A Rotation Problem PhL 7.3.12

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1. Statement of the problem:

Given vectors $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$, find vector $d\boldsymbol{\Omega}$ such that

$$R(d\boldsymbol{\Omega}) = R(\boldsymbol{\theta})^{-1}R(\boldsymbol{\theta}+d\boldsymbol{\theta})$$

We know that some $R(d\boldsymbol{\Omega})$ exists because, since rotations form a group, the product on the right is *some* rotation. We anticipate that it is a small rotation, since the two items on the right are so close to being inverses of each other. In terms of generators, the above reads

$$\exp(-i\, d\boldsymbol{\Omega} \bullet \mathbf{J}) = \exp(+i\, \boldsymbol{\theta} \bullet \mathbf{J})\ \exp(-i\, [\boldsymbol{\theta}+d\boldsymbol{\theta}] \bullet \mathbf{J})$$

I have been using the following simple notation for the vector parameters

$$\boldsymbol{\theta} = \theta\mathbf{n} \qquad d\boldsymbol{\theta} = d\varphi\, \mathbf{m} \qquad d\boldsymbol{\Omega} = d\Omega\, \mathbf{r}$$

$$[\boldsymbol{\theta}+d\boldsymbol{\theta}] = S\mathbf{s} = \theta\mathbf{n} + d\varphi\, \mathbf{m}$$

where $\mathbf{n},\mathbf{m},\mathbf{r},\mathbf{s}$ are all unit vectors.

It seems to me that there should be some easy way to solve this problem, but so far I have been unable to find it. I will review the various methods I have tried but which have failed. In any solution, one must somehow deal with the complicated object $\exp(-i\, [\boldsymbol{\theta}+d\boldsymbol{\theta}] \bullet \mathbf{J})$.

Comment: If we use Euler angles, I know that we can solve the *general* problem

$$R(\psi_3,\theta_3,\varphi_3) = R(\psi_1,\theta_1,\varphi_1)\ R(\psi_2,\theta_2,\varphi_2)$$

and this is probably written down somewhere in my notes or in Tinkham or what have you. So we know our little problem is solvable. It's just that I wanted an easy vector way to do it.

2. Plan A: Too Ugly

In this plan, I do a brute force attempt to expand $\exp(-i\, [\boldsymbol{\theta}+d\boldsymbol{\theta}] \bullet \mathbf{J})$

$$\exp(-i\,[\boldsymbol{\theta}+\mathbf{d\theta}] \bullet \mathbf{J}) = \Sigma_{n=0}^{\infty}\ (-i[\boldsymbol{\theta}+\mathbf{d\theta}] \bullet \mathbf{J})^n/n!$$

The quantity of interest then is

$$(-i[\boldsymbol{\theta}+\mathbf{d\theta}] \bullet \mathbf{J})^n = (-i[\boldsymbol{\theta}+\mathbf{d\theta}]_k\, J_k)^n$$

Define the vector

$$\mathbf{q} \equiv -i[\boldsymbol{\theta}+\mathbf{d\theta}]$$

Then the quantity of interest is

$$(q_k J_k)^n$$

Here then are the first few powers:

$$(q_a J_a)^1 = q_a J_a$$

$$(q_a J_a)^2 = q_a J_a\, q_b J_b = q_a q_b\, J_a\, J_b$$

$$(q_a J_a)^3 = q_a q_b q_c\ J_a\, J_b\, J_c$$

$$(q_a J_a)^4 = q_a q_b q_c q_d\ J_a\, J_b\, J_c\, J_d$$

$$= (-i)^4\ [\theta+d\theta]_a\, [\theta+d\theta]_b\, [\theta+d\theta]_c\, [\theta+d\theta]_d\, J_a\, J_b\, J_c\, J_d$$

$$= (-i)^4\ \theta_a\theta_b\theta_c\theta_d\ J_a\, J_b\, J_c\, J_d + (-i)^4\ d\theta_a\theta_b\theta_c\theta_d\ J_a\, J_b\, J_c\, J_d + \text{etc}$$

This seems a very messy path to follow so I did not pursue it further. The messy part is really taken care of elsewhere in Plan B to follow.

3. Plan B: Works but is a mess.

Now in document "exp nJ calculation.doc" I show this fact in 3D,

$$\exp(-i\theta\,\mathbf{n}\bullet\mathbf{J})_{ab} = \delta_{ab} + (1 - C_\theta)\,(\delta_{ab} - n_a n_b) - n_i \varepsilon_{iac}\,\{ S_\theta\,(\delta_{cb} - n_c n_b)\}$$

which I translate to say

$$\exp(-iS\,\mathbf{s}\bullet\mathbf{J})_{ab} = \delta_{ab} + (1 - C_S)\,(\delta_{ab} - s_a s_b) - s_j \varepsilon_{jae}\,\{ S_S\,(\delta_{eb} - s_e s_b)\}$$

So this seems to at least give some reasonable expansion for the ugly $\exp(-i\,[\boldsymbol{\theta}+\mathbf{d\theta}] \bullet \mathbf{J})$ object. Then we get

$$[\exp(-i\,\mathbf{d\Omega} \bullet \mathbf{J})]_{ab} = [\exp(+i\,\boldsymbol{\theta} \bullet \mathbf{J})]_{ad}\ [\exp(-i\,[\boldsymbol{\theta}+\mathbf{d\theta}] \bullet \mathbf{J})]_{db}$$

$$[\delta_{ad} + (1 - C_\theta)(\delta_{ad} - n_a n_d) - n_i \varepsilon_{iac} \{ S_\theta (\delta_{cd} - n_c n_d)\}] *$$

$$[\delta_{db} + (1 - C_s)(\delta_{db} - s_d s_b) - s_j \varepsilon_{jde} \{ S_s (\delta_{eb} - s_e s_b)\}]$$

$$= \delta_{ab} + [\text{other terms}]_{ab}$$

Expanding the LHS assuming small angle gives [since $i (J_k)_{ab} = \varepsilon_{kab}$]

$$\delta_{ab} + (-i\, d\boldsymbol{\Omega} \bullet \mathbf{J})_{ab} = \delta_{ab} + (-i\, d\Omega_k J_k)_{ab} = \delta_{ab} + (-i\, d\Omega_k (J_k)_{ab} = \delta_{ab} - d\Omega_k \varepsilon_{kab}$$

So the implication is that this must be true

$$- d\Omega_k \varepsilon_{kab} = [\text{other terms}]_{ab} \qquad // = -d\Omega\, r_k\, \varepsilon_{kab}$$

Now define these two symmetric matrices

$$N_{ab} = (\delta_{ab} - n_a n_b)$$

$$S_{ab} = (\delta_{ab} - s_a s_b)$$

Then we can write

$$[\delta_{ad} + (1 - C_\theta) N_{ad} - n_i \varepsilon_{iac} S_\theta N_{cd}] *$$

$$[\delta_{db} + (1 - C_s) S_{db} - s_j \varepsilon_{jde} S_s S_{eb}]$$

$$= \delta_{ab} + [\text{other terms}]_{ab}$$

There are 8 terms in "other terms" and I will just list them

$\delta_{ad}(1 - C_s) S_{db}$
$\delta_{ad} (- s_j \varepsilon_{jde} S_s S_{eb})$
$(1 - C_\theta) N_{ad} \delta_{db}$
$(1 - C_\theta) N_{ad}(1 - C_s) S_{db}$
$(1 - C_\theta)(- s_j \varepsilon_{jde} S_s S_{eb})$
$(- n_i \varepsilon_{iac} S_\theta N_{cd}) \delta_{db}$
$(- n_i \varepsilon_{iac} S_\theta N_{cd}) (1 - C_s) S_{db}$
$(- n_i \varepsilon_{iac} S_\theta N_{cd}) (- s_j \varepsilon_{jde} S_s S_{eb})$

Since the sum of these terms is supposed to be equal to $- d\Omega_k \varepsilon_{kab}$, the implication is that the above some is antisymmetric under a↔b, but this fact is totally non-obvious. But let's assume it is in fact true. Then we can compute for example

$- d\Omega_1 \, \varepsilon_{123} = - d\Omega_1 =$

$\delta_{2d}(1 - C_s) S_{d3} = (1 - C_s) S_{23}$
$\delta_{2d} (- s_j \varepsilon_{jde} S_s S_{e3}) = (- s_j \varepsilon_{j2e} S_s S_{e3})$
$(1 - C_\theta) N_{2d} \delta_{d3} = (1 - C_\theta) N_{23}$
$(1 - C_\theta) N_{2d}(1 - C_s) S_{d3}$
$(1 - C_\theta)(- s_j \varepsilon_{jde} S_s S_{e3})$
$(- n_i \varepsilon_{i2c} S_\theta N_{cd}) \delta_{d3} = (- n_i \varepsilon_{i2c} S_\theta N_{c3})$
$(- n_i \varepsilon_{i2c} S_\theta N_{cd}) (1 - C_s) S_{d3}$
$(- n_i \varepsilon_{i2c} S_\theta N_{cd}) (- s_j \varepsilon_{jde} S_s S_{e3})$

It is an ugly mess, but at least it is something for $-d\Omega_1$. Similarly we can compute the others and then we have computed $d\boldsymbol{\Omega}$ which was the goal. Somehow we think this will be proportional to dφ but that too is not very obvious. To pursue this we would have to compute more things, such as

$S^2 = \theta^2 + 2 \theta \, d\varphi \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}} = \theta^2 (1 + 2 \, d\varphi/\theta \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}})$
$S = \theta (1 + 2 \, d\varphi/\theta \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}})^{1/2} \approx \theta (1 + d\varphi/\theta \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}}) \approx \theta + d\varphi \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}}$

$S\mathbf{s} = \theta \mathbf{n} + d\varphi \, \mathbf{m}$
$\mathbf{s} = \theta \mathbf{n}/S + d\varphi \, \mathbf{m}/S$
$\mathbf{s} \approx \theta \mathbf{n}/(\theta + d\varphi \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}}) + d\varphi \, \mathbf{m}/\theta$
$\mathbf{s} \approx \mathbf{n}/(1 + d\varphi/\theta \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}}) + d\varphi \, \mathbf{m}/\theta$
$\mathbf{s} \approx \mathbf{n}(1 - d\varphi/\theta \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}}) + d\varphi \, \mathbf{m}/\theta$
$\mathbf{s} \approx \mathbf{n} - d\varphi/\theta \, \mathbf{n} \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}} + d\varphi \, \mathbf{m}/\theta$
$\mathbf{s} \approx \mathbf{n} - (d\varphi/\theta)[\mathbf{n} \, \hat{\mathbf{n}} \bullet \hat{\mathbf{m}} + \mathbf{m}]$

I could go on for a long time, but since I have an answer, let it be.

4. Digression On Irrelevant Fact.

When I started on this, I thought there was some reason to compute

$R(\boldsymbol{\theta}) \, R(d\boldsymbol{\theta}) \, R(-\boldsymbol{\theta})$.

But now I don't see that this has any bearing at all on the problem at hand. But since I did this calculation, I will record it here. I did it like this

$R(\boldsymbol{\theta}) \, R(d\boldsymbol{\theta}) \, R(-\boldsymbol{\theta}) = \exp(- i \theta \, \hat{\mathbf{n}} \bullet \mathbf{J}) \exp(- i \, d\varphi \, \hat{\mathbf{m}} \bullet \mathbf{J}) \exp(+ i \theta \, \hat{\mathbf{n}} \bullet \mathbf{J})$

Now expand

$$\exp(- i \, d\varphi \, \hat{\mathbf{m}} \bullet \mathbf{J}) \approx 1 - i \, d\varphi \, \hat{\mathbf{m}} \bullet \mathbf{J}$$

Then have

$$R(\boldsymbol{\theta})\, R(d\boldsymbol{\theta})\, R(-\boldsymbol{\theta}) = \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\,[1 - i\,d\varphi\,\hat{\mathbf{m}}\bullet\mathbf{J}]\,\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})$$

$$= 1 - id\varphi\,\hat{\mathbf{m}}\bullet\{\ \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\,\mathbf{J}\,\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\ \}$$

Now use the sandwich rule for the thing in "Evaluate n dot J rotation of J.doc"

$$\exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\,\mathbf{J}\,\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J}) = \mathbf{J}\cos\theta + \mathbf{J}\times\hat{\mathbf{n}}\,\sin\theta + \hat{\mathbf{n}}\,(\hat{\mathbf{n}}\bullet\mathbf{J})\,(1-\cos\theta)$$

and so we then have

$$R(\boldsymbol{\theta})\, R(d\boldsymbol{\theta})\, R(-\boldsymbol{\theta}) = 1 - id\varphi\,\hat{\mathbf{m}}\bullet\{\ \mathbf{J}\cos\theta + \mathbf{J}\times\hat{\mathbf{n}}\,\sin\theta + \hat{\mathbf{n}}\,(\hat{\mathbf{n}}\bullet\mathbf{J})\,(1-\cos\theta)\ \}$$

$$\approx \exp(-id\varphi\,\hat{\mathbf{m}}\bullet\{\ \mathbf{J}\cos\theta + \mathbf{J}\times\hat{\mathbf{n}}\,\sin\theta + \hat{\mathbf{n}}\,(\hat{\mathbf{n}}\bullet\mathbf{J})\,(1-\cos\theta)\ \})$$

Now we know that $\hat{\mathbf{m}}\bullet\mathbf{J}\times\hat{\mathbf{n}} = \mathbf{J}\bullet\hat{\mathbf{n}}\times\hat{\mathbf{m}}$ so the above becomes

$$= \exp(-id\varphi\,\{\ \hat{\mathbf{m}}\bullet\mathbf{J}\cos\theta + \mathbf{J}\bullet\hat{\mathbf{n}}\times\hat{\mathbf{m}}\sin\theta + \hat{\mathbf{m}}\bullet\hat{\mathbf{n}}\,(\hat{\mathbf{n}}\bullet\mathbf{J})\,(1-\cos\theta)\ \})$$

$$= \exp(-id\varphi\,\{\ \hat{\mathbf{m}}\cos\theta + \hat{\mathbf{n}}\times\hat{\mathbf{m}}\sin\theta + \hat{\mathbf{m}}\bullet\hat{\mathbf{n}}\,(\hat{\mathbf{n}}\,)\,(1-\cos\theta)\ \}\bullet\mathbf{J}\,)$$

I then defined **u** as

$$\mathbf{u} \equiv \hat{\mathbf{m}}\cos\theta + \hat{\mathbf{n}}\times\hat{\mathbf{m}}\sin\theta + \hat{\mathbf{m}}\bullet\hat{\mathbf{n}}\,(\hat{\mathbf{n}}\,)\,(1-\cos\theta)$$

I showed that **u** was a unit vector like so:

$$\mathbf{u}\bullet\mathbf{u} = [\hat{\mathbf{m}}\cos\theta + \hat{\mathbf{n}}\times\hat{\mathbf{m}}\sin\theta + \hat{\mathbf{m}}\bullet\hat{\mathbf{n}}\,(\hat{\mathbf{n}}\,)\,(1-\cos\theta)]^2$$

$$= \cos^2\theta + (\hat{\mathbf{n}}\times\hat{\mathbf{m}})^2\sin^2\theta + (\hat{\mathbf{m}}\bullet\hat{\mathbf{n}})^2(1-\cos\theta)^2$$
$$+ 2\cos\theta\sin\theta\,\hat{\mathbf{m}}\bullet\hat{\mathbf{n}}\times\hat{\mathbf{m}} + 2\cos\theta(1-\cos\theta)(\hat{\mathbf{m}}\bullet\hat{\mathbf{n}})^2 + 2\sin\theta(1-\cos\theta)\,\hat{\mathbf{n}}\times\hat{\mathbf{m}}\bullet\hat{\mathbf{n}}\,(\hat{\mathbf{m}}\bullet\hat{\mathbf{n}})$$

$$= \cos^2\theta + (\hat{\mathbf{n}}\times\hat{\mathbf{m}})^2\sin^2\theta + (\hat{\mathbf{m}}\bullet\hat{\mathbf{n}})^2(1-\cos\theta)^2$$
$$+ 2\cos\theta(1-\cos\theta)(\hat{\mathbf{m}}\bullet\hat{\mathbf{n}})^2$$

But $(\hat{\mathbf{n}}\times\hat{\mathbf{m}})^2 = \sin^2\psi$ where ψ is the angle between these two vectors, and $(\hat{\mathbf{m}}\bullet\hat{\mathbf{n}})^2 = \cos^2\psi$ so

$$= \cos^2\theta + \sin^2\psi\sin^2\theta + \cos^2\psi(1-\cos\theta)^2 + 2\cos\theta(1-\cos\theta)\cos^2\psi$$

$$= \cos^2\theta + \sin^2\psi\sin^2\theta + \cos^2\psi(1+\cos^2\theta-2\cos\theta) + 2\cos\theta\cos^2\psi - 2\cos^2\theta\cos^2\psi$$

$$= \cos^2\theta + \sin^2\psi\sin^2\theta + \cos^2\psi(1-\cos^2\theta-2\cos\theta) + 2\cos\theta\cos^2\psi$$

$= \cos^2\theta + \sin^2\psi \sin^2\theta + \cos^2\psi (1 - \cos^2\theta)$

$= \cos^2\theta + \sin^2\psi \sin^2\theta + \cos^2\psi \sin^2\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1$

And the conclusion then was

$R(\theta\hat{\mathbf{n}}) R(d\varphi\, \hat{\mathbf{m}}) R(-\theta\hat{\mathbf{n}}) \approx \exp(-id\varphi\, \hat{\mathbf{u}} \bullet \mathbf{J}) = R(d\varphi\, \hat{\mathbf{u}})$

where $\hat{\mathbf{u}} \equiv \hat{\mathbf{m}} \cos\theta + \hat{\mathbf{n}} \times \hat{\mathbf{m}} \sin\theta + \hat{\mathbf{m}} \bullet \hat{\mathbf{n}} (\hat{\mathbf{n}}) (1 - \cos\theta)$

So basically the sandwich here just changes the axis of the small rotation from $\hat{\mathbf{m}}$ to $\hat{\mathbf{u}}$.

5. The problem with order reversed

Given vectors $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$, find vector $d\boldsymbol{\Omega}'$ such that

$R(d\boldsymbol{\Omega}') = R(\boldsymbol{\theta}+d\boldsymbol{\theta}) R(\boldsymbol{\theta})^{-1} \quad \leftrightarrow \quad R(\boldsymbol{\theta}+d\boldsymbol{\theta}) = R(\boldsymbol{\theta}) R(d\boldsymbol{\Omega}')$

This problem can be solved in a similar fashion, but we expect that $d\boldsymbol{\Omega}' \neq d\boldsymbol{\Omega}$.