Schiff Bog Down on deriving page 143 A

Derive page 143 A. Well, this was a messy one, you never know until you try to derive something.

For r = 0 using 21.12 we get F =1 (ie, keep only first term in F) and

$$\text{RHS (21.20)} = (2\ell+1)/2*\{\, v^{-1/2}\,\Gamma(1+in)e^{-n\pi/2}\} \;\; [\;\; \int_{-1}^{1} dz\, P_\ell(z)\, e^{ikrz}]$$

Maple won't do integrals like this even if you say ℓ = integer. GR does not give orthogonal range integrals on $P_\ell(z)$ for some reason, you have to look it up as a Gegenbauer as we did above. So

$$P_\ell(z) = C_\ell^{1/2}(z)$$

Our integral is GR page 830 7.321 with $\nu = 1/2$ where we find that

$$\int_{-1}^{1} dz\, P_\ell(z)\, e^{iaz} = \pi\sqrt{2}\; i^\ell\Gamma(1+\ell)/\,[\,\ell!\;\Gamma(1/2)\,]\; a^{-1/2}\; J_{\ell+1/2}(a)$$

but now we want the small-a limit of this Bessel function,

$$J_{\ell+1/2}(a) = (2a/\pi)^{1/2}\, j_\ell(a) \rightarrow (2a/\pi)^{1/2}\, a^\ell/(2\ell+1)!! \qquad \text{Schiff page 85}$$

So our integral is then

$$\pi\sqrt{2}\; i^\ell\Gamma(1+\ell)/\,[\,\ell!\,\sqrt{\pi}\,] * \; a^{-1/2}\; (2a/\pi)^{1/2}\, a^\ell/(2\ell+1)!!$$

$$= 2\; i^\ell\Gamma(1+\ell)/[\ell!\;(2\ell+1)!!] * a^\ell \;\; = 2\;\Gamma(1+\ell)/[\ell!\;(2\ell+1)!!] *(\,ia)^{\,\ell}$$

and we then seem to have

$$\text{RHS (21.20)} = \;(2\ell+1)/2*\{\, v^{-1/2}\,\Gamma(1+in)e^{-n\pi/2}\} \;\{\;\; 2\;\Gamma(1+\ell)/[\ell!\;(2\ell+1)!!] *(\,ikr)^{\,\ell}\;\;\}$$

So this is our limit of $R_\ell(r)$ for small r where we used the parabolic formula 21.12 and kept only the F=1 term. Now we want to compare this with the small-r version or R obtained in the spherical world. From page 142 A and 21.18 we get, again setting this F = 1,

$$R_\ell(r) = r^\ell\;\; e^{ikr}\; C_\ell * 1 = r^\ell\;\; C_\ell$$

This would suggest then that, comparing our last two equations,

$$C_\ell \;\; = \;(2\ell+1)/2*\{\, v^{-1/2}\,\Gamma(1+in)e^{-n\pi/2}\} \;\{\;\; 2\;\Gamma(1+\ell)/[\ell!\;(2\ell+1)!!] *(\,ik)^{\,\ell}\;\;\}$$

$= (\,ik)^{\ell}\, e^{-n\pi/2}\, \Gamma(1+in) / v^{-1/2} \;\; * \;\; \{ \;\; (2\ell+1)/2 * \;\; 2\, \Gamma(1+\ell\,) / [\ell!\; (2\ell+1)!!] \;\; \}$

This maybe has the potential to become page 143 A, but not very obvious how.

1. How do I get rid of the !! thing? Luckily, Wolfram produces this result (web search on "double factorial")

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

which appears on page 938 of GR, now that I know where to look for it! So I can say

$1/(2\ell+1)!! = \sqrt{\pi}\; 2^{-\ell} \;/\; \Gamma(\ell+1/2)$

2. Next, Schiff is showing $(2\ell)!$ in his result A, how am I going to relate that to anything? Well, look at GR page 938 which has this doubling formula:

$\Gamma(2x) = 2^{2x-1}/\sqrt{\pi} \;\; * \Gamma(x)\, \Gamma(x+1/2) \;\; = \Gamma(2x+1)/2x$

which says

$\Gamma(2x+1) = (2x)! \;\; = \;\; 2x\; 2^{2x-1}/\sqrt{\pi} \;\; * \Gamma(x)\, \Gamma(x+1/2) = \;\; x \;\; 2^{2x}\, \Gamma(x)\, \Gamma(x+1/2)/\pi$

or

$(2\ell)! \;\; = \;\; \ell \;\; 2^{2\ell}\, \Gamma(\ell)\, \Gamma(\ell+1/2)/\pi$

So we have some way to get rid of that.

3. Next, Schiff shows $\Gamma(\ell+1+in)$ whereas I have only $\Gamma(1+in)$. How do we relate these guys? Here I am pretty much stuck, maybe he has it wrong. So let's now assemble what we have:

$C_\ell \qquad = (\,ik)^{\ell}\, e^{-n\pi/2}\, \Gamma(1+in) / v^{-1/2} \;\; * \;\; \{ \;\; (2\ell+1)/2 * \;\; 2\, \Gamma(1+\ell\,) / [\ell!\; (2\ell+1)!!] \;\; \}$

$1/(2\ell+1)!! = \sqrt{\pi}\; 2^{-\ell} \;/\; \Gamma(\ell+1/2)$

$1 \;\; = [\;\; \ell \;\; 2^{2\ell}\, \Gamma(\ell)\, \Gamma(\ell+1/2)/\pi \;] \;/\; (2\ell)!$

which gives (notice that the $\Gamma(\ell+1/2)$ factors cancel)

$C_\ell \;\; = (2ik)^{\ell}\, e^{-n\pi/2}\, \Gamma(1+in) / [\; v^{-1/2}\, (2\ell)!\;]$

$\qquad * \sqrt{\pi} \;\; * \;\; [\;\; \ell \;\; \Gamma(\ell) / \pi \;] * \;\; \{ \;\; (2\ell+1)/2 * \;\; 2\, \Gamma(1+\ell\,) / [\ell!\;] \;\; \}$

Let's process this last factor a bit and leave the first part as is:

$= \sqrt{\pi}\ \ *\ [\ \ell\ \Gamma(\ell)/\pi\ ]\ *\ \{\ (2\ell+1)/2\ *\ 2\,\Gamma(1+\ell\,)/[\ell!\ ]\ \}$

$= (1/\sqrt{\pi}\,)\ \{\ \ell\ \Gamma(\ell)/\ell!\}\ (2\ell+1)\ \Gamma(1+\ell\,)$

$= (1/\sqrt{\pi}\,)\ (2\ell+1)\ \Gamma(1+\ell\,)\ \ = (1/\sqrt{\pi}\,)\ (2\ell+1)\ \ell!$

I don't know what else to do here. So my result for A is this

$C_\ell\ = (2ik)^\ell\ e^{-n\pi/2}\ \Gamma(1+in)\ /\ [\ v^{-1/2}\ (2\ell)!\ ]\quad x\quad (1/\sqrt{\pi}\,)\ (2\ell+1)\ \ell!$

while Schiff's answer is this

$C_\ell\ = (2ik)^\ell\ e^{-n\pi/2}\ \Gamma(\ell+1+in)\ /\ [\ v^{-1/2}\ (2\ell)!\ ]$

so if these are to be equal, we must have this weird result

$\Gamma(\ell+1+in)/\ \Gamma(1+in) = (1/\sqrt{\pi}\,)\ (2\ell+1)\ \ell!$

which cannot possibly be true since the LHS is a function of n. For example, evaluate both sides at $\ell = 1$ and we get

$LHS = \Gamma(1+1+in)/\ \Gamma(1+in) = (in+1)!\ /\ (in)! = (in+1)$ $\qquad\qquad$ $RHS\ = 3/\sqrt{\pi}$

4. Vague thought. I think the spherical side of things is pretty solid, but what about higher terms in 21.2? Not clear what happens because you are doing that z integral. So we have

$F(-in,\ 1,\ 2ikr\ \sin^2(\theta/2)) = 1 + (-in)\ ikr(1-z)/1\ \ = 1 + nkr(1-z)\ .$

So the contribution of the second term will be,

$$\int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}\ \{\ nkr(1-z)\ \} =\ nkr\ [\ \int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}(1-z)\ \ ]$$

We already know the first integral with the "1" integrand term, it is this, after small r is taken

$$\int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}\quad = 2\ \Gamma(1+\ell\,)/[\ell!\ (2\ell+1)!!]\ *(\ ikr)^{\,\ell}$$

The "z" integral will be this

$$\int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}(-z)\ =\ \int_{-1}^{1} dz\ P_\ell(z)\ (-z/ikz)\ \partial_r e^{ikrz}\ = (-1/ik)\ \partial_r\ \{\ \int_{-1}^{1} dz\ P_\ell(z)\ e^{ikrz}\}$$

$= (-1/ik)\,[2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ik)^{\ell}\;*\,\partial_r\,r^{\ell} =$

$= (-1/ik)\,[2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ik)^{\ell}\;\ell\,r^{\ell-1}$

So our second term contribution will be

$nkr\,[\;\int_{-1}^{1} dz\,P_\ell(z)\,e^{ikrz}(1-z)\;]$

$= nkr\;\{\,2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ikr)^{\ell} + (-1/ik)\,[2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ik)^{\ell}\;\ell\,r^{\ell-1}\;\}$

The first term is ~ r r^{ℓ} and we can throw it out in the r→0 limit, but the second term contributes to our overall result:

$= nkr\;(-1/ik)\,[2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ik)^{\ell}\;\ell\,r^{\ell-1}\;\}$

$= i\,n\;[2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ikr)^{\ell}\;\ell$

If we then combine the first and second term contribution (ie, first and second term in F),

$\{\;2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ikr)^{\ell}\;\} + \;i\,n\;[2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ikr)^{\ell}\;\ell$

$=\;2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,(\,ikr)^{\ell}\{\,1 + in\ell\,\}$

This suggests to me that ALL terms in 21.12 are going to contribute in this limit and that is what explains the discrepancy between my result and Schiff's for page 141 A. To really do the full series, I would need this integral

$$\int_{-1}^{1} dz\,P_\ell(z)\,e^{ikrz}\,z^{n}$$

But following earlier, we have

$z\,e^{ikrz} = z\,(1/ikz)\,\partial_r\,e^{ikrz} = (1/ik)\,\partial_r\,e^{ikrz}$

$z^{n}\,e^{ikrz} = (1/ik)^{2}\,\partial_r{}^{n}\,e^{ikrz}$

So again, we had this small-r behavior

$\int_{-1}^{1} dz\,P_\ell(z)\,e^{ikrz} \quad = 2\,\Gamma(1+\ell)/[\ell!\,(2\ell+1)!!]\,*(\,ikr)^{\ell}$

Therefore we have

$$\int_{-1}^{1} dz\, P_\ell(z)\, e^{ikrz}\, z^n = (1/ik)^2\ 2\,\Gamma(1+\ell)/[\ell!\ (2\ell+1)!!]\ \partial_r{}^n(\ ikr)^{\ell}$$

OK, now express the F in 21.12 using 21.6 A and we have

$$F(-in,\ 1,\ ikr[1-z]\ ) = \Sigma_s\ \Gamma(-in+s)\ \Gamma(1)\ (1-z)^n\ /\ [\ \Gamma(-in)\ \Gamma(1+s)\ s!]$$

OK, I am sure this is the hard way to go, better to use in integral representation of F to allow the integral to be done.