**Addition of angular momenta** PhL 2.12.09

My underlying notes for this subject are in my hand-scribbled Physics 547 lecture notes, which I summarized in a file on 1.10.08, see 547 binder and also stuff in my matrix binder last section. I have never written this up in a definitive fashion, and I don't really know any place that presents it the way I do (but surely someone does this, and perhaps even in a book I own).

This document now overlaps with another on "spherical harmonics" which I store in the curvilinear folder for historical reasons.

Contents:


**What commutes with what? ..... 2**
What is meant by $J = j_1 + j_2$? ..... 2
States in the space $\mathcal{H}^{(j1}\otimes^{j2)}$ ..... 2
Angular momentum generators in $\mathcal{H}^{(j1}\otimes^{j2)}$ ..... 3
Do $j_1$ and $j_2$ commute? ..... 5
Do $J = j_1 + j_2$ and $j_1$ commute? ..... 5
If $J = L + S$, does $J^2$ commute with $L^2$ ..... 6
Theorem: the following operators mutually commute: $\{ J^2, S^2, L^2, L\bullet S , J \}$ ..... 8
Is $L\bullet S$ invariant under L spatial rotations? (no) ..... 8
Is $L\bullet S$ invariant under full J rotations? (yes) ..... 8
**Summary of Direct Product Space Rules ..... 9**
**LS and jj Coupling with Multiple Particles ..... 9**
About L-S coupling as discussed Messiah page 704. ..... 9
The Multiple Particle Direct Product Space ..... 10
Summary of L-S coupling: ..... 14
Summary of j-j coupling: ..... 16
**Clebsch-Gordon Coefficients ..... 18**
Review of Results on the Meaning of combining angular momenta ..... 24
Comments on combining angular momenta: ..... 25
How would you compute the C-G coefficients? ..... 26
**Recursion Relations ..... 29**
**The Group Representation Matrices for SU(2) ..... 32**
The Rotation Operators and Addition Theorems ..... 33
Orthogonality and Completeness of the D functions. ..... 36
Matrix elements of the generators. ..... 37
Coordinate Space and Spherical Harmonics and Fancy Formulas ..... 38
**Combining Three Angular momenta. ..... 42**
**A Tour of Books I own on these subjects ..... 44**
1. Messiah has several different sections. ..... 44
2. Tinkham. ..... 44
3. Schiff in his Symmetry Chapter 7 ..... 45
4. M&M have very little ..... 46

5. My Group Theory Books ..... 46

---

**What commutes with what?**

**What is meant by $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$?**

(1) My starting question is this: what is meant by the following equations:

$\mathbf{J} = \mathbf{L} + \mathbf{S}$ $\qquad\qquad$ $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$

We have in mind that we are somehow "combining" or "adding" two angular momentum operators. They might be **L** and **S** for the same particle, or they might be the $\mathbf{j}_i$ for two different particles to get, in either case, the total **J**.

I think what we really mean is this:

$\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$ *means*

$$\mathbf{J}^{(j1\otimes j2)} = \mathbf{j}_1 \otimes 1 + 1 \otimes \mathbf{j}_2 \equiv \mathbf{j}_1 * \mathbf{j}_2$$

The sum really exists in the following *direct product* Hilbert Space

$$\mathcal{H}^{(j1\otimes j2)} = \mathcal{H}^{(j1)} \otimes \mathcal{H}^{(j2)}$$

and a better notation for the above would be this

$$J_i{}^{(j1\otimes j2)} = J_i{}^{(j1)} \otimes 1 + 1 \otimes J_i{}^{(j2)} \qquad i = 1,2,3$$

$$\mathbf{J}^{(j1\otimes j2)} = \mathbf{J}^{(j1)} \otimes 1 + 1 \otimes \mathbf{J}^{(j2)} \qquad \text{// vector notation}$$

Notice that the "1" operator is not a vector, so is not bolded. And notice that the + sign is adding two operators in the space $\mathcal{H}^{(j1\otimes j2)}$. We could write it as $+_{(j1\otimes j2)}$ but that would be a little overbearing. The superscript tells what "space" the operator lives in. This is implicit in the location of the operator with respect to the ⊗ symbol, but we just add extra info anyway.

**States in the space $\mathcal{H}^{(j1\otimes j2)}$**

(2) We would describe a "state" in this direct product space as follows:

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \equiv |j_1 m_1; j_2 m_2\rangle \in \mathcal{H}^{(j1\otimes j2)}$$

Then following our direct product space rules for operators and states, we would say

$J_3^{(j1\otimes j2)}\ |j_1m_1; j_2m_2\rangle = \{J_3^{(j1)} \otimes 1 + 1 \otimes J_3^{(j2)}\}\ \{|j_1m_1\rangle \otimes |j_2m_2\rangle\}$

$= \{J_3^{(j1)} \otimes 1\}\{|j_1m_1\rangle \otimes |j_2m_2\rangle\} + \{1 \otimes J_3^{(j2)}\}\{|j_1m_1\rangle \otimes |j_2m_2\rangle\}$

$= \{J_3^{(j1)}|j_1m_1\rangle\} \otimes 1|j_2m_2\rangle + 1|j_1m_1\rangle \otimes \{J_3^{(j2)}|j_2m_2\rangle\}$ // Rule 1

$= \{m_1|j_1m_1\rangle\} \otimes |j_2m_2\rangle + |j_1m_1\rangle \otimes \{m_2|j_2m_2\rangle\}$

$= m_1\{|j_1m_1\rangle \otimes |j_2m_2\rangle\} + m_2\{|j_1m_1\rangle \otimes |j_2m_2\rangle\}$ // Rule 2

$= (m_1 + m_2)\ |j_1m_1\rangle \otimes |j_2m_2\rangle$

where we have a few of these "direct product space rules"

**Rule 1**: $A \otimes B\ \{|a\rangle \otimes |b\rangle\} = A|a\rangle \otimes B|b\rangle$

**Rule 2:** $\{\alpha|a\rangle\} \otimes |b\rangle = |a\rangle \otimes \{\alpha|b\rangle\} = \alpha\ \{|a\rangle \otimes |b\rangle\}$ action of a scalar

We see from the above result that in some sense we have "added" the $J_z$ components. We might abbreviate the above equation like this:

$J_3\ |j_1m_1; j_2m_2\rangle = (m_1+m_2)\ |j_1m_1; j_2m_2\rangle$

where we understand that $J_3$ really means $J_3^{(j1\otimes j2)}$. We might loosely also say

$J_3\ |j_1m_1; j_2m_2\rangle = (J_3^{(j1)} + J_3^{(j2)})\ |j_1m_1; j_2m_2\rangle = (m_1+m_2)\ |j_1m_1; j_2m_2\rangle$

but you can see that things become ambiguous when you do this and don't show the true notation. Here is another example of this looseness:

$J_3\ |LM_L;SM_S\rangle = (L_3 + S_3)\ |LM_L;SM_S\rangle = (M_L + M_S)\ |LM_L;SM_S\rangle$

and we are loosely thinking here that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ when we write this out.

**Angular momentum generators in $\mathcal{H}^{(j1\otimes j2)}$**

(3) **Theorem**: The angular momentum generators given by

$J_i^{(j1\otimes j2)} = J_i^{(j1)} \otimes 1 + 1 \otimes J_i^{(j2)}$

form a representation of the same Lie Algebra that the $J_i^{(j1)}$ do. We mean always that the $j_{1,2}$ representations are irreducible, but the direct product representation is *reducible* we know. Here is a proof that the direct product operators obey the required Lie Algebra:

$$[\ J_i^{(j1\otimes j2)}, J_j^{(j1\otimes j2)}] = J_i^{(j1\otimes j2)} J_j^{(j1\otimes j2)} - J_j^{(j1\otimes j2)} J_i^{(j1\otimes j2)}$$

The first term here can be written out as

$$J_i^{(j1\otimes j2)} J_j^{(j1\otimes j2)} = \{ J_i^{(j1)} \otimes 1 + 1 \otimes J_i^{(j2)}\}\{ J_j^{(j1)} \otimes 1 + 1 \otimes J_j^{(j2)}\}$$

$$= \{ J_i^{(j1)} J_j^{(j1)} \otimes 1 + J_i^{(j1)} \otimes J_j^{(j2)} + J_j^{(j1)} \otimes J_i^{(j2)} + 1 \otimes J_i^{(j2)} J_j^{(j2)} \}$$

where we just used the following rule four times

**Rule 3:** $A\otimes B\ C\otimes D = (AC)\otimes(BD)$

When we now subtract from our result the same thing with i↔j, the middle two terms cancel, and the remaining terms form commutators on the respective sides, so we get

$$J_i^{(j1\otimes j2)} J_j^{(j1\otimes j2)} - J_j^{(j1\otimes j2)} J_i^{(j1\otimes j2)}$$

$$= \{ [J_i^{(j1)}, J_j^{(j1)}]\} \otimes 1 + 1 \otimes \{[J_i^{(j2)}, J_j^{(j2)}]\}$$

$$= \{i\varepsilon_{ijk} J_k^{(j1)}\} \otimes 1 + 1 \otimes \{ i\varepsilon_{ijk} J_k^{(j2)} \}$$

At this point, we need another rule which I will write as a set of rules

**Rule 4:** $( A + B + C ...) \otimes 1 = A \otimes 1 + B \otimes 1 + C \otimes 1$

or: $(\Sigma_i A_i) \otimes 1 = \Sigma_i (A_i \otimes 1)$

and then

**Rule 5:** $\{ \Sigma_i \alpha_i A_i\} \otimes 1 = \Sigma_i \alpha_i \{ A_i \otimes 1 \}$

Proof of Rule 5: $\{ \Sigma_i \alpha_i A_i\} \otimes 1 = \Sigma_i ((\alpha_i A_i) \otimes 1)$ // by Rule 4

$= \Sigma_i [\alpha_i \{ A_i \otimes 1\} ]$ // by Rule 2

So using Rule 5, then, we can say

$$\{i\varepsilon_{ijk} J_k^{(j1)}\} \otimes 1 = \{\Sigma_{jk}\, i\varepsilon_{ijk} J_k^{(j1)}\} \otimes 1 = \Sigma_{jk}\, i\varepsilon_{ijk} \{ J_k^{(j1)} \otimes 1 \}$$

so our final result above becomes

$$\{i\varepsilon_{ijk} J_k^{(j1)}\} \otimes 1 + 1 \otimes \{ i\varepsilon_{ijk} J_k^{(j2)} \}$$

$= \Sigma_{jk}\, i\varepsilon_{ijk} \{ \; J_k^{(j1)} \otimes 1 + \; 1 \otimes J_k^{(j2)} \; \} \; = i\varepsilon_{ijk}\, J_k^{\,(j1 \otimes j2)}$

and this completes our proof of this fact:

$[ \; J_i^{(j1 \otimes j2)}, J_j^{(j1 \otimes j2)} ] \; = \; i\varepsilon_{ijk}\, J_k^{\,(j1 \otimes j2)}$

and therefore $J_i^{(j1 \otimes j2)}$ are a representation of the SU(2) generators.

**Do $\mathbf{j}_1$ and $\mathbf{j}_2$ commute?**

(4) Question: do $\mathbf{j_1}$ and $\mathbf{j_2}$ commute?

These are operators in different Hilbert Spaces, so the question is vague as posed. You cannot even multiply them together to form a commutator. What this question really means is this:

do $J_i^{(j1)} \otimes 1$ and $1 \otimes J_j^{(j2)}$ commute in the space $\mathcal{H}^{(j1 \otimes j2)}$ ?

Let's find out:

$[ \quad J_i^{(j1)} \otimes 1, 1 \otimes J_j^{(j2)} \; ]$

$= J_i^{(j1)} \otimes 1 \; 1 \otimes J_j^{(j2)} \; - 1 \otimes J_j^{(j2)} \; J_i^{(j1)} \otimes 1$

$= J_i^{(j1)} \otimes J_j^{(j2)} \; - \; J_i^{(j1)} \otimes J_j^{(j2)}$ // by Rule 3

$= 0$ // since both terms are the same.

So yes, " $\mathbf{j_1}$ and $\mathbf{j_2}$ commute". More generally, any two operators

$A \otimes 1$ and $1 \otimes B$

commute, where position tells in which space an operator acts.

**Do $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$ and $\mathbf{j}_1$ commute?**

(5) Question: does $\mathbf{J} = \mathbf{j_1} + \mathbf{j_2}$ commute with $\mathbf{j_1}$ ?

We rephrase the question: Does

$\mathbf{J}^{(j1 \otimes j2)} = \mathbf{J}^{(j1)} \otimes 1 + 1 \otimes \mathbf{J}^{(j2)}$

commute with

$\mathbf{J}^{(j1)} \otimes 1$

in the Hilbert Space $\mathcal{H}^{(j1\otimes j2)}$ ? The answer is no. We can see that $\mathbf{J}^{(j1)} \otimes 1$ will commute with the second term $1 \otimes \mathbf{J}^{(j2)}$, but not with the first term. So:

$$[\mathbf{J}_i{}^{(j1\otimes j2)}, \mathbf{J}_j{}^{(j1)} \otimes 1] = [\mathbf{J}_i{}^{(j1)} \otimes 1, \mathbf{J}_j{}^{(j1)} \otimes 1] = [\mathbf{J}_i{}^{(j1)}, \mathbf{J}_j{}^{(j1)}] \otimes 1$$

$$=( i\varepsilon_{ijk}\, \mathbf{J}_k{}^{(j1)}) \otimes 1 = i\varepsilon_{ijk}\, \{\mathbf{J}_k{}^{(j1)} \otimes 1\}$$

and the summary would be to say this:

$$[\,(J)_i, (j_1)_j] = [\,(j_1)_i, (j_1)_j] = i\varepsilon_{ijk}\,(j_1)_k \neq 0$$

Thus, for example, we may conclude that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ does not commute with $\mathbf{L}$.

**If $\mathbf{J} = \mathbf{L} + \mathbf{S}$, does $J^2$ commute with $L^2$**

(6) Does $J^2$ commute with $L^2$ if $\mathbf{J} = \mathbf{L} + \mathbf{S}$ ?

We can write in shorthand that

$$J^2 = L^2 + S^2 + 2\mathbf{L}\bullet\mathbf{S}$$

First, note that $\mathbf{L}\bullet\mathbf{S} = \mathbf{S}\bullet\mathbf{L}$ so we can combine them into one term as shown, this according to subsection (4) above. Now let's write this out in more precise notation:

$$[\mathbf{J}^{(j1\otimes j2)}]^2 \equiv \Sigma_i\, [\mathbf{J}_i{}^{(j1\otimes j2)}]^2 = \Sigma_i\, [\mathbf{J}_i{}^{(j1)} \otimes 1 + 1 \otimes \mathbf{J}_i{}^{(j2)}]^2$$

$$= \Sigma_i\, [\mathbf{J}_i{}^{(j1)} \otimes 1 + 1 \otimes \mathbf{J}_i{}^{(j2)}]\, [\mathbf{J}_i{}^{(j1)} \otimes 1 + 1 \otimes \mathbf{J}_i{}^{(j2)}]$$

$$= \Sigma_i\, \{\, [\mathbf{J}_i{}^{(j1)}]^2 \otimes 1 + 2\, \mathbf{J}_i{}^{(j1)} \otimes \mathbf{J}_i{}^{(j2)} + 1 \otimes [\mathbf{J}_i{}^{(j2)}]^2 \,\}$$

$$= \{\, (\Sigma_i\, [\mathbf{J}_i{}^{(j1)}]^2) \otimes 1 + 2\, \Sigma_i\, (\mathbf{J}_i{}^{(j1)} \otimes \mathbf{J}_i{}^{(j2)}) + 1 \otimes (\Sigma_i\, [\mathbf{J}_i{}^{(j2)}]^2) \,\}$$

or

$$[\mathbf{J}^{(L\otimes S)}]^2 = \{\, L^2 \otimes 1 + 2\, \Sigma_i\, (\mathbf{L}_i \otimes \mathbf{S}_i) + 1 \otimes S^2 \,\}$$

So now we see that our shorthands really stand for:

$$L^2 = L^2 \otimes 1 \qquad S^2 = 1 \otimes S^2 \qquad \mathbf{L}\bullet\mathbf{S} = \Sigma_i\, (\mathbf{L}_i \otimes \mathbf{S}_i)$$

Now we were asking about $[J^2, L^2]$, in shorthand. Continuing in shorthand, we have

$$[J^2, L^2] = [L^2 + S^2 + 2\mathbf{L}\bullet\mathbf{S}, L^2]$$

We can see that the first two terms commute, albeit for different reasons. Things in different spaces (so the other space has just a 1) always commute as noted above with A and B. And of course something always commutes with itself in the same space. So we are left with:

$1/2\ [J^2, L^2] = [\ \mathbf{L}\bullet\mathbf{S}, L^2]$

$= [\Sigma_i\ (\mathbf{L}_i \otimes \mathbf{S}_i)\ , L^2 \otimes 1] = \Sigma_i\ \ [\mathbf{L}_i \otimes \mathbf{S}_i,\ L^2 \otimes 1]$

The commutator here is

$\mathbf{L}_i \otimes \mathbf{S}_i\ \ L^2 \otimes 1\ - L^2 \otimes 1\ \mathbf{L}_i \otimes \mathbf{S}_i$

$\mathbf{L}_i\ L^2 \otimes \mathbf{S}_i\ - L^2\ \mathbf{L}_i \otimes \mathbf{S}\ \ = 0$

because we know $[L^2,\mathbf{L}] = 0$.

Now let's do this again all in shorthand:

$[J^2,L^2\ ] = 2[\ \mathbf{L}\bullet\mathbf{S}, L^2] = 2\ [L_iS_i,L^2] = 2\ [L_i,L^2]\ S_i = 0$

The operative rule is that things which are in one space commute with those in the other space.

**Conclusion**: When we have $\mathbf{J} = \mathbf{L}+\mathbf{S}$, the quantities $J^2$, $S^2$ and $L^2$ all mutually commute.

Also, we know that $[\ \mathbf{L}\bullet\mathbf{S}, L^2] = [\ \mathbf{L}\bullet\mathbf{S}, S^2] = 0$. What about $[\ \mathbf{L}\bullet\mathbf{S}, J^2]$?

$2[\ J^2, \mathbf{L}\bullet\mathbf{S}] = [\ J^2, J^2 - L^2 - S^2] = 0 + 0 + 0 = 0.$

**Fact:** If $\mathbf{J} = \mathbf{L} + \mathbf{S}$, the quantities $J^2\ L^2\ S^2$ and $\mathbf{L}\bullet\mathbf{S}$ all mutually commute. We just showed this above.

**Fact:** The vector operators **L** and **S** do *not* commute with $\mathbf{L}\bullet\mathbf{S}$, although $L^2$ and $S^2$ do.

$[\ L_i, \mathbf{L}\bullet\mathbf{S}] = [\ L_i, L_jS_j] = [\ L_i, L_j]\ S_j \neq 0$

**Fact**: $[\ \mathbf{J}, \mathbf{L}\bullet\mathbf{S}\ ] = 0$ (proven in (9) below). So we can now extend out statement:

When we have $\mathbf{J} = \mathbf{L}+\mathbf{S}$, the quantities $J^2\ L^2\ S^2$ and $\mathbf{L}\bullet\mathbf{S}$ all mutually commute. Vectors **S** and **L** commute with each other. Vector **J** does not commute with **S** or with **L** but does commute with $\mathbf{L}\bullet\mathbf{S}$. Let's write this all out:

$\mathbf{J} = \mathbf{L}+\mathbf{S}$ $\quad [J^2, S^2] = 0$ $\quad [J^2, L^2] = 0$ $\quad [L^2, S^2] = 0$

$[\ S_i, L_j] = 0$ $\quad [J_i, L_j] = i\varepsilon_{ijk}L_k$ $\quad [J_i, S_j] = i\varepsilon_{ijk}S_k$

$[J_i, \mathbf{L}\bullet\mathbf{S}] = 0$ $\quad [\ L_i, \mathbf{L}\bullet\mathbf{S}] = i\varepsilon_{ijk}L_kS_j$ $\quad [\ S_i, \mathbf{L}\bullet\mathbf{S}] = i\varepsilon_{ijk}S_kL_j$

$[J^2, \mathbf{L}\bullet\mathbf{S}] = 0$ $\quad$ $[L^2, \mathbf{L}\bullet\mathbf{S}] = 0$ $\quad$ $[S^2, \mathbf{L}\bullet\mathbf{S}] = 0$

$[J_i, L^2] = 0$ $\quad$ $[J_i, S^2] = 0$ $\quad$ $[J_i, J^2] = 0$

So we can extend our statement above:

**Theorem: the following operators mutually commute: { $J^2$, $S^2$, $L^2$, $\mathbf{L}\bullet\mathbf{S}$ , J }**

**Theorem 1:** when $\mathbf{J} = \mathbf{L} + \mathbf{S}$ , these five quantities all mutually commute with each other:

$$\{ J^2, S^2, L^2, \mathbf{L}\bullet\mathbf{S} , \mathbf{J} \}$$

**Is L•S invariant under L spatial rotations? (no)**

(7) Is the quantity **L•S** invariant under spatial rotations?

It *looks* like it is, it looks like a normal dot product that is a scalar. But for it to be so, you would have to have $[\mathbf{L}\bullet\mathbf{S}, L_i] = 0$, but we know this is not true. So the answer is that it is NOT invariant under spatial rotations because it does not commute with the generators of such spatial rotations! One way to understand this is to realized that $[\mathbf{S}, L_i] = 0$ which says that **S** "does not move" when you do a spatial rotation. In fact, **S** is a spatial-rotation "scalar'. This is certainly a little confusion since **S** has three components and you want to say it is a "vector". The better term is to say that **S** is *invariant* under spatial rotations. Thus, when you do a spatial rotation and you examine what **L•S** does, you have **L** rotating and **S** staying put, so the dot product changes, and so the result **L•S** changes and is not invariant.

**Is L•S invariant under full J rotations? (yes)**

(8) Is the quantity **L•S** invariant under full **J** rotations?

The answer is yes, and it is because we will now show that $[\mathbf{L}\bullet\mathbf{S}, \mathbf{J}] = 0$. The shorthand proof is this:

$[\mathbf{L}\bullet\mathbf{S}, \mathbf{J}] = [\mathbf{L}\bullet\mathbf{S}, \mathbf{L}+\mathbf{S}]$

$[ L_iS_i, L_j + S_j] = [ L_iS_i, L_j] + [ L_iS_i, S_j] = S_i [ L_i, L_j] + L_i[S_i, S_j]$

$= S_i \; i\varepsilon_{ijk} L_k + L_i \; i\varepsilon_{ijk} S_k = i\varepsilon_{ijk}\{ S_iL_k + S_kL_i\} =$ (antisym) (sym) = 0.

**Effect of an L•S term in a Hamiltonian**

**Corollary**: If a Hamiltonian contains an **L•S** term, it does not commute with **L** or **S**, but it does commute with **J**. Such a Hamiltonian is NOT invariant under spatial rotations (or spin-only rotations).

**Diagonality of Hamiltonian Matrix elements when H commutes with vector L.**

(9) About state labeling. Suppose some Hamiltonian H commutes with vector **L** . Then we are inclined to label the eigenkets of H by some $|\gamma\ \ell m\rangle$ or say $|\gamma\ L\ M_L\rangle$. We know for sure that

$$\langle\gamma' L'\ M_{L'}|H\ |\gamma\ L\ M_L\rangle = \delta_{LL'}\delta_{M_L M_{L'}}\langle\gamma' L\ |||\ H\ |||\ L\gamma\rangle\ = \delta_{LL'}\delta_{M_L M_{L'}}\ H^{(L)}{}_{\gamma'\gamma}$$

where we end up with some sort of reduced matrix element which does not depend on the $M_L$. If H commutes with vector **L**, it is easy to show this fact, I won't repeat it here. So the idea here is that commutation with vector **L** allows states to be labeled with good quantum numbers L and $M_L$. If we didn't have the internal commutation problem, we would like to label the states by some $m_x$, $m_y$ and $m_z$ corresponding to $L_{x,y,z}$ but the best we can do is $L^2$ and $L_z$ so they are the labels we use.

Note Added: the above equation has two features.

First, if $L \neq L'$, you get zero. You can prove this by starting with $[L^2, H] = 0$ and you get $\{ L'(L'+1) - L(L+1)\} \langle\gamma' L'\ M_{L'}|H\ |\gamma\ L\ M_L\rangle\ = 0$. You can prove the same thing for the $M_L$ starting with $[L_3, H] = 0$.

The second feature is that the resulting non-zero matrix elements which are $\langle\gamma' L\ M_L|H\ |\gamma\ L\ M_L\rangle$ do not depend at all on $M_L$. I show this elsewhere using the raising and lowering operators.

**Fact:** The idea of $[H,\mathbf{L}] = 0$ is directly related to labeling energy eigenkets as $|\gamma\ \ell m\rangle$ which form subspaces which are $(2\ell+1)$ fold degenerate for each $\ell$ . Both $\ell$ and m are "good quantum numbers".

**Summary of Direct Product Space Rules**

**Rule 1**: $A \otimes B\ \{ |a\rangle \otimes |b\rangle\}\ =\ \{A\ |a\rangle\} \otimes \{B\ |b\rangle\}$ // operator on ket

**Rule 2:** $\{\alpha|a\rangle\} \otimes |b\rangle\ = |a\rangle \otimes\{\alpha\ |b\rangle\}\ = \alpha\ \{ |a\rangle \otimes |b\rangle\}$ action of a scalar

**Rule 3:** $A\otimes B\ \ C\otimes D\ \ =\ (AC)\otimes(BD)$ // product of operators

**Rule 4:** $(\ A + B + C\ ...) \otimes 1\ = A \otimes 1\ + B \otimes 1 + C \otimes 1$ $(\Sigma_i A_i) \otimes 1 =\ \Sigma_i\ (A_i \otimes 1)$

**Rule 5:** $\{\ \Sigma_i \alpha_i A_i\} \otimes 1\ = \Sigma_i\ \alpha_i\ \{\ A_i \otimes 1\ \}$

**Rule 6:** $\{\langle a'| \otimes \langle b'|\}\ \ \{|a\rangle \otimes|b\rangle\}\ \ =\ \langle a'\ |\ a\rangle\langle b'\ |\ b\rangle$ // direct product space IP

**LS and jj Coupling with Multiple Particles**

**About L-S coupling as discussed Messiah page 704.** We start there with $V_1$ as our perturbation and this does not even involve spin, it is not a function of spin, it is therefore fully a spatial and spin rotational scalar and $V_1$ commutes with vector **L** and vector **S** and we label states $|\gamma\ L\ M_L;\ S\ M_S\rangle$ and all four items are good quantum numbers and the matrix element of $V^1$ (and $H_C$ with $V_1$) is completely diagonal in all four quantum numbers as shown top of page 702 in A. The reduced matrix element is a function of L and S so he gets $H^{(L,S)}{}_{\gamma'\gamma}$ in analogy with the above. For carbon's two valence electrons, the 15 states get partitioned into 9+5+1 as we group things into totally antisymmetric states where each group has a

well defined pair of values (L,S). The matrix elements of $V_1$ are basically the $\varepsilon_1^{(LS)}$ so there are three values of this energy correction which breaks the 15 states into the 9+5+1. If we were to organize the 15 states into the appropriate linear combinations which make the three spectroscopic groups, then the 15 x 15 matrix would be in block diagonal form and would in fact be completely diagonal. The diagonal might contain 9 equal values for the top part, then 5 equal diagonal values, then the single value in the lower right corner.

If we were now to turn on the $V_2$ **L•S** type perturbation, we can look at one of our diagonal subspaces just described and see what happens to it. We can think of the 9x9 matrix with diagonal value $\varepsilon_1$ as being the starting point for doing *first order* perturbation theory with $V_2$. We want to ask what happens to those 9 states. In the existing basis, the matrix for $V_2$ will not be diagonal, and we have to go diagonalize it to find the $\varepsilon_1$ energies caused by turning on $V_2$. Since **L•S** and therefore H no longer commute with vector **L** and vector **S**, the state labeling $|\gamma\ L\ M_L;\ S\ M_S\rangle$ is no longer meaningful (as an eigenstate of H). We know from above that $S^2$ and $L^2$ commute with **L•S**, so in this sense S and L are still good quantum numbers, but $M_L$ and $M_S$ are no longer good quantum numbers. So we could blindly do our $\varepsilon_1$ diagonalization problem to find the solution. BUT, we know that vector **J** commutes with **L•S** and thus with $V_2$ . We know that J and $M_J$ are good quantum numbers and matrix elements with these labels will be diagonal in both these numbers. Consider the 9 states that are degenerate before $V_2$ is turned on. These states have S=1 and L=1 and there are 3x3 = 9 states. We know we can reorganize these states into J multiplets with J = 2, 1 and 0 giving 5+3+1 = 9 states. We can label these states $|\gamma\ LS;J\ M_J\rangle$. These are some linear combinations of our starting 9 states. The little 9x9 piece of the $V_2$ matrix will then be fully diagonal again with these new states, and the first 5 diagonal elements will have value $\varepsilon_1(J=2)$ , the next three will be $\varepsilon_1(J=1)$ and so on. This then fully explains everything about the lower half of the figure on page 703 where we trace our 15 states through the two separate applications of first order perturbation theory. The $^3P_2$ has J=2 and represents 5 degenerate states.

What happens to the 5 states and the 1 state. These both had S = 0, so we have J = L and the 5 states then stay degenerate, though they might move a little, and of course the 1 state stays itself but might move a little. The picture shows no movement however.

Caveat: In this section, I have assumed that the Hamiltonian had **L•S**, but this is not exactly the form that appears. To get this detangled and redo things, we have to worry about: [ Also, later Messiah uses the WET to show that you can imagine **L•S** in there anyway. ]

**The Multiple Particle Direct Product Space**

(1) For starters, assume a system of two particles 1 and 2 and suppose the Hamiltonian has this form:

$$H = \Sigma_{i=1,2}\ [\ p_i^2/2m_i + V_i(r_i)\ ]$$

where we allow each particle to see its own potential. What do we mean by the "sum" shown above? I think we have to say it is this, where this is a whole new use of the direct product, separate from that used in the combination of angular momenta above,

$$H = H^{(1)} \otimes 1 + 1 \otimes H^{(2)}$$

H is now an operator in the space $\mathcal{H}^{(1\otimes 2)}$. We suspect that the eigenkets of H will look like this

$$| \psi^{(1)}{}_i > \otimes | \varphi^{(2)}{}_j >$$

where we allow as $\psi_i$ might $\neq \varphi_i$ if the potentials are different (but in our Messiah application they will be the same). What then does the SS SE look like?

$$\{ H^{(1)} \otimes 1 + 1 \otimes H^{(2)} \} \{ | \psi^{(1)}{}_i > \otimes | \varphi^{(2)}{}_j > \} = E^{(12)}{}_{ij} \{ | \psi^{(1)}{}_i > \otimes | \varphi^{(2)}{}_j > \}$$

We can then use Rule 1 above to get

$$\{ H^{(1)} | \psi^{(1)}{}_i > \otimes 1 | \varphi^{(2)}{}_j > \} + \{ 1 | \psi^{(1)}{}_i > \otimes H^{(2)} | \varphi^{(2)}{}_j > \} = E^{(12)}{}_{ij} \{ | \psi^{(1)}{}_i > \otimes | \varphi^{(2)}{}_j > \}$$

$$= \{ E_i{}^{(1)} | \psi^{(1)}{}_i > \otimes 1 | \varphi^{(2)}{}_j > \} + \{ 1 | \psi^{(1)}{}_i > \otimes E_j{}^{(2)} | \varphi^{(2)}{}_j > \}$$

$$= (E_i{}^{(1)} + E_j{}^{(2)}) \{ | \psi^{(1)}{}_i > \otimes | \varphi^{(2)}{}_j > \}$$

where we took a quick path since we have done the internal steps in detail above. Thus, the energy of our two particle state is

$$E^{(12)}{}_{ij} = E_i{}^{(1)} + E_j{}^{(2)} \qquad\qquad \mathcal{H}^{(1\otimes 2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$$

This is certainly the result we want to see for our two "independent particles". They don't interact with each other, I think this is what $H = H^{(1)} \otimes 1 + 1 \otimes H^{(2)}$ is telling us.

If the Hamiltonian does contain an interaction term, then I think this whole point of view falls apart. But in our Messiah situation page 701 A, the interaction stuff is treated as a perturbation, so at the $H_0$ level we can do our direct product thing as above. The idea extends to any number of particles, let's just write something for N=3:

$$H = H^{(1)} \otimes 1 \otimes 1 + 1 \otimes H^{(2)} \otimes 1 + 1 \otimes 1 \otimes H^{(3)}$$

$$E^{(123)}{}_{ijk} = E_i{}^{(1)} + E_j{}^{(2)} + E_k{}^{(3)} \qquad\qquad \mathcal{H}^{(1\otimes 2\otimes 3)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)}$$

$$\text{ket} = | \psi^{(1)}{}_i > \otimes | \varphi^{(2)}{}_j > \otimes | \chi^{(3)}{}_k >$$

We have not shown spin indices in the states, but if these are electrons, we can add S $M_S$ = 1/2, m to the labels of each state. These states are the "base states" of the $H_0$ level of any perturbation theory we might choose to do! We will be linearly combining them to meet our needs.

Messiah talks about this direct product space idea on page 586 for N particles, so I don't think I am totally off my rocker here.

(2) Assuming that $H^{(1)}$ is a rotational invariant, and commutes with $L^{(1)}{}_i$ (the spatial angular momentum for particle 1) and also $S^{(1)}{}_i$, we can really expand things in terms of labels. For N = 2 we get

$$| \psi^{(1)}{}_i ; L^{(1)} M^{(1)}{}_L ; S^{(1)} M^{(1)}{}_S > \otimes | \varphi^{(2)}{}_i ; L^{(2)} M^{(2)}{}_L ; S^{(2)} M^{(2)}{}_S >$$

In each ket, we have "full $LM_L SM_S$ labeling" because $H^{(i)}$ commutes with vector $\mathbf{L}^{(1)}$ *and* with vector $\mathbf{S}^{(1)}$.

If $H^{(1)}$ commutes with vector $\mathbf{L}^{(1)}$ and $H^{(2)}$ commutes with vector $\mathbf{L}^{(2)}$, then consider this processing in the shorthand notation:

$$[\mathbf{L},H] = [\mathbf{L}^{(1)} + \mathbf{L}^{(2)}, H^{(1)} + H^{(2)}] = 0$$

so we certainly know that we could use $LM_L$ as good quantum numbers if we wanted to. An exactly analogous statement could be made for spin, so we could use $SM_S$ as good quantum number labels.

We know that we need to form totally antisymmetric kets for multiple electrons. We usually do this by separately combining $S^{(1)} \otimes S^{(2)}$ to get possible S values, and similarly for L, where no superscript now indicates a "total" operator. This means messy linear combinations of the above states. So we now need to mix together our two direct product concepts.

Aside: The symbol $\otimes$ is generic and to know exactly what a direct product is, you would have to write out some equations in detail, but I think there is never any ambiguity about the meaning of $\otimes$. It is generic in the sense that + is generic, so I don't think it is meaningful to try to color code the different $\otimes$ symbols. If we write a+b+c+d = a+d+b+c, we don't code the plus signs. The word I am looking for is (a+b)+c = a + (b+c) "associative property" Stakgold p 96, which is built into any vector space. So I am claiming that $\otimes$ is generic in the sense that it is associative.

So write the above as: (we expose the internal $\otimes$ in each ket)

$$\{|\psi^{(1)}{}_i; L^{(1)}M^{(1)}{}_L\rangle \otimes |S^{(1)}M^{(1)}{}_S\rangle\} \otimes \{|\varphi^{(2)}{}_i; L^{(2)}M^{(2)}{}_L\rangle \otimes |S^{(2)}M^{(2)}{}_S\rangle\}$$

where our HS structure could now be written

$$\mathcal{H}^{(1\otimes 2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} = \mathcal{H}^{(L1\otimes S1)} \otimes \mathcal{H}^{(L2\otimes S2)}$$

$$= \mathcal{H}^{(L1)} \otimes \mathcal{H}^{(S1)} \otimes \mathcal{H}^{(L2)} \otimes \mathcal{H}^{(S2)}$$

so it really is the direct product of four spaces. We then reorder the spaces like this (this is the associative idea mentioned above)

$$= \mathcal{H}^{(L1)} \otimes \mathcal{H}^{(L2)} \otimes \mathcal{H}^{(S1)} \otimes \mathcal{H}^{(S2)}$$

$$= \mathcal{H}^{(L1\otimes L2)} \otimes \mathcal{H}^{(S1\otimes S2)}$$

$$|\psi^{(1)}{}_i; L^{(1)}M^{(1)}{}_L;\ \varphi^{(2)}{}_i; L^{(2)}M^{(2)}{}_L\rangle \otimes |S^{(1)}M^{(1)}{}_S;\ S^{(2)}M^{(2)}{}_S\rangle$$

Where the $\otimes$ separates two non-interacting spaces, the orbital space and the spin space. We are now free to linearly combine the spin basis functions to get our desired total spin S states, and the same for L. For example:

$$|S=1,M_S=0\rangle = \{|1/2\ \ 1/2;\ \ 1/2\ \ -1/2\rangle - |1/2\ \ -1/2;\ \ 1/2\ \ 1/2\rangle\}/\sqrt{2}$$

and yes, things are very messy. We need a nice notation now for the spin states. We can use what we did above and say

$$| S_1\ S_2\ S\ M_S >$$

and associated with this state is the idea that the Hamiltonian TOTAL must commute with $S_1^2$ and $S_2^2$ and $S^2$ and with vector **S** (see Theorem 1 above). So doing this also for L's, we can express our *reorganized* states as follows:

$$| L_1\ L_2\ L\ M_L > \otimes | S_1\ S_2\ S\ M_S > \ \approx\ | L\ M_L > \otimes | S\ M_S > \approx |\gamma\ L\ S\ M_L M_S>$$

where $L_1$ represents really $L_1^2$ and so on. We now pick combinations which are overall antisymmetric and these are the allowed states. This is what we did on page 702 Messiah, and the labels like $L_1$ are suppressed and we only see L $M_L$ and S $M_S$ , and we get the shorthand as shown above. So this is the underlying machinery for the 2-electron carbon atom discussion in Messiah.

This is of course the L-S coupling approach. We then turn on the spin-orbit coupling and we have to change to $|\gamma\ L\ S\ J\ M_J>$ states to re-diagonalize ourselves, as noted above. So these states could be written as

$$|\gamma\ L\ S\ J\ M_J> = |\gamma\ L_1\ L_2\ L\ ;\ S_1\ S_2\ S;\ J\ M_J>$$

if you really want to be long-winded.

Our only assumption for going to this last result is that the "spin-orbit term" somehow does not commute with total vector **L** and total vector **S** , but does commute with **J**. Thus would be true if the interaction term were **L•S** . But the actual term is more like this:

$$V_2 = \Sigma_i [\ (\mathbf{L}^{(i)} \bullet \mathbf{S}^{(i)})\ g(r_i)\ ] = \mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}\ g(r_1) + \mathbf{L}^{(2)} \bullet \mathbf{S}^{(2)}\ g(r_2)$$

$H = H_C + V_1$ = commutes with **L** and **S** .

So, let's now show that **L** does *not* commute with $V_2$ . In particular we have

$$[\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}, \mathbf{L}\ ] = [\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}, \mathbf{L}^{(1)} + \mathbf{L}^{(2)}\ ] = [\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}, \mathbf{L}^{(1)}]$$

Let's start over in easier notation

$$[\ L_{1i} S_{1i}, L_j] \ = \ [\ L_{1i} S_{1i}, L_{1j} + L_{2j}] \ = [\ L_{1i} S_{1i}, L_{1j}] \ = S_{1i}\ i\ \varepsilon_{ijk}\ L_{1k}$$

So we get

$$[V_2,\ L_j] = i\ \varepsilon_{ijk}\ S_{1i}\ L_{1k}\ g(r_1) + i\ \varepsilon_{ijk}\ S_{2i}\ L_{2k}\ g(r_2)$$

$$= i\varepsilon_{ijk}\ \{\ g(r_1)\ S_{1i}\ L_{1k} + g(r_2)\ S_{2i}\ L_{2k}\ \}$$

and this is not zero even if the g's were constant. So this is why the $V_2$ in this form "breaks the LS symmetry" and we have to reorganize into $|\gamma\ L_1\ L_2\ L\ ;\ S_1\ S_2\ S;\ J\ M_J>$ states.

And now lets verify that $V_2$ still commutes with **J** :

$$[V_2,\ J_j] = [\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}\ g(r_1) + \mathbf{L}^{(2)} \bullet \mathbf{S}^{(2)}\ g(r_2)\ ,\ L_j + S_j\ ]$$

Let's examine

$$[\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}, L_j] = [\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}, L^{(1)}{}_j] = S^{(1)}{}_i\ [\ L^{(1)}{}_i, L^{(1)}{}_j] = i\varepsilon_{ijk}\ S^{(1)}{}_i\ L^{(1)}{}_k$$

$$\Rightarrow\ [\mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}, S_j] = i\varepsilon_{ijk}\ L^{(1)}{}_i\ S^{(1)}{}_k$$

$$\Rightarrow\ \text{sum of these two terms } = i\varepsilon_{ijk}[S^{(1)}{}_i\ L^{(1)}{}_k + L^{(1)}{}_i\ S^{(1)}{}_k] = 0$$

So this verifies that $[V_2,\ \mathbf{J}] = 0$ so we are allowed to use $J\ M_J$ as good quantum numbers when $V_2$ is turned on.

So I think this clears up the haziness that was present in my L-S Messiah coupling discussion above. And of course you extend the idea to any number of particles. The Slater determinant tells you how to make antisymmetric combinations of things, but I don't want to sidetrack on that right now.

**Summary of L-S coupling:**

(1) Start at the $H_0$ level with $H_0 = H_C$ Messiah page 701 A. This $H_C$ commutes with a *lot* of operators: $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \mathbf{L}, \mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \mathbf{S}$ . Were we to define $\mathbf{J}^{(1)} = \mathbf{L}^{(1)} + \mathbf{S}^{(1)}$ and same for $\mathbf{J}^{(2)}$, then we could add $\mathbf{J}^{(1)}, \mathbf{J}^{(2)}, \mathbf{J}$ to the list. Everything is degenerate! In the example, N = 15 at this point.

(2) Now **turn on perturbation** $\mathbf{V}_1$ shown in (40). Due to the $|r_i - r_j|$ factors, we are no longer invariant under the separate rotations $\mathbf{L}^{(1)}$, $\mathbf{L}^{(2)}$ but we are still invariant under **L**. We also lose $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ from our group of morning commuters because of the loss of the L's. So $H_0 + V_1$ still commutes with the following operators: $\mathbf{L}, \mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \mathbf{S}, \mathbf{J}$, so we reduced from 9 commuting vectors to 5. At this point, due to the fact that **L** and **S** are still good, we can label states $|\gamma LM_L SM_S>$. Within each of these manifolds, all the states with different M values are degenerate, so we say

$$<\gamma' L' M_L' S' M_S'|\ H_o + V_1\ |\gamma LM_L SM_S> = \delta_{L',L}\ \delta_{ML',ML}\ \delta_{S',S}\ \delta_{MS',MS}\ <\gamma' LS||\ H_o + V_1\ ||\gamma LS>$$

(3) We can say that the states $|\gamma LM_L SM_S>$ all belong to a degenerate "spectroscopic multiplet" of degeneracy N = (2S+1)(2L+1). So think of the N states of one of these multiplets as the starting point for doing more perturbation theory when we turn on $V_2$ below. The effect of $V_1$ is to shift all the N states of our multiplet by the same amount $\varepsilon_1(V_1) = <\gamma' LS||\ V_1\ ||\gamma LS>$ so all multiplet states "move together" under the effect of $V_1$.

(4) We keep only multiplets that are overall antisymmetric if we are dealing with electrons. So this will rule out certain LS combinations.

(5) If we wanted, we could rearrange the states of our degenerate multiplet. Each multiplet could be decomposed into a sum of $JM_J$ multiplets using C-G coefficients. We might write this idea as

$$L \otimes S = J^{(L+S)} \oplus J^{(L+S-1)} \oplus .... \oplus J^{(|L-S|)}$$

We would write these kets as $|\gamma LS\ JM_J\rangle$ within each subspace on the right side. When states are reorganized in this manner, $L^2$ and $S^2$ are still good labels, but such states are no longer eigenstates of $L_3$ and $S_3$ so we no longer try to use $M_L M_S$ numbers.

But at this point, we have no motivation to do this decomposition. If we did it, all states in all the J subspaces would have the same energy and there is nothing to be gained.

(6) Now **turn on** $\mathbf{V}_2 = \Sigma_i [ (\mathbf{L}^{(i)} \bullet \mathbf{S}^{(i)}) g(r_i) ]$ as a perturbation, and let's examine one of our LS multiplets. We find that $[ \mathbf{L}, V_2] \neq 0$ just because we know $[ \mathbf{L}_i, \mathbf{L}^{(n)}{}_j] \neq 0$. Similarly $[\mathbf{S}, V_2] \neq 0$. So our LS multiplet state labels are no longer "good". However, if we reorganize the states as in (5) above, the state labels L S J and $M_J$ are all "good" because $[ \mathbf{J}, V_2] = [ L^2, V_2] = [ S^2, V_2] = 0$. Then we can say, within one of our initial LS groups of states,

$$\langle\gamma'LSJ'M_{J'}| H_o + V_1 + V_2 |\gamma'LSJM_J\rangle = \delta_{J',J}\ \delta_{M_{J'},M_J} \langle\gamma'LSJ|| H_o + V_1 + V_2 ||\gamma LSJ\rangle$$

Here we are only looking inside one specific LS multiplet, and within that, one specific J value which arises in the L ⊗ S sum shown above. We now see that we have smaller multiplets which remain degenerate. For each J, (2J+1) states are degenerate. A given J-multiplet within our LS group moves by this much due to $V_2$ : $\varepsilon_1(V_2) = \langle\gamma'LSJ|| V_2 ||\gamma LSJ\rangle$. So this group has now shifted $\varepsilon_1(V_1) + \varepsilon_1(V_2)$ from the starting position under just $H_0$.

(7) In the items above, we have assumed that there is no need for label γ (which is the case for the two valence electrons in a carbon atom). If there were some other labels, each time we do the $\varepsilon_1$ type perturbation theory, we would have do a level 1 diagonalization of something like this

$$\langle\gamma'LS|| V_1 ||\gamma LS\rangle = (V_1)^{(LS)}{}_{\gamma'\gamma} \qquad \Rightarrow \qquad (V_1)^{(LS)}{}_{\gamma\gamma'}\ \psi_{\gamma'} = \varepsilon_1\ \psi_\gamma$$

and so we diagonalize with respect to whatever these γ indices refer to, this tells us the right linear combinations of states and the right values of $\varepsilon_1$ and in this case you would not end up with the entire multiplet having the same $\varepsilon_1$ value! Let's ignore this complication for now. Without this complication, our $\varepsilon_1$ problems for $V_1$ and $V_2$ as done above are immediately diagonal.

(8) The L-S method just described works well if in some sense $V_2 << V_1 << H_0$ so it makes sense to do the two level 1 perturbation theories in the order shown above. Look at the picture Messiah page 703. The idea is that the second column splittings are small compared to the first column splittings, so the general method of grouping is logical.

(9) Messiah's carbon example has # particles = 2, $L^{(1)} = L^{(2)} = 1$ and $S^{(1)} = S^{(2)} = 1/2$.

**Summary of j-j coupling:**

(1) Start at the $H_0$ level with $H_0 = H_C$ Messiah page 701 A. This $H_C$ commutes with a *lot* of operators: $\mathbf{L}^{(1)}$, $\mathbf{L}^{(2)}$, $\mathbf{L}$, $\mathbf{S}^{(1)}$, $\mathbf{S}^{(2)}$, $\mathbf{S}$. Were we to define $\mathbf{J}^{(1)} = \mathbf{L}^{(1)} + \mathbf{S}^{(1)}$ and same for $\mathbf{J}^{(2)}$, then we could add $\mathbf{J}^{(1)}$, $\mathbf{J}^{(2)}$, $\mathbf{J}$ to the list. Everything is degenerate! In the example, N = 15 at this point.

(2) Now **turn on perturbation** $V_2$ shown in (43). Due to the $\mathbf{L}^{(i)} \bullet \mathbf{S}^{(i)}$ factors, we are no longer invariant under the following: $\mathbf{L}^{(1)}$, $\mathbf{L}^{(2)}$, $\mathbf{L}$, $\mathbf{S}^{(1)}$, $\mathbf{S}^{(2)}$, $\mathbf{S}$. However, we maintain invariance under $\mathbf{J}^{(1)}$, $\mathbf{J}^{(2)}$, $\mathbf{J}$. { To see this, note that for example $[\mathbf{J}^{(1)}, \mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}] = 0$ as shown in (8) above } . So $H_0 + V_2$still commutes with the following operators: $\mathbf{J}^{(1)}$, $\mathbf{J}^{(2)}$, $\mathbf{J}$, so we reduced from 9 commuting vectors to 3. We might add that things like $[L_1^2, \mathbf{L}^{(1)} \bullet \mathbf{S}^{(1)}] = 0$ still.

(2a) In light of this, we go back to the original ordering of our direct product spaces shown above,

$$\mathcal{H}^{(1\otimes 2)} = \mathcal{H}^{(L1\otimes S1)} \otimes \mathcal{H}^{(L2\otimes S2)}$$

but we expand each of these two spaces into J spaces,

$$L_1 \otimes S_1 = \Sigma_{\oplus}\, J_1$$
$$L_2 \otimes S_2 = \Sigma_{\oplus}\, J_2$$

and then reorganize states so our kets then have this form

$$|\psi^{(1)}{}_i;\, L_1S_1;\, J_1M_{J1}> \otimes |\varphi^{(2)}{}_j;\, L_2S_2;\, J_2M_{J2}> \equiv |\gamma\, J_1M_{J1};\, J_2M_{J2}>$$

where as usual $L_1$ refers to $L_1^2$ and $S_1$ refers to $S_1^2$ and so on. All labels are now "good" labels. Let's now suppress the labels like $L_1$ and the $\psi$ stuff and just abbreviate things as on the right above. Since our $H_o + V_2$ commutes with *vector* $\mathbf{J}^{(1)}$ *and* $\mathbf{J}^{(2)}$, we may conclude that

$$<\gamma'\, J_1'M_{J1}';\, J_2'M_{J2}'|\, H_o + V_2\, |\gamma\, J_1M_{J1};\, J_2M_{J2}>$$

$$= \delta_{J1',J1}\, \delta_{MJ1',MJ1}\, \delta_{J2',J2}\, \delta_{MJ2',MJ2}\, <\gamma'\, J_1J_2||\, H_o + V_2\, ||\, \gamma\, J_1\, J_2>$$

and see note above concerning a simpler version of this as to why this is so. So this means that we have a set of $(2J_1+1)(2J_2+1)$ states which are degenerate in the $(J_1J_2)$ group.

(3) We can say that the states $|\gamma\, J_1M_{J1};\, J_2M_{J2}>$ all belong to a degenerate "spectroscopic multiplet" of degeneracy $N = (2J_1+1)(2J_2+1)$. So think of the N states of one of these multiplets as the starting point for doing more perturbation theory when we turn on $V_1$ below. The effect of $V_2$ is to shift all the N states of our multiplet by the same amount $\varepsilon_1(V_2) = <\gamma'\, J_1J_2||\, V_2\, ||\, \gamma\, J_1\, J_2>$ so all multiplet states "move together" under the effect of $V_2$.

(4) We keep only multiplets that are overall antisymmetric if we are dealing with electrons. So this will rule out certain $J_1 J_2$ combinations (I think)

(5) If we *wanted*, we could rearrange the states of our degenerate $J_1 J_2$ multiplet. Each multiplet could be decomposed into a sum of $JM_J$ multiplets using C-G coefficients. We might write this idea as

$$J_1 \otimes J_2 = \Sigma_{\oplus} J$$

We would write these kets as $|\gamma J_1 J_2\ JM_J\rangle$ within each subspace on the right side. When states are reorganized in this manner, $J_1^2$ and $J_2^2$ are still good labels, but such states are no longer eigenstates of $(J_1)_3$ and $(J_2)_3$ so we no longer try to use $M_{J1}\ M_{J2}$ numbers.

But at this point, we have no motivation to do this decomposition. If we did it, all states in all the J subspaces would have the same energy and there is nothing to be gained.

(6) Now **turn on $V_1$ as a perturbation**, and let's examine one of our $J_1 J_2$ multiplets. We find that

$$[\ \mathbf{J}^{(1)}, V_1\ ] = [\ \mathbf{L}^{(1)}, V_1\ ] \neq 0$$

So see why this is so, look at (40) for $V_1$. We have the factor $|\mathbf{r}_1 - \mathbf{r}_2|$ appearing. We know that the action of $L^{(1)}$ is to rotate $\mathbf{r}_1$ but not $\mathbf{r}_2$, so both $|\mathbf{r}_1 - \mathbf{r}_2|$ and $\mathbf{r}_1 \bullet \mathbf{r}_2$ are not invariant under an $L^{(1)}$ rotation! This is a key point, and you really need to think of the direct product 2-particle space and the meaning of $L^{(1)}$ to be reminded that $L^{(1)}$ does nothing to the particle-2 space.

So, once we turn on $V_1$, we no longer have invariance under the vectors $\mathbf{J}_1$ and $\mathbf{J}_2$, but we still have invariance under vector $\mathbf{J}$.

$$[\ \mathbf{J}, V_1\ ] = [\ \mathbf{J}^{(1)} + \mathbf{J}^{(2)}\ , V_1\ ] = [\ \mathbf{L}^{(1)} + \mathbf{L}^{(2)}\ , V_1\ ] = [\ \mathbf{L}\ , V_1\ ] = 0$$

Thus our states $|\ \gamma\ J_1 M_{J1};\ J_2 M_{J2}\rangle$ are no longer "good", but if we reorganize the states as in (5) above, the state labels in $|\gamma J_1 J_2\ JM_J\rangle$ are all "good". Then we can say

$$\langle \gamma' J_1 J_2 J' M_{J'}|\ H_o + V_1 + V_2 |\ \gamma'\ J_1 J_2 J M_J\rangle = \delta_{J',J}\ \delta_{MJ',MJ} \langle \gamma'\ J_1 J_2 J ||\ H_o + V_1 + V_2\ || \gamma L\ J_1 J_2\rangle$$

where here we are only looking inside a specific $J_1 J_2$ multiplet and within that, one specific J value which arises in the $J_1 \otimes J_2$ sum shown above. We now see that we have smaller multiplets which remain degenerate. For each J, (2J+1) states are degenerate. A given J-multiplet within our $J_1 J_2$ group moves by this much due to $V_1$ : $\varepsilon_1(V_1) = \langle \gamma'\ J_1 J_2\ J ||\ V_1\ || \gamma\ J_1 J_2\ J\rangle$. So this group has now shifted $\varepsilon_1(V_1) + \varepsilon_1(V_2)$ from the starting position under just $H_0$.

(7) In the items above, we have assumed that there is no need for label $\gamma$ (which is the case for the two valence electrons in a carbon atom). If there were some other labels, each time we do the $\varepsilon_1$ type perturbation theory, we would have do a level 1 diagonalization of something like this

$$\langle \gamma'\ J_1 J_2 ||\ V_1\ || \gamma\ J_1 J_2\rangle = (V_1)^{(J1J2)}_{\gamma'\gamma} \quad \Rightarrow \quad (V_1)^{(J1J2)}_{\gamma\gamma'}\ \psi_{\gamma'} = \varepsilon_1\ \psi_{\gamma}$$

and so we diagonalize with respect to whatever these $\gamma$ indices refer to, this tells us the right linear combinations of states and the right values of $\varepsilon_1$ and in this case you would not end up with the entire multiplet having the same $\varepsilon_1$ value! Let's ignore this complication for now. Without this complication, our $\varepsilon_1$ problems for $V_1$ and $V_2$ as done above are immediately diagonal.

(8) The j-j method just described works well if in some sense $V_1 << V_2 << H_0$ so it makes sense to do the two level 1 perturbation theories in the order shown above. Messiah provides no pictures for this example that I have seen yet, will keep an eye out.

(9) Messiah's carbon example has # particles = 2, $L^{(1)} = L^{(2)} = 1$ and $S^{(1)} = S^{(2)} = 1/2$.

**Clebsch-Gordon Coefficients**

Consider this example of "combining" angular momenta: [ This combining $1 \otimes 2$ is usually referred to as "the addition of angular momenta".]

$$D^{(1\otimes 2)} \approx D^{(3)} \oplus D^{(2)} \oplus D^{(1)}$$
$$\mathbf{J}^{(1\otimes 2)} \approx \mathbf{J}^{(3)} \oplus \mathbf{J}^{(2)} \oplus \mathbf{J}^{(1)}$$
$$1 \otimes 2 \approx 3 \oplus 2 \oplus 1$$

where we show first the "rotation matrices" in the "canonical" representation, and under that we show the same idea for the generator matrices, and under that we show an economical notation which stands for either line above. In the first line, things are really functions of some Euler angles $\varphi,\theta,\varphi'$ not shown. The second line is true for each of the three components of angular momentum, so we bold it to suggest it is a vector equation. This is the first appearance of the "direct sum" notation $\oplus$ in this document.

First of all, what exactly do the above equations even **mean**?

**The Left Side**

Many words are needed before giving an answer. If you wrote out, say, the generators $\mathbf{J}^{(1\otimes 2)}$, each of the three generators will be a 15x15 matrix. You can think of this matrix as follows:

$$< 1\ m_1' ; 2\ m_2' | \mathbf{J}^{(1\otimes 2)} | 1\ m_1 ; 2\ m_2> = [\mathbf{J}^{(1\otimes 2)}]_{m_1'm_2', m_1m_2}$$

where $m_1m_2$ can assume 15 distinct values, since $m_1 = -1,0,1$ and $m_2 = -2,-1,0,1,2$ . In general, the matrix would be N x N where $N = (2j_1+1)(2j_2+1)$.

This matrix is a (reducible) matrix representation of the su(2) Lie Algebra (the J's) or SU(2) Lie Group. It is easiest to think of the generators! We know exactly what this 15x15 matrix looks like:

$$< 1\ m_1' ; 2\ m_2' | J_i^{(1\otimes 2)} | 1\ m_1 ; 2\ m_2>$$

$$= < 1\ m_1' | \otimes < 2\ m_2' | \ J_i^{(j1)} \otimes 1 + 1 \otimes J_i^{(j2)} \ | 1\ m_1> \otimes | 2\ m_2>$$

$= < 1\ m_1' | \otimes < 2\ m_2' |\ \ \{J_i^{(j1)} | 1\ m_1>\} \otimes | 2\ m_2> \ \ + | 1\ m_1> \otimes \{J_i^{(j2)} | 2\ m_2>\}$ // Rule 1

$= < 1\ m_1' | J_i^{(j1)} | 1\ m_1> \ < 2\ m_2' | 2\ m_2> \ \ + < 1\ m_1' | 1\ m_1> \ < 2\ m_2' | J_i^{(j2)} | 2\ m_2>$

where we use Rule 6 not really mentioned until now

**Rule 6:** $\{<a'| \otimes <b'|\} \ \ \{|a> \otimes |b>\} \ = \ <a' | a> <b' | b>$

and this really defines the inner product for the space $H^{1\otimes 2}$ . We now continue

$< 1\ m_1' ; 2\ m_2' | \mathbf{J}^{(1\otimes 2)} \ 1\ m_1 ; 2\ m_2> =$

$< 1\ m_1' | J_i^{(j1)} | 1\ m_1> \delta_{m2',m2} \ + \delta_{m1',m1} < 2\ m_2' | J_i^{(j2)} | 2\ m_2>$

$= [J_i^{(j1)}]_{m1',m1} \delta_{m2',m2} \ + \ \delta_{m1',m1} [J_i^{(j2)}]_{m2',m2}$

$= [J_i^{(1\otimes 2)}]_{m1'm2',\ m1m2}$

We presumably know what the generator matrices like $[J_i^{(j1)}]_{m1',m1}$. We can look them up somewhere. And this then tells us exactly what numbers we have in our 15x15 matrix whose elements are $[J_i^{(1\otimes 2)}]_{m1'm2',\ m1m2}$. For i = z, the result is very simple:

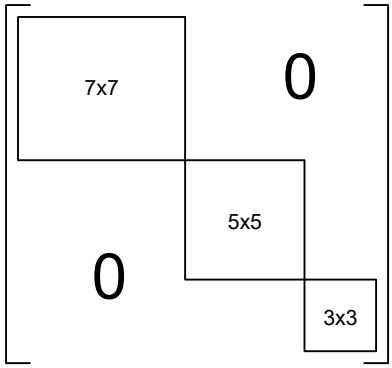
$[J_z^{(1\otimes 2)}]_{m1'm2',\ m1m2} \ = \ [J_i^{(j1)}]_{m1',m1} \delta_{m2',m2} \ + \ \delta_{m1',m1} [J_i^{(j2)}]_{m2',m2}$

$= (m_1 + m_2) \delta_{m1',m1} \delta_{m2',m2}$ = completely diagonal.

But the others x and y are not diagonal. It is not easy to draw the two matrices being summed here. Once you select some order in which to write the indices along the left and top of your matrix, one of the two terms might have a nice block diagonal form of five 3x3 matrices, but the other term will be completely scrambled. In general, it is probably best just to think of these things as fully populated in principle 15x15 matrices, though we can see that the z case is very simple.

**The Right Side**

Now here is the big claim from group theory which I am not proving here. The claim is that there exists a 15x15 unitary matrices U which brings our 15x15 generator matrices into block diagonal form, and when this is done, the blocks will look like this:




Now THIS matrix is what we mean by the direct sum notation (D = block Diagonal)

$$J_i^D \equiv J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)} \qquad \text{// a 15x15 matrix}$$

The direct sum notation just means you have a block diagonal form as shown. The dimension of the matrix you get by direct summing some square matrices like this is the sum of the dimensions of each matrix in the sum. In this case we have 7 + 5 + 3 = 15. We have not bothered to state the well-known "triangle rule" for determining what is in the direct sum.

We could define certain "padded" matrices by filling with zeros as needed. For example, we could define the 15x15 matrix $J_i^{(2p)}$ to be all zero except in the 5x5 area shown above. If we pad each matrix on the right in this way, then we can write the above as follows:

$$J_i^D = J_i^{(3p)} + J_i^{(2p)} + J_i^{(1p)}$$

and now we have regular + signs because we are adding 15x15 matrices in the normal way.

Now that we know the exact meaning of the direct product sum, and now that we have a completely unambiguous definition of $J_i^D$, let' ask again our initial question:

What is the meaning of $J_i^{(1\otimes 2)} \approx J_i^D$ ?

The answer is this: This means that there is a 15x15 unitary matrix U which brings $\mathbf{J}^{(1\otimes 2)}$ to the block diagonal form shown, so the real equation with an = sign is this:

$$U^{(1\otimes 2)} J_i^{(1\otimes 2)} U^{(1\otimes 2)-1} = J^D{}_i \qquad i = 1,2,3 \qquad (*)$$

Note: All the objects appearing here are *matrices*, not operators! This is not an operator equation in some Hilbert Space

Here we say $U^{(1\otimes 2)}$ because this unitary matrix 15x15 is specific to our (1⊗2) situation. The elements of this matrix are going to be the Clebsch-Gordon coefficients, but we need to do a little fiddling to get to that point.

**How the basis states in the |jm> basis might be ordered.**

First of all, looking at the direct sum matrix $J^D{}_i$ shown above, the implication is that the *rows and columns are being labeled in a specific manner* such as this: (we show J $M_J$)

3,3; 3,2; 3,1; 3,0; 3,$\overline{1}$; 3,$\overline{2}$; 3,$\overline{3}$ ;  2,2; 2,1; 2,0; 2,$\overline{1}$; 2,$\overline{2}$;  1,1; 1,0; 1,$\overline{1}$;

where $\overline{2}$ means -2, etc. Any one of these index pairs corresponds to some state |j,m> and we have just shown every possibility for our example. So we can think of {jm} in the order shown above as the 15 indices for our matrices. For example, here are the matrix elements of $J^D{}_i$:

$$<j',m' \mid J^D{}_i|j,m> = \delta_{j',j} <j,m' \mid J^D{}_i|j,m>$$

where the $\delta_{j',j}$ expresses the block diagonal form.

**Statement that both bases are complete.**

Now part of the group theory result just claimed is that the space 1⊗2 can be spanned by two sets of kets:

$|1\ m_1\ ;\ 2\ m_2>$  for $m_1$ = -1 to 1  and $m_2$ = -2,2  (15 kets)

or

$|j,m>$  j = 1,2,3 and $m_j$ = -j,j  3 + 5 + 7 = 15 kets

We therefore can represent "unity" in two different ways in our Hilbert Space

$$1 = \Sigma_{m1,m2} |1\ m_1\ ;\ 2\ m_2><1\ m_1\ ;\ 2\ m_2| = \Sigma_{m1,m2} |m_1m_2><m_1m_2|$$

$$1 = \Sigma_j \Sigma_m |j,m><j,m|$$

where the sum on j gives j = 1,2,3 , etc. I just mention this in passing since we might use it.

---

Rewrite our unitary matrix transformation from above:

$$U^{(1\otimes2)}\ J_i{}^{(1\otimes2)}\ U^{(1\otimes2)-1} = J_i{}^D$$

The above matrix equation can be written out in detail as follows:

$$U_{jm,m1m2}\ \ J_i{}^{(1\otimes2)}{}_{m1m2,n1n2}\ \ U^{\dagger}{}_{n1n2,j'm'} = J_i{}^D{}_{jm,j'm'}$$

with implied sums on both $m_i$ and both $n_i$. <u>This is what the above equation means.</u> The matrix U has a different index labeling for its two indices. The thing $U_{jm,m1m2}$ is a number. Again, there are no Hilbert Space operators here (we will put hats over them if we encounter any). In particular, there does not exist any Hilbert Space operator $\hat{U}$. The things $U_{jm,m1m2}$ are just a set of *numbers*. These numbers form a

matrix 15x15 which converts us from one "basis" to another in a certain Hilbert Space, but there is no operator $\hat{U}$ which exists in this Hilbert Space. I want to stress this strongly because I wasted 4 hours thinking there was such an operator.

Now looking at our identity forms above, we know that

$$|m_1m_2\rangle = \Sigma_j \Sigma_m |j,m\rangle\langle j,m|m_1m_2\rangle = \Sigma_j \Sigma_m \langle j,m|m_1m_2\rangle |j,m\rangle$$

so we know that the numbers $\langle j,m|m_1m_2\rangle$ convert us from one basis to the other. Therefore we may make this identification:

$$U_{jm,m1m2} = \langle j,m|m_1m_2\rangle = \langle j,m \mid j_1m_1; j_2m_2\rangle$$

These numbers ARE the Clebsch-Gordon coefficients, also known as the Wigner coefficients. Using this notation, we can write our matrix equation above as

$$\langle j,m|m_1m_2\rangle \; J_i{}^{(1\otimes 2)}{}_{m1m2,n1n2} \; \langle n_1n_2|j',m'\rangle = J_i{}^{D}{}_{jm,j'm'}$$

There are still no operators anywhere. Now, however, we will express each J object in terms of an operator

$$J_i{}^{(1\otimes 2)}{}_{m1m2,n1n2} = \langle m_1m_2| \hat{J}_i{}^{(1\otimes 2)} | n_1n_2\rangle$$

$$= J_i{}^{D}{}_{jm,j'm'} = \langle j,m| \hat{J}^{D}{}_i |j',m'\rangle$$

where $\hat{J}_i{}^{(1\otimes 2)}$ and $\hat{J}^{D}{}_i$ are both operators in the same Hilbert space. We have two bases which span this Hilbert space which has dimension 15. Thus we can say

$$\langle j,m|m_1m_2\rangle \langle m_1m_2| \hat{J}_i{}^{(1\otimes 2)} | n_1n_2\rangle\langle n_1n_2|j',m'\rangle = \langle j,m| \hat{J}^{D}{}_i |j',m'\rangle$$

If we now recognize our two expressions of unity in the above equation, we find that

$$\langle j,m| \hat{J}_i{}^{(1\otimes 2)} |j',m'\rangle = \langle j,m| \hat{J}^{D}{}_i |j',m'\rangle$$

and since this is true for a complete set of basis elements on both sides, we may conclude that

$$\hat{J}_i{}^{(1\otimes 2)} = \hat{J}^{D}{}_i \equiv \hat{A}_i$$

Thus, these two Hilbert Space operators are the same operator which I now call $\hat{A}_i$. We can compute the matrix elements of $\hat{A}_i$ in either basis we want:

$$\langle j,m| \hat{A}_i |j',m'\rangle = \langle j,m| \hat{J}^{D}{}_i |j',m'\rangle = \delta_{j',j} \langle j,m| \hat{J}^{(j)}{}_i |j,m'\rangle = [\hat{A}_i]_{jm,j'm'}$$

$$\langle m_1 m_2 | \hat{A}_i | n_1 n_2 \rangle = \langle m_1 m_2 | \hat{J}_i^{(1\otimes 2)} | n_1 n_2 \rangle = [\hat{A}_i]_{m1m2,n1n2}$$

$$= [J_i^{(j1)}]_{m1,n1}\, \delta_{m2,m2} + \delta_{m1,n1}\, [J_i^{(j2)}]_{m2,n2}$$

Although the operator $\hat{A}_i$ is the same in both cases, these two matrices are different matrices

$$[\hat{A}_i]_{jm,j'm'} \neq [\hat{A}_i]_{m1m2,n1n2}$$

These are both 15x15 matrices, but they are unequal. They are equivalent under the similarity matrix U. We rewrite this from above

$$U_{jm,m1m2}\ J_i^{(1\otimes 2)}{}_{m1m2,n1n2}\ U^{\dagger}{}_{n1n2,j'm'} = J_i^{D}{}_{jm,j'm'}$$

$$U_{jm,m1m2}\ [\hat{A}_i]_{m1m2,n1n2}\ U^{\dagger}{}_{n1n2,j'm'} = [\hat{A}_i]_{jm,j'm'}$$

If you were to *insist* on having an operator $\hat{U}$ in this same Hilbert Space, you would be forced to say that $\hat{U} = 1$. Then you can have the operator equation $\hat{U}^{(1\otimes 2)}\, \hat{J}_i^{(1\otimes 2)}\, \hat{U}^{(1\otimes 2)-1} = \hat{J}_i^{D}$ and this of course is consistent with our statement above that $\hat{J}_i^{(1\otimes 2)} = \hat{J}^{D}{}_i$.

So now let's return to our original equation and clarify things. We started with these notations

$$D^{(1\otimes 2)}(R) \approx D^{(3)}(R) \oplus D^{(2)}(R) \oplus D^{(1)}(R)$$
$$J_i^{(1\otimes 2)} \approx J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)}$$
$$1\otimes 2 \approx 3 \oplus 2 \oplus 1$$

and asked what they mean. We have now installed some rotation angles R in the first line, and gotten rid of vector notation on the second line. We now focus only on the second line.

We first learned that we can interpret this line as follows:

$$U^{(1\otimes 2)}\, J_i^{(1\otimes 2)}\, U^{(1\otimes 2)-1} = J_i^{(3p)} + J_i^{(2p)} + J_i^{(1p)} \equiv J_i^{D}$$

where every object is a 15x15 *matrix*. The "p" notation is used to pad a matrix with 0's to get it to be 15x15, where we carefully install the submatrix $J_i^{(j)}$ in the right location within the 15x15 matrix. Perhaps we agree to always work down from largest j to smallest as we did in our example. We define above the matrix $J_i^{D}$ to be the sum of the three J's as shown, so that $J_i^{D}$ has the block Diagonal form as shown in our picture above. We could write out the above matrix equation explicitly as follows:

$$U_{jm,m1m2}\ J_i^{(1\otimes 2)}{}_{m1m2,n1n2}\ U^{\dagger}{}_{n1n2,j'm'} = J_i^{D}{}_{jm,j'm'}$$

We next learned that we could invent Hilbert Space operators $\hat{J}_i^{(1\otimes 2)}$ and $\hat{J}^{D}{}_i$ simply by interpreting two of the matrices above as expectation value type matrix elements in a single Hilbert space spanned by

two different bases. We found that these operators are the same operator and wrote $\hat{J}_i^{(1\otimes 2)} = \hat{J}^D{}_i \equiv \hat{A}_i$ and we also found that $U^{(1\otimes 2)}{}_{jm,m_1m_2} = \langle jm|m_1m_2\rangle$ and we gave these numbers the formal name Clebsch-Gordon Coefficients. So we then have this second interpretation of line 2 above as operators:

$$\hat{J}_i^{(1\otimes 2)} = \hat{J}^D{}_i$$

But the matrix $J_i{}^D{}_{jm,j'm'}$ was just a linear combination of three added items, so we could extend our operator notation as follows:

$$J_i{}^D{}_{jm,j'm'} = \langle j,m|\hat{J}^D{}_i|j',m'\rangle = \langle j,m|\hat{J}_i^{(3p)} + \hat{J}_i^{(2p)} + \hat{J}_i^{(1p)}|j',m'\rangle$$

$$= \delta_{j,3}\,\delta_{j,j'}\langle 3,m|\hat{J}_i^{(3p)}|3,m'\rangle + \delta_{j,2}\,\delta_{j,j'}\langle 2,m|\hat{J}_i^{(2p)}|2,m'\rangle + \delta_{j,1}\,\delta_{j,j'}\langle 1,m|\hat{J}_i^{(1p)}|1,m'\rangle$$

$$= \delta_{j,j'}\{\,\delta_{j,3}\langle 3,m|\hat{J}_i^{(3)}|3,m'\rangle + \delta_{j,2}\langle 2,m|\hat{J}_i^{(2)}|2,m'\rangle + \delta_{j,1}\langle 1,m|\hat{J}_i^{(1)}|1,m'\rangle\,\}$$

Is there some nice way to handle things like $\hat{J}_i^{(3p)}$?

$$\hat{J}_i^{(3p)} = \Sigma_m|3m\rangle\langle 3m|\hat{J}_i^{(3)}\,\Sigma_m|3m'\rangle\langle 3m'|$$

$$= \hat{P}^{(3)}\,\hat{J}_i^{(3)}\,\hat{P}^{(3)}$$

Messiah to the rescue, wow zowie woo hoo. Here $\hat{J}_i^{(3)}$ is only defined on this subspace, and we have our two projector protectors to make this work! So now I want to say

$$\hat{J}_i^{(1\otimes 2)} = \hat{P}^{(3)}\,\hat{J}_i^{(3)}\,\hat{P}^{(3)} + \hat{P}^{(2)}\,\hat{J}_i^{(2)}\,\hat{P}^{(2)} + \hat{P}^{(1)}\,\hat{J}_i^{(1)}\,\hat{P}^{(1)}$$

and then this is our "operator interpretation" . I think we could do this in the matrix world as well.

$P^{(3)}$ = a 15x15 matrix with 7 1's at the start of the diagonal, else all 0.

but perhaps not that useful.

**Review of Results on the Meaning of combining angular momenta**

**1. Symbolic Form**

$$J_i^{(1\otimes 2)} \approx J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)}$$

**2. Matrix forms:**

$$U^{(1\otimes 2)}\,J_i^{(1\otimes 2)}\,U^{(1\otimes 2)-1} = J_i^{(3p)} + J_i^{(2p)} + J_i^{(1p)} = [J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)}]$$

$$= P^{(3)}\, J_i^{(3)}\, P^{(3)} + P^{(2)}\, J_i^{(2)}\, P^{(2)} + P^{(1)}\, J_i^{(1)}\, P^{(1)}$$

$$U_{jm,m1m2}\;\; J_i{}^{(1\otimes 2)}{}_{m1m2,n1n2}\;\; U^{\dagger}{}_{n1n2,j'm'} \;=\; [J_i{}^{(3)} \oplus J_i{}^{(2)} \oplus J_i{}^{(1)}]_{jm,j'm'}$$

$U_{jm,m1m2} = \langle jm|m_1 m_2\rangle$ = Clebsch-Gordon (Wigner) coefficients

where the rightmost notation just tells us to build up the block form from components shown.

$$[J_i{}^{(3)} \oplus J_i{}^{(2)} \oplus J_i{}^{(1)}]_{jm,j'm'}$$

$$= \delta_{j,3}\, J_i{}^{(3)}{}_{jm,j'm'} + \delta_{j,2}\, J_i{}^{(2)}{}_{jm,j'm'} + \delta_{j,1}\, J_i{}^{(1)}{}_{jm,j'm'}$$

$$= \delta_{j,j'}\, \{\, \delta_{j,3}\, J_i{}^{(3)}{}_{jm,jm'} + \delta_{j,2}\, J_i{}^{(2)}{}_{jm,jm'} + \delta_{j,1}\, J_i{}^{(1)}{}_{jm,jm'}$$

so there is no confusion at all about what this direct sum matrix is! The projection matrices accomplish this as well.

**3. Operator form:**

$$\hat{J}_i{}^{(1\otimes 2)} \;=\; \hat{P}^{(3)}\, \hat{J}_i{}^{(3)}\, \hat{P}^{(3)} + \hat{P}^{(2)}\, \hat{J}_i{}^{(2)}\, \hat{P}^{(2)} + \hat{P}^{(1)}\, \hat{J}_i{}^{(1)}\, \hat{P}^{(1)}$$

And write them all again

$$J_i{}^{(1\otimes 2)} \;\approx\; J_i{}^{(3)} \oplus J_i{}^{(2)} \oplus J_i{}^{(1)} \qquad \text{// symbolic}$$

$$U^{(1\otimes 2)}\, J_i{}^{(1\otimes 2)}\, U^{(1\otimes 2)-1} \;=\; [J_i{}^{(3)} \oplus J_i{}^{(2)} \oplus J_i{}^{(1)}] \qquad \text{// matrix}$$

$$\hat{J}_i{}^{(1\otimes 2)} \;=\; \hat{P}^{(3)}\, \hat{J}_i{}^{(3)}\, \hat{P}^{(3)} + \hat{P}^{(2)}\, \hat{J}_i{}^{(2)}\, \hat{P}^{(2)} + \hat{P}^{(1)}\, \hat{J}_i{}^{(1)}\, \hat{P}^{(1)} \qquad \text{// operator}$$

**Comments on combining angular momenta:**

1. Whenever you see some statement like **J** = **L** + **S** (such as Messiah p 552) and you hear the phrase "addition of angular momenta", what is really meant is this:

$$J_i{}^{(L\otimes S)} \;=\; L_i \otimes 1 + 1 \otimes S_i \qquad\qquad i = 1,2,3 \qquad\qquad (J_i = L_i * S_i \text{ in my notation})$$

and the total $J_i{}^{(L\otimes S)}$ exists only in a direct product Hilbert space $\mathcal{H}^{(L\otimes S)}$. When you do "manipulations", if you always keep one symbol consistently to the left of the other, you can manipulate with a shorthand notation where it is understood that $L_i$ means $L_i \otimes 1$, and $L_2 S_3$ means $L_2 \otimes S_3$, for example.

Probably a better phrase is "combining angular momenta" because so doing really involves both concepts of "multiplication" as in L ⊗ S and "addition" as in L + S.

2. The direct product $J_i^{(L\otimes S)}$ representation shown above forms a *reducible* representation of the SU(2) Lie Algebra. You can change basis from the "separable" direct product basis $LM_L SM_S$ to the JM basis and express the above J as a sum of irreducible representations of SU(2). This idea is expressed in several different notations all discussed above

$J_i^{(L\otimes S)} \approx J_i^{(L+S)} \oplus J_i^{(L+S-1)} \oplus ..... \oplus J_i^{(|L-S|)}$ // symbolic

$L_i \otimes S_i \approx (L+S) \oplus (L+S-1) \oplus ..... \oplus (|L-S|)$ // symbolic

$\hat{J}_i^{(L\otimes S)} = \hat{P}^{(L+S)} \hat{J}_i^{(L+S)} \hat{P}^{(L+S)} + \hat{P}^{(L+S-1)} \hat{J}_i^{(L+S-1)} \hat{P}^{(L+S-1)} + .....$ // operator

$U^{(L\otimes S)} J_i^{(L\otimes S)} U^{(L\otimes S)-1} = J_i^{(L+S)} \oplus J_i^{(L+S-1)} \oplus ..... \oplus J_i^{(|L-S|)}$ // matrix

$U_{jm,m1m2} J_i^{(L\otimes S)}{}_{m1m2,n1n2} U^{\dagger}{}_{n1n2,j'm'} = [J_i^{(L+S)} \oplus J_i^{(L+S-1)} ..... ]_{jm,j'm'}$

3. In the matrix forms shown above, the matrix U changes you from one basis to the other, and it is a unitary real matrix. U does not correspond to any operator in any Hilbert Space. The elements of the matrix U are usually written in this manner

$U_{jm,m1m2} = \langle jm|Lm_1Sm_2\rangle$ = Clebsch-Gordon coefficients = Wigner coefficients

$|jm\rangle = |Lm_1Sm_2\rangle \langle Lm_1Sm_2| jm\rangle$ // sum on both $m_i$

$|Lm_1Sm_2\rangle = |jm\rangle\langle jm|Lm_1Sm_2\rangle$ // sum on j then m within j space

These last show exactly how the two bases are related. Both have dimension (2L+1)(2S+1). Notice that only the matrix forms above "expose" the C-G coefficients. You don't see them in the operator form or in the symbolic forms. When you actually have to calculate something, you usually use the matrix form.

4. So if we combine things, we might say

$$J_i^{(L\otimes S)} = \underset{1}{L_i + S_i} = \underset{2}{L_i \otimes 1 + 1 \otimes S_i} = \underset{3}{L_i * S_i} \approx \underset{4}{L_i \otimes S_i} \approx \underset{5}{(L+S) \oplus (L+S-1) \oplus ..... \oplus (|L-S|)}$$

where 1 = shorthand notation, 2 = precision notation, 3 = Phil notation, 4 = symbolic notation and where 5 = the reduction in symbolic notation. There are lots of confusing notations all associated with this same concept! But one thing you would never say is $L \oplus S$ .

**How would you compute the C-G coefficients?**

(1) The first fact we know is this:

$\langle jm|Lm_1Sm_2\rangle = \delta_{m,m1+m2} \langle j(m_1+m_2)|Lm_1Sm_2\rangle$

Proof: First, note this from above:

$$\hat{J}_3^{(L\otimes S)} = \Sigma_J \hat{P}^{(J)} \hat{J}_3^{(J)} \hat{P}^{(J)}$$

Then let's compute the following object two ways:

$$\langle jm| \hat{J}_3^{(L\otimes S)} | Lm_1Sm_2\rangle$$

The first way, we let the operator act to the right and use $\hat{J}_3^{(L\otimes S)} = \hat{L}_i \otimes 1 + 1 \otimes \hat{S}_i$ and, as was shown in gory detail above, you get $(m_1+ m_2) \langle jm| Lm_1Sm_2\rangle$. The second way, we let the (Hermitian) operator act to the left using the operator expansion shown above

$$\langle jm| \hat{J}_3^{(L\otimes S)} | Lm_1Sm_2\rangle = \Sigma_J \langle jm| \hat{P}^{(J)} \hat{J}_3^{(J)} \hat{P}^{(J)} | Lm_1Sm_2\rangle$$

The action $\langle jm| \hat{P}^{(J)} = \delta_{j,J}\langle jm|$ so this becomes

$$= \langle jm| \hat{J}_3^{(j)} \hat{P}^{(j)} | Lm_1Sm_2\rangle = m \langle jm| Lm_1Sm_2\rangle$$

So we find that

$$(m_1+ m_2) \langle jm| Lm_1Sm_2\rangle = m \langle jm| Lm_1Sm_2\rangle$$

which says either $m = m_1+m_2$ or the coefficient must vanish, **QED**.

(2) Question: How can you "visualize" the matrix $\langle jm|Lm_1Sm_2\rangle$ ?

We have to decide how we want to "list things off" for rows and columns. Consider the case 2 + 1 that we worked with earlier. One way to list the $m_1m_2$ would be

| 3 | 2 | 1 | 0 | -1 | -2 | -3 |
|---|---|---|---|---|---|---|
| $m_1m_2 =$ 21; | 20, 11; | $2\bar{1}$, 10, 01; | $1\bar{1}$, 00, $\bar{1}1$ ; | $\bar{1}0$, | $0\bar{1}$ , $\bar{2}1$ ; | $\bar{2}0$,$\bar{1}\bar{1}$ ; | $\bar{2}\bar{1}$ |

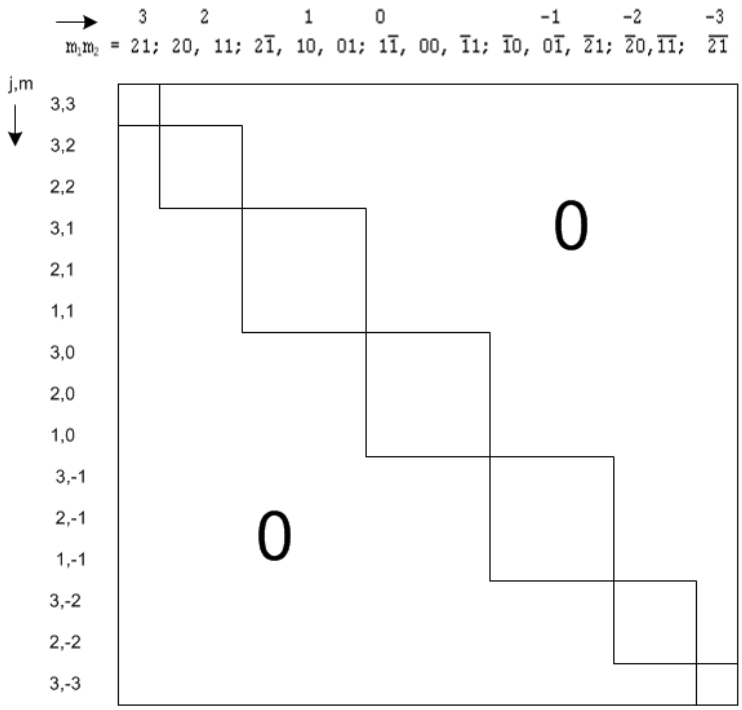
Counting, we have all 15 states somewhere. So we have grouped these direct product states according to the sum $m_1+m_2$. Here I repeat in Courier in order to cut and paste to a picture

```
        3     2         1      0              -1      -2      -3
m1m2 = 21; 20, 11; 21bar, 10, 01; 1bar, 00, 1bar1; 1bar0, 0bar, 2bar1; 2bar0,1bar1bar;  2bar1bar
       x   x   x   x   x   x   x   x   x   x   x   x   x   x   x
```

So this will be across the top, which labels the columns

$$U_{jm,m1m2} = \langle jm|Lm_1Sm_2\rangle$$

The first index is the "row index" and that is why we list the jm vertically. Here then is our 15x15 matrix visualized:




Because of our rule that $m = m_1+m_2$, all entries are 0 except in the squares shown! These are the submatrices that Schiff refers to at the bottom of page 215. If you add up all the squares that are non-zero, you find the total number of C-G coefficients for this example:

$2 + 8 + 27 = 37$

(3) Remember that the jm states are linear combinations of the direct product states. But in the upper left corner, the state jm = 33 can only be a linear combination of one state which is 21 because this is the only state that has $m_1+m_2 = 3$. If we want to maintain normalization, we say

$| jm = 33\rangle = | m_1m_2 = 21\rangle$

These states must be the same. The normalization of any state is supposed to be 1:

$\langle jm = 33| jm = 33\rangle = 1$

because we really want all states in either jm or direct product to be orthonormalized. I did not mention this above, but mention it now. That is why we don't add some constant to $| jm = 33\rangle = | m_1m_2 = 21\rangle$. Therefore we have computed **our very first CG coefficient**:

$\langle j=L+S\ m=L+S|L=2\ m_1=2S=1m_2=1\rangle = 1$

You could of course make this a phase, but simplest thing is to just call it 1. In the general case we have

$$\langle j=L+S,\ m=L+S | L=L, m_1=2;\ S=S,\ m_2=1\rangle = 1$$

In the general case the upper left corner will always be a 1x1 matrix and this works.

$$\langle L+S,\ L+S | 2,2;\ 1,1\rangle = 1 \qquad = \langle 3,3 |\ 2,2\ ;\ 1,1\rangle$$

**Recursion Relations**

To compute all the C-G coefficients, you have to use little recursion relations. I will derive them right here:

$$J_{\pm}|j,m\rangle = \sqrt{j(j+1)-m(m\pm1)}\ |j,m\pm 1\rangle$$

$$\langle j,m\ |\ J_{\mp} = \sqrt{j(j+1)-m(m\pm1)}\ \langle j,m\pm 1\ |$$

$$\langle JM|\ J_+J_-|\ j_1m_2;\ j_2m_2\rangle \qquad\qquad J_+J_- = J^2 - J_z^2 + J_z$$

$$= \sqrt{J(J+1)-M(M-1)}\ \ \langle JM-1|\ J_-|\ j_1m_2;\ j_2m_2\rangle$$

But

$$J_-|\ j_1m_1;\ j_2m_2\rangle = \sqrt{j_1(j_1+1)-m_1(m_1-1)}\ |\ j_1m_1-1;\ j_2m_2\rangle + \sqrt{j_2(j_2+1)-m_2(m_2-1)}\ |\ j_1m_1;\ j_2m_2-1\rangle$$

Thus we find that

$$\langle JM-1|\ J_-|\ j_1m_2;\ j_2m_2\rangle = \sqrt{j_1(j_1+1)-m_1(m_1-1)}\ \langle JM-1|\ j_1m_1-1;\ j_2m_2\rangle$$

$$+\sqrt{j_2(j_2+1)-m_2(m_2-1)}\ \langle JM-1|\ j_1m_1;\ j_2m_2-1\rangle$$

And thus we know that

$$\langle JM|\ J_+J_-|\ j_1m_2;\ j_2m_2\rangle = \sqrt{J(J+1)-M(M-1)}\ \ \sqrt{j_1(j_1+1)-m_1(m_1-1)}\ \langle JM-1|\ j_1m_1-1;\ j_2m_2\rangle$$

$$+\sqrt{J(J+1)-M(M-1)}\ \ \sqrt{j_2(j_2+1)-m_2(m_2-1)}\ \langle JM-1|\ j_1m_1;\ j_2m_2-1\rangle$$

But using $J_+J_- = J^2 - J_z(J_z-1)$ and acting to the left we pick up $[\sqrt{J(J+1)-M(M-1)}\ ]^2$ so that

$$\langle JM|\ J_+J_-|\ j_1m_2;\ j_2m_2\rangle = [\sqrt{J(J+1)-M(M-1)}\ ]^2 \quad \langle JM|\ j_1m_2;\ j_2m_2\rangle$$

Comparing, we find this relationship

$$\sqrt{J(J+1)-M(M-1)}\ \langle JM|\, j_1 m_1;\, j_2 m_2\rangle = \sqrt{j_1(j_1+1)-m_1(m_1-1)}\ \langle JM-1|\, j_1 m_1-1;\, j_2 m_2\rangle$$

$$+\sqrt{j_2(j_2+1)-m_2(m_2-1)}\ \langle JM-1|\, j_1 m_1;\, j_2 m_2-1\rangle$$

which agrees with **Messiah page 1057 (C.19)**.

Were we to start instead with $J_- J_+ = J^2 - J_z(J_z+1)$, we get a very similar result :

$$\sqrt{J(J+1)-M(M+1)}\ \langle JM|\, j_1 m_1;\, j_2 m_2\rangle = \sqrt{j_1(j_1+1)-m_1(m_1+1)}\ \langle JM+1|\, j_1 m_1+1;\, j_2 m_2\rangle$$

$$+\sqrt{j_2(j_2+1)-m_2(m_2+1)}\ \langle JM+1|\, j_1 m_1;\, j_2 m_2+1\rangle$$

which is **C.18**. We just replace all m-1 with m+1 everywhere they appear. Messiah gives another relation which relates adjacent values of J, but I am not sure this one is really needed to solve for all the coefficients.

In our example above of $2\otimes 1$, there are 37 non-vanishing C-G coefficients. So we have 37 starting points for each of our two equations above. So we have 74 relations with 2 or 3 terms in each one. That seems to suggest 74 equations in 37 unknowns, but some must duplicate each other.

Obviously, any relationship with one vanishing term is useful. For example:

Example from our chart: The upper left corner coefficient is 1. Come down on the diagonal one box. This box has these values for things

J=3, M=2 m1=2 m2=0

Then look at the above + relation above and plug in

$$\sqrt{3(3+1)-2(2+1)}\ \langle 32|\, j_1 2;\, j_2 0\rangle = \sqrt{2(2+1)-2(2+1)}\ \langle 33|\, 2,2+1;\, 10\rangle$$

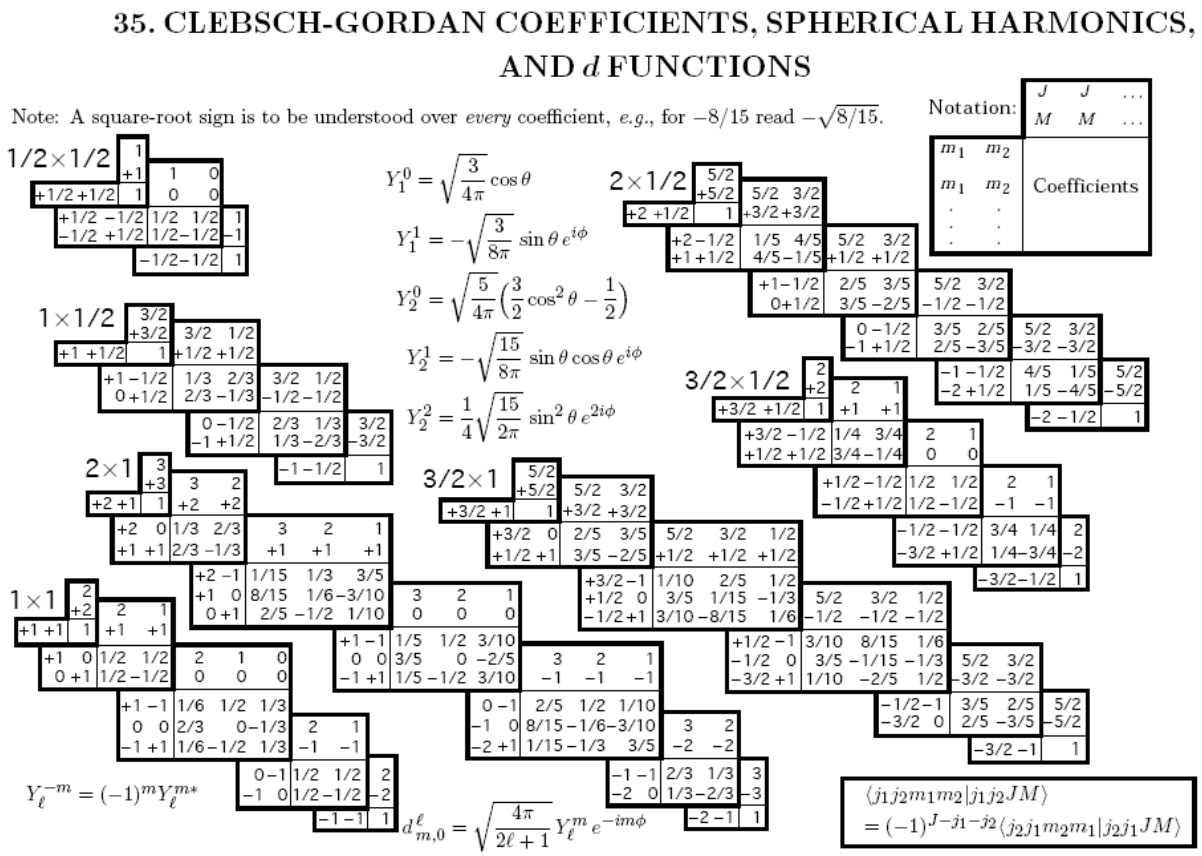
$$+\sqrt{1(1+1)-0(0+1)}\ \langle 33|\, 22;\, 10+1\rangle$$

$$\sqrt{12-6}\ \langle 32|\, 22;\, 10\rangle = \sqrt{2}\ \langle 33|\, 22;\, 11\rangle$$

$$\langle 32|\, 22;\, 10\rangle = (\sqrt{2}/\sqrt{6})\ \langle 33|\, 22;\, 11\rangle = (1/\sqrt{3})$$

and amazingly enough, this agrees with the PDG tables! Look for the 2 x 1 section below and notice that it has the same structure as that which I drew above (except, unfortunately, the rows and columns are switched). You are supposed to put a $\sqrt{}$ over every coefficient and put the minus sign of course outside the radical if there is a minus sign. You see that our <3,2|20> box says 1/3 meaning $(1/\sqrt{3})$ as I just computed above.

Messiah shows a very fancy recursion relation C.20 that mixes different values of J. I don't know how this one is done, and it is a big mess, and is not really needed.




Notice that the PDG has the rows and columns swapped from my drawing. There are more in their table which I will not bother to copy here.

**Comments**: I am pretty sure I could write down a method to compute every coefficient. And we do have a closed formula 5-47 Tinkham page 121 but it is pretty ugly but you could use it in a computer program to print out things like the cases above. There are online C-G calculators on the web, for example

http://personal.ph.surrey.ac.uk/~phs3ps/cgjava.html

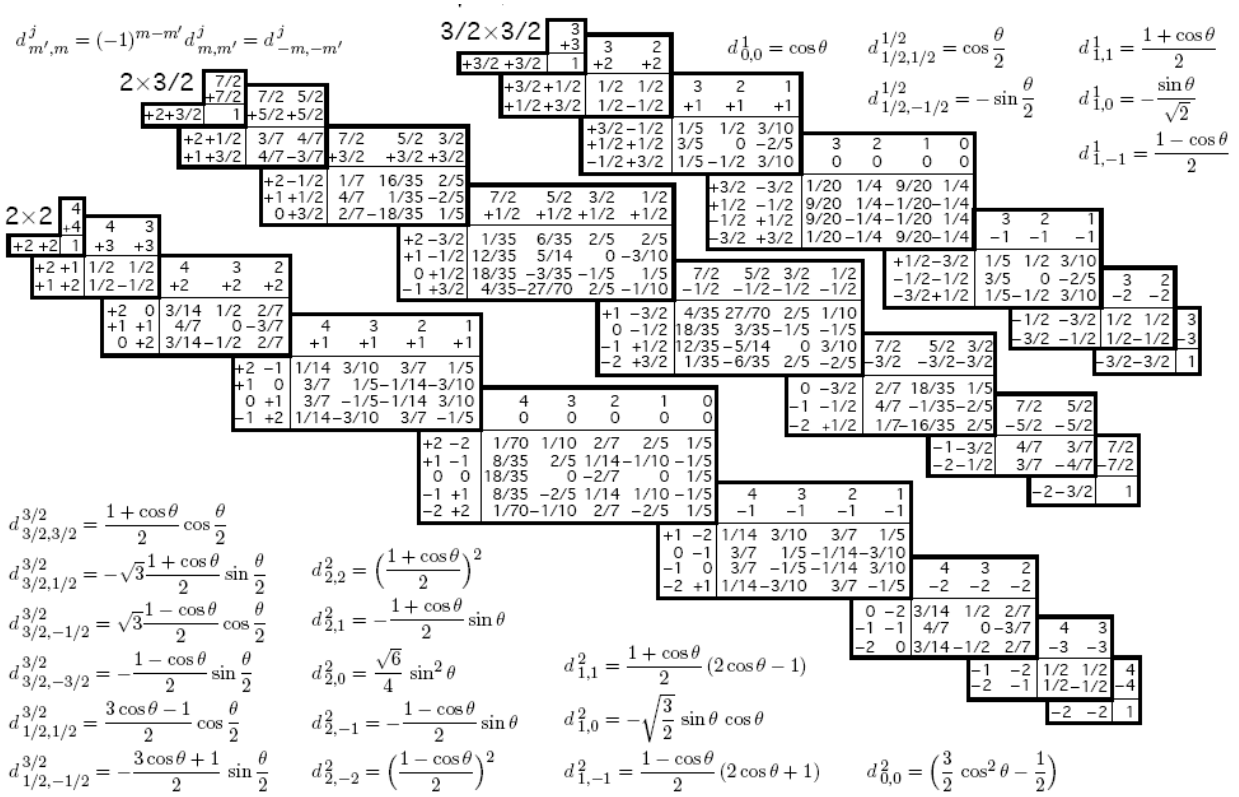
# Java™ Clebsch-Gordan Coefficient Calculator

<j1= 2 m1= 2 j2= 1 m2= 0 | j= 3 m= 2 > Go! Clear

(1/3)*sqrt(3)

approx: 0.5773502691896257

Here is the other half of the above page,



**The Group Representation Matrices for SU(2)**

Recall from above:

$$D^{(1\otimes 2)} \approx D^{(3)} \oplus D^{(2)} \oplus D^{(1)}$$

$$\mathbf{J}^{(1\otimes 2)} \approx \mathbf{J}^{(3)} \oplus \mathbf{J}^{(2)} \oplus \mathbf{J}^{(1)}$$

$$1\otimes 2 \approx 3 \oplus 2 \oplus 1$$

The PDG charts are also showing elements of the actual $D^{J}(\varphi,\theta,\varphi')_{mm'} = e^{-im\varphi}\, e^{-im'\varphi'} d^{J}_{mm'}(\theta)$ which are the SU(2) group representation matrices. We can write, by the way

$$[D^{j1\otimes j1}(R)]_{m1m2,m1'm2'} = D^{j1}(R)_{m1,m1'}\, D^{j2}(R)_{m2,m2'}$$

$$D^{j1\otimes j1}(R) = D^{j1}(R) \otimes D^{j1}(R)$$

which has a simpler form that what we get for the generators,

$$J_i^{(j1\otimes j2)} = J_i^{(j1)} \otimes 1 + 1 \otimes J_i^{(j2)} = J_i^{(j1)} * J_i^{(j2)}$$

Here is a little proof of the above based on theorem $e^{a*b} = e^{a} \otimes e^{b}$,

$$\exp[\, -i\theta\, J_i^{(j1\otimes j2)}] = \exp[\, -i\theta(\, J_i^{(j1)} * J_i^{(j2)})] = \exp[\, -i\theta(\, J_i^{(j1)})] \otimes \exp[\, -i\theta(\, J_i^{(j2)})]$$

or more generally

$$\exp[ -i\theta\, \hat{\mathbf{n}} \bullet \mathbf{J}^{(j1\otimes j2)}] = \exp[ -i\theta \hat{\mathbf{n}} \bullet ( \mathbf{J}^{(j1)} * \mathbf{J}^{(j2)})] = \exp[ -i\theta \hat{\mathbf{n}} \bullet \mathbf{J}^{(j1)}] \otimes \exp[ -i\theta \hat{\mathbf{n}} \bullet \mathbf{J}^{(j2)}]$$

Suppose we take the first form and do Euler angles like so

$$\exp[ -i\varphi\, J_3^{(j1\otimes j2)}] \exp[ -i\theta\, J_2^{(j1\otimes j2)}] \exp[ -i\varphi'\, J_3^{(j1\otimes j2)}] = D^{(j1\otimes j2)}(\varphi,\theta,\varphi')$$

$$= \exp[ -i\varphi ( J_3^{(j1)})] \otimes \exp[ -i\varphi ( J_3^{(j2)})]$$

$$\exp[ -i\theta ( J_2^{(j1)})] \otimes \exp[ -i\theta ( J_2^{(j2)})]$$

$$\exp[ -i\varphi' ( J_3^{(j1)})] \otimes \exp[ -i\varphi' ( J_3^{(j2)})] \qquad \text{// product of three } \mathcal{H}^{(j1\otimes j2)} \text{ elements}$$

$$= \exp[ -i\varphi ( J_3^{(j1)})] \exp[ -i\theta ( J_2^{(j1)})] \exp[ -i\varphi' ( J_3^{(j1)})] \otimes$$
$$\exp[ -i\varphi ( J_3^{(j2)})] \exp[ -i\theta ( J_2^{(j2)})] \exp[ -i\varphi' ( J_3^{(j2)})]$$

$$= D^{(j1)}(\varphi,\theta,\varphi') \otimes D^{(j2)}(\varphi,\theta,\varphi')$$

**The Rotation Operators and Addition Theorems**

I have not mentioned up to this point the notation of Hilbert Space rotation operators, but we did have generators as operators. Recall for these generators that

$$J_i^{(1\otimes 2)} \approx J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)}$$

$$U^{(1\otimes 2)}\, J_i^{(1\otimes 2)}\, U^{(1\otimes 2)-1} = J_i^{(3p)} + J_i^{(2p)} + J_i^{(1p)} = [J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)}]$$

$$U_{jm,m1m2}\; J_i^{(1\otimes 2)}{}_{m1m2,n1n2}\; U^{\dagger}{}_{n1n2,j'm'} = [J_i^{(3)} \oplus J_i^{(2)} \oplus J_i^{(1)}]_{jm,j'm'}$$

$U_{jm,m1m2} = \langle jm|m_1 m_2\rangle$ = Clebsch-Gordon coefficients

$$\hat{J}_i^{(1\otimes 2)} = \hat{P}^{(3)} \hat{J}_i^{(3)} \hat{P}^{(3)} + \hat{P}^{(2)} \hat{J}_i^{(2)} \hat{P}^{(2)} + \hat{P}^{(1)} \hat{J}_i^{(1)} \hat{P}^{(1)}$$

The rotation operators are exponentiated generators as shown above. This fact is what lets us rewrite all of the above items in terms of the rotations. For example:

$$R_i^{(1\otimes 2)}(\theta) \approx R_i^{(3)}(\theta) \oplus R_i^{(2)}(\theta) \oplus R_i^{(1)}(\theta)$$

Let's suppress the rotation angle argument so we can quickly restate all the above forms:

$$R_i^{(1\otimes 2)} \approx R_i^{(3)} \oplus R_i^{(2)} \oplus R_i^{(1)}$$

$$U^{(1\otimes 2)}\, R_i^{(1\otimes 2)}\, U^{(1\otimes 2)-1} = R_i^{(3p)} + R_i^{(2p)} + R_i^{(1p)} = [R_i^{(3)} \oplus R_i^{(2)} \oplus R_i^{(1)}]$$

$$U_{jm,m1m2}\; R_i^{(1\otimes 2)}{}_{m1m2,n1n2}\; U^{\dagger}{}_{n1n2,j'm'} = [R_i^{(3)} \oplus R_i^{(2)} \oplus R_i^{(1)}]_{jm,j'm'}$$

$U_{jm,m1m2} = \langle jm|m_1m_2\rangle$ = Clebsch-Gordon coefficients

$$\hat{R}_i^{(1\otimes 2)} = \hat{P}^{(3)}\, \hat{R}_i^{(3)}\, \hat{P}^{(3)} + \hat{P}^{(2)}\, \hat{R}\hat{J}_i^{(2)}\, \hat{P}^{(2)} + \hat{P}^{(1)}\, \hat{R}_i^{(1)}\, \hat{P}^{(1)} \qquad (*)$$

If we use the Euler triplet φ,θ,φ' ( I will us 2=y for the central angle, but could be 2=x instead), then matrix elements of the rotation operators are the D functions:

$$\langle jm|\, \hat{R}_i^{(j)} |jm'\rangle = D^{(j)}{}_{mm'}$$

$$\hat{R}_i^{(j)}|jm\rangle = 1^j\, \hat{R}_i^{(j)}|jm\rangle = |jm'\rangle\langle jm'|\hat{R}_i^{(j)}|jm\rangle = D^{(j)}{}_{m'm}\,|jm'\rangle$$

with the usual unfriendly matrix index order. So we know the effect of a rotation operator on a state. We could write this out for the direct product representation, but the result is obvious and was quoted above

$$D^{j1\otimes j1}{}_{m1m2,m1'm2'} = D^{j1}{}_{m1,m1'}\, D^{j2}{}_{m2,m2'} \qquad (**)$$

We can use the operator form of the combination rule (*) above to get an interesting result. On the LHS, if we take the matrix element in the direct product space, we get the LHS of (**). Let's look at the same matrix element of one of the terms on the RHS of (*):

$$\langle m1m2|\hat{P}^{(3)}\, \hat{R}_i^{(3)}\, \hat{P}^{(3)}|m1'm2'\rangle =$$

$$\langle m1m2|3m\rangle\langle 3m|\, \hat{R}_i^{(3)}|3m'\rangle\langle 3m'|m1'm2'\rangle = \langle m1m2|3m\rangle D_i^{(3)}{}_{mm'}\langle 3m'|m1'm2'\rangle$$

So we get this result including all terms (suppress the i subscript as well now)

$$D^{j1\otimes j1}{}_{m1m2,m1'm2'} = D^{j1}{}_{m1,m1'}\, D^{j2}{}_{m2,m2'}$$

$$= \Sigma_j \langle m1m2|jm\rangle D_i^{(j)}{}_{mm'}\langle jm'|m1'm2'\rangle$$

Well, this is nothing very new. If we go back to our matrix form above

$$U^{(1\otimes 2)}\, R_i^{(1\otimes 2)}\, U^{(1\otimes 2)-1} = [R_i^{(3)} \oplus R_i^{(2)} \oplus R_i^{(1)}]$$

we can rewrite this as

$$R_i^{(1\otimes 2)} = U^{(1\otimes 2)-1}\ [R_i^{(3)} \oplus R_i^{(2)}\ \oplus R_i^{(1)}]\ U^{(1\otimes 2)}$$

The interesting new feature is that $R_i^{(1\otimes 2)} = R_i^{(1)} \otimes R_i^{(2)}$ so we have

$$R_i^{(1)} \otimes R_i^{(2)} = U^{(1\otimes 2)-1}\ [R_i^{(3)} \oplus R_i^{(2)}\ \oplus R_i^{(1)}]\ U^{(1\otimes 2)}$$

where everything here is a matrix. When we write out the matrix elements, we get the above:

$$D^{j1}{}_{m1,m1'}\ D^{j2}{}_{m2,m2'} = \Sigma_j \langle m1m2|jm\rangle D^{(j)}{}_{mm'} \langle jm'|m1'm2'\rangle$$

This starts with some rotation i subscript (not shown) but is easily generalized to the Euler product. Of course we could also write out the original form

$$U_{jm,m1m2}\ \ R_i^{(1\otimes 2)}{}_{m1m2,n1n2}\ \ U^{\dagger}{}_{n1n2,j'm'} = [R_i^{(3)} \oplus R_i^{(2)}\ \oplus R_i^{(1)}]_{jm,j'm'}$$

$$\langle jm|m1m2\rangle\ D^{j1}{}_{m1,m1'}\ D^{j2}{}_{m2,m2'} \langle m1'm2'|j'm'\rangle = \langle jm|\ [R_i^{(3)} \oplus R_i^{(2)}\ \oplus R_i^{(1)}]|j'm'\rangle$$

But we have our block diagonal form and we could insert projector matrices to show this:

$$R_i^{(3)} \oplus R_i^{(2)}\ \oplus R_i^{(1)} = P^{(3)} R_i^{(3)}\ P^{(3)} + P^{(2)} R_i^{(2)}\ P^{(2)} + P^{(1)} R_i^{(1)}\ P^{(1)}$$

then we get

$$\langle jm|\ [R_i^{(3)} \oplus R_i^{(2)}\ \oplus R_i^{(1)}]|j'm'\rangle$$

$$= \langle jm|\ P^{(3)} R_i^{(3)}\ P^{(3)} + P^{(2)} R_i^{(2)}\ P^{(2)} + P^{(1)} R_i^{(1)}\ P^{(1)}\ |j'm'\rangle$$

$$= \langle jm\ |3m'\rangle\langle 3m'|R_i^{(3)}\ |3m''\rangle\langle 3m''|\ j'm'\rangle + \text{etc}$$

$$= \delta_{j,3}\ \delta_{j',3} \langle 3m|R_i^{(3)}\ |3m'\rangle + \text{etc} = \delta_{j,3}\ \delta_{j',3}\ D^{(3)}{}_{mm'}$$

So our result is then this:

$$\langle jm|m1m2\rangle\ D^{j1}{}_{m1,m1'}\ D^{j2}{}_{m2,m2'} \langle m1'm2'|j'm'\rangle =$$

$$\delta_{j,3}\ \delta_{j',3}\ D^{(3)}{}_{mm'} + \delta_{j,2}\ \delta_{j',2}\ D^{(2)}{}_{mm'} + \delta_{j,1}\ \delta_{j',1}\ D^{(1)}{}_{mm'}$$

$$= \Sigma_{j''}\ \delta_{j,j''}\ \delta_{j',j''}\ D^{(j'')}{}_{mm'}$$

So here is a summary of the two directions of our addition theorem or whatever you want to call it:

$$D^{j1}{}_{m1,m1'}\ D^{j2}{}_{m2,m2'} = \Sigma_j \langle m1m2|jm\rangle D^{(j)}{}_{mm'} \langle jm'|m1'm2'\rangle$$

$$\langle jm|m1m2\rangle\ D^{j1}{}_{m1,m1'}\ D^{j2}{}_{m2,m2'} \langle m1'm2'|j'm'\rangle = \Sigma_{j''}\ \delta_{j,j''}\ \delta_{j',j''}\ D^{(j'')}{}_{mm'}$$

and we can write our matrix form rule as follows:

$$U^{(1\otimes 2)}\, R_i^{(1\otimes 2)}\, U^{(1\otimes 2)-1} \;=\; P^{(3)} R_i^{(3)}\, P^{(3)} + P^{(2)} R_i^{(2)}\, P^{(2)} + P^{(1)} R_i^{(1)}\, P^{(1)}$$

where everything here is a matrix.

**Orthogonality and Completeness of the D functions.**

Tinkham has what appears to be a very general rule for continuous groups as page 100 (5-11). In his notation Γ is a representation, so I think this would apply to our D functions as follows:

$$\int d\varphi \int d(\cos\theta) \int d\varphi' \; D^{(j)*}_{m,n} D^{(j')}_{m',n'} = \delta_{j,j'}\delta_{m,m'}\delta_{n,n'} (1/(2j+1)) * \int d\varphi \int d(\cos\theta) \int d\varphi' \, 1$$

The integral on the right is $2\pi * 4\pi = 8\pi^2$ so rewrite as

$$\int d\varphi \int d(\cos\theta) \int d\varphi' \; D^{(j)*}_{m,n} D^{(j')}_{m',n'} = 8\pi^2/(2j+1) * \delta_{j,j'}\delta_{m,m'}\delta_{n,n'}$$

I don't see this stated as such in Tinkham, however. The above property is orthogonality of the D functions because we are doing an integral of the function variables which here are angles. As a special case of the above, we have

$$\int d\varphi \int d(\cos\theta) \int d\varphi' \; D^{(j)*}_{m,n} D^{(j)}_{m',n'} = 8\pi^2/(2j+1) * \delta_{m,m'}\delta_{n,n'}$$

As a second special case starting from the original, let's set n = n' = 0 to get

$$\int d\varphi \int d(\cos\theta) \int d\varphi' \; D^{(j)*}_{m,0} D^{(j')}_{m',0} = 8\pi^2/(2j+1) * \delta_{j,j'}\delta_{m,m'}$$

But now there is no φ' dependence in the integrand so $\int d\varphi' = 2\pi$ and we then get

$$\int d\varphi \int d(\cos\theta) \; D^{(j)*}_{m,0} D^{(j')}_{m',0} = 4\pi/(2j+1) * \delta_{j,j'}\delta_{m,m'}$$

The D matrices are completely determined by group theory and a closed form expression of them is given as Tinkham page (5-35) and the example j=1 is shown page 111 where rows and columns are labeled from larger to smaller m values, as you would expect.

Let's now "just try something" and see where it leads:

$$R^{-1}R = 1$$

$$\langle jm|\, R^{-1}|j'm'\rangle\langle j'm'|\, R|j''m''\rangle = \delta_{j,j''}\,\delta_{m,m''}$$

$$\delta_{j,j'}\, D^{(j)-1}{}_{mm'}\, \delta_{j',j''}\, D^{(j')}{}_{m'm''} = \delta_{j,j''}\,\delta_{m,m''} = \delta_{j,j'}\,\delta_{j',j''}\, D^{(j)-1}{}_{mm'}\, D^{(j')}{}_{m'm''}$$

$$= \delta_{j,j}\delta_{j,j''}\, D^{(j)-1}{}_{mm'}\, D^{(j)}{}_{m'm''} = \delta_{j,j''}\, D^{(j)-1}{}_{mm'}\, D^{(j)}{}_{m'm''}$$

Therefore we learn that

$$\Sigma_{m'} D^{(j)-1}{}_{mm'}\, D^{(j)}{}_{m'm''} = \delta_{m,m''}$$

I think this is some kind of "completeness" property of the D functions for each j. Here is Tinkham page 103

$$D^{(j)}(\varphi,\theta,\varphi')_{mm'} = e^{-im\varphi}\, d^{(j)}(\theta)_{mm'}\, e^{-im'\varphi'}$$

so there is no confusion about our definitions here.

**Matrix elements of the generators.**

Side Question: Is there a simple formula for the *generator* matrices of the "spherical representations"? The form I know for the Lorentz Group I would say is more "Cartesian". Well here is the answer:

$$J_i{}^{(j)}{}_{m,m'} = \langle jm|\, J_i|jm'\rangle$$

$$J_x{}^{(j)}{}_{m,m'} = \langle jm|\, J_x|jm'\rangle = 1/2\ \langle jm|\, J_+ + J_-|jm'\rangle =$$

$$= \langle jm|\ \{\sqrt{j(j+1)-m(m+1)}\,|jm'+1\rangle + \sqrt{j(j+1)-m(m-1)}\,|jm'-1\rangle$$

$$= \sqrt{j(j+1)-m(m+1)}\,\delta_{m,m'+1} + \sqrt{j(j+1)-m(m-1)}\ \delta_{m,m'-1}$$

So here is the answer to this question:

$$J_x{}^{(j)}{}_{m,m'} = \sqrt{j(j+1)-m(m+1)}\,\delta_{m,m'+1} + \sqrt{j(j+1)-m(m-1)}\ \delta_{m,m'-1}$$

$$iJ_y{}^{(j)}{}_{m,m'} = \sqrt{j(j+1)-m(m+1)}\,\delta_{m,m'+1} - \sqrt{j(j+1)-m(m-1)}\ \delta_{m,m'-1}$$

$$J_z{}^{(j)}{}_{m,m'} = m\,\delta_{m,m'}$$

So $J_z$ is diagonal and the other two have elements on the both first off-diagonals.

**Coordinate Space and Spherical Harmonics and Fancy Formulas**

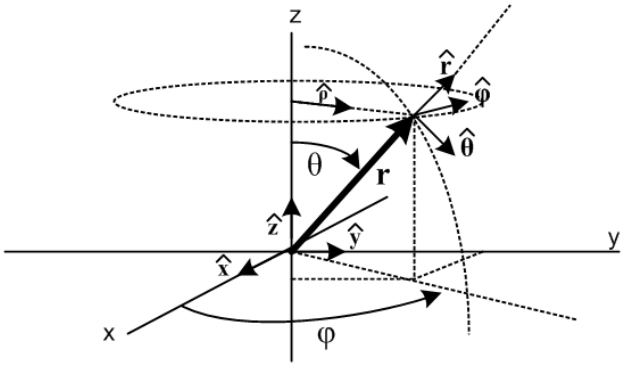
This has not been mentioned at all to this point! Let's first just think of a single j and not combination of angular momenta. We can write:

$$\langle \mathbf{r} \mid jm \rangle = \psi_{jm}(\mathbf{r})$$

where r = x,y,z = r,θ,φ (position of a single particle in QM!) Obviously we want to use the spherical coordinates here. We can write the "position states" of a particle as |r,θ,φ>. A particle located on the +z axis is at position |r,0,0>. We can say:

$$|r,\theta,\varphi\rangle = R_z(\varphi)R_y(\theta)|r,0,0\rangle$$

with my usual picture:




How might we normalize these position states? If we start in Cartesian, we have'

$$\langle \mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r}-\mathbf{r}') \qquad\qquad 1 = \int d^3r \ |\mathbf{r}\rangle\langle\mathbf{r}|$$

Thus we are going to have

$$1 = \int r^2 dr \int d\Omega \ | r,\theta,\varphi \rangle\langle r,\theta,\varphi |$$

$$\langle r,\theta,\varphi \mid r',\theta',\varphi' \rangle = \delta(r-r')/r^2 \ \delta(\Omega-\Omega') = \delta(r-r')/r^2 \ \delta(\cos\theta - \cos\theta')\delta(\varphi-\varphi')$$

which are consistent with each other.

If we are ONLY thinking about rotations, we can imagine our particle is confined to a spherical shell of some radius r which is a constant of the problem. In this case, we choose these states

$$| r,\theta,\varphi \rangle = \delta(r-a)/a \ |\theta,\varphi\rangle$$

where we have now defined some new states |θ,φ> by the above. We then find

$$1 = \int r^2 dr \int d\Omega \; | r,\theta,\varphi \rangle\langle r,\theta,\varphi | \; = \int r^2 dr \int d\Omega \; \delta(r-a)/a \; \delta(r-a)/a \; | \theta,\varphi \rangle\langle \theta,\varphi |$$

$$= \int r^2 dr \int d\Omega \; \delta(r-a)/a^2 \; | \theta,\varphi \rangle\langle \theta,\varphi | \; = \int d\Omega \; | \theta,\varphi \rangle\langle \theta,\varphi |$$

$$\langle r,\theta,\varphi \; | \; r',\theta',\varphi' \rangle \; = \delta(r-r')/r^2 \; \delta(\Omega-\Omega') = \; \delta(r-r')/r^2 \; \delta(\cos\theta - \cos\theta')\delta(\varphi-\varphi')$$

$$= \delta(r-a)/a \; \langle \theta,\varphi | \; \delta(r'-a)/a \; | \theta',\varphi' \rangle \; = \delta(r-r')/r^2 \; \langle \theta,\varphi | \; \theta',\varphi' \rangle$$

which takes us to these familiar relations

$$1 = \int d\Omega \; | \theta,\varphi \rangle\langle \theta,\varphi | \qquad \text{and} \qquad \langle \theta,\varphi | \; \theta',\varphi' \rangle = \delta(\cos\theta - \cos\theta')\delta(\varphi-\varphi')$$

and then in this "angle space" we will write:

$$|\theta,\varphi\rangle = R_z(\varphi) R_y(\theta) |0,0\rangle$$

and we can then compute

$$\langle \theta\varphi | jm \rangle = \langle 00 | \; R_y^{-1}(\theta) \; R_z^{-1}(\varphi) | jm \rangle \; = \langle 00 | jm' \rangle\langle jm' | \; R_z^{-1}(\varphi'=0) R_y^{-1}(\theta) \; R_z^{-1}(\varphi) | jm \rangle$$

$$= \langle 00 | jm' \rangle \; D^{(j)-1}{}_{m',m} = \Sigma_{m'} D^{(j)*}{}_{mm'}(\varphi,\theta,0) \; \langle 00 | jm' \rangle$$

Now, can we evaluate $\langle 00|jm\rangle$ somehow? Consider:

$$\langle 00 | R_z(\varphi) | jm \rangle \; = \langle 00 | jm \rangle \; = \; \langle 00 | e^{-im\varphi} | jm \rangle \; = e^{-im\varphi} \langle 00 | jm \rangle$$

by taking our operator in both directions. Here we assume that the position state $|00\rangle$ does not pick up a phase when you z-rotate it. This suggests that

$$\langle 00 | jm \rangle = \delta_{m,0} \langle 00 | j0 \rangle$$

Then our result above becomes

$$\langle \theta\varphi | jm \rangle = \Sigma_{m'} D^{(j)*}{}_{mm'}(\varphi,\theta,0) \; \langle 00 | jm' \rangle \; = D^{(j)*}{}_{m0}(\varphi,\theta,0) \; \langle 00 | j0 \rangle$$

Now recall our orthogonality specialized result from above

$$\int d\varphi \int d(\cos\theta) \; D^{(j)*}{}_{m,0} \, D^{(j')}{}_{m',0} \; = 4\pi/(2j+1) * \delta_{j,j'} \delta_{m,m'}$$

$$\Rightarrow \; \int d\varphi \int d(\cos\theta) \; \langle \theta\varphi | jm \rangle^* \; \langle \theta\varphi | j'm' \rangle \; = 4\pi/(2j+1) * |\langle 00 | j0 \rangle|^2 \; \delta_{m,m'} \; \delta_{j,j'}$$

But we can rewrite the LHS as follows:

$$\int d\Omega \langle jm|\theta\varphi\rangle\langle\theta\varphi|j'm'\rangle = \langle jm|j'm'\rangle = \delta_{j,j'}\delta_{m,m'}$$

and comparing we may conclude that

$$|\langle 00|j0\rangle|^2 = (2j+1)/4\pi$$

and we then have from above that

$$\langle\theta\varphi|jm\rangle = \sqrt{(2j+1)/4\pi}\;\; D^{(j)*}_{m0}(\varphi,\theta,0) = \sqrt{(2j+1)/4\pi}\;\; d^{j}_{m0}(\theta)\, e^{+im\varphi}$$

Since we orthonormalized both the $|jm\rangle$ and the $|\theta\varphi\rangle$ states, it should not surprise us to find that this factor is not a free parameter but is forced by the scales of our other states.

If we then define the spherical harmonics like so:

$$Y_{jm}(\theta,\varphi) = Y_j^m(\theta,\varphi) \equiv \langle\theta\varphi|jm\rangle = \sqrt{(2j+1)/4\pi}\;\; D^{(j)*}_{m0}(\varphi,\theta,0) = \sqrt{(2j+1)/4\pi}\;\; d^{j}_{m0}(\theta)\, e^{+im\varphi}$$

we have our usual condition

$$\int d\Omega\, Y_{jm}(\theta,\varphi)^*\, Y_{j',m'}(\theta,\varphi) = \delta_{j,j'}\delta_{m,m'}$$

which is the Jackson and Messiah normalization. The above between $d^{j}_{m0}$ and $Y_{jm}$ is confirmed in the PDG page above.

**Example**: Let's look at Tinkham page 111 with his j=1 example. The first index is the row index, so $D^{(j)}_{m0}$ would be the middle column of the matrix shown. So

$$D^{(1)}_{10} = -1/\sqrt{2}\,\sin\theta\, e^{-i\varphi} \qquad D^{(1)}_{00} = \cos\theta \qquad D^{(1)}_{-10} = 1/\sqrt{2}\,\sin\theta\, e^{+i\varphi}$$

$$d^{(1)}_{10} = -1/\sqrt{2}\,\sin\theta \qquad d^{(1)}_{00} = \cos\theta \qquad d^{(1)}_{-10} = 1/\sqrt{2}\,\sin\theta$$

which would seem to say that

$$Y_{11}(\theta,\varphi) = Y_1^1(\theta,\varphi) = \langle\theta\varphi|11\rangle = \sqrt{3/4\pi}\;\; D^{(1)*}_{10}(\varphi,\theta,0) = -\sqrt{3/8\pi}\;\; \sin\theta\, e^{+i\varphi}$$

This agrees with Jackson p 66 and with the PDG sheet above.

Now let's go back to our direct product D function result above

$$D^{j1}_{m1,m1'}\, D^{j2}_{m2,m2'} = \Sigma_{jmm'}\, \langle m1m2|jm\rangle D^{(j)}_{mm'}\langle jm'|m1'm2'\rangle$$

and specialize by setting $m_{1'} = m_{2'} = 0$ so we get

$$D^{j1}_{m1,0}\, D^{j2}_{m2,0} = \Sigma_{jmm'} \langle m1m2|jm\rangle D^{(j)}_{mm'} \langle jm'|00\rangle$$

$$= \Sigma_{jm} \langle m1m2|jm\rangle D^{(j)}_{m0} \langle j0|00\rangle$$

But from above we just showed that

$$Y_{jm}(\theta,\varphi) = \sqrt{(2j+1)/4\pi}\;\; D^{(j)*}_{m0}(\varphi,\theta,0)$$

$$\Rightarrow \qquad D^{(j)}_{m0}(\varphi,\theta,0) = \sqrt{4\pi/(2j+1)}\;\; Y^{*}_{jm}(\theta,\varphi)$$

So the above equation becomes

$$\sqrt{4\pi/(2j_1+1)}\;\; Y^{*}_{j1,m1}(\theta,\varphi)\, \sqrt{4\pi/(2j_2+1)}\;\; Y^{*}_{j2,m2}(\theta,\varphi)$$

$$= \Sigma_{jm} \langle m1m2|jm\rangle \sqrt{4\pi/(2j+1)}\;\; Y^{*}_{j,m}(\theta,\varphi) \langle j0|00\rangle$$

We can then remove all the asterisks since else is real to get

$$Y_{j1,m1}(\theta,\varphi)\, Y_{j2,m2}(\theta,\varphi) = \sqrt{(2j_1+1)(2j_2+1)}\,/4\pi\;\; \Sigma_{jm} \langle m1m2|jm\rangle \sqrt{4\pi/(2j+1)}\;\; Y_{j,m}(\theta,\varphi) \langle j0|00\rangle$$

$$= \sqrt{(2j_1+1)(2j_2+1)\,/4\pi}\;\; \Sigma_{jm} \langle m1m2|jm\rangle \sqrt{1/(2j+1)}\;\; Y_{j,m}(\theta,\varphi) \langle j0|00\rangle$$

$$= \sqrt{(2j_1+1)(2j_2+1)\,/4\pi}\;\; \Sigma_{j} \langle m1m2|j\ (m_1+m_2)\rangle \sqrt{1/(2j+1)}\;\; Y_{j,m1+m2}(\theta,\varphi) \langle j0|00\rangle$$

$$= \sqrt{(2j_1+1)(2j_2+1)\,/4\pi}\;\; \Sigma_{j} \langle j_1j_2m_1m_2|j\ (m_1+m_2)\rangle \sqrt{1/(2j+1)}\;\; Y_{j,m1+m2}(\theta,\varphi) \langle j0|j_1j_200\rangle$$

$$= \Sigma_{j} \sqrt{(2j_1+1)(2j_2+1)\,/[4\pi(2j+1)]}\;\; \langle j_1j_2m_1m_2|j\ (m_1+m_2)\rangle\, Y_{j,m1+m2}(\theta,\varphi) \langle j0|j_1j_200\rangle$$

and very happily this agrees with a formula I found on the web and pasted into my "spherical harmonics" document. I have left the C-G coefficients in their "natural order" here.

Now, what happens if you multiply the above by $Y_{j3,m3}(\theta,\varphi)$ and apply $\int d\Omega$ ? On the RHS we pick up

$$\int d\Omega\; Y_{j3,m3}(\theta,\varphi)\, Y_{j,m1+m2}(\theta,\varphi) = (-1)^{m3} \int d\Omega\; Y^{*}_{j3,-m3}(\theta,\varphi)\, Y_{j,m1+m2}(\theta,\varphi)$$

$$= (-1)^{m3}\;\; \delta_{j3,j}\, \delta_{-m3,m1+m2}$$

and let's now assume that $j_3$ is in the range of $j_1 \otimes j_2$. Then we get

$$\int d\Omega\, Y_{j1,m1}(\theta,\varphi)\, Y_{j2,m2}(\theta,\varphi)\, Y_{j3,m3}(\theta,\varphi)$$

$$= \sqrt{(2j_1+1)(2j_2+1)/[4\pi(2j_3+1)]}\ \langle j_1 j_2 m_1 m_2 | j_3\ (-m_3)\rangle \langle j_3 0 | j_1 j_2 00\rangle$$

$$= \sqrt{(2j_1+1)(2j_2+1)\,(2j_3+1)/4\pi}\ \langle j_1 j_2 m_1 m_2 | j_3\ (-m_3)\rangle\, (-1)^{m3} \langle j_3 0 | j_1 j_2 00\rangle\, (2j_3+1)^{-1}$$

Now using page 1056 Messiah, we know that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{j1-j2} / \sqrt{2j_3+1}\ \langle j_1 j_2 00 | J0\rangle$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\,j1-j2-m3} / \sqrt{2j_3+1}\ \langle j_1 j_2\ m_1 m_2 | j_3 (-m_3)\rangle$$

So then

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{-m3}\,(2j_3+1)^{-1}\langle j_1 j_2\ m_1 m_2 | j_3 (-m_3)\rangle \langle j_1 j_2 00 | J0\rangle$$

And so we have shown that

$$\int d\Omega\, Y_{j1,m1}(\theta,\varphi)\, Y_{j2,m2}(\theta,\varphi)\, Y_{j3,m3}(\theta,\varphi) =$$

$$\sqrt{(2j_1+1)(2j_2+1)\,(2j_3+1)/4\pi}\ \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

which agrees with another downloaded formula I found, also appears as C.16 in Messiah, and is manifestly symmetric, the benefit of those 3j symbols. If $j_3$ is out of range, I know the RHS = 0 and I guess the LHS = 0 as well. You would prove this just using the double Y formula above.

**Combining Three Angular momenta.**

Here are just some of the ideas that come to mind here:

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle$ in $H^{(j1\otimes j2\otimes j3)}$

$$J_i^{(j1\otimes j2\otimes j3)} = J^{(j1)}{}_i \otimes 1 \otimes 1 + 1 \otimes J^{(j2)}{}_i \otimes 1 + 1 \otimes 1 \otimes J^{(j3)}{}_i$$

$$R_i^{(j1\otimes j2\otimes j3)} = R_i^{(j1)} \otimes R_i^{(j2)} \otimes R_i^{(j3)}$$

You could first do this:

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \Sigma_{j \in j_1 \otimes j_2} \langle jm|j_1 j_2 m_1 m_2\rangle |jm\rangle \qquad m = m_1 + m_2$

Then bring in the third part of the direct product space

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle = \Sigma_{j \in j_1 \otimes j_2} \langle jm|j_1 j_2 m_1 m_2\rangle |jm\rangle \otimes |j_3 m_3\rangle$

and then write

$|jm\rangle \otimes |j_3 m_3\rangle = \Sigma_{j' \in j \otimes j_3} \langle j'm'|jj_3 m m_3\rangle |j'm'\rangle \qquad m' = m + m_3 = m_1 + m_2 + m_3$

so we end up with

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle = \Sigma_{j \in j_1 \otimes j_2} \Sigma_{j' \in j \otimes j_3} \langle jm|j_1 j_2 m_1 m_2\rangle\langle j'm'|jj_3 m m_3\rangle |j'm'\rangle$

$= \Sigma_{j \in j_1 \otimes j_2} \Sigma_{j' \in j \otimes j_3} \langle j(m_1 + m_2)|j_1 j_2 m_1 m_2\rangle\langle j'(m_1 + m_2 + m_3)|jj_3(m_1 + m_2)\rangle |j'(m_1 + m_2 + m_3)\rangle$

For example, let $j_1 = 1 \;\; j_2 = 2 \;\; j_3 = 1$. Then the outer sum is j = 1,2,3. So define,'

$f(j1,j2,j3,j,j') = \langle jm|j_1 j_2 m_1 m_2\rangle\langle j'm'|jj_3 m m_3\rangle |j'm'\rangle$

and write out the above double sum thing in manual detail,

$\Sigma_{j \in j_1 \otimes j_2} \Sigma_{j' \in j \otimes j_3} \; f(j1,j2,j3,j,j')$

$= \Sigma_{j=1,2,3} \Sigma_{j' \in j \otimes j_3} \; f(j1,j2,j3,j,j')$

$= \Sigma_{j' \in 1 \otimes j_3} \; f(j1,j2,j3,1,j') + \Sigma_{j' \in 2 \otimes j_3} \; f(j1,j2,j3,2,j') + \Sigma_{j' \in 3 \otimes j_3} \; f(j1,j2,j3,3,j')$

$= \Sigma_{j'=0,1,2} f(j1,j2,j3,1,j') + \Sigma_{j'=1,2,3} \; f(j1,j2,j3,2,j') + \Sigma_{j'=2,3,4} f(j1,j2,j3,3,j')$

Now if I write $\langle jm|j_1 j_2 m_1 m_2\rangle\langle j'm'|jj_3 m m_3\rangle |j'm'\rangle = f(j1,j2,j3,1,j') = g(j1,j2,j3,1,j')|j'm'\rangle$ I get

$= \Sigma_{j'=0,1,2} g(j1,j2,j3,1,j') |j'm'\rangle + \Sigma_{j'=1,2,3} \; g(j1,j2,j3,2,j') |j'm'\rangle + \Sigma_{j'=2,3,4} g(j1,j2,j3,3,j') |j'm'\rangle$

$= \Sigma_{j'=0,1,2} g(j1,j2,j3,1,j') |j'm'\rangle + \Sigma_{j'=1,2,3} \; g(j1,j2,j3,2,j') |j'm'\rangle + \Sigma_{j'=2,3,4} g(j1,j2,j3,3,j') |j'm'\rangle$

$= \quad |0m'\rangle \; g(j1,j2,j3,1,0)$

$+ |1m'\rangle \{ g(j1,j2,j3,1,1) + g(j1,j2,j3,2,1) \}$

$+ |2m'\rangle \{ g(j1,j2,j3,1,2) + g(j1,j2,j3,2,2) + g(j1,j2,j3,3,2) \}$

$+ |3m'\rangle \{ g(j1,j2,j3,2,3) + g(j1,j2,j3,3,3) \}$

+ |4m'> g(j1,j2,j3,3,4)

So OK, the things you see above are the "triple coupling Clebsch coefficients. This subject is pretty messy and involves the **6j symbols** or the **Racah W coefficients**. You can couple 4 together and use the **9j symbols** in Messiah page 1066. I suspect this is presented better elsewhere. I don't care much about this right now, just wanted to take a quick look at it.

**A Tour of Books I own on these subjects**

**1. Messiah has several different sections.**

**Chapter 13** is dedicated to the subject of angular momentum, about 75 pages! After long and detailed sections on the basics, we get to "addition of angular momenta" on page 555. He does not use any direct product notation here, just regular + signs. Page 560 has the C-G coefficients, and their recursion relations. As always, theory is interspersed with applications. Page 566 starts into adding 3 angular momenta. Then on page 569 we start the tensor operator stuff including the WE theorem. Then problems.

**Appendix C** then seems to start all over with the vector addition idea, giving all the fancy formulas. I derived several of these things above, such as C.16 and C17 and C.18 and C.19. You can change J just using a $J_-$ both directions, but I did not do this above. The normal add 2 gives the C-G and 3j symbols. Add 3 and you have the Racah and 6j. Add 4 and you have the 9j symbols. He then starts talking about the rotations on page 1068 instead of just the generators. He refers to Tinkham's D matrices as R matrices in this section. The signs in exponents all seem PL-standard. Page 1072 has the DD formula and its reverse. The famous Wigner formula is stated as C.72 for the R matrices. On page 1074 he does a case showing how you "rotate" the $Y_{jm}$. Then he is on to tensor operators again with fancy cases.

**Appendix D** is on group theory in a very general way (I have not read this stuff in full). He then wanders into the Permutation group and the Young Tableaux. It is all here for when I need it.

Messiah of course happily uses bra/ket notation everywhere.

**2. Tinkham.**

His whole book is on QM group theory. Chapter 5 page 94 is on SO(3). He derives the Wigner formula for the D functions. On page 115 he starts his "vector addition" section. But for me, a weakness of this book is no use of bra/ket notation. Hilbert Space does not appear in the index. His equation 5-41a on page 117 is like my

$$R_i^{(1)} \otimes R_i^{(2)} = U^{(1\otimes 2)-1} \; [R_i^{(3)} \oplus R_i^{(2)} \; \oplus R_i^{(1)}] \; U^{(1\otimes 2)}$$

above, so it would appear that his A and my U are the same. He uses M for the direct sum matrix.

Since he has no Hilbert Space, he cannot say things like R |jm> to express a rotation. Instead he does things like this: (see 5-15 page 102)

$$P_R \, \varphi_{jm}(\mathbf{r}) = P_R \, \langle \mathbf{r}|jm\rangle = \langle \mathbf{r} \,|R\,|jm\rangle \; = \varphi_{jm}(R^{-1}\mathbf{r}) \; = \; \langle \mathbf{r} \,|R\,|jm\rangle = \langle \mathbf{r}|jm'\rangle\langle jm'|R|\, jm\rangle$$

$= \varphi_{jm'}(\mathbf{r})\, D^{j}_{m'm}(R)$ $= \hat{R}\,\langle \mathbf{r}|jm\rangle$ in one notation I sometimes use

This is confirmed way back on page 32 where $P_R$ is first used. He likes to think of it as making a new function like this

$$P_R\, \varphi_{jm}(\mathbf{r}) = \varphi_{jm}(R^{-1}\mathbf{r}) = \psi_{jm}(\mathbf{r}) = [P_R\varphi]_{jm}(\mathbf{r})$$

He wants it to act on functions rather than the coordinates. If he had kets to act on, he could ignore the coordinates. He of course never shows the coordinate, so you can pretty much make this identification:

$\psi_M^J(\mathbf{r}) = \langle \mathbf{r}|JM\rangle$ where we just ignore the r. $\psi_M^J = |JM\rangle$

But then you wonder how to interpret (5-42b). He is forced to this clumsy notation

$\langle \mathbf{r}|\ |j_1m_1\rangle \otimes |j_2m_2\rangle = u_{m1}(\mathbf{r})\, v_{m2}(\mathbf{r})$ $|j_1m_1\rangle \otimes |j_2m_2\rangle = u_{m1}\, v_{m2}$

What he is trying to say in 5-42b is simply this:

$|j_1m_1; j_2m_2\rangle = |JM\rangle\langle JM\,|j_1m_1; j_2m_2\rangle$ $\langle JM\,|j_1m_1; j_2m_2\rangle = A_{JM.m1m2} = U_{JM,m1m2}$

Then to add to the confusion he writes

$$\langle JM\,|j_1m_1; j_2m_2\rangle = A_{JM.m1m2} = U_{JM,m1m2} = \delta_{M,m1+m2}\, A_{J(m1+m2),m1m2} = \delta_{M,m1+m2}\, a_{Jm1m2}$$

and now we have $a_{Jm1m2}$ flying around. Then on page 121 we have still new notation

$$A^{j1j2J}_{m1m2M} = a^{(j1j2)}_{m1m2} = \langle J(m_1+m_2)|\, j_1j_2m_1m_2\rangle$$

Again, the bra-ket notation simply shines clarity light on what you are doing, and not using it obscures what you are doing! His book is 1964, similar to Messiah. He does seem to derive that Wigner formula for the CG coefficients with some group fiddling on page 120.

Then he is off on the 3j symbols, but first we have a V symbol on page 122 so now we have about 4 different notations for our C-G friends. And then we get into tensor operators with no commutators and no kets! But wait, a commutator appears on page 130. And he does that Levitt dipole thing, but he never uses the word "quadrupole". Finally on page 131 we see some bras and kets, but he uses ( | just to be obnoxious.

Then he is off on the Racah's with an actual application.

Tinkham is a Harvard emeritus right now. He wrote two books on superconductivity, and this book I have went Dover in 2003 says wiki.

**3. Schiff in his Symmetry Chapter 7**

Page 194 starts rotations and we are happy with bras and kets. He shows some generator matrices on page 203 and you see the first off-diagonal stuff with $J_y$ all imaginary. He shows the angle differential operators for a few things. Then page 212 starts "combination of angular momentum states", a light touch

on this stuff, no Wigner formula, no 3j symbols. Then onto tensor operators and a quick arm waving proof on the WET. I have read and annotated this entire chapter. But I don't see any check marks on equations, so I must have done a scan only.

**4. M&M have very little**

This 1943/1956 book has a chapter on group theory, but hardly mentions $Y_{jm}$. Clebsch is not in the index. This huge multi-topic book has no room for this kind of stuff.

**5. My Group Theory Books**

I have lots of these and they all have lots of good stuff, but now is not the time.