Spherical harmonics and Clebsch-Gordon

This is just a collection of oddball facts, will add more some day.

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1. Spherical harmonic conventions.

(1) Jackson $Y_{\ell m}$. Let's start with green Jackson page 64. His $Y_{\ell m}$ are given in 3.53 in terms of the P_ℓ^m which in turn he defines in 3.49 and 3.50. His 3.49 P_ℓ^m agrees with AS p 334, I doubt there are multiple conventions for the associated Legendre functions. The scale of the $Y_{\ell m}$ is selected so this will be true:

$$\int d\Omega\, Y_{\ell m}(\theta,\varphi)^*\, Y_{\ell' m'}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

as he shows in 3.55. So the above is a good test to make sure people have scaled their $Y_{\ell m}$ properly. This also fits with our desire to say

$$Y_{\ell m}(\theta,\varphi) = \langle \theta\varphi | \ell m \rangle$$

$$1 = \int d\Omega\, |\theta\varphi\rangle\langle\theta\varphi| \qquad\qquad 1 = \Sigma_{\ell m}\, |\ell m\rangle\langle \ell m|$$

$$\langle \theta'\varphi' | \theta\varphi\rangle = \delta(\cos\theta' - \cos\theta)\delta(\varphi'-\varphi) \qquad\qquad \langle \ell' m' | \ell m\rangle = \delta_{\ell'\ell}\,\delta_{m'm}$$

Jackson does not do CB since he is doing E&M, so we want to hand-off now to Messiah.

(2) Messiah $Y_{\ell m}$ .Messiah Vol 1 Appendix B page 494 B.93 gives his definition of the $Y_{\ell m}$. We compare this to Jackson page 3.53, and sure enough we have a difference. But first, let's make sure they both agree on the P_ℓ^m.

Jackson P_ℓ^m : 3.49 page 64, valid for $m \geq 0$ // shows $(-1)^m$

Jackson says his form follows MO and Condon Shortley

Messiah P_ℓ^m : B.72 page 493 // does not have this $(-1)^m$ factor

Thus we can say:

P_ℓ^m(Jackson) $= (-1)^m P_\ell^m$(Messiah)

Now let's look at Messiah's definition of the $Y_{\ell m}$ on page 495 B.93. His formula says

$Y_{\ell m}$(Messiah) = [...] $^{1/2}$ $(-1)^m P_\ell^m$(Messiah) $e^{im\varphi}$

$=$ [...] $^{1/2}$ P_ℓ^m(Jackson) $e^{im\varphi}$ $= Y_{\ell m}$(Jackson)

so these two authors agree on their definitions of $Y_{\ell m}$. Normalization same B.87.

Warning: There are other normalizations in use says wiki: Some people do this,

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m\, Y_{\ell'}^{m'*}\, d\Omega = \delta_{\ell\ell'}\,\delta_{mm'}\,.$$

and some people do this:

The magnetics community, in contrast, uses Schmidt semi-normalized harmonics

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

which have the normalization

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m\, Y_{\ell'}^{m'*}\, d\Omega = \frac{4\pi}{(2\ell+1)}\delta_{\ell\ell'}\,\delta_{mm'}\,.$$

In quantum mechanics this normalization is often used, too, and is there named Racah's normalization after Giulio Racah.

2. Clebsch-Gordon Coefficients.

(3) Messiah CG discussion. This is Vol 2 Appendix C page 1054. Right off the bat he uses this symbol

<j1 j2 m1 m2| J M>

which he refers to as C-G coefficients, and he says Condon-Shortley used exactly this symbol. He then adopts the same "phase convention" as Condon-Shortley. This amounts to saying first the implied states like |j1 m1> and |J M> have the usual behavior on application of $J_\pm$, AND this coefficient is positive:

<j1 j2 j1 J-j_1 | J J> > 0 (and real).

He then at once relates this thing to the Wigner 3-J symbol on page 1056 top. He then lists off the nice symmetries of the 3j symbol, which makes it a better animal that the C-S symbol.

Messiah does not then go on to state the case I am right now interested in which is $\ell \otimes 1$, so let's continue our voyage. At least we have related our bra-ket form to the Wigner 3j symbols.

(4) Condon and Shortley. Once again I want to zero in on the $Y_{\ell m}$. This is a 1935 early book which uses this notation:

$\Phi(m) = 1/\sqrt{2\pi}\ e^{im\varphi}$

$\Theta(\ell m)$ = (17) on page 52 = $\sqrt{2\pi}$ [...]$^{1/2}$ P_ℓ^m(Jackson)

Thus we find that

$\Theta(\ell m)\ \Phi(m)$ = [...]$^{1/2}$ P_ℓ^m(Jackson) $e^{im\varphi}$ = $Y_{\ell m}$ of either Jackson or Messiah

but C-S don't use the notation $Y_{\ell m}$. We then come to page 75 where we see the C-G symbol as shown above but writes it out more:

<j1 j2 m1 m2| j1 j2 J M>

CS then quote a huge closed form formula for this thing on page 75 which in includes a sum. CS go on to give their famous tables and table 2^3 is of interest to me right now which has $j_2 = 1$, and this particular table they quote directly from Wigner's book Gruppentheorie p 206. [The Wigner reference appears on page 11 after a search. This thing has be reprinted in English at least once.

E. Wigner, “Gruppentheorie und ihre Anwendungen auf die Quantentheorie der Atomspektren”, (Braunschweig: Vieweg; 1931).
// recall Reimers going to Braunschweig !

Wigner, Eugene Paul. *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra, Expanded and Improved ed.* New York: Academic Press, 1959. A little dense

So the table is showing the <j1 j2 m1 m2| j1 j2 J M>object, not the 3-j symbol.

(5) Tinkham gives the same fancy Wigner summation formula on page 121, but the reader is not very clear what the symbols A and a are as defined by Tinkham, you have to dig around a bit. On page 124 Tinkham replicates the same table just noted above from C-S for $j_2 = 1$.

Clebsch-Gordan coefficients are related to 3-jm symbols which have more convenient symmetry relations.

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 - j_2 + m_3} \sqrt{2j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}.$$

Relation to Wigner D-matrices

$$\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\gamma D^J_{MK}(\alpha,\beta,\gamma)^* D^{j_1}_{m_1 k_1}(\alpha,\beta,\gamma) D^{j_2}_{m_2 k_2}(\alpha,\beta,\gamma) = \frac{8\pi^2}{2J+1} \langle j_1 m_1 j_2 m_2 | JM \rangle \langle j_1 k_1 j_2 k_2 | JK \rangle.$$

Relation with spherical harmonic functions [edit]

The D-matrix elements with second index equal to zero, are proportional to spherical harmonics, normalized to unity and with Condon and Shortley phase convention,

$$D^{\ell}_{m0}(\alpha,\beta,\gamma)^* = \sqrt{\frac{4\pi}{2\ell+1}} Y^m_\ell(\beta,\alpha).$$

3. Derivation of Messiah Vol 1 page 495 (B.90)

I need this result for the next section.

I suppose there might be a recursion relation among the adjacent P_ℓ^m . Messiah B.78 gives the result

$(2\ell+1)z\, P_\ell^m(z) = (\ell+1-m)\, P_{\ell+1}^m(z) + (\ell+m)\, P_{\ell-1}^m(z)$

or

$(2\ell+1)z\, P_\ell^m(z)\, e^{im\varphi} = (\ell+1-m)\, P_{\ell+1}^m(z)\, e^{im\varphi} + (\ell+m)\, P_{\ell-1}^m(z)\, e^{im\varphi}$

which, since it has constant m, is valid in either Jackson or Messiah phase conventions. To convert this recursion thing to $Y_{\ell m}$ we have to add the ugly factors,

$Y_{\ell m}(\theta,\varphi) = [\,(2\ell+1)(\ell-m)!/(4\pi\,(\ell+m)!]^{1/2}\, P_\ell^m(z)e^{im\varphi}$ // Jackson p 65 3.53

So do the required edits

$Y_{\ell+1,m}(\theta,\varphi) = [\,(2\ell+3)(\ell+1-m)!/(4\pi\,(\ell+1+m)!]^{1/2}\, P_{\ell+1}^m(z)e^{im\varphi}$

$Y_{\ell-1,m}(\theta,\varphi) = [\,(2\ell-1)(\ell-1-m)!/(4\pi\,(\ell-1+m)!]^{1/2}\, P_{\ell-1}^m(z)e^{im\varphi}$

Then we get

$(2\ell+1)z[\,(2\ell+1)(\ell-m)!/(4\pi\,(\ell+m)!]^{-1/2}\, Y_{\ell m}(\theta,\varphi)$

$= (\ell+1-m)\,[\,(2\ell+3)(\ell+1-m)!/(4\pi\,(\ell+1+m)!]^{-1/2}\, Y_{\ell+1,m}(\theta,\varphi)$

$+ (\ell+m)\,[\,(2\ell-1)(\ell-1-m)!/(4\pi\,(\ell-1+m)!]^{-1/2}\, Y_{\ell-1,m}(\theta,\varphi)$

Now move the radical to the right side

$(2\ell+1)z\, Y_{\ell m}(\theta,\varphi) =$

$(\ell+1-m)\,[\,(2\ell+3)(\ell+1-m)!/(4\pi\,(\ell+1+m)!]^{-1/2}\,[\,(2\ell+1)(\ell-m)!/(4\pi\,(\ell+m)!]^{1/2}\, Y_{\ell+1,m}(\theta,\varphi)$

$+ (\ell+m)\,[\,(2\ell-1)(\ell-1-m)!/(4\pi\,(\ell-1+m)!]^{-1/2}[\,(2\ell+1)(\ell-m)!/(4\pi\,(\ell+m)!]^{1/2}\, Y_{\ell-1,m}(\theta,\varphi)$

$$= (\ell+1-m)\ [\ \{(2\ell+1)(\ell-m)!\ 4\pi\ (\ell+1+m)!\}\ /\ \{4\pi\ (\ell+m)!\ (2\ell+3)(\ell+1-m)!\}\ ]^{1/2}\ Y_{\ell+1,m}(\theta,\varphi)$$

$$+ \quad (\ell+m)\ [\ \{(2\ell+1)(\ell-m)!\ 4\pi\ (\ell-1+m)!\}\ /\ \{4\pi\ (\ell+m)!\ (2\ell-1)(\ell-1-m)!\}\ ]^{1/2}\ Y_{\ell-1,m}(\theta,\varphi)$$

$$= (\ell+1-m)\ [\ \{(2\ell+1)\ (\ell+1+m)\ \}\ /\ \{\ (2\ell+3)(\ell+1-m)\}\ ]^{1/2}\ Y_{\ell+1,m}(\theta,\varphi)$$

$$+ \quad (\ell+m)\ [\ \{(2\ell+1)(\ell-m)\ \}\ /\ \{\ (\ell+m)\ (2\ell-1)\ \}\ ]^{1/2}\ Y_{\ell-1,m}(\theta,\varphi)$$

$$= \ [\ \{(2\ell+1)\ (\ell+1+m)\ (\ell+1-m)\ \}\ /\ \{\ (2\ell+3)\}\ ]^{1/2}\ Y_{\ell+1,m}(\theta,\varphi)$$

$$+ \quad [\ \{(2\ell+1)\ (\ell+m)\ (\ell-m)\ \}\ /\ \{\ (2\ell-1)\ \}\ ]^{1/2}\ Y_{\ell-1,m}(\theta,\varphi)$$

Now divide both sides by $(2\ell+1)$

$$z\ Y_{\ell m}(\theta,\varphi) \ =$$

$$= \ [\ \{\ (\ell+1+m)(\ell+1-m)\ \}\ /\ \{\ (2\ell+1)(2\ell+3)\}\ ]^{1/2}\ Y_{\ell+1,m}(\theta,\varphi)$$

$$+ \quad [\ \{\ (\ell+m)(\ell-m)\ \}\ /\ \{\ (2\ell+1)(2\ell-1)\ \}\ ]^{1/2}\ Y_{\ell-1,m}(\theta,\varphi)$$

and this agrees finally with Messiah B.90 page 495.

4. The General Product of Two Spherical Harmonics

Using the same CG coefficients as discussed above, write:

$$[|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle] \ = |\ j_1 m_1; j_2 m_2\rangle = \quad \Sigma_{jm}\ |jm\rangle\langle jm\ |\ j_1 m_1; j_2 m_2\rangle$$

If we close with $\langle\theta\varphi|$, I think what happens is this (this is complete BS, but keep it as a lesson)

$$[\langle\theta\varphi| \otimes \langle\theta\varphi|\ ]\ [|j_1\ m_1\rangle \otimes |j_2\ m_2\rangle] \ = \langle\theta\varphi|\ \ [\Sigma_{jm}\ |jm\rangle\langle jm\ |\ j_1 m_1; j_2 m_2\rangle\ ] \qquad \text{// wrong}$$

$$\langle\theta\varphi|j_1\ m_1\rangle\langle\theta\varphi|j_2\ m_2\rangle = \ \Sigma_{jm}\langle\theta\varphi|jm\rangle\langle jm\ |\ j_1 m_1; j_2 m_2\rangle$$

$$Y_{jm}(\theta,\varphi)\ Y_{j'm'}(\theta,\varphi) = \Sigma_{j"m"}\ Y_{j"m"}(\theta,\varphi)\langle j"m"\ |\ jm; j'm'\rangle \qquad \text{// wrong}$$

So this is a sort of product of spherical harmonics rule. Tinkham has something like this on page 117 as (5-42b) but his notation is hard to follow.

Unfortunately, this formula is wrong, and my arm-waving derivation is also wrong. The correct formula from the web involves *two* C-G coefficients like this:

$$Y_{\ell_1 m_1}(\theta,\phi)Y_{\ell_2 m_2}(\theta,\phi) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell 0|\ell_1\ell_2 00\rangle \langle \ell(m_1+m_2)|\ell_1\ell_2 m_1 m_2\rangle Y_{\ell(m_1+m_2)}(\theta,\phi)$$

and applying it in my case (a little more general),

$$Y_{1m}(\theta,\phi)Y_{\ell_i m_i}(\theta,\phi) \;=\; \sum_{\ell=|\ell_i-1|}^{\ell_i+1} \sqrt{\frac{3(2\ell_i+1)}{4\pi(2\ell+1)}} \langle \ell 0|\ell_i 100\rangle \langle \ell(m+m_i)|\ell_i 1 m_i m\rangle Y_{\ell(m_i+m)}(\theta,\phi)$$

In the above, if we set m=0, we get these coefficients: $\langle \ell 0|\ell_i 100\rangle\langle \ell m_i|\ \ell_i 1 m_i 0\rangle$. I want to show that the term with $\ell = \ell_i$ vanishes. It would have these factors: $\langle\ \ell_i\ 0|\ell_i 100\rangle\langle\ \ell_i\ m_i|\ \ell_i 1 m_i 0\rangle$. So in my normal notation I would have

$Y_{10}(\theta,\varphi)\ Y_{\ell m}(\theta,\varphi)$, term proportional to $Y_{\ell m}(\theta,\varphi)$: $\quad \alpha \quad \langle \ell\ 0|\ell 100\rangle \quad * \quad \langle \ell\ m_i|\ \ell 1 m_i 0\rangle$
$\langle jm \mid j_1 j_2 m_1 m_2\rangle$

If we look at the CG table on page 124, we are in the middle row and we have $m_2 = 0$ so the middle column as well. But our first factor has m=0 so the first coefficient above is 0. Finally, I am getting a result that is consistent with the recursion relation B.90 derived above. I was stumped by this seeming contradiction for a while.

5. An interesting matrix element from Messiah page 697.

Consider: $\quad \langle \ell' m' \mid \cos\theta \mid \ell m\rangle$

Here we have to realize that θ a coordinate space *operator*, the way **r** would be if we had it inside some matrix element. So let's make it a cap to remind us of that fact.

$$\langle \ell' m' \mid \cos\Theta \mid \ell m\rangle = \langle \ell' m' \mid \cos\Theta \int d\Omega\ |\theta\varphi\rangle\langle\theta\varphi|\ell m\rangle$$

$$= \int d\Omega \langle \ell' m' \mid \cos\Theta\ |\theta\varphi\rangle\langle\theta\varphi|\ell m\rangle \;=\; \int d\Omega \langle \ell' m' \mid \cos\theta\ |\theta\varphi\rangle\langle\theta\varphi|\ell m\rangle$$

$$= \int d\Omega \cos\theta \langle \ell' m'\ |\theta\varphi\rangle\langle\theta\varphi|\ell m\rangle$$

$$= \int d\Omega \cos\theta\ Y_{\ell' m'}(\theta,\varphi)^*\ Y_{\ell m}(\theta,\varphi)$$

We could use the full triple-Y formula to evaluate this, but instead let's use B.90 which says

$$z\ Y_{\ell m}(\theta,\varphi) \;=\; [\ \{\ (\ell+1+m)(\ell+1-m)\ \}\ /\ \{\ (2\ell+1)(2\ell+3)\}\ ]^{1/2}\ Y_{\ell+1,m}(\theta,\varphi)$$

$$+\ \ [\ \{\ (\ell+m)(\ell-m)\ \}\ /\ \{\ (2\ell+1)(2\ell-1)\ \}\ ]^{1/2}\ Y_{\ell-1,m}(\theta,\varphi)$$

Then we just have Y orthogonality integrals left and we can say

$$\langle \ell' m' | \cos\Theta | \ell m \rangle = [\ \{ (\ell+1+m)(\ell+1-m) \} / \{ (2\ell+1)(2\ell+3) \}\ ]^{1/2}\ \delta_{\ell+1,\,\ell'}\delta_{m,m'}$$

$$+\ [\ \{ (\ell+m)(\ell-m)\ \} / \{\ (2\ell+1)(2\ell-1)\ \}\ ]^{1/2}\ \delta_{\ell-1,\,\ell'}\delta_{m,m'}$$

To pick off the second term, let's choose $\ell' = \ell - 1$, so then we have

$$\langle \ell\text{-}1\ m | \cos\Theta | \ell\ m \rangle = [\ \{ (\ell+m)(\ell-m)\ \} / \{\ (2\ell+1)(2\ell-1)\ \}\ ]^{1/2}$$

$$= [\ (\ell^2 - m^2) / (4\ell^2-1)\ ]^{1/2} \qquad \textbf{// agrees with Messiah (33) p 697}$$

and since it is real, and position operator Θ is Hermitian (real eigenvalues), we can flip the thing around to get the form that Messiah shows.

6. Collection of formulas where we know the normalization used!

Wolfram has a lot on this subject, but you have to poke around a lot.

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\,\overline{Y}_{l'}^{m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$
$$= \int_0^{2\pi}\int_{-1}^{1} Y_l^m(\theta,\phi)\,\overline{Y}_{l'}^{m'}(\theta,\phi)\,d(\cos\theta)\,d\phi$$
$$= \delta_{m m'}\,\delta_{l l'}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_{l_1}^{m_1}(\theta,\phi)\,Y_{l_2}^{m_2}(\theta,\phi)\,Y_{l_3}^{m_3}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Here is a pretty good site http://quantummechanics.ucsd.edu/ph130a/130_notes/node210.html

7. These are all from http://mathworld.wolfram.com/SphericalHarmonic.html

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\,\overline{Y}_{l'}^{m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$
$$= \int_0^{2\pi}\int_{-1}^{1} Y_l^m(\theta,\phi)\,\overline{Y}_{l'}^{m'}(\theta,\phi)\,d(\cos\theta)\,d\phi$$
$$= \delta_{m m'}\,\delta_{l l'}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_{l_1}^{m_1}(\theta,\phi)\,Y_{l_2}^{m_2}(\theta,\phi)\,Y_{l_3}^{m_3}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

$$Y_{n_1}^{m_1}(\vartheta,\varphi)\,Y_{n_2}^{m_2}(\vartheta,\varphi) == \sqrt{\frac{(2\,n_1+1)\,(2\,n_2+1)}{4\,\pi}}$$

$$\sum_{k=\max\left(\left|n_1-n_2\right|,\left|m_1+m_2\right|\right)}^{n_1+n_2} \frac{1}{\sqrt{2\,k+1}}\,Y_k^{m_1+m_2}(\vartheta,\varphi)\,\langle n_1\,n_2\,0\,0 \mid n_1\,n_2\,k\,0\rangle\,\langle n_1\,n_2\,m_1\,m_2 \mid n_1\,n_2\,k\,m_1+m_2\rangle$$

This last seems to agree with the previous version of this above.

Note: I have derived all formulas above, see "addition of angular momenta" doc. 2/17/09