Evaluate n dot J rotation of J PhL maybe 1.10.08

Problem 1: Is there some way to evaluate this thing:

$$\mathbf{Q} = \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;\;\mathbf{J}\;\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})$$

We can always write this:

$$\exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J}) = \exp(-i\,[\,\beta_x J_x + \beta_y J_y + \beta_z J_z\,]) \qquad \text{where } \beta_i = \theta n_i$$

but we cannot factor this into the product of three rotations because the operators don't commute!

Comment: I give two evaluations here, the second uses Campbell-Hausdorff and is more substantial. The first arrives at the answer by computing it in a special frame, then rotating out of that frame to a general frame. I think both methods are valid.

Method #1: I think we can do it by using some coordinate system gibberish. As we write this, we already have in mind that the components of J and of n are x,z,y components and $\hat{\mathbf{n}}$ is some arbitrary direction. Suppose we define a new set of coordinate axis x', y' and z' such that $\hat{\mathbf{n}}$ points in the z' direction, say. I think we have to be very specific about now we define these new coordinates. But for now, let's assume we have found a rotation such that

$$R\,\hat{\mathbf{n}} = \hat{\mathbf{z}}' \qquad R \text{ is a 3x3 matrix}$$

Now the equation $\mathbf{Q} = \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;\;\mathbf{J}\;\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})$ says vector = vector, and we can look at this in any coordinate system we want. If we have a vector **V** in the unprimed system, then we know that

$$\mathbf{V}' = R\mathbf{V}$$

For example, if $\mathbf{V} = \hat{\mathbf{n}}$, we find that $\mathbf{V}' = R\hat{\mathbf{n}} = \hat{\mathbf{z}}'$. So, lets start with

$$\mathbf{Q} = \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;\;\mathbf{J}\;\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})$$

and apply R to both sides. This R ix just a 3x3 matrix, not an operator, so I think it can pass through the exponential operators which are 3x3-space rotational scalars [huh??]. If so, we would have

$$\mathbf{Q}' = \exp(-i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;\;\mathbf{J}'\;\exp(+i\,\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})$$

and we can write out the scalars in the primed system as well

$$\mathbf{Q}' = \exp(-i\,\theta\,\hat{\mathbf{z}}'\bullet\mathbf{J}')\;\;\mathbf{J}'\;\exp(+i\,\theta\,\hat{\mathbf{z}}'\bullet\mathbf{J}')$$

$$= \exp(-i\,\theta\;\;J'_3)\;\mathbf{J}'\exp(+i\,\theta\;\;J'_3)$$

To evaluate this, use component k as follows:

$$Q'_k = \exp(-\,i\,\theta\;\; J'_3\,)\; J'_k \exp(+\,i\,\theta\;\; J'_3\,)$$

$$= J'_k \cos\theta + \varepsilon_{3k\ell} J'_\ell \sin\theta$$

according to Levitt top page 144. It agrees if we use k = 1 because then things are in standard A,B,C order and we have $\varepsilon_{3k\ell} = \varepsilon_{31\ell} = \delta_{\ell 2}$ so yes, a plus sign as Levitt shows. With k = 2 we will get a minus sign as expected. BUT: Levitt's result assumes that $k \neq 3$. If we want to make this result be true also for k=3 as well, we can add a little correction term,

$$= J'_k \cos\theta + \varepsilon_{3k\ell} J'_\ell \sin\theta \;\; + \;\; \delta_{k3}\,(1 - \cos\theta)\, J_k' \qquad \text{// no summation on k in last term!}$$

Then when k = 3, the we have $\varepsilon_{33\ell} = 0$ and then the cosines cancel and we get the right answer J_3.

However, in order to cast the above into vector notation , we have to cast the last term into something that is in fact a vector with component k and we don't have that as written above. So we write the factor of interest as,

$$\delta_{k3}\;\; J_k' \;=\; \delta_{k3}\;\; J_3' \;\; = \;\; \hat{\mathbf{z}}'_k \;\; (\; \hat{\mathbf{z}}' \bullet \mathbf{J}') \qquad \text{// since } \hat{\mathbf{z}}'_k = \delta_{k3} \text{ and } (\; \hat{\mathbf{z}}' \bullet \mathbf{J}') = J_3$$

Then we can write this in vector notation like so:

$$\mathbf{Q}' \;\; = \;\; \mathbf{J}' \cos\theta \;\; - \;\; (\; \hat{\mathbf{z}}' \;\; \mathrm{x} \;\; \mathbf{J}' \;)\;\; \sin\theta \;\; + \;\; \hat{\mathbf{z}}' \;(\; \hat{\mathbf{z}}' \bullet \mathbf{J}\,')\;\; (1 - \cos\theta)$$

(Detail: the kth component of $\hat{\mathbf{z}}'$ x $\mathbf{J}'$ is $\varepsilon_{km\ell}\, \hat{\mathbf{z}}'_m\, J'_\ell$ but $\hat{\mathbf{z}}'_m = \delta_{m3}$ so get $\varepsilon_{k3\ell} J'_\ell$.)

Now, this appears to be a 3-space vector equation. We should be able to write this in the unprimed system as follows:

$$\mathbf{Q} \;\; = \;\; \mathbf{J} \cos\theta \;\; - \;\; \hat{\mathbf{n}} \;\; \mathrm{x}\, \mathbf{J} \;\; \sin\theta \;\; + \hat{\mathbf{n}}\;(\; \hat{\mathbf{n}} \bullet \mathbf{J}\;)\;(1 - \cos\theta)$$

and this then appears to be our "evaluation". It is much simpler than I thought it would be.
So here then is our final result. [see written in more ways at the end of this document]

$$\mathbf{Q} = \exp(-\,i\,\theta\;\hat{\mathbf{n}} \bullet \mathbf{J})\;\; \mathbf{J}\; \exp(+\,i\,\theta\;\hat{\mathbf{n}} \bullet \mathbf{J}) \;\; = \mathbf{J} \cos\theta \;\; + \;\; \mathbf{J} \times \hat{\mathbf{n}} \;\; \sin\theta + \;\; \hat{\mathbf{n}}\;(\hat{\mathbf{n}} \bullet \mathbf{J})\;\; (1 - \cos\theta)$$

Note: the vector covariant form for the correction term was not obvious until I found it in the C-H method below.

Method #2: Let's try a separate proof using Campbell-Hausdorff. We have

$$\mathbf{Q} = \exp(-\,i\,\theta\;\hat{\mathbf{n}} \bullet \mathbf{J})\;\; \mathbf{J}\; \exp(+\,i\,\theta\;\hat{\mathbf{n}} \bullet \mathbf{J})$$

so define

$A = i\,\theta\,\hat{\mathbf{n}} \bullet \mathbf{J} = i\theta\, n_i\, J_i$ and
$B = J_k$

We now need to compute a series of commutators.

$$C_1 = [B,A] = [\, J_k \,, i\,\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}\,] = i\,\theta\,\hat{\mathbf{n}} \bullet [\, J_k, \mathbf{J}]$$

But we know that

$$[\, J_k, \mathbf{J}\,] = i\,\mathbf{J} \times \hat{\mathbf{k}}$$ where $\hat{\mathbf{k}}$ is the unit vector along the k axis, k = 1,2,3

so we get

$$[B,A] = [\, J_k \,, i\,\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}\,] = i\,\theta\,\hat{\mathbf{n}} \bullet [\, J_k, \mathbf{J}] = i\,\theta\,\hat{\mathbf{n}} \bullet (\,\mathbf{J} \times \hat{\mathbf{k}}\,)\, i$$

But the cyclic rule says that

$$\hat{\mathbf{n}} \bullet (\,\mathbf{J} \times \hat{\mathbf{k}}\,) = \hat{\mathbf{k}} \bullet (\,\hat{\mathbf{n}} \times \mathbf{J}\,)$$

so we have so far that:

$$C_1 = [B,A] = i\,\theta\;\hat{\mathbf{k}} \bullet (\,\hat{\mathbf{n}} \times \mathbf{J}\,)\, i$$

and with some interest we see an $(\,\hat{\mathbf{n}} \times \mathbf{J}\,)$ factor appearing on the right.

Now, let's do the next commutator:

$$C_2 = [\,[B,A]\,, A\,] = [\; i\,\theta\;\hat{\mathbf{k}} \bullet (\,\hat{\mathbf{n}} \times \mathbf{J}\,),\; i\,\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}\,]$$

We really cannot maintain vector notation here so need now to resort to summations, so lets go back to the start:

$$C_0 = B = J_k$$

$$C_1 = [B,A] = [J_k, i\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}\,] = i\theta\, n_i\, [J_k, J_i\,] = i\theta\, n_i\, \varepsilon_{kij}\, J_j\, i = -\theta\, n_i\, \varepsilon_{kij}\, J_j$$

Then the next one is:

$$C_2 = [\,[B,A]\,, A\,] = [\, -\,\theta\, n_i\, \varepsilon_{kij}\, J_j \;,\; i\theta\, n_n\, J_n] = -i\,(\theta)^2\, n_i\, \varepsilon_{kij}\, n_n\, [J_j\,, J_n] = (\theta)^2\, n_i\, \varepsilon_{kij}\, n_n\, \varepsilon_{jns} J_s$$

$$= (\theta)^2\, n_i\, n_n \varepsilon_{kij}\, \varepsilon_{jns}\, J_s = (\theta)^2\, n_i\, n_n\, \varepsilon_{jki}\, \varepsilon_{jns}\, J_s = (\theta)^2\, n_i\, n_n\, J_s\, [\;\delta_{kn}\, \delta_{is} - \delta_{ks}\delta_{in}]$$

$$= (\theta)^2\, n_i\, n_n\, J_s\, \delta_{kn}\, \delta_{is} - (\theta)^2\, n_i\, n_n\, J_s\, \delta_{ks}\delta_{in}$$

$$= (\theta)^2\, n_i\, n_k\, J_i - (\theta)^2\, n_i\, n_i\, J_k = (\theta)^2\, n_i\, n_k\, J_i - (\theta)^2\, J_k$$

$= (\theta)^2 (n_k n_i J_i - J_k)$

Then the next one is

$C_3 = [(\theta)^2 (n_k n_i J_i - J_k) , i\theta n_n J_n] = i (\theta)^3 [(n_k n_i J_i - J_k) , n_n J_n]$

$= i (\theta)^3 \{ n_k n_i n_n i \varepsilon_{ins} J_s - n_n i \varepsilon_{kns} J_s \} = (\theta)^3 n_n \varepsilon_{kns} J_s = (\theta)^3 n_i \varepsilon_{kij} J_j = -(\theta)^2 C_1$

Here the first term vanishes because ε is contracted on two indices with a symmetric tensor n n.

Let's do just one more to get the pattern; // $n_i n_n \varepsilon_{kij} \varepsilon_{jns} J_s$

$C_4 = [(\theta)^3 n_n \varepsilon_{kns} J_s, i\theta n_r J_r] = i (\theta)^4 n_n \varepsilon_{kns} n_r i \varepsilon_{srt} J_t = - (\theta)^4 n_i n_n \varepsilon_{kij} \varepsilon_{jns} J_s$

$= - (\theta)^2 C_2$

Here is what we have found so far:

$C_0 = J_k$ $\qquad$ $C_1 = -\theta n_i \varepsilon_{kij} J_j$

$C_2 = (\theta)^2 (n_k n_i J_i - J_k)$ $\qquad$ $C_3 = -(\theta)^2 C_1 = \theta^3 n_i \varepsilon_{kij} J_j$

$C_4 = - (\theta)^2 C_2 = - \theta^4 (n_k n_i J_i - J_k)$

Noted added 2.14.17: The conjecture is then that for n = 2,4,6...

$C_n = (-1)^{n/2-1} \theta^n (n_k n_i J_i - J_k) = - (-1)^{n/2} \theta^n (n_k n_i J_i - J_k)$

And for n = 1,3,,5...

$C_n = (-1)^{(n+1)/2} \theta^n n_i \varepsilon_{kij} J_j = - (-1)^{(n-1)/2} \theta^n n_i \varepsilon_{kij} J_j$

Then

$e^{-A} B e^{A} = B + [B,A]/1! + [[B,A],A]/2! + [[[B,A],A],A]/3! + ...$ (G.3.1)

$= B + C_1/1! + C_2/2! + C/3! + ...$

In our case we have

$\exp(- i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) J_k \exp(+ i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) = J_k + C_1/1! + C_2/2! + C/3! + ...$

$= J_k - \Sigma_{n=1,3,5..} (-1)^{(n-1)/2} \theta^n n_i \varepsilon_{kij} J_j/n! - \Sigma_{n=2,4,6..} (-1)^{n/2} \theta^n (n_k n_i J_i - J_k)/n!$

$= J_k - n_i \varepsilon_{kij} J_j \sum_{n=1,3,5..} (-1)^{(n-1)/2} \theta^n/n! - (n_k n_i J_i - J_k) \sum_{n=2,4,6..} (-1)^{n/2} \theta^n/n!$

$= J_k - n_i \varepsilon_{kij} J_j [\theta - \theta^3/3! + ...] - (n_k n_i J_i - J_k) [-\theta^2/2! + \theta^4/4! + ...]$

$= J_k - n_i \varepsilon_{kij} J_j \sin\theta - (n_k n_i J_i - J_k) (\cos\theta - 1)$

$= J_k - n_i \varepsilon_{kij} J_j \sin\theta - n_k n_i J_i(\cos\theta - 1) + J_k(\cos\theta - 1)$

$= J_k\cos\theta - n_i \varepsilon_{kij} J_j \sin\theta - n_k n_i J_i(\cos\theta - 1)$

$= J_k\cos\theta - n_i \varepsilon_{kij} J_j \sin\theta - n_k (\mathbf{n}\bullet\mathbf{J})(\cos\theta - 1)$

$= J_k\cos\theta - \varepsilon_{kij} J_j n_i \sin\theta - n_k (\mathbf{n}\bullet\mathbf{J})(\cos\theta - 1)$

$= J_k\cos\theta + \varepsilon_{kji} J_j n_i \sin\theta + n_k (\mathbf{n}\bullet\mathbf{J})(1-\cos\theta)$

and this is the correct result comparing to the box below.

So the result is going to be this:

answer $= C_0 + C_1/1! + C_2/2! + C_3/3! + C_4/4! + ...$

$= J_k + (\theta)^2 (n_k n_i J_i - J_k) /2! - (\theta)^4 (n_k n_i J_i - J_k)/4! + ...$

$+ - \theta n_i \varepsilon_{kij} J_j /1! + (\theta)^3 n_i \varepsilon_{kij} J_j/3! + ...$

The second series of the odd terms is this:

$- n_i \varepsilon_{kij} J_j [\theta - \theta^3/3! + ...] = - n_i \varepsilon_{kij} J_j \sin\theta = - \varepsilon_{kij} n_i J_j \sin\theta$

$= - [\mathbf{n} \times \mathbf{J}]_k \sin\theta$

and this agrees exactly with my expected second term in the Q expression above.

What about the first series of even terms?

$J_k + (\theta)^2 (n_k n_i J_i - J_k) /2! - (\theta)^4 n_k n_i J_i - J_k)/4! + ...$

Let's add and subtract ($n_k n_i J_i$) to get

$= n_k n_i J_i + (J_k - n_k n_i J_i) + (\theta)^2 (n_k n_i J_i - J_k) /2! - (\theta)^4 (n_k n_i J_i - J_k)/4! + ...$

$= n_k n_i J_i + (J_k - n_k n_i J_i) [1 - (\theta)^2/2! + \theta^4 + ...]$

$= n_k n_i J_i + (J_k - n_k n_i J_i) \cos\theta$

$= n_k \mathbf{n}\bullet\mathbf{J} + (J_k - n_k \mathbf{n}\bullet\mathbf{J}) \cos\theta$

This makes the final result seem to be this:

$Q_k = \exp(- i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) \; J_k \exp(+ i \theta \hat{\mathbf{n}} \bullet \mathbf{J})$

$= n_k \hat{\mathbf{n}} \bullet\mathbf{J} + (J_k - n_k \hat{\mathbf{n}} \bullet\mathbf{J}) \cos\theta - [\hat{\mathbf{n}} \times \mathbf{J}]_k \sin\theta$

$= J_k \cos\theta - [\hat{\mathbf{n}} \times \mathbf{J}]_k \sin\theta + n_k \hat{\mathbf{n}} \bullet\mathbf{J} (1 - \cos\theta)$

So this agrees exactly with the boxed result found by Method #1 !

Let's test this result with n in the z direction:

$J_k \cos\theta - [\hat{\mathbf{z}} \times \mathbf{J}]_k \sin\theta + \delta_{k3} J_3 (1-\cos\theta)$

and $[\hat{\mathbf{z}} \times \mathbf{J}]_k = \varepsilon_{kij} \hat{\mathbf{z}}_i J_j = \varepsilon_{kij} \delta_{i3} J_j = \varepsilon_{k3j} J_j$ giving the result as

$J_k \cos\theta - \varepsilon_{k3j} J_j \sin\theta + \delta_{k3} J_3 (1-\cos\theta)$

Now, suppose k = 1. We then get

$J_1 \cos\theta - \varepsilon_{13j} J_j \sin\theta = J_1 \cos\theta + J_2 \sin\theta$

and this is the correct result Levitt p 144. Now let's try k = 3 to activate the last term.

$J_k \cos\theta - \varepsilon_{k3j} J_j \sin\theta + \delta_{k3} J_3 (1-\cos\theta)$

$= J_3 \cos\theta - \varepsilon_{33j} J_j \sin\theta + J_3 (1 - \cos\theta) = J_3 \cos\theta + J_3 (1 - \cos\theta) = J_3$

which again is the correct answer! Now we see the need for that extra term.

So I will now write this out in a few different ways: // remember that $\hat{\mathbf{n}}$ is a unit vector

(1) $\mathbf{Q} = \exp(- i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) \; \mathbf{J} \exp(+ i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) = \mathbf{J} \cos\theta + \mathbf{J} \times \hat{\mathbf{n}} \sin\theta + \hat{\mathbf{n}} (\hat{\mathbf{n}} \bullet \mathbf{J}) (1 - \cos\theta)$

(2) $Q_k = \exp(- i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) \; J_k \exp(+ i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) = J_k \cos\theta + [\mathbf{J} \times \hat{\mathbf{n}}]_k \sin\theta + \hat{n}_k (\hat{\mathbf{n}} \bullet \mathbf{J}) (1 - \cos\theta)$

(3) $Q_k = \exp(- i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) \; J_k \exp(+ i \theta \hat{\mathbf{n}} \bullet \mathbf{J}) = J_k \cos\theta + \varepsilon_{kms} J_m n_s \sin\theta + n_k (\hat{\mathbf{n}} \bullet \mathbf{J}) (1 - \cos\theta)$

To get the third version we used these facts:

$[\mathbf{J} \times \hat{\mathbf{n}}]_k = \varepsilon_{kms} J_m n_s$ and $\hat{\mathbf{n}}_k = n_k$ in our notation

Addendum 1: After writing the above, I realized that you can write the above in this way (rightmost entry)

$$\mathbf{Q} = RJR^{-1} = \mathbf{J}\cos\theta + \mathbf{J} \times \hat{\mathbf{n}} \sin\theta + \hat{\mathbf{n}}(\hat{\mathbf{n}} \bullet \mathbf{J})(1 - \cos\theta) = R^{-1}\mathbf{J}$$

where $R = R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J})$ and $R = R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J})$ = 3x3 Cartesian rotation matrix. We therefore can "read off" what this matrix R must be.

Use the component form to say

$$J_k \cos\theta + \varepsilon_{kms} J_m n_s \sin\theta + n_k(\hat{\mathbf{n}} \bullet \mathbf{J})(1 - \cos\theta) = [R^{-1}\mathbf{J}]_k = [R^{-1}]_{k\ell} J_\ell$$

Rewrite the LHS like so:

$$[\cos\theta\, \delta_{k\ell} + \varepsilon_{k\ell s} n_s \sin\theta + n_k n_\ell (1 - \cos\theta)]\, J_\ell$$

Therefore we have

$$[R_{\hat{n}}(-\theta)]_{k\ell} = \cos\theta\, \delta_{k\ell} + \varepsilon_{k\ell s} n_s \sin\theta + n_k n_\ell (1 - \cos\theta)$$

or

$$[R_{\hat{n}}(\theta)]_{k\ell} = \cos\theta\, \delta_{k\ell} + \varepsilon_{ks\ell} n_s \sin\theta + n_k n_\ell (1 - \cos\theta)$$

I verified that this gives the right matrix for $\hat{n} = \hat{x}$.

Addendum 2. Proof of theorem: $[R_{\hat{n}}(\theta)]_{ij} = (1/2)\,\mathrm{tr}\,[\sigma_i R_{\hat{n}}(\theta)\sigma_j R_{\hat{n}}^{\dagger}(\theta)]$

From work above we know that , since $\boldsymbol{\sigma}$ is a vector operator,

$$R_{\hat{n}}(\theta)\sigma_j R_{\hat{n}}^{\dagger}(\theta) = R\sigma_j R^{-1} = [R^{-1}]_{j\ell}\,\sigma_\ell$$

Therefore:

$$(1/2)\,\mathrm{tr}\,[\sigma_i R_{\hat{n}}(\theta)\sigma_j R_{\hat{n}}^{\dagger}(\theta)] = (1/2)\,\mathrm{tr}\,[\sigma_i [R^{-1}]_{j\ell}\,\sigma_\ell] = [R^{-1}]_{j\ell}\,\{(1/2)\,\mathrm{tr}\,[\sigma_i\sigma_\ell]\}$$

$$= [R^{-1}]_{j\ell}\,\delta_{i\ell} = [R^{-1}]_{ji} = [R^{\dagger}]_{ji} = R_{ij} \qquad \text{QED.}$$

So restate the theorem in general terms: // this appeared in my 547 notes.

$$R_{ij} = (1/2)\,\mathrm{tr}\,[\,\sigma_i R \sigma_j R^{\dagger}]$$