**This is the Title** PhL 3.26.05

Note that page numbering is turned on in this template.

$(J_i)^n = J_i$ odd powers $= -i\,\varepsilon_{ijk}$

$(J_i)^n = (J_i)^2$ even powers $= \delta_{jk}(1 - \delta_{ij})$

$(J_i)^{n+2} = (J_i)^n$ all powers

$(K_i{}^n) = (-1)^{(n-1)/2}\ K_i$ odd powers $= (-1)^{(n-1)/2}\ i\,(g_i{}^\alpha g_{0\beta} - g_0{}^\alpha g_{i\beta})$

$(K_i{}^n) = (-1)^{n/2+1}\ (K_i)^2$ even powers $= (-1)^{n/2+1}\ (\delta_i{}^\alpha g_{i\beta} - \delta_0{}^\alpha g_{0\beta})$

$(K_i{}^{n+2}) = -(K_i{}^n)$ all powers!

$\exp(-i[\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}]) = 1 - i[\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}] + (-i)^2/2\,[\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}]^2 + \ldots$

$[\mathbf{r}\bullet\mathbf{J} + \mathbf{b}\bullet\mathbf{K}]^2 = (r_iJ_i + b_iK_i)^2 = (r_iJ_i + b_iK_i)(r_aJ_a + b_aK_a)$

$= r_ir_aJ_iJ_a + b_ir_aK_iJ_a + r_ib_aJ_iK_a + b_ib_aK_iK_a$ ????

**r•J + b• K =**

**Note: On 11.19.16 I found a sign error in the second term of my final result below. Everything is now corrected. I hope I did not use the wrong result in any of my docs.**

I searched all my PDFs for exp(- and did not find anything.

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Calculation of $\exp(-i\theta\mathbf{n}\bullet\mathbf{J})$ where **n** is a unit vector.

1. Powers of **n•J**

$(n\bullet J)^2{}_{ab} = n_iJ_in_jJ_j = n_in_j(-i\varepsilon_{iac})(-i\varepsilon_{jcb}) = -n_in_j\varepsilon_{cia}\varepsilon_{cbj} = n_in_j\varepsilon_{cia}\varepsilon_{cjb}$

$= n_in_j(\delta_{ij}\delta_{ab} - \delta_{ib}\delta_{aj}) = (\delta_{ab} - n_an_b) \equiv T_{ab}$ $\Rightarrow (n\bullet J)^2 = T$

Notice that ( on this line I use $n_cn_c = 1$, so I am assuming **n** is a unit vector)

$T_{ac}T_{cb} = (T^2)_{ab} = (\delta_{ac} - n_an_c)(\delta_{cb} - n_cn_b) = \delta_{ab} - n_an_b - n_an_b + n_an_b = (\delta_{ab} - n_an_b) = T_{ab}$

which says $T^2 = T$. Now, moving right along

$(n\bullet J)^3{}_{ab} = (n\bullet J)_{ac}\,(n\bullet J)^2{}_{cb} = (n\bullet J)_{ac}\,T_{cb}$ => $(n\bullet J)^3 = (n\bullet J)\,T = T\,(n\bullet J)$

$(n\bullet J)^4 = (n\bullet J)^2\,(n\bullet J)^2 = T\,T = T^2 = T$

$(n\bullet J)^5 = (n\bullet J)T$ **ok**

The conclusion is then that

$(n\bullet J)^n = (n\bullet J)\,T$ for n = 3,5,7,....
$(n\bullet J)^n = T$ for n = 2,4,6...

Now write

$\exp(-i\theta n\bullet J) = \cos(\theta n\bullet J) - i\sin(\theta n\bullet J)$ // how is this justified? Maybe as all series

The first term is this:

$\cos(\theta n\bullet J) = 1 - (\theta n\bullet J)^2/2! + (\theta n\bullet J)^4/4! - ...$

$= 1 - T\,(\theta^2/2! - \theta^4/4! + ...)$

$= 1 + T\,(-\theta^2/2! + \theta^4/4! + ...)$

$= 1 + T\,(1 - \theta^2/2! + \theta^4/4! + ... - 1)$

$= 1 + T(\cos\theta - 1)$

The second term sin is this

$\sin(\theta n\bullet J) = (\theta n\bullet J) - (\theta n\bullet J)^3/3! + (\theta n\bullet J)^5/5! -$

$= (\theta n\bullet J) + (n\bullet J)T\,(-\theta^3/3! + \theta^5/5! - ...)$

$= (\theta n\bullet J) + (n\bullet J)T\,(\theta - \theta^3/3! + \theta^5/5! - ... - \theta)$

$= (\theta n\bullet J) + (n\bullet J)T\,(\sin(\theta) - \theta)$

Thus, our result is

$\exp(-i\theta n\bullet J) = \cos(\theta n\bullet J) - i\sin(\theta n\bullet J) =$

$= 1 + T\,(\cos(\theta) - 1) - i\,\{(\theta n\bullet J) + (n\bullet J)T\,(\sin(\theta) - \theta)\}$

$$= 1 + T\,(\cos(\theta) - 1) \;-i\;(n\bullet J)\{\,\theta + T\,(\sin(\theta) - \theta)\}$$

We can write this out in specific components as

$$\exp(-i\theta n\bullet J)_{ab} = \delta_{ab} + (C_\theta - 1)\,T_{ab} \;- i\,(n\bullet J)_{ac}\,\{\,\theta\,\delta_{cb} + (S_\theta - \theta)\,T_{cb}\}$$

where

$$(n\bullet J)_{ac} = n_i(J_i)_{ac} = n_i(-i\varepsilon_{iac})$$

$$T_{ab} = (\delta_{ab} - n_a n_b)$$

So we can write the whole thing out in fine detail as:

$$\exp(-i\theta n\bullet J)_{ab} = \delta_{ab} + (\,(C_\theta - 1)\,(\delta_{ab} - n_a n_b) \;- i\,n_i(-i\varepsilon_{iac})\,\{\,\theta\,\delta_{cb} + (S_\theta - \theta)\,(\delta_{cb} - n_c n_b)\}$$

$$= \delta_{ab} + \;(C_\theta - 1)\,(\delta_{ab} - n_a n_b) \;- n_i\varepsilon_{iac}\,\{\,\theta\,\delta_{cb} + (S_\theta - \theta)\,(\delta_{cb} - n_c n_b)\}$$

The terms linear in θ seem unusual. These terms are:

$$- n_i\varepsilon_{iac}\,\{\,\theta\,\delta_{cb} - \theta\,(\delta_{cb} - n_c n_b)\} \;=\; -\theta\,n_i\,\varepsilon_{iac}\,[\delta_{cb} - \delta_{cb} + n_c n_b\,]$$

$$= \;-\theta\,n_i\,\varepsilon_{iac}\,n_c n_b = -\theta\,\varepsilon_{iac}\,n_i\,n_c\;n_b \;= 0 \text{ by symmetry!}$$

This leaves us with

$$\exp(-i\theta n\bullet J)_{ab} = \delta_{ab} + \;(C_\theta - 1)\,(\delta_{ab} - n_a n_b) \;- n_i\varepsilon_{iac}\,\{\,S_\theta\,(\delta_{cb} - n_c n_b)\}$$

where we have set the linear θ terms to zero. The very last term also vanishes by the same symmetry argument, so we can simplify further to get

$$= \delta_{ab} + (C_\theta - 1)\,(\delta_{ab} - n_a n_b) \;- in_{i\,(}-i\varepsilon_{iab})\,S_\theta$$

$$= \delta_{ab} + \;(C_\theta - 1)\,(\delta_{ab} - n_a n_b) + S_\theta\,[\,-i(n\bullet J)_{ab}]$$

which says

$$\exp(-i\theta n\bullet J) \;= 1 + \;\;(C_\theta - 1)\,T \;+ S_\theta[\,-i(n\bullet J)]$$

It is now a simple matter to just construct the matrices where$(\delta_{ab} - n_a n_b) \;= T_{ab}$ and $(n\bullet J)_{ac} \;= n_i(-i\varepsilon_{iac})$

$$T = \begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix}$$

which of course is symmetric

and then use $(n\bullet J)_{ac} = n_i(-i\varepsilon_{iac}) = -i\, n_i\varepsilon_{iac} = -i\, n_i\varepsilon_{aci}$ to get

$$(n\bullet J) = i\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \quad \Rightarrow \quad -i(n\bullet J) = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

so our final result becomes

$$\exp(-i\theta n\bullet J) = 1 + (C_\theta - 1)\,T + S_\theta[-i(n\bullet J)]$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (C_\theta - 1)\begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} + S_\theta\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

which is a pretty good closed form I think! I see no benefit to combining all into one matrix.

Now, special cases:

n = (1,0,0)

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (C_\theta - 1)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + S_\theta\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{pmatrix}$$ which is correct for $R_x(\theta)$

For each axis case, the T matrix has a very simple form, and the rightmost matrix is the generator matrix with the i's set to 1.

n = (0,1,0)

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (C_\theta - 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + S_\theta\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{pmatrix}$$ which is correct for $R_y(\theta)$

**Conclusion**: We have computed the general 3x3 rotation matrix in a symmetrical closed form:

$$R_{\mathbf{n}}(\theta) = \exp(-i\theta\,\mathbf{n}\bullet\mathbf{J}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (C_\theta - 1)\begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} + S_\theta \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$