Momentum and Hilbert Space PhL 2.10.09

I recently read Stakgold's Chapter 2 about Hilbert Spaces (HS) (and other things), and want to clarify the connection to quantum mechanics (QM) because right now things seem hazy. Stakgold is mainly interested in comparing E^n (dimension n) and L_2 (infinite dimensional) because Stakgold is interested in handling differential and integral equations, he is not interested in QM per se.

1. Linear momentum. Consider the 1D situation and P|p> = p|p>. This is an EV problem, P is a Hermitian operator, the spectrum is the entire real axis.

How does this operator P fit into the Stakgold world? I am now reviewing the $E^{(n)}$ in my Stakgold notes. If you think of <i|P|j> = P_{ij} as a matrix and P as an operator, we can fit this in. But we know in the |p> basis there seem to be an infinite number of kets, so we suspect that P is operating in some $E^{(\infty)}$ space. Would we ever consider an equation Px = f in the sense of Ax=f ? I have never asked that question before in my life, not sure what the answer is. We *do* ask about the EV problem Px = px and of course in this case the eigenvectors are x = |p> . What about the adjoint of P. <P*i|j> = <i|Pj>. We suspect that at least in our usual |p> basis that P = P* so must be true in any basis, so P is Hermitian which Stakgold calls Symmetric. Is there a nullspace for operator P such that Px = 0 ? I think there is no non-trivial nullspace and this is a full rank operator. In a dim=N space, we set det(A)=0 to see if Ax=f has a solution, and we set det(A-λI) = 0 to get the eigenvalues of Ax = λx and since we get a polynomial, we have only a point spectrum which is real for Hermitian A. We also get the notion of eigenmanifolds which are orthogonal. What about norm of P? Think max of ||P |p>||/|| |p>|| = something like p, and I would say that since p can have any real value, P is an unbounded operator. What about an inverse P^{-1}? Again, something I have never thought about. Relates to the Px=f equation and its solution. I draw a blank. Maybe since there is no nullspace for P, we just think of P^{-1} as 1/P and use it that way.

OK, now let's move into Stakgold's section 2.8 where he starting talking ∞ dim HS. Back to the nullspace question: Ax = 0. What is the "0" vector in our |p> world? Never thought of that either. We could imagine a general |0> vector such that, for example, <x|0> = 0 for all x. This would not be the same as the |p=0> momentum vector which happens to have p = 0. Note <x|p> = e^{ipx} so <x|p=0> = 1. So OK, I guess we can just imagine some |0> ket such that |i> + |0> = |i>.

I am hoping that P is the case of the "continuous spectrum". In Stak notes this is case 3, which says that (P-λI)|i> = 0 "has only the trivial solution"

Now lets think of P = d/dx as on Stak page 174. But then we are talking d/dx: $L_2 \rightarrow L_2$ so we now have a space of functions. He shows that d/dx is "closed", and A* = - d/dx. On page 192 he considers the EV equation d/dx f = λf ad f = $e^{\lambda x}$ is EV and every λ is an eigenvalue (page bottom). When you add BC's, this changes. Then Bf = (d/dx - λ)f = 0 has a non-trivial solution (and so B^{-1} does not exist)! I think this is case 2 and you would still call the continuous range of λ the "**point spectrum**". We know that by changing the BV defining the domain, we can change this fact. Notice that you need p = ± id/dx to get a Hermitian operator due to the sign change of parts integration.

So the abstract operator P is far distant from the Stakgold discussion, but I think we can say that the operator has a continuous point spectrum.

You would not say that $|p_x= 6\rangle = 3\ |p_x= 2\rangle$. These two vectors are orthogonal in fact, as we just stated $\langle p_x|p_x'\rangle = \delta(p_x - p_x)$. So the ket $|p_x\rangle$ for each real value of p_x is a distinct vector in our HS, so this is an infinite dimensional HS.

How do you add kets? $|p_x= 2\rangle + 5.4\ |p_x= 3\rangle$ is in our HS I guess, but not an eigenket.

The only way I know to understand operator P is to think of it as id/dx acting on L_2. Then I can show it is Hermitian, I can think of eigenfunctions e^{ipx} and write them as <x|p>. Every ket with a different real value of p is a separate "basis ket", so basis is infinite dimensional.

Now think of 3D momentum. We have kets of the form $|p_x, p_y, p_z\rangle = |\mathbf{p}\rangle$. Can we have the idea of a subspace here? What about the kets of the form $|p_x, 0,0\rangle$. We can talk about $5|p_x, 0,0\rangle$ if we want, so scale it all you want. Can we add kets? Well think $e^{ipx.x}\, e^{ipy.y}\, e^{ipz.z} = \langle \mathbf{r}|\mathbf{p}\rangle$. So I don't see an associated subspace idea here. If you have $L_2(x,y,z)$, you could also talk $L_2(x)$ as a different domain and range of an operator.

2. Angular Momentum. We could define this in the usual sense with r x ∇ and discuss this operator in the Stakgold sense. But just call it J for the moment. If we say $\mathbf{J}|\mathbf{j}\rangle = \mathbf{j}\,|\mathbf{j}\rangle$ there are no eigenkets. This is intrinsic 3D. We can diagonalize J^2 and J_z using the |jm> kets. Some HS I guess is spanned by these things, the space of angles $|\theta\varphi\rangle$ for example (which might describe the position of a particle). We can talk about $\langle j'm'|A|jm\rangle = A_{j'm',j,m}$ as a matrix. If A is a rotational scalar, it commutes with **J** . What does that tell us?

$$0 = \langle j'm'|[J^2,A]|jm\rangle = j'(j'+1)\langle j'm'|A|jm\rangle - j(j+1)\langle j'm'|A|jm\rangle \quad \Rightarrow \quad j = j' \text{ else } 0$$

$$0 = \langle j'm'|[J_3,A]|jm\rangle = m'\langle j'm'|A|jm\rangle - m\langle j'm'|A|jm\rangle \quad \Rightarrow \quad m = m' \text{ else } 0$$

But then the matrix for A is completely diagonal

$$\langle j'm'|A|jm\rangle = \delta_{j,j'}\delta_{m,m'}\langle jm|A|jm\rangle$$

I think we know even more.

$$J_+|jm\rangle = [j(j+1)-m(m+1)]^{1/2}\,|jm+1\rangle = k_{jm}|jm+1\rangle$$
$$\langle jm|J_- = [j(j+1)-m(m+1)]^{1/2}\,\langle jm+1| = k_{jm}\langle jm+1|$$

Then

$$(k_{jm})^2\langle jm+1|A|jm+1\rangle = \langle jm|J_- A J_+|jm\rangle$$

$$= \langle jm|A J_- J_+|jm\rangle = \langle jm|A\{J^2 - J_z^2 - J_z\}|jm\rangle = \langle jm|A|jm\rangle k_{jm}^2$$

and therefore

$$\langle jm+1|A|jm+1\rangle = \langle jm|A|jm\rangle = \langle j \| A \| j\rangle$$

where we write it in the Wigner-Eckart sense that it has no dependence on m. So our matrix then has this form

$$A_{j'm',jm} = \delta_{j,j'}\delta_{m,m'}\langle j \| A \| j\rangle$$

Now I want to say that we have different orthogonal subspaces for different values of j. What does that mean? There is some HS that is pretty vague, but which has this unity

$$1 = \Sigma_{j=0,1/2,3/2...} \; [\; \Sigma_{m\,=\,-j,-j+1\;...+j} \; |jm><jm| \; \} \;]$$

$$= \Sigma_j \; 1^{(j)} \qquad \text{where } 1^{(j)} \; = \; \Sigma_{m\,=\,-j,-j+1\;...+j} \; |jm><jm|$$

Maybe I am claiming here that the operators J^2 and J_z have a "spectrum" of eigenkets. We have a set of finite matrices where are the irreducible representations of the generators of the rotation group. The space we are talking about in the physical ang mom world has this fact

$$1 = \int d\Omega \; |\theta\varphi><\theta\varphi| \; = \Sigma_{jm} \, |jm><jm| \; = \oplus_j \; |jm><jm| \; = \oplus_j \; 1^{(j)}$$

so somehow this suggests that our total space is a sum of fixed-j subspaces. We know that the kets of different subspaces are orthogonal.

I don't even know what questions to ask here, so let's put this on hold.

Suppose you have a problem with a Hamiltonian H and you know that $[H,\mathbf{J}] = 0$. We then use j and m as two of the "good quantum numbers" of our solution kets $|\, jm;\alpha>$ where α are "other" quantum numbers such as energy or perhaps of other symmetries like isospin, say. Then we know that

$<j'm',\alpha' \,|\, H \,|\, jm;\alpha> \; = 0$ no matter what, if $j' \neq j$.

$<jm',\alpha' \,|\, H \,|\, jm;\alpha> \; = 0$ no matter what, if $m' \neq m$.

So it is the spectrum of operator H that we can represent I think as a direct sum of subspaces.

Suppose $H = -k\mathbf{S}\bullet\mathbf{B}$. Then we know that $[H,S^2] = 0$. But for general **B**, we cannot say that we have $[H,S_z] = 0$. So this Hamiltonian is not a spin-rotational scalar, because we don't have $[H,\mathbf{S}] = 0$. But still, we could make the first claim above, albeit not the second claim. But if $\mathbf{B} = B\hat{\mathbf{z}}$, we can then make both claims.

So back to $|\, jm;\alpha>$. Think of this as the state of some physical system. We can define J and J_z somehow for such a system. This ket is an eigenket of three operators J^2, J_z and H and perhaps others like isospin that we group into index α. So what exactly is "the Hilbert Space" here? It is the eigenstates of H. But it happens that these can be selected to also be eigenkets of J^2 and J_z where these are certain operators for our system.

3. My Confusion. Before you can talk about an operator like P or J or A, you have to say what your Hilbert Space is. It might be the QM Hilbert Space of some "physical system" like "a particle" or "system of particles" such as a rigid rotor or like a cloud of electrons. Then once you have such a system, you can talk about a space of kets for the Hamiltonian of that system, so the eigenstates of H span the Hilbert Space of those kets. The scalar product of the HS is then the contraction of bras of H with kets of H. Then you can come along and talk about operators which represent "observables", like P or like L or S or J. If P

and H commute, then the HS kets can be labeled by p. It is $H = P^2/2m$ that yields $|p,\alpha\rangle$ where α are some other quantum numbers. If there are no others, you just write $|p\rangle$. And if H commutes with J (total ang mom) as it does for $H = L^2/2I$ with a spinless system, then the eigenstates of H may be labeled by jm and you have states $|jm,\alpha\rangle$. I think the fact that $|jm\rangle$ is an eigenvector of operator J^2 and J_z really has nothing to do with QM a priori. The $|jm\rangle$ are really eigenvectors of a 2j+1 dimensional matrix representation of the Lie Algebra SO(3), just a math thing, no physical system required. You enter the QM realm when you have a ket $|jm,\alpha\rangle$ for a physical system whose H commutes with J^2 and J_z.

4. More comments. I have always tried to think of the QM Hilbert Space as being completely detached from the coordinate representation, but maybe I have gone too far on this. After all, the coordinate representation is "the real thing" we are after in QM. Once we have it, we can then talk about alternative representations and then abstract things as much as possible. You need the coordinate representation to think about things a la Stakgold.

(a) Linear Momentum P. For example, we know that P is the generator of translations in coordinate space:

$$P = -i\hbar\partial_x = -i\hbar D \qquad e^{-iaP}f(x) = f(x-a) \qquad (1 - iaP)f(x) = f(x) - af'(x) = (1 - a\partial_x)f(x)$$

where I think the signs are right. But now we have a function space to "work with" for domain and range of the operator P -- that is, we have a Hilbert Space with a well-defined scalar product. In QM, the HS is always something close to L_2 since $\langle\psi|\psi\rangle$ is then probability -- we would never use the L_1 norm. Once we have this "representation" for P, we know exactly what we mean by the scalar product, and we can show that $P = P^\dagger$ for example using parts integration.

The operator D as studied by Stakgold is closed (continuous in some sense) and unbounded in his operator classification. The operator has an inverse, which is "integration", so we can talk about D^{-1} in this sense, and D^{-1} is bounded. D has a nullspace, since f(x) = C is a non-zero function which maps to 0. The adjoint operator D* only exists if we have, in terms of Ax=y, the fact that y(b) = y(a) = 0 (see p 175). This means that D* has no nullspace, and that means that the range of D is all of L_2 , according to the Alternative Theorem. D is itself not Hermitian (it is anti-Hermitian without the i of P). If no other domain BC's, D has a point spectrum which is the entire complex plane for $D\psi(x) = \lambda\psi(x)$.

Classifying Closed Operators. These are the only possibilities, one regular and three singular:

(1) A is regular (meets the three criteria listed above)

(2) A^{-1} does not exist, nontrivial Ax=0; [violates criterion (a)]

(3) A^{-1} exists *and* A^{-1} is unbounded [violates criterion (c) and (b)]

In this case, we know from our drawing that $R_A \neq \overline{R}_A = H$

(4) A^{-1} exists but $\overline{R_A} \neq H$. [violates criterion (b)] $R_A \neq H$

For our operator D, D^{-1} exists, so that rules out (2). D^{-1} is bounded, that rules out (3). The range of D is all of H as we just claimed above, so that rules out (4). This, D is a "regular" operator.

When we want to think about the spectrum of A, however, we have to apply the above classification not to A, but to $B = (A - \lambda I)$. Since every complex and real λ is an eigenvalue (no extra BC's on the domain of D), B is in (2) above, and this means that every λ is in the point spectrum. I wish Stakgold had

considered the operator P = iD as one of his examples. It would be Hermitian, so I think the conclusion would be that eigenvalues must be real, so then the entire real axis is the point spectrum.

Note that we don't ever think about the operator P^{-1} in quantum mechanics, this being integration instead of differentiation. We use P more as something which labels eigenstates as noted above, and the differential equation concern is more with H, not with P. Even for the free particle, our operator of concern is ∇^2 and not D, and Stakgold surely has lots to say somewhere about ∇^2 in his chapters on differential equations.

Once we have analyzed things in the coordinate representation $|\mathbf{r}\rangle$, we can then think of $|\mathbf{p}\rangle$ as an alternative representation, but the properties of operators will be independent of representation. For example, we have

$$1 = \int d^3r\, |\mathbf{r}\rangle\langle\mathbf{r}| = \int d^3p\, |\mathbf{p}\rangle\langle\mathbf{p}| \qquad \langle\mathbf{r}|\mathbf{r}'\rangle = \delta^3(\mathbf{r}-\mathbf{r}') \qquad \langle\mathbf{p}|\mathbf{p}'\rangle = \delta^3(\mathbf{p}-\mathbf{p}')$$

and either form can act as the unit operator in the Hilbert Space that starts out being the familiar one of the coordinate representation. The p-representation is of course useful when P commutes with H. One usually does not have R commuting with H in simple situations.

(b) Angular Momentum L For example, we know that L is the generator of rotations in coordinate space. We know how to write all the generators in terms of θ and φ variables, so we have a Hilbert Space to "work with" for angular momentum L. If we think about the "1D" angular momentum situation, we would be talking $L_z = \pm i\partial_\varphi$ as our operator. This would again be parts-integration Hermitian, but now there are extra boundary conditions matching φ = 0 with φ = 2π, and this is what causes the spectrum of this operator to be discrete. If we have $L_z f = \lambda f$, and we assume the implied Hilbert Space, we end up with the spectrum λ = 0, ±1,±2 and so on. So this is a clear distinction with the spectrum of operator P_x considered above. When we go into 3D, we cannot diagonalize all three generators, so we settle for L^2 and L_z and we end up with each of these operators having a discrete real spectrum (and we think about this in the coordinate space representation which angles θ and φ). Surely one could do a Stakgold analysis of these two operators, but I won't do that here.

As above, once we have the θφ situation figured out, we can switch to an ℓm basis,

$$1 = \int d\Omega\ |\theta\varphi\rangle\langle\theta\varphi| = \Sigma_{\ell m}\, |\ell m\rangle\langle\ell m| \qquad \langle\theta'\varphi'|\theta\varphi\rangle = \delta(C_{\theta'} - C_\theta)\delta(\varphi'-\varphi) \qquad \langle\ell' m'|\ell m\rangle = \delta_{\ell'\ell}\,\delta_{m'm}$$