**Rotation Matrices** PhL 1.23.08

**1. Two dimensions as in SU(2):**

generators: $J_i = (1/2)\sigma_i$ rotations: $R_{\hat{n}}(\theta) = \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J})$

$$J_x = (1/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_y = (1/2)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad J_z = (1/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_x = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad R_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad R_z = \begin{pmatrix} \exp(-i\frac{\theta}{2}) & 0 \\ 0 & \exp(+i\frac{\theta}{2}) \end{pmatrix}$$

$R_{\hat{n}}(\theta) = \exp(-i\theta\hat{\mathbf{n}}\bullet\mathbf{J}) = \exp(-i\theta/2\hat{\mathbf{n}}\bullet\boldsymbol{\sigma}) = \cos(\theta/2) - i\,\hat{\mathbf{n}}\bullet\boldsymbol{\sigma}\sin(\theta/2)$ since $(\hat{\mathbf{n}}\bullet\boldsymbol{\sigma})^2 = 1$

**2. Three dimensions Cartesian as in SO(3):**

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(J_i)_{jk} = -i\,\varepsilon_{ijk}$. $[J_i, V_j] = i\varepsilon_{ijk}V_k$ $R\ \mathbf{V}\ R^{-1} = R^{-1}\mathbf{V}$

Special case of the above: $R\ \mathbf{J}\ R^{-1} = R^{-1}\mathbf{J}$ for any $\mathbf{J}$ representation

$[R_{\hat{n}}(\theta)]_{k\ell} = \cos\theta\,\delta_{k\ell} + \varepsilon_{ks\ell}\,n_s\,\sin\theta + n_k\,n_\ell\,(1 - \cos\theta)$

$R_{ij} = (1/2)\,\mathrm{tr}\,[\,\sigma_i R \sigma_j R^\dagger]$

$R_y(\beta) = U\,d^{(1)}(\beta)\,U^{-1}$

$R_z(\alpha)\,R_y(\beta)\,R_z(\gamma) = U\,D^{(1)}(\alpha,\beta,\gamma)\,U^{-1}$ $\qquad U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}$ $\qquad \langle k|1,m\rangle = U_{km}$

Transcription of the 3D rotation matrices:

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here is a 2D $R_{\mathbf{z}}(\theta)$ matrix

$$R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$