Confusion about rotation operators PhL 1.19.08

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See "Summary of this Document" at the end of this document! No need to read stuff before that summary unless you want to see derivations.

CONTEXT: The states $|\alpha\rangle$ are the $|\ \tfrac{1}{2}\ ;\pm\ \tfrac{1}{2}\ \rangle$ states which represent a spin-1/2 being up or down, and the letter **I** refers to the Pauli matrices as normalized in Levitt, which is to say, $\mathbf{I} = \tfrac{1}{2}\,\boldsymbol{\sigma}$. Thus, the matrices I are the 2x2 generators of the rotation group.

Contents of this document:

1. Vector Operators: Wrong Version. Did not realize that $\langle\alpha|\ \mathbf{I}\ |\alpha\rangle = \langle\alpha'|\ \mathbf{I'}\ |\alpha'\rangle$ nor did I understand what the primes on things really meant in terms of rotations. Am keeping this section for history only.

2. Comments on the above: "Parsed" the words of Messiah on his page 525, and was able then to express in bracket and operator notation three "situations": observer rotates forward, experiment rotates backward, and both rotate forward.

3. Vector Operators: Correct Version. Armed with the information of part 2 above, I redid part 1 here. All the contradictions went away, everything makes sense.

4. Exact details on rotating vector operators. By assuming that $\mathsf{R}\ \mathbf{I}\ \mathsf{R}^{-1} = \mathsf{R}^{-1}\ \mathbf{I}$, I evaluate the generators of R and find them to be $(\mathsf{J}_i)_{jk} = -i\ \varepsilon_{ijk}$. I then write down the three generators and the three finite rotations, and these rotations agree with my historical "active rotation matrices" from days of old. I have the general rotation at the end of a different document.

5. Theorem: I restate as a theorem the fact that $\mathsf{R}\ \mathbf{I}\ \mathsf{R}^{-1} = \mathsf{R}^{-1}\ \mathbf{I}$. But of course I now know that this is just the definition of how a Cartesian vector operator must transform, and **I** is just an example. As an application, I derive the "substitution formulas" shown page 287 of Levitt involving the density matrix.

6. **Matrices**. Here I write down explicit matrices for the three rotations R and the three generators J.

7. **Summary**. Here I summarize the important results from all the previous sections, and I add more notes on how to tell what the spin-1/2 density matrix does under a rotation.

Comments: I was reading Messiah and was trying to understand how you do things like "rotate the observation instruments" or "rotate the experiment" in the language of bras, kets and operators.

In my first section below, called "wrong version", I made a total mess because I did not realize that if you prime everything, the object $\langle\alpha|\ \mathbf{I}\ |\alpha\rangle$ does not change at all.

In the second section below "comments on the above", I looked harder at Messiah and was able to understand the three "situations" described there.

1. Vector Operators: Wrong Version

I will first present this as I originally did, then comment at the end why it is wrong"

Now write

$$\mathbf{M} = \langle\alpha|\ \mathbf{I}\ |\alpha\rangle$$

Note that **M** is a vector like **A** above. So in the rotated frame we have

$$\mathbf{M'} = \langle\alpha'|\ \mathbf{I'}\ |\alpha'\rangle \ = R\ \mathbf{M}$$

Certainly we still have $|\alpha'\rangle = R\ |\alpha\rangle$ as in the previous example, so we now have

$$R\mathbf{M} \ = \ \langle\alpha|\ R^{-1} \quad \mathbf{I'} \quad R\ |\alpha\rangle$$

This equation is self consistent provided

$$R^{-1}\ \mathbf{I'}\ R = R\ \mathbf{I}$$

because we then get

$$R\mathbf{M} \ = \ \langle\alpha|\ R^{-1}\ \mathbf{I'} \quad R\ |\alpha\rangle \ = \ \langle\alpha| \ \ R\ \mathbf{I}\ \ |\alpha\rangle \ = \ R \ \ \langle\alpha|\ \mathbf{I}\ |\alpha\rangle = R\ \mathbf{M}$$

Now as a special case of the above equation, we can write

$$R^{-1}\ I'_k\ R = [R\ \mathbf{I}]_k \ = (R)_{k\ell}\ \ I_\ell$$

We can then move the operators to the other side to get

$$I'_k \ = (R)_{k\ell}\ R I_\ell\ R^{-1}$$

and we conclude that the vector operator I must transform under rotations as follows:

$$\mathbf{I'} = R\ (R\ \mathbf{I}\ R^{-1})$$

2. Comments on the above:

(1) Luckily for me, this subject with the same sense of rotation is discussed very carefully in Messiah starting on page 525. He first discusses the frame change which I called S' = R^{-1}S. In the unprimed frame, the unit vectors are the $\mathbf{a}_i$ where i is not a component index, it is a unit vector label. In the primed system the unit vectors are $\mathbf{A}_i$. Now before going on, I have to clarify what he is saying here, it is confusing. I would have said it this way:

"Start with some unit vectors $\mathbf{a}^{(i)}$ where we have $\mathbf{a}^{(i)}{}_k = \delta_{ik}$. Now define some new rotated unit vectors according to $\mathbf{A}^{(j)} = R\,\mathbf{a}^{(j)}$. In components we get $\mathbf{A}^{(j)}{}_i = R_{ik}\,\mathbf{a}^{(j)}{}_k = R_{ij}$. Now consider that $\mathbf{a}^{(i)} \bullet \mathbf{A}^{(j)} = \delta_{ik}\ \mathbf{A}^{(j)}{}_k = \mathbf{A}^{(j)}{}_i = R_{ij}$ which agrees with Messiah. Now the next step is this:

$$\mathbf{A}^{(j)} = (\mathbf{a}^{(i)} \bullet \mathbf{A}^{(j)})\,\mathbf{a}^{(i)} = \mathbf{a}^{(i)} R_{ij} = (R^{-1})_{ji}\ \mathbf{a}^{(i)}$$

Here we express A as a sum of its projections on the unit vectors a. We then replicate Messiah's other result, and also "my result" which says you involve the "inverse rotation" if you use unit vector labels in place of vector component labels. Now, we wrote $\mathbf{A}^{(j)} = R\ \mathbf{a}^{(j)}$ but this is how *any* vector would transform, so we say that $\mathbf{V}' = R\mathbf{V}$ which agrees top page 526."

Now in light of this, I think it would make more sense to say " let's transform to a primed frame S' = RS " because the unit vectors in the primed frame are rotated by R compared to this in the unprimed frame. The fact that $\mathbf{A}^{(j)} = (R^{-1})_{ji}\ \mathbf{a}^{(i)}$ is true but not relevant because it is not a standard rotation form.

(2) Now, suppose in the unprimed frame we have some NMR sample (the experiment) and we observe (the observer, or measurement apparatus) some result $\mathbf{M} = \langle\alpha|\ \mathbf{I}\ |\alpha\rangle$. If we rotate the experiment but not the observer, or vice versa, then we will measure some $\mathbf{M}' \neq \mathbf{M}$. But if we rotate both at the same time, then I think we will have $\mathbf{M}' = \mathbf{M}$. For example, suppose in the unprimed system we measure $\mathbf{M} = 5\,\hat{z}$. Then surely in the rotated system we will measure $\mathbf{M}' = 5\,\hat{z}'$.

But now this confuses me. How can we say $\mathbf{M}' = \mathbf{M}$ and at the same time $\mathbf{V}' = R\mathbf{V}$ **?** Is Messiah on page 529 talking only about a scalar operator Q??

Alas, it is always the simplest questions in QM that are the hardest to understand! At least I have clarified that idea that we rotate our unit vectors.

(3) Another chunk of this same Messiah section seems to make some sense. We should have

$$\langle \mathbf{r}'|\psi'\rangle = \langle \mathbf{r}|\psi\rangle \quad \text{where } \mathbf{r}' = R\mathbf{r}$$

In the new frame, the new rotated wavefunction ψ' evaluated at the rotated position $\mathbf{r}'$ should be the same as the unrotated wavefunction evaluated at the unrotated position. Thus you would say

$$\psi'(R\mathbf{r}) = \psi(\mathbf{r}) \qquad \text{or} \qquad \psi'(\mathbf{r}) = \psi(R^{-1}\mathbf{r})$$

which agrees with top of Messiah page 528. Based on this fact I think we can conclude these facts:

$|\psi'\rangle = R|\psi\rangle$ and $|\mathbf{r}'\rangle = R|\mathbf{r}\rangle$

because then it is trivial to obtain $\langle\mathbf{r}'|\psi'\rangle = \langle\mathbf{r}|\psi\rangle$. The item on the left agrees with Messiah (50) on page 528. This means the right item must be correct. And we can write it as

$$R|\mathbf{r}\rangle = |\mathbf{r}'\rangle = |R\mathbf{r}\rangle$$

which might be useful. With what we have so far, we know that $\langle\alpha|\ \mathbf{I}\ |\alpha\rangle = \langle\alpha'|\ R\mathbf{I}R^{-1}\ |\alpha'\rangle$. I have always thought that you should, at this point, say that $R\mathbf{I}R^{-1} = \mathbf{I}'$ and then $\langle\alpha|\ \mathbf{I}\ |\alpha\rangle = \langle\alpha'|\ \mathbf{I}'\ |\alpha'\rangle$, but then we are back to the issue of $\mathbf{M} = \mathbf{M}'$.

This needs to be resolved now or I am stuck forever. Suppose α is an up spin state, while α' is an up state relative to a rotated axis. And $\mathbf{I}'_z$ is the rotated z spin operator. It seems awfully reasonable to claim for example that $\langle\alpha|\ \mathbf{I}_z\ |\alpha\rangle = \langle\alpha'|\ \mathbf{I}'_z\ |\alpha'\rangle$. The rotated operator points in the $\hat{z}'$ direction and our expectation value should be unchanged. It is the number 1/2 in either case.

$$\langle\alpha|\ \mathbf{I}_z\ |\alpha\rangle = \langle\alpha'|\ \mathbf{I}'_z\ |\alpha'\rangle = 1/2$$

Now suppose I write the above as

$$\langle\alpha|\ \mathbf{I}\bullet\hat{z}\ |\alpha\rangle = \langle\alpha'|\ \mathbf{I}'\bullet\hat{\mathbf{z}}'\ |\alpha'\rangle = 1/2$$

Then I could remove the vectors to get

$$\langle\alpha|\ \mathbf{I}\ |\alpha\rangle\ \bullet\ \hat{z} = \langle\alpha'|\ \mathbf{I}'\ |\alpha'\rangle\bullet\hat{\mathbf{z}}' = \langle\alpha'|\ \mathbf{I}'\ |\alpha'\rangle\bullet R\hat{z}$$

But this is now incompatible with $\langle\alpha|\ \mathbf{I}\ |\alpha\rangle = \langle\alpha'|\ \mathbf{I}'\ |\alpha'\rangle$. I need to look elsewhere for help, Messiah has petered out.

Tinkham page 125 makes this unambiguous statement:

If **I** is a vector operator, then it must transform like this: (his 5-55)

$$\mathbf{I}' = R\ \mathbf{I}\ R^{-1} = R^{-1}\mathbf{I}$$

I am hard-pressed to argue with this. It maintains the desire that $\mathbf{I}' = R^{-1}\mathbf{I}\ R$ is how you transform an operator to a rotated frame.

Now, let's see if I can escape from the trap I have put myself into.

<u>Situation #1:</u> Start with $\mathbf{M} = \langle\alpha|\ \mathbf{I}\ |\alpha\rangle$. Rotate the experiment "forward" but not the observing equipment, and call this new situation the primed situation. Then we have

$$\mathbf{M}' = \langle\alpha|\ \mathbf{I}'\ |\alpha\rangle = \langle\alpha|\ R\ \mathbf{I}\ R^{-1}\ |\alpha\rangle = \langle\alpha|\ R^{-1}\mathbf{I}\ |\alpha\rangle = R^{-1}\langle\alpha|\ \mathbf{I}\ |\alpha\rangle = R^{-1}\mathbf{M}.$$

Our observation is taken in state α but we have not moved our observing equipment, it is still thinking in terms of the z axis, α is the same state it was before and after we rotate the experiment. So this gives a reasonable consistency with Tinkham's claim.

Situation #2: Start with $\mathbf{M} = \langle\alpha|\ \mathbf{I}\ |\alpha\rangle$. This time, rotate the observation equipment "backwards" but not the experiment, and call this new situation the double primed situation. Because we rotate backwards, we have that $|\alpha''\rangle = R^{-1}|\alpha\rangle$. Then we have

$$\mathbf{M''} = \langle\alpha''|\ \mathbf{I}\ |\alpha''\rangle = \langle\alpha|\ R\ \ \mathbf{I}\ \ R^{-1}\ |\alpha\rangle\ =\ \langle\alpha|\ R^{-1}\ \mathbf{I}\ |\alpha\rangle\ = R^{-1}\langle\alpha|\ \mathbf{I}\ |\alpha\rangle\ = R^{-1}\mathbf{M}.$$

Our observation is taken in state α'' relative to a new z axis, so we expect our results to change. So this also gives a reasonable consistency with Tinkham's claim.

Now, we expect the two situations above to give exactly the same observation results, so we are not surprised to find that $\mathbf{M'} = \mathbf{M''}$.

Situation #3. This time, rotate both the experiment and the observation system "forward" and call this the ` situation (tick). In this case, our observation states will be $|\alpha`\rangle = R|\alpha\rangle$ and we have

$$\mathbf{M`} = \langle\alpha`|\ \mathbf{I}\ `\ |\alpha`\rangle\ \ =\ \ \langle\alpha\ |R^{-1}\ \ \ \ R\ \mathbf{I}\ R^{-1}\ \ \ \ R|\alpha\rangle\ \ =\ \ \langle\alpha|\ \mathbf{I}\ |\alpha\rangle\ = \mathbf{M}$$

Again, we expect this result that $\mathbf{M`} = \mathbf{M}$. I think this is what Messiah meant in the middle of page 529. So yes, his claim is true for a vector **Q** as well as a scalar Q.

So, now we see the error in our "Wrong Version" of analyzing the vector situation above. We know that

$$\langle\alpha`|\ \mathbf{I}\ `\ |\alpha`\rangle\ =\ \langle\alpha|\ \mathbf{I}\ |\alpha\rangle$$

when we rotate everything. This is what happens when you take a fixed experiment and a fixed piece of observation equipment and you leave them both put and you just change your frame of reference. So let's rewrite the above as follows:

3. Vector Operators: Correct Version

Now write

$$\mathbf{M} = \langle\alpha|\ \mathbf{I}\ |\alpha\rangle$$

So in the rotated frame we have

$$\mathbf{M}\ ` = \langle\alpha`|\ \mathbf{I}\ `\ |\alpha`\rangle\ \ = \langle\alpha|\ \mathbf{I}\ |\alpha\rangle\ = \mathbf{M}$$

Certainly we still have $|\alpha`\rangle = R\ |\alpha\rangle$ as in the previous example, so we now have

$$\mathbf{M} = \langle\alpha| R^{-1} \quad \mathbf{I}\,` \quad R\,|\alpha\rangle$$

This equation is self consistent provided

$$R^{-1}\,\mathbf{I}\,`\,R = \mathbf{I} \qquad \text{meaning} \quad \mathbf{I}\,` = R\,\mathbf{I}\,R^{-1} \quad \text{as in Tinkham}$$

because we then get

$$\mathbf{M}\,` = \langle\alpha| R^{-1}\ \mathbf{I}' \quad R\,|\alpha\rangle = \langle\alpha| \ \mathbf{I}\ |\alpha\rangle = \mathbf{M}$$

and we conclude that the vector operator I must transform under rotations as follows:

$$\mathbf{I}' = R\,\mathbf{I}\,R^{-1}$$

I am sure this is all correct. Here is further confirmation from Messiah page 529 top:

$$\rho` = \Sigma_{\boldsymbol{\alpha}}\,|\alpha`\rangle p_{\boldsymbol{\alpha}}\langle\alpha`| = \Sigma_{\boldsymbol{\alpha}}\,R\,|\alpha\rangle p_{\boldsymbol{\alpha}}\langle\alpha|R^{-1} = R\rho R^{-1}.$$

Then we have

$$\mathbf{M}` = \mathrm{tr}(\rho`\,\mathbf{I}\,`) = \mathrm{tr}(\,R\rho R^{-1}\,R\mathbf{I}R^{-1}) = \mathrm{tr}(\rho\mathbf{I}) = \mathbf{M}$$

But what *words* do we now use to explain that fact that, although **M** is a vector (and not even an operator), when we go into our rotated frame of reference, we don't get $\mathbf{M}\,` = R^{-1}\mathbf{M}$, but instead we get $\mathbf{M}\,` = \mathbf{M}$? Well, in the tick situation, we have rotated the vectors that describe our physical experiment (such as the locations of pieces of the experimental equipment) but we have also rotated the coordinate system in which we measure those vector. After doing both of these things, there is no change. We have

$$\mathbf{M}\,` = M_{\mathbf{i}}` \ \hat{\mathbf{n}}\,`_{\mathbf{i}} = \mathbf{M} = M_{\mathbf{i}}\,\hat{\mathbf{n}}_{\mathbf{i}}$$

Now if we were to only rotate the physical experiment but maintain our same coordinate system, then we would have the $\mathbf{M}' = R^{-1}\mathbf{M}$ situation #1 and then the vectors transform in the usual vector manner. This also happens in situation #2 where we just observe the same experiment from a rotated frame of reference, then the vectors describing locations of pieces of the equipment "rotate as usual".

OK, I think I am now happy with this painful situation.

4. Exact details on rotating vector operators.

In the above, we arrive at this conclusion

$$\mathbf{I}' = R\ \mathbf{I}\,R^{-1} = R^{-1}\,\mathbf{I}$$

and this appears in Tinkham. I would like to verify that this is correct with my usual formulas for rotations. Let's assume something for R, and see what R comes out to be!

Let R = exp(-iθ $\hat{n}$ • **I**) and let R = exp(-iθ $\hat{n}$ • J) Then do differential to get

$$(1 - i\theta\, \hat{n} \bullet \mathbf{I})\mathbf{I}(1 + i\theta\, \hat{n} \bullet \mathbf{I}) = (1 + i\theta\, \hat{n} \bullet \mathrm{J})\, \mathbf{I}$$

The θ term then says

$$-i\theta\, [\hat{n} \bullet \mathbf{I}, \mathbf{I}] = i\theta\, \hat{n} \bullet \mathrm{J}\, \mathbf{I}$$

But we know that $[\hat{n} \bullet \mathbf{I}, \mathbf{I}] = i\, \mathbf{I} \times \hat{n}$ so we then have

$$+\theta\, \mathbf{I} \times \hat{n} = i\theta\, \hat{n} \bullet \mathrm{J}\, \mathbf{I} \qquad \Rightarrow \qquad \mathbf{I} \times \hat{n} = i\, \hat{n} \bullet \mathrm{J}\, \mathbf{I} \;\Rightarrow\; 0 = \hat{n} \times \mathbf{I} + i\, \hat{n} \bullet \mathrm{J}\, \mathbf{I}$$

In component k this says

$$0 = \varepsilon_{ijk} n_j I_k + i\, n_j\, (J_j)_{ik} I_k = [\, \varepsilon_{ijk} + i\, (J_j)_{ik}\,]\, n_j I_k$$

But since k is arbitrary and so is $\hat{n}$, we must have

$$0 = \varepsilon_{ijk} + i\, (J_j)_{ik} \;\Rightarrow\; i\, (J_j)_{ik} = -\varepsilon_{ijk} = +\varepsilon_{jik} \;\Rightarrow\; (J_j)_{ik} = -i\, \varepsilon_{jik}$$

where indices are in the same order. So write again in easier choice

$$(J_i)_{jk} = -i\, \varepsilon_{ijk}$$

Now here is our question: is this result compatible with our 547 notes where I say this:

$$R_z(\phi) = \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{so generator becomes} \qquad J_z = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix}$$

$$R_x(\phi) = \begin{pmatrix} 1 & & \\ & C & -S \\ & S & C \end{pmatrix} \qquad \text{so generator becomes} \qquad J_x = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}$$

$$R_y(\phi) = \begin{pmatrix} C\theta & & S\theta \\ & 1 & \\ -S\theta & & C\theta \end{pmatrix} \qquad \text{so generator becomes} \qquad J_y = \begin{pmatrix} & & i \\ & & \\ -i & & \end{pmatrix}$$

You see here that the +i position corresponds to the sinθ position. Well, here we go. First index is row, second is column. So looking at J_z we have $(J_z)_{21} = i$. Our formula says $(J_z)_{21} = -i\, \varepsilon_{321} = +i$ so that one agrees. Next look at J_x to get: $(J_x)_{32} = i$. Our formula says $(J_1)_{32} = -i\, \varepsilon_{132} = +i$, again OK. Finally we have $(J_y)_{13} = i$ and formula says $(J_y)_{13} = -i\, \varepsilon_{213} = +i$, so all agree.

I have now proved the following theorem:

5. Theorem:

$R\ \mathbf{I}\ R^{-1} = \mathsf{R}^{-1}\ \mathbf{I}$ where $R = \exp(-i\theta\ \hat{n} \bullet \mathbf{I})$ and let $\mathsf{R} = \exp(-i\theta\ \hat{n} \bullet \mathsf{J})$ are the "usual rotations" we always use. The rotations R are those shown on my "active rotations" page and also above.

Application: Let's look at Levitt page 296. We are talking in general about pulses $(\beta)_{\phi}$ which are controlled by the rotations $R_{\phi}(\beta) = R_z(\phi)\ R_x(\beta)R_z(-\phi) \equiv R$. as given on page 264 (9.32) which I seem to have a red check on so I checked it. Now from (10.27) page 287 we have $\rho' = R\rho R^{-1}$ which is in our standard form. Suppose we start off with $\rho = 1/2 + \mathbf{M}\bullet\mathbf{I}$ where M is normalized in some manner. then we know that $\rho' = R\rho R^{-1} = 1/2 + \mathbf{M}\bullet R\mathbf{I}R^{-1} = 1/2 + \mathbf{M}\bullet \mathsf{R}^{-1}\ \mathbf{I}$. So we can "see what happens to the density matrix" by making the substitution $\mathbf{I} \rightarrow \mathsf{R}^{-1}\ \mathbf{I}$ in ρ.

As our first example, suppose $\phi = 0$ so we are doing $(\beta)_x$. Then $R = R_x(\beta)$ so we want $I \rightarrow \mathsf{R}_x(-\beta)\ \mathbf{I}$ and I do this on paper and I get exactly as shown in (10.33) so it gets a red check.

Now lets try $\phi = \pi/2$. Then we have

$$R = R_z(\pi/2)\ R_x(\beta)R_z(-\pi/2)$$

To compute this rotation, we can sandwich the z rotations around I_x and use this fact

$$R_z(\pi/2)\ I_x\ R_z(-\pi/2) = [\mathsf{R}^{-1}\ \mathbf{I}]_x = [\mathsf{R}_z(-\pi/2)\ \mathbf{I}]_x = I_y \qquad \text{// did this on paper}$$

So we really have $R = R_y(\beta)$, but we already know that of course, just checking. Now we use our rule that $\mathbf{I} \rightarrow \mathsf{R}^{-1}\ \mathbf{I}$ in ρ and we get $\mathbf{I} \rightarrow \mathsf{R}_y(-\beta)\mathbf{I}$ and my paper result exactly agrees with (10.34), so another red check.

Next, let's try $\phi = \pi$. We know we end up with $R = R_{-x}(\beta) = R_x(-\beta)$ so we end up with just a sign change between (10.33) and (10.35). Then a similar sign change between (10.34) and (10.36). So we now have red checks on all equations on page 296.

Now move to page 297. This time we have $R = R_z(\phi)$ which makes $\mathsf{R}_z(-\phi)$ I and paper then shows that (10.37) is exactly right for this one. The next box is exactly the same with $\phi \rightarrow \Omega$. So now at last I know what is going on for these two Levitt pages.

6. Side Trip. Let's try to get our matrices working better.

$$\begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{pmatrix} \quad \begin{pmatrix} C & 0 & S \\ 0 & 1 & 0 \\ -S & 0 & C \end{pmatrix} \quad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathsf{R_x}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad \mathsf{R_y}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad \mathsf{R_z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathsf{J_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \mathsf{J_y} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad \mathsf{J_z} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

I started a Word software notes on the equation editor, control-e is there, it works fine, see above.

7. Summary of this Document.

A. Interpretation: Consider $\mathbf{M} = \langle\alpha|\, \mathbf{I}\, |\alpha\rangle$. **M** is measured in the unprimed frame of reference, and the system being observed (we call the experiment) is also described in this frame.

(a) If we maintain the definition of our observation state $|\alpha\rangle$, but we rotate the physical experiment "forward" by rotation R, we get this result:

$$\mathbf{M'} = \langle\alpha|\, \mathbf{I'}\, |\alpha\rangle = \langle\alpha|\, \mathsf{R}\ \mathbf{I}\ \mathsf{R^{-1}}\ |\alpha\rangle = \langle\alpha|\ \mathsf{R^{-1}}\ \mathbf{I}\ |\alpha\rangle = \mathsf{R^{-1}}\langle\alpha|\, \mathbf{I}\, |\alpha\rangle = \mathsf{R^{-1}}\mathbf{M}.$$

(b) If we maintain the experiment's orientation, but rotate our observation state "backwards" to $|\alpha''\rangle = \mathsf{R^{-1}}|\alpha\rangle$, we get

$$\mathbf{M''} = \langle\alpha''|\, \mathbf{I}\, |\alpha''\rangle = \langle\alpha|\, \mathsf{R}\quad \mathbf{I}\quad \mathsf{R^{-1}}\ |\alpha\rangle = \langle\alpha|\, \mathsf{R^{-1}}\ \mathbf{I}\ |\alpha\rangle = \mathsf{R^{-1}}\langle\alpha|\, \mathbf{I}\, |\alpha\rangle = \mathsf{R^{-1}}\mathbf{M}.$$

As expected, **M''** = **M'** and these are a rotated version of **M**

(c) If we rotate the experiment forward, and also rotate the observation state forward so $|\alpha'\rangle = \mathsf{R}|\alpha\rangle$, then we get this result

$$\mathbf{M'} = \langle\alpha'|\, \mathbf{I'}\, |\alpha'\rangle = \langle\alpha|\mathsf{R^{-1}}\ \mathsf{R}\ \mathbf{I}\ \mathsf{R^{-1}}\ \mathsf{R}|\alpha\rangle = \langle\alpha|\, \mathbf{I}\, |\alpha\rangle = \mathbf{M}$$

This is equivalent to an overall change in reference frame in which both the experiment and the observation are described. This is not the same as the **M'** defined in item (a) above, but we want to use the prime on it anyway. Different animals.

B. **Theorem:** $\mathsf{R}\ \mathbf{I}\ \mathsf{R^{-1}} = \mathsf{R^{-1}}\ \mathbf{I}$ where $\mathsf{R} = \exp(-i\theta\, \hat{n} \bullet \mathbf{I})$ and let $\mathsf{R} = \exp(-i\theta\, \hat{n} \bullet \mathsf{J})$ are the "usual active rotations" we always use. The rotations R are those shown on my "active rotations" page and also above. This is the way *any vector operator* must transform under rotation.

C. The 3D generator matrix elements (which make the above theorem true) are given by

$(J_i)_{jk} = -i\,\varepsilon_{ijk}$ or

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and these generate our usual active rotation matrices

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

D. The density matrix is a scalar operator and transforms $\rho' = R\rho R^{-1}$, which is how every scalar operator must transform Notice that

$\mathbf{M} = \mathrm{tr}(\rho\mathbf{I})$

If we do case (c) above, and rotate both the experiment and the observer state, we get

$\mathbf{M}' = \mathrm{tr}(\rho'\mathbf{I}') = \mathrm{tr}(R\rho R^{-1} R\ \mathbf{I}\ R^{-1}) = \mathrm{tr}(\rho\mathbf{I}) = \mathbf{M}$

which is similar to result (c) above, but with statistics added.

E. Suppose we start off with $\rho = 1/2 + k\ \mathbf{M}\bullet\mathbf{I}$ where **M** is normalized as shown. Then we know that $\rho' = R\rho R^{-1} = 1/2 + \mathbf{M}\bullet R\mathbf{I}R^{-1} = 1/2 + \mathbf{M}\bullet R^{-1}\mathbf{I}$. So we can "see what happens to the density matrix" by making the substitution $\mathbf{I} \rightarrow R^{-1}\ \mathbf{I}$ in ρ to get ρ'. For various rotations R, this is what Levitt shows on page 296. These are substitutions you should make to see what each kind of rotation R does to the density matrix.

For example, for $(\pi/2)_x$ you have $R = R_x(\pi/2)$ and we replace $\mathbf{I} \rightarrow R_x(\pi/2)^{-1}\mathbf{I}$ in ρ to get ρ':

$$\begin{pmatrix} Ix \\ Iy \\ Iz \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-\pi/2) & -\sin(-\pi/2) \\ 0 & \sin(-\pi/2) & \cos(-\pi/2) \end{pmatrix} \begin{pmatrix} Ix \\ Iy \\ Iz \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} Ix \\ Iy \\ Iz \end{pmatrix} = \begin{pmatrix} Ix \\ Iz \\ -Iy \end{pmatrix}$$

so in particular, $I_z \rightarrow -I_y$ which we know is the right answer.

Suppose $\rho = 1/2 + k\ \mathbf{M}\bullet\mathbf{I}$ where $\mathbf{M} = (0,0,M_z)$. Then we get

$$\rho' = 1/2 + k\ \mathbf{M}\bullet R\mathbf{I}R^{-1} = 1/2 + k\ \mathbf{M}\bullet R^{-1}\mathbf{I} = 1/2 + k\ (0,0,M_z)\bullet(I_x\ I_z\ -I_y) = 1/2 - kM_zI_y$$

But here is an easier way to do this same thing if $\mathbf{M} = M_z\hat{\mathbf{z}}$ **:**

$$\rho' = 1/2 + k\ \mathbf{M}\bullet R\mathbf{I}R^{-1} = 1/2 + k\ \mathbf{M}\bullet (R^{-1}\mathbf{I}) = 1/2 + k\ (R\ \mathbf{M})\bullet\ \mathbf{I} = 1/2 + k\ M_z\ (R\ \hat{\mathbf{z}}\ )\bullet\ \mathbf{I}$$

$= 1/2 + k\, M_z\, (-\hat{y}) \bullet \mathbf{I} = 1/2 - kM_z\, I_y$

In this mode of thinking, you need only "compute" $R_x(\pi/2)\hat{z} = -\hat{y}$ which is just the trivial active rotation of a vector which you can do in your head or quickly with a simple picture.