A simple theorem about basis vectors PhL 7.19.12

Just skip to part (b) where this theorem is quickly proven:

$\mathbf{e'}_n = R\mathbf{e}_n \quad \Rightarrow \quad \mathbf{e'}_n = R^{-1}{}_{nm}\,\mathbf{e}_m$ or $\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \\ \dots \end{pmatrix} = R^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \dots \end{pmatrix}$

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(a) The Theorem for N = 2

Let's assume that two sets of basis vectors are related this way (N=2)

$\mathbf{e'}_n = R_z(\varphi)\mathbf{e}_n \qquad R_z(\varphi)$ = the usual active form = $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

The **theorem** is that you can write

$$\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \end{pmatrix} = R_z(-\varphi)\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$$

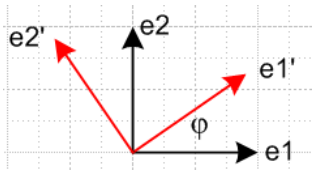
which is a *shorthand notation* for

$\mathbf{e'}_n = R_z(-\varphi)_{nm}\,\mathbf{e}_m \qquad$ for n = 1,2

which is the same as

$\mathbf{e'}_1 = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2$
$\mathbf{e'}_2 = -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2$

Proof #1:



Write

$\mathbf{e'}_1 = (\mathbf{e}_1 \bullet \mathbf{e'}_1)\,\mathbf{e}_1 + (\mathbf{e}_2 \bullet \mathbf{e'}_1)\,\mathbf{e}_2$
$\mathbf{e'}_2 = (\mathbf{e}_1 \bullet \mathbf{e'}_2)\,\mathbf{e}_1 + (\mathbf{e}_2 \bullet \mathbf{e'}_2)\,\mathbf{e}_2$

The drawing shows that

$(\mathbf{e}_1 \bullet \mathbf{e}'_1) = \cos\varphi$ $\qquad$ $(\mathbf{e}_2 \bullet \mathbf{e}'_1) = \cos(\pi/2-\varphi) = \sin\varphi$ = picture shows positive

$(\mathbf{e}_2 \bullet \mathbf{e}'_2) = \cos\varphi$ $\qquad$ $(\mathbf{e}_1 \bullet \mathbf{e}'_2) = \cos(\pi/2+\varphi) = -\sin\varphi$

Therefore

$\mathbf{e}'_1 = \cos\varphi\ \mathbf{e}_1 + \sin\varphi\ \mathbf{e}_2$

$\mathbf{e}'_2 = -\sin\varphi\ \mathbf{e}_1 + \cos\varphi\ \mathbf{e}_2$ $\qquad$ QED.

Proof #2: Here is what we want to verify:

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = R_z(-\varphi)\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \qquad (1)$$

and here is what we know

$$\mathbf{e}'_n = R_z(\varphi)\mathbf{e}_n \qquad n = 1,2 \qquad (2)$$

Write out (1) this way

$$\mathbf{e}'_n = R_z(-\varphi)_{nm}\,\mathbf{e}_m \qquad (3)$$

Install (2) as the LHS of (3) to get

$$R_z(\varphi)\mathbf{e}_n = R_z(-\varphi)_{nm}\,\mathbf{e}_m \qquad (4)$$

Take the ith component

$$R_z(\varphi)_{ij}(\mathbf{e}_n)_j = R_z(-\varphi)_{nm}\,(\mathbf{e}_m)_i \qquad (5)$$

But we know that $(\mathbf{e}_n)_j = \delta_{n,j}$ when evaluated in the black frame, so we then have

$$R_z(\varphi)_{ij}\,\delta_{n,j} = R_z(-\varphi)_{nm}\,\delta_{m,i} \qquad (6)$$

$$R_z(\varphi)_{in} = R_z(-\varphi)_{ni} \qquad (7)$$

But this is true since

$$R_z(-\varphi)_{ni} = R_z(\varphi)^{-1}_{ni} = R(\varphi)^{T}_{ni} = R(\varphi)_{in} \qquad (8)$$

Proof #3. Now let's reverse Proof 2 :

$$R_z(\varphi)_{in} = R_z(-\varphi)_{ni} \qquad (1)$$

$$R_z(\varphi)_{ij}\,\delta_{n,j} = R_z(-\varphi)_{nm}\,\delta_{m,i} \qquad (2)$$

$$R_z(\varphi)_{ij}(\mathbf{e}_n)_j = R_z(-\varphi)_{nm}\,(\mathbf{e}_m)_i \qquad (3)$$

$$R_z(\varphi)\mathbf{e}_n = R_z(-\varphi)_{nm}\,\mathbf{e}_m \qquad (4)$$

$$\mathbf{e}'_n = R_z(-\varphi)_{nm}\,\mathbf{e}_m \qquad (5)$$

$$\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\end{pmatrix} = R_z(-\varphi)\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix} \qquad (6)$$

(b) The Theorem for general N

The **theorem** is that, if this is true for how Cartesian basis vectors are related in two frames,

$\mathbf{e}'_n = R\mathbf{e}_n$ n = 1,2....N , // $(\mathbf{e}_n)_i = \delta_{n,i}$

then the following is true

$$\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\\ \dots\end{pmatrix} = R^{-1}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \dots\end{pmatrix}$$

which is just a *shorthand notation* for

$\mathbf{e}'_n = R^{-1}{}_{nm}\,\mathbf{e}_m$ n = 1,2...N

Proof A:

$R_{in} = R^{-1}{}_{ni}$ // since $R^{-1}{}_{ni} = R^T{}_{ni} = R_{in}$ (1)

$R_{ij}\,\delta_{n,j} = R^{-1}{}_{nm}\,\delta_{m,i}$ // rewrite (2)

$R_{ij}(\mathbf{e}_n)_j = R^{-1}{}_{nm}\,(\mathbf{e}_m)_i$ // ID the basis vectors (3)

$R\mathbf{e}_n = R^{-1}{}_{nm}\,(\mathbf{e}_m)$ // write above as vector equation (4)

$\mathbf{e}'_n = R^{-1}{}_{nm}\,\mathbf{e}_m$ // use known fact on LHS (5)

$$\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\end{pmatrix} = R^{-1}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix}$$ // QED (6)

Proof B:

Since the $\mathbf{e}_n$ are a Cartesian basis, and since $\mathbf{a}\bullet\mathbf{b}$ is a scalar under rotations, we may write

$$\delta_{n,k} = \mathbf{e}_n \bullet \mathbf{e}_k = \mathbf{e}'_n \bullet \mathbf{e}'_k \qquad (1)$$

Take the known true fact $\mathbf{e}'_n = R\mathbf{e}_n$ and dot both sides into $\mathbf{e}_m$ to get

$$\mathbf{e}_m \bullet \mathbf{e}'_n = \mathbf{e}_m \bullet R\mathbf{e}_n = (\mathbf{e}_m)_i \, R_{ij} (\mathbf{e}_n)_j = \delta_{m,i} \, R_{ij} \delta_{n,j} = R_{mn} \qquad (2)$$

Now here is our claimed theorem statement :

$$\mathbf{e}'_n = R^{-1}_{nm} \, \mathbf{e}_m \qquad n = 1,2...N \qquad (3)$$

To verify this is true, it suffices to show that it is true when dotted into *all* the $\mathbf{e}'_k$ vectors since they form a complete basis. So we need to show that this is true

$$\mathbf{e}'_n \bullet \mathbf{e}'_k = R^{-1}_{nm} \, \mathbf{e}_m \bullet \mathbf{e}'_k \qquad n = 1,2...N \quad \text{and} \quad k = 1,2...N \qquad (4)$$

But LHS is $\delta_{n,k}$ from (1), and we know that $\mathbf{e}_m \bullet \mathbf{e}'_k = R_{mk}$ from (2), so we get

$$\delta_{n,k} = R^{-1}_{nm} \, R_{mk} = (R^{-1}R)_{nk} \qquad \text{QED} \qquad (5)$$