Angular Momentum Formulas PhL 1.23.08

$\hbar = 1$ everywhere

$[J_i, J_k] = i\varepsilon_{jk\ell}J_\ell$ $\quad [J_z, J_\pm] = \pm J_\pm$ $\quad J_\pm = J_x \pm iJ_y$ $\quad J_\mp J_\pm = J^2 - J_z^2 \mp J_z$

$\mathbf{J \times J} = i\mathbf{J}$ $\quad J_x = \frac{1}{2}(J_+ + J_-)$ $\quad J_y = \frac{1}{2i}(J_+ - J_-)$ $\quad J_\pm|m\rangle = \sqrt{j(j+1)-m(m\pm1)}\,|m\pm 1\rangle$

$\hat{L}_\pm \langle\theta\phi|\ell m\rangle = [\hat{L}_\pm, \langle\theta\phi|\ell m\rangle]$ // equation in C , not space of functions, so $\langle\theta\phi|\ell m\rangle \hat{L}_\pm 1 = 0$

$\hat{L}_\pm \langle\theta\phi|\ell m\rangle = \langle\theta\phi|L_\pm|\ell m\rangle = \langle\theta\phi|\sqrt{\ell(\ell+1)-m(m\pm1)}\,|\ell\, m\pm1\rangle = \sqrt{\ell(\ell+1)-m(m\pm1)}\ \langle\theta\phi|\ell\, m\pm1\rangle$

$Y_{\ell m}(\theta,\phi) = \langle\theta\phi|\ell m\rangle$ // spherical harmonics

$a\bullet b = a_z b_z + (1/2)[a_- b_+ + a_+ b_-]$ given that $a_\pm \equiv a_x \pm ia_y$ and $b_\pm \equiv b_x \pm ib_y$, and **a b** vectors

Tensor Operators

$[J_\pm, A_{\ell,m}] = \sqrt{j(j+1)-m(m\pm1)}\ A_{\ell,m\pm1}$ $\quad [J_3, A_{\ell,m}] = m A_{\ell,m}$

Cartesian Vector Operators

$\mathbf{J \times V} = i\mathbf{V}$

$R = R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J})$ $\quad RVR^{-1} = R^{-1}\mathbf{V}$ $\quad [J_i, V_j] = i\varepsilon_{ijk}V_k$

$R = R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{\mathbf{n}}\bullet J)$ $\quad [J_i]_{jk} = -i\,\varepsilon_{ijk}$ $\quad [J_\pm, V_j] = [i\varepsilon_{xjk} \mp \varepsilon_{yjk}]V_k$

Spherical Vector Operators and relation to Cartesian

$[J_+, V_{1,m}] = \sqrt{2-m(m+1)}\ V_{1,m+1}$ $\quad [J_\pm, V_x] = \pm V_{1,0}$

$[J_-, V_{1,m}] = \sqrt{2-m(m-1)}\ V_{1,m-1}$ $\quad [J_\pm, V_y] = +i\,V_{1,0}$

$[J_z, V_{1,m}] = m\,V_{1,m}$ $\quad [J_\pm, V_z] = +\sqrt{2}\,V_{1,\pm1}$

$[J_+, V_{1,1}] = 0$ $\quad [J_+, V_{1,-1}] = \sqrt{2}\,V_{1,0}$ $\quad [J_+, V_{1,0}] = \sqrt{2}\,V_{1,1}$

$[J_-, V_{1,1}] = \sqrt{2}\,V_{1,0}$ $\quad [J_-, V_{1,-1}] = 0$ $\quad [J_-, V_{1,0}] = \sqrt{2}\,V_{1,-1}$

$V_{1,1} = -(1/\sqrt{2})(V_x + iV_y)$ $\quad V_x = -(1/\sqrt{2})[V_{1,1} - V_{1,-1}]$

$V_{1,-1} = (1/\sqrt{2})(V_x - iV_y)V_y = -(1/\sqrt{2}\,i)[V_{1,1} + V_{1,-1}]$

$V_{1,0} = V_z$ $\quad V_z = V_{1,0}$

Angular Momentum differential operators in Spherical Coordinates

$L_x = i[S_\phi \partial_\theta + \cot\theta\, C_\phi \partial_\phi]$ $\quad L_\pm = e^{\pm i\phi}[\pm\partial_\theta + i\,\cot\theta\,\partial_\phi]$

$L_y = -i[C_\phi \partial_\theta - \cot\theta\, S_\phi \partial_\phi]$ $\quad L^2 = -[\partial_\theta^2 + \cot\theta\,\partial_\theta + (1/\sin\theta)^2\partial_\phi^2]$

$L_z = -i\,\partial_\phi$ $\quad L^2 = -[(1/S\theta)\partial_\theta[S\theta\,\partial_\theta] + (1/\sin\theta)^2\partial_\phi^2] = -r^2\nabla^2_\Omega$