Angular momentum operators in spherical coordinates PhL 1.23.08

From Saxon page 262 we have

$$\mathbf{p} = -i\nabla$$

and so

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -i\, \mathbf{r} \times \nabla$$

But in spherical coordinates we know that (see T.K page)

$$\nabla = \hat{r}\, \partial_r + \hat{\theta}\, (1/r)\partial_\theta + \hat{\phi}\, (1/rS\theta)\partial_\phi \qquad = ip_r\hat{r} + ip_\theta\, \hat{\theta} + ip_\varphi\hat{\phi}$$

Therefore (note" in MM ponderings, cross product is form invariant under rotations!)

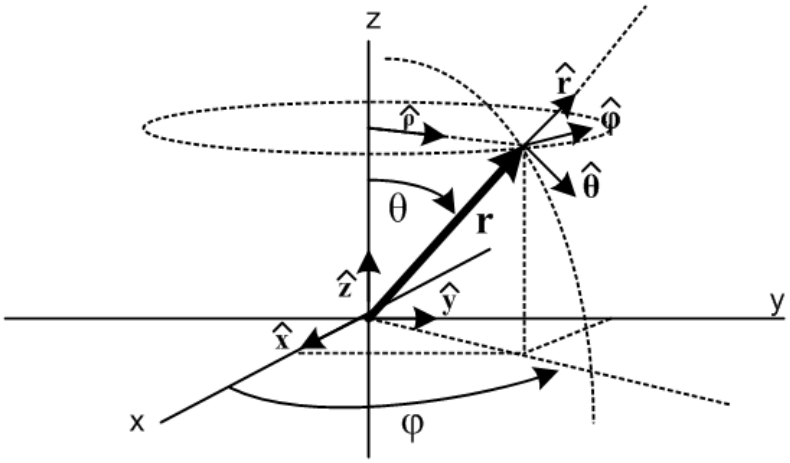
$$i\mathbf{L} = \mathbf{r} \times \nabla = \hat{r}x\hat{\theta}\, \partial_\theta + \hat{r}x\hat{\phi}\, (1/S\theta)\partial_\phi \quad = \quad \hat{\phi}\, \partial_\theta - \hat{\theta}\, (1/S\theta)\partial_\phi$$

From this we could say that

$$iL_\phi = \partial_\theta$$
$$iL_\theta = -\, (1/S\theta)\partial_\phi$$
$$iL_r = 0$$

which seems to suggest that $\nabla = \hat{r}\, \partial_r + \hat{\theta}\, (1/r)\, iL_\phi - \hat{\phi}\, iL_\theta$

These seem deceptively simple, but I think they are correct. [supported by the L^2 calculation later on in this doc.] These are dimensionless operators. Consider the picture below, but imagine the unit vector triad translated to the origin. A function f(r,θ,φ) has to change in the θ direction to "have some angular momentum about the $\hat{\phi}$ axis" because you can see that sort of implies "rotation" about the $\hat{\phi}$ axis (moved to the origin), put right hand thumb in $\hat{\phi}$. Similarly, your function f(r,θ,φ) has to change in the φ direction to "have some angular momentum about the $\hat{\theta}$ axis" because you can see that sort of implies "rotation" about the $\hat{\theta}$ axis, put right hand thumb in $\hat{\theta}$. You can see that the sign is negative because at the equator, $\hat{\theta}$ points in the $-\hat{z}$ direction which is the normal direction associated with the φ rotation direction shown. At this equator we expect to get $iL_\theta = -\partial_\phi$ because the situation is exactly analogous to the $iL_\phi = \partial_\theta$ situation. As we move toward the north pole, I guess you need a huge amount of ∂_ϕ amplitude to generate your iL_θ because $\hat{\theta}$ is moving into the φ rotation plane, so to speak. As for $iL_r = 0$, you would need some ψ Euler angle to get some action for angular momentum about this axis, thinking Goldstein. But we don't have any ψ action here! I guess we could sort of add it and then say $iL_r = \partial_\psi$.



We usually deal with the Cartesian components, so repeat the above,

$$\mathbf{L} = -i\,[\ \hat{\phi}\,\partial_\theta - \hat{\theta}\,(1/S\theta)\partial_\phi]$$

Then (see another T.K. page for the various spherical coordinate dot products)

$$L_x = -\,i\,[\ \hat{x}\bullet\hat{\phi}\,\partial_\theta - \hat{x}\bullet\hat{\theta}\,(1/S\theta)\partial_\phi] = -i\,[\,(-S\phi)\,\partial_\theta - (C\theta C\phi\,)(1/S\theta)\partial_\phi] = i\,[\,S\phi\,\partial_\theta + \cot\theta\,C\phi\,\partial_\phi]$$

$$L_y = -\,i\,[\ \hat{y}\bullet\hat{\phi}\,\partial_\theta - \hat{y}\bullet\hat{\theta}\,(1/S\theta)\partial_\phi] = -i\,[\,(C\phi)\,\partial_\theta - (C\theta S\phi\,)(1/S\theta)\partial_\phi] = i\,[\,-C\phi\,\partial_\theta + \cot\theta\,S\phi\,\partial_\phi]$$

$$L_z = -\,i\,[\ \hat{z}\bullet\hat{\phi}\,\partial_\theta - \hat{z}\bullet\hat{\theta}\,(1/S\theta)\partial_\phi] = -i\,[\,(0)\,\partial_\theta - (-S\theta\,)(1/S\theta)\partial_\phi] = -i\,[\partial_\phi]$$

The raising and lower operators are:

$$L_\pm = L_x \pm i\,L_y = i\,[\,S\phi\,\partial_\theta + \cot\theta\,C\phi\,\partial_\phi] \pm i\,\{\ i\,[\,-C\phi\,\partial_\theta + \cot\theta\,S\phi\,\partial_\phi]\}$$
$$= i\,[\,S\phi\,\partial_\theta + \cot\theta\,C\phi\,\partial_\phi] \pm \{\,[\ C\phi\,\partial_\theta - \cot\theta\,S\phi\,\partial_\phi]\}$$

$$= \{\,iS\phi \pm C\phi\}\,\partial_\theta + \{\,i\cot\theta\,C\phi \mp \cot\theta\,S\phi\,\}\,\partial_\phi$$

$$= \{\,iS\phi \pm C\phi\}\,\partial_\theta + \cot\theta\,\{\,iC\phi \mp S\phi\,\}\,\partial_\phi$$

$$= \{\,iS\phi \pm C\phi\}\,\partial_\theta + i\,\cot\theta\,\{\,C\phi \pm iS\phi\,\}\,\partial_\phi$$

$$= \pm\,\{\,C\phi \pm iS\phi\}\,\partial_\theta + i\,\cot\theta\,\{\,C\phi \pm iS\phi\,\}\,\partial_\phi$$

$$= e^{\pm i\phi}\ [\,\pm\partial_\theta + i\,\cot\theta\ \partial_\phi\,] \qquad \text{// agrees with "multipole expannsions.doc}$$

Next, a calculation which takes a *lot* of lines to carry out, at least by this method:

$$L^2 = L_x^2 + L_y^2 + L_z^2 =$$

$$\{ i [S_\phi \partial_\theta + \cot\theta C_\phi \partial_\phi]\}^2 + \{ i [- C_\phi \partial_\theta + \cot\theta S_\phi \partial_\phi]\}^2 + \{ -i [\partial_\phi]\}^2$$

$$= - [S_\phi \partial_\theta + \cot\theta C_\phi \partial_\phi]^2 - [- C_\phi \partial_\theta + \cot\theta S_\phi \partial_\phi]^2 - [\partial_\phi]^2$$

So now have to be careful about what the derivatives act on, so write it out as is:

$$-L^2 = [S_\phi \partial_\theta + \cot\theta C_\phi \partial_\phi]^2 + [- C_\phi \partial_\theta + \cot\theta S_\phi \partial_\phi]^2 + [\partial_\phi]^2$$

$$= [S_\phi^2 \partial_\theta^2 + \cot^2\theta C_\phi^2 \partial_\phi^2 + S_\phi \partial_\theta \cot\theta C_\phi \partial_\phi + \cot\theta C_\phi \partial_\phi S_\phi \partial_\theta] f(\theta,\phi)$$

$$+ [C_\phi^2 \partial_\theta^2 + \cot^2\theta S_\phi^2 \partial_\phi^2 - C_\phi \partial_\theta \cot\theta S_\phi \partial_\phi - \cot\theta S_\phi \partial_\phi C_\phi \partial_\theta] + \partial_\phi^2$$

where each differential operator acts on *everything to its right* (imagine some $f(\theta,\phi)$ sitting there). If we ignore the four cross terms for the moment, we get

$$= \partial_\theta^2 + \cot^2\theta \partial_\phi^2 + \partial_\phi^2 = \partial_\theta^2 + (1/\sin\theta)^2 \partial_\phi^2$$

Now come the cross terms 1,2,3,4:

1 $S_\phi \partial_\theta \cot\theta C_\phi \partial_\phi = (S_\phi C_\phi \partial_\phi) \partial_\theta [\cot\theta f(\theta)] = (S_\phi C_\phi \partial_\phi) [\csc^2\theta + \cot\theta \partial_\theta] f$
So get $1 = (S_\phi C_\phi \partial_\phi) [\csc^2\theta + \cot\theta \partial_\theta]$

2 $\cot\theta C_\phi \partial_\phi S_\phi \partial_\theta = (\cot\theta \partial_\theta) C_\phi \partial_\phi [S_\phi f] = (\cot\theta \partial_\theta) C_\phi [C_\phi + S_\phi \partial_\phi] f$
So get $2 = (C_\phi [C_\phi + S_\phi \partial_\phi]) (\cot\theta \partial_\theta) = (C_\phi^2 + S_\phi C_\phi \partial_\phi)(\cot\theta \partial_\theta)$

3 $- C_\phi \partial_\theta \cot\theta S_\phi \partial_\phi = - (S_\phi C_\phi \partial_\phi) [\csc^2\theta + \cot\theta \partial_\theta]$

4 $- \cot\theta S_\phi \partial_\phi C_\phi \partial_\theta = - (\cot\theta\partial_\theta) S_\phi \partial_\phi [C_\phi f] = - (\cot\theta\partial_\theta) S_\phi [-S_\phi + C_\phi\partial_\phi]$
$= -S_\phi [-S_\phi + C_\phi\partial_\phi](\cot\theta\partial_\theta) = [S_\phi^2 - S_\phi C_\phi\partial_\phi] (\cot\theta\partial_\theta)$

So combine these four terms to get

$$1+2+3+4 = (S_\phi C_\phi \partial_\phi) [\csc^2\theta + \cot\theta \partial_\theta] + (C_\phi^2 + S_\phi C_\phi \partial_\phi)(\cot\theta \partial_\theta)$$
$$- (S_\phi C_\phi \partial_\phi) [\csc^2\theta + \cot\theta \partial_\theta] + (S_\phi^2 - S_\phi C_\phi\partial_\phi) (\cot\theta\partial_\theta)$$

$$= \{ (C_\phi^2 + S_\phi C_\phi \partial_\phi) + (S_\phi^2 - S_\phi C_\phi\partial_\phi)\} (\cot\theta\partial_\theta) = (\cot\theta\partial_\theta)$$

So our total result is now

$$-L^2 = \partial_\theta^2 + (1/\sin\theta)^2\,\partial_\phi^2 + (\cot\theta\partial_\theta) = \partial_\theta^2 + \cot\theta\,\partial_\theta + (1/\sin\theta)^2\,\partial_\phi^2$$

You can write the first two terms as $(1/S_\theta)\,\partial_\theta\,[\,S_\theta\,\partial_\theta\,]$ so we get

$$L^2 = -\partial_\theta^2 - \cot\theta\,\partial_\theta - (1/\sin\theta)^2\,\partial_\phi^2$$

$$L^2 = -(1/S_\theta)\,\partial_\theta\,[\,S_\theta\,\partial_\theta\,] - (1/\sin\theta)^2\,\partial_\phi^2 \qquad \text{// agrees with multipole}$$

Summary of all results:

$$L_x = i\,[\,S_\phi\,\partial_\theta + \cot\theta\,C_\phi\,\partial_\phi] \qquad L_\pm = e^{\pm i\phi}\,[\,\pm\partial_\theta + i\,\cot\theta\,\partial_\phi\,]$$
$$L_y = -i\,[\,C_\phi\,\partial_\theta - \cot\theta\,S_\phi\,\partial_\phi] \qquad L^2 = -[\,\partial_\theta^2 + \cot\theta\,\partial_\theta + (1/\sin\theta)^2\,\partial_\phi^2]$$
$$L_z = -i\,\partial_\phi \qquad L^2 = -[\,(1/S_\theta)\,\partial_\theta\,[\,S_\theta\,\partial_\theta\,] + (1/\sin\theta)^2\,\partial_\phi^2\,]$$

Exercise: Compute $\mathbf{L}^2$ entirely in spherical coordinates.

$$-L^2 = -\mathbf{L}\bullet\mathbf{L} = +i\mathbf{L}\bullet i\mathbf{L} = (\hat{\boldsymbol{\phi}}\,\partial_\theta - \hat{\boldsymbol{\theta}}\,(1/S_\theta)\partial_\phi)\bullet(\hat{\boldsymbol{\phi}}\partial_\theta - \hat{\boldsymbol{\theta}}\,(1/S_\theta)\partial_\phi)$$

$$= \hat{\boldsymbol{\phi}}\bullet\partial_\theta\,(\hat{\boldsymbol{\phi}}\partial_\theta - \hat{\boldsymbol{\theta}}\,(1/S_\theta)\partial_\phi) - \hat{\boldsymbol{\theta}}\bullet(1/S_\theta)\partial_\phi\,(\hat{\boldsymbol{\phi}}\partial_\theta - \hat{\boldsymbol{\theta}}\,(1/S_\theta)\partial_\phi)$$

Now we have to account for derivatives of the unit vectors, which we know are these

$$\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}} \qquad \partial\hat{\boldsymbol{\phi}}/\partial\theta = 0 \qquad \partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}} \qquad \partial\hat{\boldsymbol{\rho}}/\partial\theta = 0$$
$$d\hat{\boldsymbol{\theta}}/\partial\varphi = C_\theta\,\hat{\boldsymbol{\varphi}} \qquad \partial\hat{\boldsymbol{\phi}}/\partial\varphi = -\hat{\boldsymbol{\rho}} = -C_\theta\,\hat{\boldsymbol{\theta}} - S_\theta\,\hat{\mathbf{r}} \qquad \partial\hat{\mathbf{r}}/\partial\varphi = S_\theta\,\hat{\boldsymbol{\varphi}} \qquad \partial\hat{\boldsymbol{\rho}}/\partial\varphi = \hat{\boldsymbol{\varphi}}$$
$$\partial\hat{\boldsymbol{\theta}}/\partial r = 0 \qquad \partial\hat{\boldsymbol{\phi}}/\partial r = 0 \qquad \partial\hat{\mathbf{r}}/\partial r = 0 \qquad \partial\hat{\boldsymbol{\rho}}/\partial r = 0$$

So we have many terms in the above to think about.

$$T1 = \hat{\boldsymbol{\phi}}\bullet\partial_\theta\,(\hat{\boldsymbol{\phi}}\partial_\theta) = \hat{\boldsymbol{\phi}}\bullet\partial_\theta\,(\hat{\boldsymbol{\phi}})\partial_\theta + \hat{\boldsymbol{\phi}}\bullet\hat{\boldsymbol{\phi}}\partial_\theta\,(\partial_\theta) = \mathbf{0} + \hat{\boldsymbol{\phi}}\bullet\hat{\boldsymbol{\phi}}\partial_\theta\,(\partial_\theta) = \partial^2_\theta$$

$$T2 = -\hat{\boldsymbol{\phi}}\bullet\partial_\theta(\,\hat{\boldsymbol{\theta}}\,(1/S_\theta)\partial_\phi) = -\hat{\boldsymbol{\phi}}\bullet\partial_\theta(\,\hat{\boldsymbol{\theta}})\,(1/S_\theta)\partial_\phi - \hat{\boldsymbol{\phi}}\bullet\,\hat{\boldsymbol{\theta}}\,\partial_\theta((1/S_\theta)\partial_\phi)$$

$$= +\hat{\boldsymbol{\phi}}\bullet\hat{\mathbf{r}}\,(1/S_\theta)\partial_\phi - \hat{\boldsymbol{\phi}}\bullet\,\hat{\boldsymbol{\theta}}\,\partial_\theta((1/S_\theta)\partial_\phi) = 0 + 0 = 0$$

$$T3 = -\hat{\boldsymbol{\theta}}\bullet(1/S_\theta)\partial_\phi\,(\hat{\boldsymbol{\phi}}\partial_\theta) = -\hat{\boldsymbol{\theta}}\bullet(1/S_\theta)\partial_\phi\,(\hat{\boldsymbol{\phi}})\partial_\theta - \hat{\boldsymbol{\theta}}\bullet(1/S_\theta)\hat{\boldsymbol{\phi}}\partial_\phi\,(\partial_\theta)$$

$$= -\hat{\boldsymbol{\theta}}\bullet(1/S_\theta)[-C_\theta\,\hat{\boldsymbol{\theta}} - S_\theta\,\hat{\mathbf{r}}]\,\partial_\theta + 0 = -\hat{\boldsymbol{\theta}}\bullet(1/S_\theta)[-C_\theta\,\hat{\boldsymbol{\theta}}] = \cot\theta\,\partial_\theta$$

$$T4 = \hat{\theta} \bullet (1/S_\theta)\partial_\phi[\,\hat{\theta}\,(1/S_\theta)\partial_\phi] = \hat{\theta} \bullet (1/S_\theta)(\partial_\phi\,\hat{\theta}\,)(1/S_\theta)\partial_\phi + \hat{\theta} \bullet (1/S_\theta)\,(1/S_\theta)\,\hat{\theta}\,\partial_\phi\partial_\phi$$

$$= \hat{\theta} \bullet (1/S_\theta)(\,C_\theta\,\hat{\varphi}\,)(1/S_\theta)\partial_\phi + \hat{\theta} \bullet (1/S_\theta)^2\,\hat{\theta}\,\partial_\phi\partial_\phi = 0 + (1/S_\theta)^2\partial^2_\phi = (1/S_\theta)^2\partial^2_\phi$$

So, adding the three non-vanishing terms, we get

$$-L^2 = \partial^2_\theta + \cot\theta\,\partial_\theta + (1/S_\theta)^2\partial^2_\phi$$

and this, happily, agrees with the first form shown above. And this seems a little easier, if you happen to know all those unit vector derivatives.

Note added 8.29.08.

How do we know that $[\,L^2\,,\,\mathbf{a}\bullet\nabla\,] = 0$, where $\mathbf{a}$ = constant vector ? Our usual answer is that $\mathbf{a}\bullet\nabla$ transforms as a rotational scalar, and L^2 commutates with any such scalar. But let's just look a bit into the details.

Let's start off computing the commutator in a primed spherical coordinate system in which $\mathbf{a}' = a\,\hat{\mathbf{r}}'$. That is, we just *choose* our primed coordinate system to line up in this way since we are free to choose it in any way we want. So we have $\hat{\mathbf{a}}' = \hat{\mathbf{r}}'$:

$$[\,L'^2\,,\,\mathbf{a}'\bullet\nabla'\,] = [\,L'^2\,,\,(a\,\hat{\mathbf{r}}')\bullet\nabla'\,] = [\,L'^2\,,\,a\,\partial_{r'}\,] = 0$$

where the last equality comes from the fact that L'^2 is a differential operator only in θ' and φ' and does not involve r' at all. For example, $[\partial_{\theta'},\partial_{r'}] = 0$. So, in this primed frame we have proven our result.

Now let's go to some unprimed frame according to $\mathbf{r} = R\,\mathbf{r}'$ where R is a 3x3 rotation matrix. Then we know that $\mathbf{r}' = R^{-1}\mathbf{r}$ which says

$$r'_i = R^{-1}_{ij}\,r_j \quad \text{and} \quad dr'_i = R^{-1}_{ij}\,dr_j \quad \text{and} \quad (dr'_i/dr_j) = R^{-1}_{ij}$$

So

$$\nabla_i = \partial_i = d/dr_i = \frac{dr_j'}{dr_i}\,\frac{d}{dr_j'} \qquad \text{by the chain rule acting on some function f}$$

$$= R^{-1}_{ji}\,d/dr'_j \qquad \text{by what was just shown above}$$

$$= R_{ij}\nabla'_j \qquad \text{since } R^{-1} = R^T \text{ , rotations are real orthogonal}$$

so we have shown that

$$\nabla = R\nabla' \qquad \text{which is the same as } \mathbf{r} = R\,\mathbf{r}'$$

In this bit of detail, we have just *proven* that the gradient vector transforms in the same manner as the position vector, in case there was any doubt about it.

Now, in this new unprimed frame we have $\mathbf{a} = R\,\mathbf{a}'$. So let's start with our primed frame result and work from there:

$$0 = [\, L'^2 \,,\, \mathbf{a}'\bullet\nabla' \,] \;= [\, L^2, (R^{-1}\mathbf{a}) \bullet (\, R^{-1}\nabla) \,] \;=\; [\, L^2, \mathbf{a} \bullet \nabla \,] \qquad \text{QED}$$

In the first step we use the fact that L^2 is the same in all rotationally related frames. We can make this proof a little more specific by using rotation operators $R = \exp(-i\gamma \mathbf{n}\bullet\mathbf{L})$ which are exponentiated differential operators in θ and φ, with rotation angle γ, and we then start again with

$$0 = [\, L'^2 \,,\, \mathbf{a}'\bullet\nabla' \,]$$

and apply $R\,R^{-1}$ around both sides to get

$$0 = \;[\, RL'^2\, R^{-1} \;\;,\, R(\mathbf{a}'\bullet\nabla')\, R^{-1}]$$

where $R\,R^{-1}$ around something is how we move an operator from the primed to unprimed frame. Thus we have

$$RL'^2\, R^{-1} \;= L^2 \qquad \text{// R does this to any scalar operator like } L'^2$$

$$\begin{aligned} R(\mathbf{a}'\bullet\nabla')\, R^{-1} \;&= \mathbf{a}'\bullet R\nabla'\, R^{-1} \;= \mathbf{a}'\bullet R^{-1}\nabla \qquad \text{// R does this to any vector operator like } \nabla' \\ &= (R\,\mathbf{a}')\bullet\;\nabla \\ &= \mathbf{a}\bullet\;\nabla \end{aligned}$$

and therefore we have

$$0 = \;[\, RL'^2\, R^{-1} \;\;,\, R(\mathbf{a}'\bullet\nabla')\, R^{-1}] \;\; = [L^2, \mathbf{a}\bullet\;\nabla]$$

and we are finally done.

Now that we are dead sure that $[L^2, \mathbf{a}\bullet\nabla] = 0$, we can examine it in spherical coordinates. We know that

$$\mathbf{a}\bullet\nabla \;= a_r\partial_r \;+ a_\theta(1/r)\partial_\theta + a_\varphi(1/r\sin\theta)\partial_\varphi$$

False Statement: "Since **a** is an arbitrary vector, each term above must be zero. "

When I made this statement, I had in mind just pulling **a** out of the commutator and writing

$$[L^2, \mathbf{a}\bullet\nabla] = \Sigma_i a_i\, [L^2, \nabla_i] \quad \text{sum i is over the spherical components,} \quad \text{(not true!)}$$

But of course we are thinking of $\mathbf{a} = a_r\hat{\mathbf{r}} \;+ a_\varphi\hat{\boldsymbol{\varphi}} \;+ a_\theta\hat{\boldsymbol{\theta}}$. But we know that L^2 is going to act on these unit vectors, so you cannot just pull **a** out in this way!

Had this been a true statement, we would then have concluded that

$[L^2, \nabla_i] = 0$ for each spherical component $\quad \nabla = \hat{r}\,\partial_r + \hat{\theta}\,(1/r)\partial_\theta + \hat{\phi}\,(1/rS\theta)\partial_\phi$

$\Rightarrow \quad 0 = [L^2, \partial_r] = [L^2, (1/r)\partial_\theta] = [L^2, (1/rS\theta)\partial_\phi]$

and the 1/r slips though L^2 so we would conclude that

$[L^2, L_\varphi] = [L^2, L_\theta] = 0$ // not true!

This looks appealing thinking of the Cartesian world, but these equations are *not true* in spherical coordinates! You can show this by direct calculation using L^2 given as angles.

Action of L^2 on unit vectors:

$\partial\hat{\theta}/\partial\theta = -\hat{r}$ $\quad \partial\hat{\phi}/\partial\theta = 0$ $\quad \partial\hat{r}/\partial\theta = \hat{\theta}$ $\quad \partial\hat{\rho}/\partial\theta = 0$

$d\hat{\theta}/\partial\varphi = C_\theta\,\hat{\phi}$ $\quad \partial\hat{\phi}/\partial\varphi = -\hat{\rho} = -C_\theta\,\hat{\theta} - S_\theta\,\hat{r}$ $\quad \partial\hat{r}/\partial\varphi = S_\theta\,\hat{\phi}$ $\quad \partial\hat{\rho}/\partial\varphi = \hat{\phi}$

$\partial\hat{\theta}/\partial r = 0$ $\quad \partial\hat{\phi}/\partial r = 0$ $\quad \partial\hat{r}/\partial r = 0$ $\quad \partial\hat{\rho}/\partial r = 0$

$-L^2\,\hat{r} = [\,\partial_\theta^2 + \cot\theta\,\partial_\theta + (1/\sin\theta)^2\,\partial_\phi^2]\,\hat{r}$

$= \partial_\theta(\partial_\theta\hat{r}) + \cot\theta(\partial_\theta\hat{r}) + (1/\sin\theta)^2\partial_\phi(\partial_\phi\hat{r})$

$= \partial_\theta(\hat{\theta}) + \cot\theta\,(\hat{\theta}) + (1/\sin\theta)^2\partial_\phi(\,S_\theta\,\hat{\phi}\,)$

$= -\hat{r} + \cot\theta\,(\hat{\theta}) + (1/\sin\theta)^2\,S_\theta\,[-C_\theta\,\hat{\theta} - S_\theta\,\hat{r}]$

$= -\hat{r} + \cot\theta\,(\hat{\theta}) - (1/\sin\theta)^2\,S_\theta\,C_\theta\,\hat{\theta} - (1/\sin\theta)^2\,S_\theta\,S_\theta\,\hat{r}$

$= -\hat{r} + \cot\theta\,(\hat{\theta}) - \cot\theta\,\hat{\theta} - \hat{r}$

$= -2\,\hat{r}$

What does this means that $L^2\,\hat{r} = 2\,\hat{r} = \ell(\ell+1)\,\hat{r}$, saying that somehow $\hat{r}$ is an eigenfunction of L^2 with angular momentum $\ell = 1$. Well, $\hat{r}$ is a rotation group vector which transforms by a 3x3 matrix which is equivalent to a 3x3 representation of the rotation group and this is the $\ell = 1$ representation. So pretty nice how that came out.

Let's try another unit vector:

$-L^2\,\hat{\theta} = [\,\partial_\theta^2 + \cot\theta\,\partial_\theta + (1/\sin\theta)^2\,\partial_\phi^2]\,\hat{\theta}$

$= \partial_\theta(\partial_\theta\hat{\theta}) + \cot\theta(\partial_\theta\hat{\theta}) + (1/\sin\theta)^2\,\partial_\phi(\partial_\phi\,\hat{\theta})$

$$= \partial_\theta(-\hat{\mathbf{r}}) + \cot\theta(-\hat{\mathbf{r}}) \quad + (1/\sin\theta)^2\, \partial_\phi(\, C_\theta\, \hat{\boldsymbol{\phi}})$$

$$= \partial_\theta(-\hat{\mathbf{r}}) + \cot\theta(-\hat{\mathbf{r}}) \quad + (1/\sin\theta)^2\, C_\theta\, [-\, C_\theta\, \hat{\boldsymbol{\theta}} \; - S_\theta \; \hat{\mathbf{r}}]$$

$$= -\hat{\boldsymbol{\theta}} \quad + \cot\theta(-\hat{\mathbf{r}}) \quad - \cot^2\theta\, \hat{\boldsymbol{\theta}} \; - \; \cot\theta \; \hat{\mathbf{r}}$$

$$= -\hat{\boldsymbol{\theta}}(1 + \cot^2\theta) \; - 2\cot\theta\hat{\mathbf{r}} \;\; = -\hat{\boldsymbol{\theta}} \;\csc^2\theta - 2\cot\theta\hat{\mathbf{r}}$$

So we find this result:

$$L^2 \;\hat{\boldsymbol{\theta}} \;= \csc^2\theta\, \hat{\boldsymbol{\theta}} + 2\cot\theta\, \hat{\mathbf{r}}$$

I have *no idea* how to interpret this, but as expected, it is non-zero.