Rotation Group Generator Matrices and their Powers PhL 1.22.08

1. The Generator Matrices. First of all, how do we build these matrices for an arbitrary representation j? The dimension of the matrices is going to be (2j+1), at least we know that much.

We always know that

$\langle m|J_3|m'\rangle = \delta_{mm'}m$ so $J_3 = \text{diag}(j,j-1,j-2...,-j)$ // examples: diag(1/2,-1/2), diag(1,0,-1)

Then we can say [where $J_\pm = J_x \pm iJ_y$ and the inverse $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/(2i)$]

$\langle m|J_\pm|m'\rangle = \sqrt{j(j+1)-m'(m'\pm1)}\,\langle m|m'\pm 1\rangle$ // Tinkham page 145

$= \sqrt{j(j+1)-m'(m'\pm1)}\ \delta_{m,m'\pm1} = \sqrt{j(j+1)-mm'}\,\delta_{m,m'\pm1}$

so these matrices have real numbers on a first off-diagonal. It follows that

$\langle m|J_x|m'\rangle = (1/2)\sqrt{j(j+1)-mm'}\,\delta_{m,m'+1} + (1/2)\sqrt{j(j+1)-mm'}\,\delta_{m,m'-1}$

$= (1/2)\sqrt{j(j+1)-mm'}\,[\,\delta_{m,m'+1} + \delta_{m,m'-1}]$

$\langle m|J_y|m'\rangle = (1/2i)\sqrt{j(j+1)-mm'}\,[\,\delta_{m,m'+1} - \delta_{m,m'-1}]$

So we now have our answers:

Generator matrices for any value of j :

$\langle m|J_z|m'\rangle = \delta_{mm'}m = (J_z)_{mm'}$

$\langle m|J_x|m'\rangle = (1/2)\sqrt{j(j+1)-mm'}\,[\,\delta_{m,m'+1} + \delta_{m,m'-1}] = (J_x)_{mm'}$

$\langle m|J_y|m'\rangle = (1/2i)\sqrt{j(j+1)-mm'}\,[\,\delta_{m,m'+1} - \delta_{m,m'-1}] = (J_y)_{mm'}$

$\langle m|J_\pm|m'\rangle = \sqrt{j(j+1)-mm'}\,\delta_{m,m'\pm1} = (J_\pm)_{mm'}$

2. Powers of the generator matrices. Now we can look at powers of these matrices:

$(J_z)^2_{mm'} = m^2\delta_{m,m'}$ and more generally $(J_z)^N_{mm'} = m^N\delta_{m,m'}$

For *general* values of j, each power of (J_z) is a brand new matrix, and you cannot write $(J_z)^N$ as a linear combination of lower powers. For example:

J_z = diag[3/2, 1/2, -1/2, -3/2]

J_z^N = diag [$(3/2)^N$, $(1/2)^N$, $(-1/2)^N$,$(-3/2)^N$] j = 3/2

J_z^N = diag [2^N, 1, 0, $(-1)^N$,$(-2)^N$] j = 2

However, in the cases of j=1/2 and j=1 we find some simplifications.

First, for j = 1/2 we get

$(J_z)^2_{mm'} = m^2\delta_{m,m'} = (1/4)\,\delta_{mm'}$ so $J_z^2 = (1/4)\,1$

$(J_z)^N = (1/2)^N\,1$ N = even
$(J_z)^{N+1} = (1/2)^N\,J_z$ N = even

In general we can say $J_i = (1/2)\,\sigma_i$ and it turns out in that $\sigma_i^2 = 1$ for i = z and x and y.

Secondly, for j = 1 we get

J_z = diag[1, 0, -1]

J_z^2 = diag[1, 0, 1]

J_z^N = diag[1, 0, 1] N even

$J_z^{N+1} = J_z$ N even so $J_z^3 = J_z$

What about powers of J_x and J_y ? Consider:

$\langle m|J_x|m'\rangle = (1/2)\sqrt{j(j+1)-mm'}\,[\,\delta_{m,m'+1} + \delta_{m,m'-1}]$

Matrix elements vanish unless m,m' differ by 1. But if they so differ, then m*m' = 0 such as 1*0 = 0. Then we find that

$\langle m|J_x|m'\rangle = (1/\sqrt{2}\,)[\,\delta_{m,m'+1} + \delta_{m,m'-1}]$ = $J_x = (1/\sqrt{2}\,)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

If we tell Maple to square this we get

$J_x^2 = (1/2)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and then $J_x^3 = (1/\sqrt{2})\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = J_x$

so we can use $J_x^3 = J_x$ to reduce powers. Similarly,

$\langle m|J_y|m'\rangle = \langle m|J_y|m'\rangle = (1/2i)\sqrt{j(j+1)-mm'}\,[\,\delta_{m,m'+1} - \delta_{m,m'-1}]$

$= (1/\sqrt{2}\,i)\,[\,\delta_{m,m'+1} - \delta_{m,m'-1}]$ $J_y = (1/\sqrt{2}\,i)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

The matrix in the brackets when cubed gives -2 times itself says Maple, so not too surprisingly we find that $J_y^3 = J_y$.

Summary of the special cases of powers of generators: "powers"

j=1/2	$J_i^3 = J_i/4$	for i = x,y or z	and $J_i^2 = (1/4)\,*1$
j=1	$J_i^3 = J_i$	for i = x, y or z	
j > 1	powers of J_z are all different (and not sure what happens for powers of J_x and J_y)		