why do the above generators generate SO(4)? Schiff shows that if you invent a fourth coordinate r_4 and a fourth momentum p_4, then the rule $L_{ij} = r_i p_j - r_j p_i$ provides our 6 generators such that

$$L_k = (1/2)\, \varepsilon_{kij}L_{ij} \qquad\qquad M'_k = L_{k4}$$

So these M' things would be our Lorentz boost generators, but here they are just more regular rotations, albeit involving the 4th dimension axis.

So fine. Now about this question: is this a geometrical symmetry for the H atom? If by "geometry" you mean the 3D geometry of our space, the answer is no. If you allow 4D geometry, then yes. So the first answer is taken and this is then classified as a "dynamical" symmetry.

Energy Levels of the H atom. In my LG notes I start with J and K and then decouple them into the L and R generators. The same thing happens here where we start with L and M' and convert them to decoupled I and K. The lincom "i"s in my Lorentz group stuff are not present here, see 30.14. The

resulting Lie Algebra is shown in 30.15. Both the I and K generators commute with H of course, since the L and M' ones did. So as in the LG case, we just treat the two I and K worlds separately. The two Casimirs are now obvious as in page 238 A, which we can rearrange to be the C and C' on page 239 top. But we have C' = 0 since L•M = 0 which means we must have $I^2 = K^2$ so i(i+1) = k(k+1) one solution of which is i = k, and the only one if these are the usual non-negative values as in 30.16. Then

$$C = I^2 + K^2 = 2\,k(k+1)\hbar^2$$

Now look at equation 30.6 which I believe, but which I did not derive. We can translate this to say

$$(-2E/\mu)\,M'^2 = 2E/\mu * (L^2 + \hbar^2) + \kappa^2 \qquad \text{recall } V(r) = -\kappa/r$$

But $4I^2 = L^2 + M'^2$ so write as

$$(-2E/\mu)\,[\,4I^2 - L^2\,] = 2E/\mu * (L^2 + \hbar^2) + \kappa^2$$

But then the L^2 stuff cancels and we have

$$(-2E/\mu)\,[\,4I^2\,] = 2E\hbar^2/\mu + \kappa^2$$

$$-E\,\{\,2/\mu * 4I^2 + 2\hbar^2/\mu\,\} = \kappa^2$$

$$-E\,\{\,4I^2/\hbar^2 + 1\,\} = \kappa^2\mu/2$$

$$-E\,\{\,4k(k+1) + 1\,\} = \kappa^2\mu/2$$

$$-E\,\{\,4k^2 + 4k\,|+\,1\,\} = \kappa^2\mu/2$$

$$-E\,\{\,(2k+1)^2\,\} = \kappa^2\mu/2$$

$$-E = \kappa^2\mu/[\,2\,(2k+1)^2\,]$$

If we set n = 2k+1 as the principle quantum number, this gives the usual Bohr atom energies!

Comment: We never used the SE here! We defined M as in 30.5, showed by algebra the results in 30.6, then showed the various SO(4) commutator results, decoupled the Lie algebra into I and K, showed that i = k was necessary , and got the eigenenergies! Symmetry completely solved this problem, at least for the energies, not for the eigenfunctions. I think this is pretty impressive.

Schiff has more to say. Reverse solving 30.14 gives **L** = **I**+**K**. But **I** and **K** are decoupled ang mom gens that commute with each other, so the triangle rule applies. But i = k, so we get $\boldsymbol{\ell}$ = 0,1,2...(2k) = 0...n-1 which is the known H atom range, so that part works out right too.

What about degeneracy? The states are really $|\,k,m_k\,;\,i=k,\,m_i\rangle$ and each m can take on 2k+1 values, so we have $(2k+1)^2 = n^2$ degeneracy, so that also comes out right. His final comment is that the states

presented in this way do not have well defined parity, and this agrees with the fact that even and odd ℓ are mixed together here.

All in all, an excellent Schiff section.

Classical Isotropic 3D HO. The known Cartesian solution gives energy 30.21. Here is my computation of the degeneracy:

Suppose $n_1 + n_2 = N$. How many ways can you pick (n_1,n_2) to add up to N, from a range starting at 0? There are (N+1) ways to pick n_1 and then only 1 way to pick n_2, so answer is (N+1),

Now suppose $N + n_3 = n$. There are (n+1) ways to pick N, then n_3 is forced and has only one way. But for each N, we know there are (N+1) ways to assemble N from n_1 and n_2. So we get

$$\text{(total ways } n_1+n_2+n_3 = n) = \Sigma_{N=0,n}\,(N+1) = \Sigma_{k=1,n+1}\; k = (n+2)(n+1)/2$$

where each of those (n+1) ways to pick N is represented by one element in the sum.

Now unfortunately the section sort of fizzles out. First Schiff sets α and β so that the SU(3) Casimir shown as C on page 212 looks like the 3D HO Hamiltonian squared shown in 30.20 p 239. Thus we end up with 30.22 and we find that [C,H] = 0 of course. Earlier, Schiff claims that the "quad tensor Q" shown bottom of page 240 is a "constant of the motion". That is to say, for any particular HO classical orbit, as the particle moves, these things all stay constant, just the way the vector **A** stayed constant for the Kepler problem. I guess the implication is that the operator versions of these things shown page 211 all commute with H, though he has not shown that here. For example, probably we have $[\, Q_{xy}, H] = 0$. So when we take the 5 Q's and the 3 L's, we have eight generators that commute with H, and thus our problem has SU(3) symmetry. Thus, if we knew a little about the representations of SU(3) which we do not right now and which Schiff does not mention, then we could determine the energy levels and degeneracies as we did in the H atom case with SO(4). Schiff just fades out here, so maybe some day if I have SU(3) powered up again, I can look at this problem. Also, I have no idea why his "quad tensor" has such strangely named components, as it if were a mixture of a Cartesian and Multipole tensor.

Comment: these two dynamical symmetry examples do not appear in Tinkham. Nor do they appear in Messiah, but the latter has a huge appendix on Group Theory and lots of details on the Permutation Group on n objects, called S_n , including those Young Tableau I dimly remember. M&M have a group chapter but no mention (I don't think) of our current problems. I think my advanced group theory books like Wybourne DO discuss these things, but I am staying away from that right now. Obviously this is a highly specialized topic and we are lucky that Schiff even did what he did with it.

This brings us roughly to the half-way point through Schiff!